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Missing and Mis-in Concept Images of Parallelograms: the Case of Tal

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Abstract
In this article, we describe a case study that was conducted within a study aiming to diagnose grade 5 students' concept images of parallelograms. The theoretical framework that we adopted for this study was that of concept definition–concept image as reported by Tall and Vinner (*Educational Studies in Mathematics* 12:151–169, 1981), a theory that is widely used in mathematics education. The occurrences during our interviews with one of the first students that we interviewed led us to identify a need to extend this theory. This manuscript suggests to add two new constructs to the theory of concept definition–concept image: *missing concept images* and *mis-in concept images*. *Missing concept image* defines a situation in which an example of a concept is erroneously categorized as a non-example of the concept. *Mis-in concept image* is the somewhat complementary case, in which a non-example of the concept is *mis(takenly) in(cluded) in* the set of examples of the concept and consequently this non-example is erroneously identified as an example of the concept. In this manuscript, we introduce these two constructs. We also describe two possible sources of students' decisions regarding their ways of sorting figures into examples and non-examples of parallelograms that were detected during the interviews. To the best of our knowledge, these sources were not reported in the related literature.

Keywords Concept definition · Concept image · Missing concept images · Mis-in concept images · Parallelograms

Introduction

The notions concept definition and concept image, coined by a number of researchers in the early 1980s (Vinner & Hershkowitz, 1980, 1983; Tall & Vinner, 1981), are

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intensively used in the community of mathematics education (e.g. Alcock & Simpson, 2011; Bingolbali & Monaghan, 2008; Nardi, 2006; Tall, 2008; Vinner, 2015; Vinner & Dreyfus, 1989; Marmur, Yan, & Zaskis, 2020; Ulusoy, 2019). In conjunction with concept definition, Tall and Vinner (1981) distinguished between formal concept definition and personal concept definition. In the context of concept image, they defined the notion evoked concept image.

In this article, we describe a case study with one grade 5 student that was conducted as part of a larger study aiming to diagnose typical concept images of parallelograms among grade 5 students. The theoretical framework that we adopted for this study was that of concept definition–concept image. However, the occurrences during the interviews led us to suggest to implement two new constructs to the theory of concept definition–concept image: *missing concept images* and *mis-in concept images*. The main aim of this paper is to introduce these constructs. Another aim is to describe two possible sources of students' decisions regarding their ways of sorting figures into examples and non-examples of parallelograms that were revealed during these interviews.

Theoretical Background

In this section, we first provide a short description of the concept definition–concept image theory and the extension that we suggest to this theory. Then, we describe four constructs that we use in the course of the study design and the interview analysis, namely, intuitive examples, unintuitive examples, intuitive non-examples, and unintuitive non-examples.

Concept Definition and Concept Image

In their classical article on concept image and concept definition, Tall and Vinner (1981, p. 152) defined the notions concept image, evoked concept image, concept definition, formal concept definition, and personal concept definition in the following manner:

"We shall use the term concept image to describe the total cognitive structure that is associated with the concept, which includes all the mental pictures and associated properties and processes. It is built up over the years through experiences of all kinds, changing as the individual meets new stimuli and matures."

"We shall call the portion of the concept image which is activated at a particular time the evoked concept image."

Concept definition is "a form of words used to specify that concept."

"A formal concept definition, …[is] a concept definition which is accepted by the mathematical community at large."

"Personal concept definition… is the form of words that the student uses for his own explanation of his (evoked) concept image."

In this paper, we share with the readers our observation that there is a need to add two constructs to the concept image–concept definition theory: *missing concept images* and *mis-in concept images*. The construct *missing concept image* defines a situation in which an example of a concept is erroneously categorized as a non-example of the concept. *Mis-in concept image* is the somewhat complementary case, in which a non-example of the concept is *mis(takenly) in(cluded) in* the set of examples of the concept and consequently this non-example is erroneously identified as an example of the concept. In the following sections, we share with the readers the observations that led us to suggest this extension to the concept image–concept definition theory.

Sorting Assignments: the Mathematics and the Intuitive Dimensions

As stated in the "Introduction" section, a main aim of our larger study was to describe possible sources of students' decisions regarding their ways of sorting figures into examples and non-examples of parallelograms. In accordance with the concept image–concept definition theory, we based the study design and the interview analysis on two dimensions: The formal, mathematics dimension and the intuitive dimension (Tirosh & Tsamir, 2008; Tsamir, Tirosh & Levenson, 2008). The mathematics dimension differentiates between figures that are examples and those that are non-examples of a certain, mathematical concept. The dichotomous distinction between examples and non-examples of a concept, determined by the formal concept definition, is an integral part of the learning and the teaching of mathematics (e.g. Klausmeier, 1992; Goldenberg & Mason, 2008; Ulusoy, 2019). In the present study, because the mathematical concept is parallelogram, the division of the figures that the students are asked to sort is into two sets: parallelograms and non-parallelograms.

The second dimension is "intuitiveness." Intuition, a kind of persistent cognition which is accepted directly and confidently as self-evident, has been shown to play an essential role in mathematical thinking processes (e.g. Fischbein, 1987). Accordingly, intuitive examples and non-examples are examples and non-examples that the tendency is to immediately identify them as such, without a feeling that there is a need to justify the related decisions. Unintuitive examples are examples that are often mistakenly identified as non-examples, and unintuitive non-examples are examples that are often erroneously identified as examples.

Notably, the mathematics and the intuitive dimensions largely differ in their nature. Contrary to the mathematical dimension, in which the distinction between examples and non-examples is dichotomous, the distinctions between intuitive and unintuitive examples (and non-examples) are not determined by a formal definition. They are based on typical students' responses to various tasks that are aimed at identifying examples and non-examples of a certain, mathematical concept.

In their article on sorting examples and non-examples of triangles (Tirosh & Tsamir, 2008; Tsamir, Tirosh & Levenson, 2008), they clarified that intuitive examples of triangles and intuitive examples of non-triangles are figures that the tendency is to correctly identify them as triangles or as non-triangles, respectively. In the same vein, unintuitive examples of triangles and unintuitive examples of non-triangles are triangles and non-triangles that are frequently incorrectly classified as non-triangles and triangles, respectively. Based on these two dimensions, the authors created four groups: intuitive examples of triangles (e.g. an isosceles triangle with a base that is parallel to the bottom of the page or the card in which it is printed) intuitive examples of non-triangles (e.g. a circle), unintuitive examples of triangles (e.g. a scalene triangle), and unintuitive examples of non-triangles (e.g. an open "triangle" — a geometric figure that is similar to an isosceles triangle, missing the critical attribute of being a closed figure).

As stated, the aim of our larger study was to diagnose Grade 5 students' concept images of parallelograms. The body of knowledge related to learners' conceptions of parallelograms includes articles reporting on ways of classifying parallelograms (e.g. Brunheira & da Ponte, 2015; Casa & Gavin, 2009; Clements & Battista, 1992; de Villiers, 2010; Fischbein & Nachlieli, 1998; Fujita & Jones, 2007; Mayberry, 1983; Feza & Webb, 2005; Petty & Jansson, 1987; Zilkova, 2014). However, the focus in these studies was mainly on learners' identification of examples of parallelograms (often, on whether the participants tended to identify specific parallelograms such as rectangles and squares as examples of parallelograms). We gathered that identifying intuitive examples of parallelograms and intuitive examples of non-parallelograms, as well as unintuitive examples of parallelograms and unintuitive examples of non-parallelograms, would significantly enrich the body of knowledge on learners' concept images of parallelograms, and thus contribute information that is needed for making related, instructional decisions.

Method

This paper reports on a case study in which two, consecutive interviews with a fifth grade student (Tal, a pseudonym) uncovered a need to add new constructs to the concept image–concept definition theory. Tal was described by her mathematics teacher as "a student slightly above the middle level of my class in her achievement in mathematics. Tal can fluently and clearly express her views." Tal was the first student that we interviewed in the larger study that exhibited, in her way of sorting the figures that were presented to her into parallelograms and non-parallelograms, both *missing* and *mis-in* concept images. These two constructs, which were added to our conversations when attempting to describe Tal's concept images of parallelograms, created a language that enabled us to more accurately define her concept images and the concept images of the other interviewees, in the larger study.

We conducted two consecutive interviews with Tal in a period of five days between the two sessions. Both of us were present in the meetings with Tal. One conducted the interview while the other wrote down what Tal did and said. The next two sections are devoted to these interviews.

The First Interview with Tal

In the first part of this section, we present the tasks that were designed for our first meeting with Tal and the reasons for employing these, specific tasks. We then describe the occurrences during our first meeting with Tal. In the third (and last) part of this section, we share with the readers our reflections on our meeting with Tal.

The Tasks

The first task that was presented to Tal was as follows: "What is a parallelogram?" By this question, we aimed to ascertain her personal definition of parallelogram and the correspondence between her definition and the formal concept definition of parallelogram. We then presented her with 12 cards, displaying different geometric figures. Tal was asked to sort the cards into two groups: parallelograms and not-parallelograms. After sorting the cards, she was asked to explain her classification.

The collection of the 12 cards that were presented to Tal in the first interview is illustrated is Fig. 1. The choice of the shapes was based on the mathematical and on the intuitiveness dimensions. Our choice of the figures that were included in the related, four groups (intuitive examples of parallelograms, intuitive non-examples of parallelograms, unintuitive examples of parallelograms, unintuitive non-examples of parallelograms) were based on two main sources. The first is our analysis of responses that were reported in studies that included information on learners' ways of sorting geometrical figures into parallelograms (references to these studies are listed in the section: "sorting assignments: the mathematics and the intuitive dimensions"). The second source is studies that implemented the mathematical and intuitiveness dimensions for classifying figures (e.g. Tirosh & Tsamir, 2008; Tsamir, Tirosh & Levenson, 2008).

We included two, intuitive examples of parallelograms (prototypical parallelograms — Card 1 and Card 3), and two examples of intuitive non-parallelograms (isosceles trapezoids — Card 2 and Card 4) in the set of the 12 cards. Two squares and one rectangle represented the unintuitive examples of parallelograms (Cards 5, 6, 7). The choices of the figures for these three groups were based on studies that included reference to sorting such figures. In the fourth group, the unintuitive examples of non-parallelograms, we included five cards. The choice of three cards: kite (Card 8), quadrilaterals with only one pair of parallel sides (Card 11), and hexagons that is visually similar to prototypical parallelograms (Card 12) was based on studies related to sorting given figures to parallelograms. The two other cards in this group are shapes with four "sides" that are similar to prototypical parallelograms but are either not closed (Card 10) or their sides are not connected with vertices (Card 9). Similar figures were identified as unintuitive non-examples in our studies on intuitive examples and non-examples of triangles (Tirosh & Tsamir, 2008; Tsamir, Tirosh & Levenson, 2008).

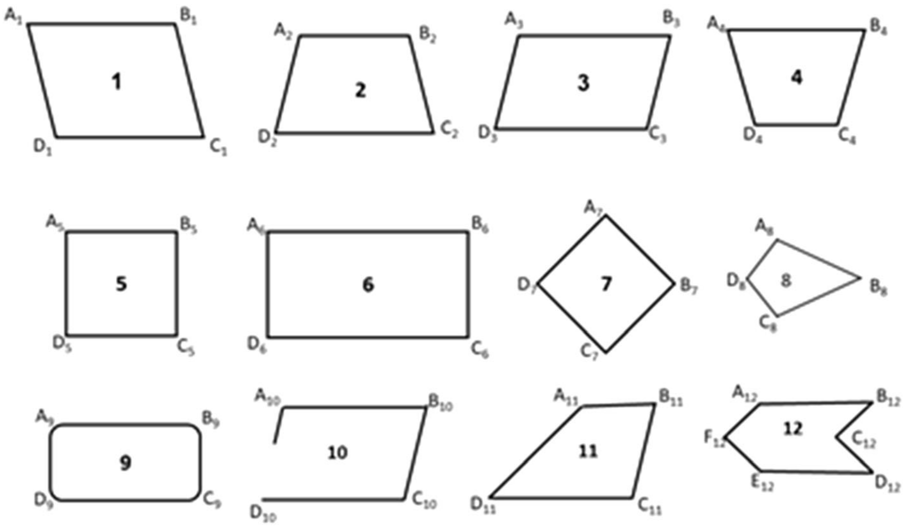

Fig. 1 Parallelograms and non-parallelograms: first interview

Each of these 12 figures was printed on a separate card. The number of each card was written on the top, right corner of each card. The vertexes were not marked with letters (we added, in Fig. 1, indications of the vertexes and wrote the number of the card inside each figure to simplify the presentation of Tal's explanations of her way of sorting the cards). The decision to first introduce a prototypical parallelogram (Card 1) aimed, on the one hand, to start the interview in a welcoming manner. On the other hand, a mis-identification of a prototypical parallelogram would have raised the hypothesis that Tal was unfamiliar with parallelograms. The second, presented card (Card 2) was an isosceles trapezoid. Again, we hypothesized, based on the existing literature, that Tal would recognize that this geometric figure is not a parallelogram. This order of presenting the cards in the first interview was followed for Card 3 and Card 4 (a prototypical parallelogram and an isosceles trapezoid). Following these four cards, unintuitive examples and unintuitive non- examples of parallelograms were presented in the order shown in Fig. 1.

The First Meeting with Tal

At the beginning of our meeting with Tal, we introduced ourselves and clarified that the meeting is about parallelograms. We then asked:

> Interviewer: Tal, do you know what is a parallelogram?
> Tal: Yes. I know. Parallelograms are quadrilaterals with… It's a very big family.
> Interviewer: Is there anything in common to the family of parallelograms?
> Tal: Yes. Four sides… and there is also this matter of parallels… too. The opposite sides should be parallel to each other.

Tal's definition of parallelogram matches the formal concept definition of parallelogram (a parallelogram is often defined as a quadrilateral with two pairs of parallel sides. In some definitions, a clarification that the quadrilateral has to be non-self-intersecting is added).

Then, we shared with Tal, one by one, the 12 cards (Fig. 1) and asked her to sort the cards into two groups: parallelograms and not parallelograms. Tal included seven cards (1, 2, 3, 4, 5, 6, 7) in the set of parallelograms and five cards (8, 9, 10, 11, 12) in the set of non-parallelograms. The following discussion was then held with Tal:

> Interviewer: You included these cards in the set of parallelograms (pointing to the cards in the parallelogram set). Can you please explain…?
> Tal: Sure. (She arranged the cards in the manner illustrated in Fig. 2). In the first row I put the squares and the rectangle. We learnt that they are parallelograms. In the second row I put the two with the sides (pointing to A_1D_1 and B_1C_1) that go in the same direction, Card 1 to the left and Card 3 to the right (showing A_1D_1 and B_1C_1 in Card 1, with her two hands, together, and then moving her hands to Card 3, showing A_3D_3 and B_3C_3 with her two hands). And in the third row those with the sides that go in… (showing with her two hands, together, the sides A_2D_2 and B_2C_2 of Card 2, starting from the longer base and illustrating that her both hands are "going in" to the shorter base, then repeating these exact actions with Card 4).

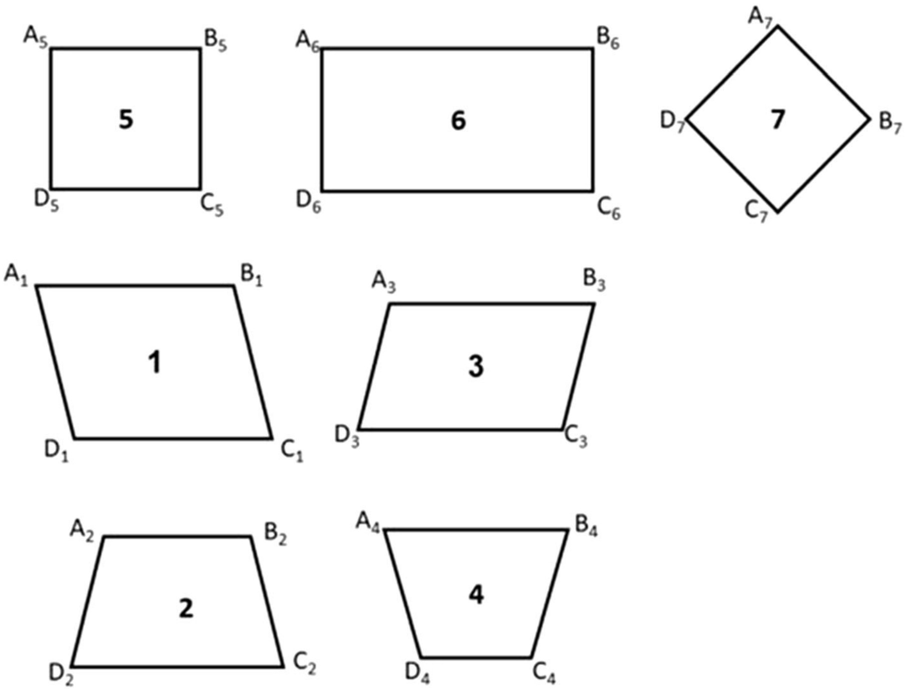

Fig. 2 Tal's arrangement of the cards that she identified as parallelograms: first interview

Interviewer: Let's talk about the non- parallelograms. What about Card 8?
Tal: It's a kite. We learnt that kites are not parallelograms.
Interviewer: But – why?
Tal: Because…because… the parallel sides are not open in the same way. You see, it's like Card 11 (taking Card 11 in her hands) … these (pointing to $A_{11}D_{11}$ and $B_{11}C_{11}$ in Card 11) - they are not parallel – they are not open in the same way. $A_{11}D_{11}$ is more open… And in Card 8, B_8C_8 is more open than A_8D_8.
Interviewer: And Card 9?
Tal: It's easy. Nine does not have corners.
Interviewer: and Card 10?
Tal: It's open (pointing) here.
Interviewer: And Card 12?
Tal: It has six sides.
Interviewer: Thank you Tal. We'll meet again soon.

Our Reflection on the First Meeting with Tal

We should note, first, that Tal correctly included all the intuitive and non-intuitive examples of parallelograms in the set of parallelograms. Yet, Tal's concept image of parallelogram includes non-examples of parallelograms. She *mis(takenly) in(cluded)* two isosceles trapezoids, non-examples of parallelograms, in the set of parallelograms.

In an attempt to identify the criteria by which Tal determined whether a particular shape is a parallelogram, the researchers related to the apparent contradiction between Tal's personal concept definition of parallelogram ("Parallelograms are quadrilaterals … and …the opposite sides should be parallel to each other") and her choice to include the isosceles trapezoids in the set of parallelograms. The phenomenon of expressing two judgements that contradict each other (i.e. compartmentalization) was described and discussed in various publications in mathematics education (e.g. Buchbinder & Zaslavsky, 2013; Tsamir & Bazzini, 2004; Wilson, 1990). Often, this phenomenon is described as resulting from a discrepancy between analytical and perceptual similarity (e.g. Fischbein, 1993). However, based on Tal's way of sorting the isosceles trapezoids and her related explanations, we posed the question: What are parallel lines according to Tal? It seems to us that although Tal's declared personal concept image matches the concept definition of parallelogram, the interpretation that Tal assigned to the expression "The opposite sides should be parallel to each other" differs from the formal meaning of that assertion. We agreed that Tal's concept images of parallel lines could shade light on her decisions regarding her way of classifying the figures into parallelograms and non-parallelograms. The first part of our second meeting with Tal was designed according to this aim.

The Second Interview with Tal

The main aims of our second interview with Tal were to acquire more information about Tal's personal concept definition and concept images of parallel lines, to learn about their possible impact on Tal's decisions regarding parallelograms and to gather more information about Tal's concept images of parallelograms. In the light of Tal's responses during our first meeting with her and our reflections on that meeting, we designed two sets of tasks that were presented to Tal during the second interview. The first set relates to parallel lines and the second to parallelograms.

In this section, we first share with the readers the parallel lines' tasks and Tal's reactions to these tasks, adding our observations related to Tal's performance in this part of the interview. Then we present the figures that were presented to Tal during the part of the interview that was associated with parallelograms, Tal's responses to this task and our reflection on this, second part of the interview.

Parallel Lines: the Tasks, Tal's Responses to the Tasks, and Some Reflections

The first task that we presented to Tal during our second meeting with her was as follows: "Tal, do you know what parallel lines are?" This question targeted at providing us with information about Tal's personal, formal definition of parallel lines.

Unfortunately, we did not find studies reporting on learners' conceptions of parallel lines, information that could assist in designing the sorting, parallel lines assignments to be presented to Tal. Thus, we based our choice of the figures on our first interview with Tal. We chose to present her with nine cards (Fig. 3), four illustrating parallel lines (Cards 1, 3, 8, 9) and five exemplifying non-parallel lines (Cards 2, 4, 5, 6, 7). We hypothesized that Tal will correctly identify Card 1, Card 3, Card 8, and Card 9 as examples and Card 2, Card 4, and Card 7 as non-examples of parallel lines, and that she will erroneously classified Card 5 and Card 6 as examples of parallel lines.

In response to the question: “Tal, do you know what parallel lines are?” Tal responded: “Yes. I know… We talked about it in class, parallel lines do not meet… the lines do not intersect.” She then included six of the nine cards in one set (Cards 1, 3, 5, 6, 8, 9) and three in the other set (Cards 2, 4, 7).

> Interviewer: Can you please explain how you sorted the cards?
> Tal: It was easy. Card 1 is how you usually draw parallel lines and Card 3 is the same, just a bit on the side. Cards 8 and 9 are similar to Card 1 but one line is shorter – but it does not matter – they are parallel. Now…In Cards 5 and 6 - the lines are shifted in the same amount – in Card 5 they go inside and in Card 6 they go out, so they are parallel. Now the other set. In Card 4 the lines intersect so they are not parallel. In Cards 2 and 7 the lines are not open in the same amount so they are not parallel.

In her definition of parallel lines, Tal mentioned that “parallel lines do not meet… the lines do not intersect,” a definition that, on face of it, seems to match the concept definition of parallel lines. However, Tal’s way of sorting Card 5 and Card 6 was consistent with our assumption that for Tal, lines (or more precisely, segments and sides of parallelograms) that inclined “in the same amount” towards each other, or against each other, are parallel lines. Here, again, the notion *Mis-in concept image* assists in describing Tal’s way of sorting the figures that were presented to her, and consequently, more precisely describe her concept image of parallel lines. Tal’s concept image of parallel lines *mis(takenly) in(cludes)* lines that inclined towards each other, or against each other.

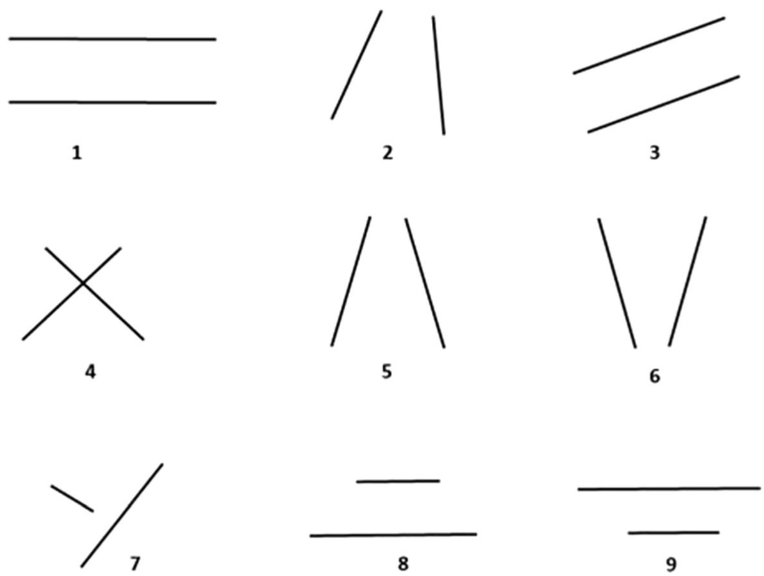

Fig. 3 Parallel and non-parallel lines: second interview

In her personal concept definition of parallel lines, Tal mentioned that "the lines do not intersect." However, her decisions about the lines that were drawn in Cards 2, 5, 6, and 7 seemed to indicate that she based her judgement on the actual drawing of the segments of the lines without recognizing that these drawings illustrated only small parts of the (infinite) lines and that these lines would eventually intersect. This observation led us to identify a need to attempt to study her concept images of lines.

Parallelograms: the Tasks, Tal's Responses to the Tasks, and Some Reflections

The second set of cards that we designed for the second meeting with Tal consisted of four parallelograms and five non-parallelograms (Fig. 4). The set of parallelograms contained a prototypical parallelogram (Card 13), a rectangle (Card 15), a rhombus (Card 19), and a parallelogram similar in appearance to a prototypical parallelogram but unlike prototypical parallelograms, in this parallelogram, none of the opposite sides was parallel to the upper and the bottom edges of the card (Card 20). The rectangle (Card 15) and one of the parallelograms (Card 20) were located in this interview (unlike in the first interview) in non-standard positions on the cards. Card 19 presented a geometrical figure (rhombus) that was not included in the first interview. The five non-parallelograms cards included a trapezoid (Card 14), a concave kite (Card 16), an isosceles trapezoid (Card 17), a convex quadrilateral (Card 18), and a concave quadrilateral (Card 21). The isosceles trapezoid (Card 17) and the convex quadrilateral (Card 18) were located in this interview (unlike in the first interview) in non-standard positions on the card. Card 16 (a concave kite) and Card 21 presented concave quadrilaterals (concaves quadrilaterals were not included in the first interview).

The order of the presentation of the cards to Tal is from Card 13 to Card 21. Much like in the first interview, we chose to first present a prototypical parallelogram (Card 13), following by a trapezoid (Card 14). We assume, based on Tal's responses in the first interview, that she would correctly identified these two figures. The other figures were presented to Tal in the order shown in Fig. 4. The last question that we asked, at the end of the second meeting with Tal, was identical to the question that was presented to her at the beginning of the first meeting, namely, "what is a parallelogram?" The aim of presenting this question at the end of the second (and last) session was to identify changes, if there were any, in Tal's personal concept definition of parallelogram.

The set of the parallelograms and non-parallelograms cards was presented to Tal immediately after the part of the meeting that dealt with parallel lines.

> Interviewer: Now we'll give you nine cards and ask you to sort them into two sets: parallelograms and non-parallelograms.
> Tal: I did it in the first meeting.
> Interviewer: Right. But today the cards are different.
> Tal: OK.
> Tal included four cards (13, 15, 17, 20) in the parallelogram set. She arranged these cards in three rows (Fig. 5).She included the other five cards (14, 16, 18, 19, 21) in the non-parallelogram set. She was then asked to explain her choices.
> Tal: OK. Look, it's just like I did in our previous meeting. In the first row I put the rectangle [Card 15], in the second row, the two with the sides that go in the

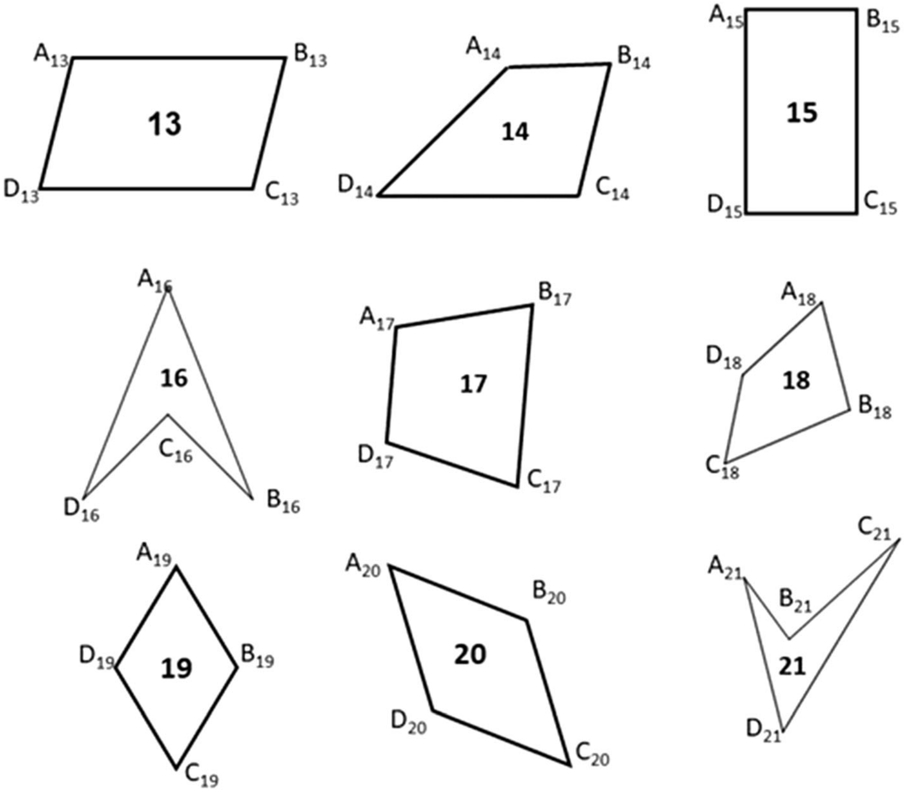

Fig. 4 Parallelograms and non-parallelograms cards: second interview

same direction [Cards 13 and 20], and in the third row the one with the sides that bend in the same amount [Card 17].

Interviewer: And what about the non-parallelograms?

Tal: In Card 14… each side bend in a different amount. Card 16 is a kite… Card 19 is a rhombus… a rhombus is similar to kite. They look alike… It's like a kite but it has four equal sides, not two and two. It's like… it's like a rectangle and a square… a rectangle has two equal sides and two equal sides and a square has four equal sides… . A kite has two equals and two equals and a rhombus has four equals. Card 18 - these two (pointing to $A_{18}D_{18}$ and $B_{18}C_{18}$) are not parallel. Card 21 is convex, and we learnt that convex quadrilaterals are not parallelograms.

Interviewer: …About Card 19… Can you please repeat? What did you say about Card 19? I'm not sure I've heard…

Tal: It's a rhombus, and… rhombus is a kind of kite… it's like rectangles and squares. You see… if I take a rectangle and … and… I can change it to a square … I …shorten these two sides (pointing to $A_{15}D_{15}$ and $B_{15}C_{15}$). And… and… and… I can change the square and make it a rectangle by making … by taking two sides and making them longer … so two sides are the same (pointing to $A_{15}D_{15}$ and $B_{15}C_{15}$) and these two… two sides (pointing to $A_{15}B_{15}$ and $C_{15}D_{15}$) are the same. Now, the rhombus… If $C_{19}D_{19}$ and $A_{19}B_{19}$ were shorter (showing

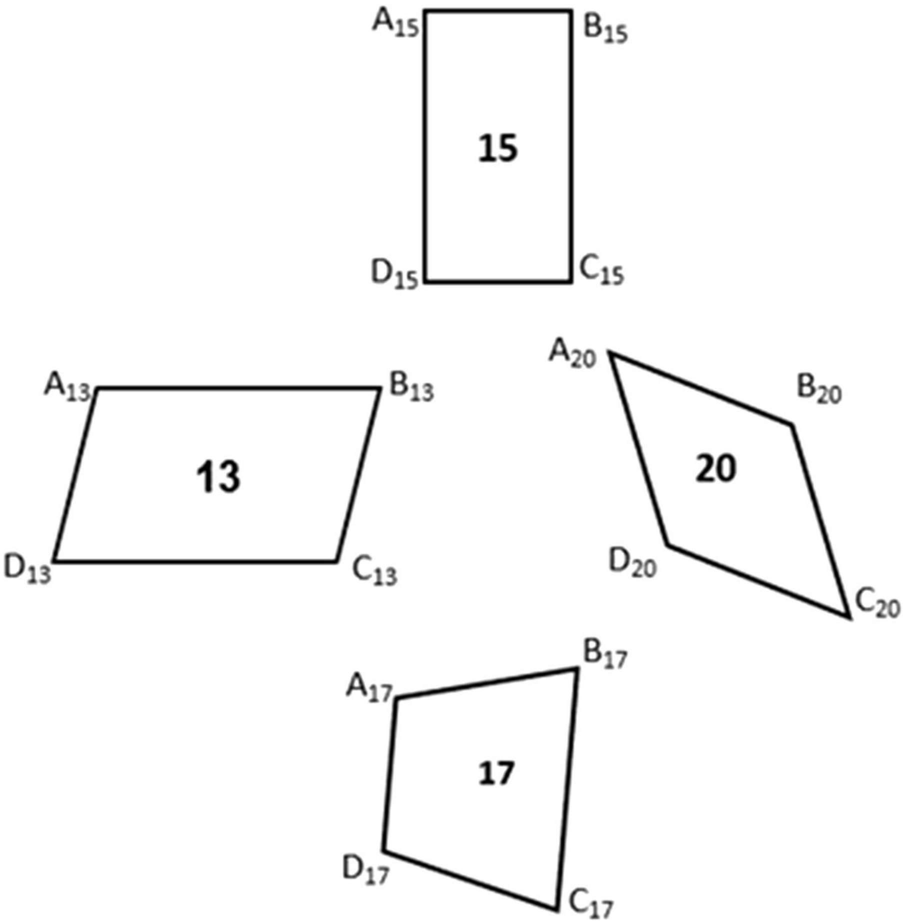

Fig. 5 Tal's arrangement of the cards that she identified as parallelograms: second interview

> with his two hands…) then… it was a kite… But all of them are the same size so it's rhombus. So… so … rectangles and squares are similar and they are together – they are parallelograms, and rhombuses and kites are similar in the same way… so they should also be together, so they are not parallelograms.
>
> Interviewer: Thanks, Tal. We have one more question. What is a parallelogram?
>
> Tal: Parallelograms? OK… Squares and rectangles and… quadrilaterals with… the opposite sides should be parallel to each other.
>
> Interviewer: Thank you very much, Tal.

Some of Tal's decisions regarding the figures that are introduced in Fig. 4 were consistent with her notion of being parallel. In accordance with this notion, she correctly identified the parallelograms in Card 13 and Card 20, and the figures in Card 14 and Card 18 as not parallelograms. Her decision to *mis*(takenly) *in*(clude) the isosceles trapezoid (Card 17) in the parallelogram set was also consistent with her notion of being parallel. However, during this interview, Tal erroneously categorized the rhombus as a non-example of parallelogram. She clarified that her (erroneous) decision not to include the rhombus in the set of parallelograms was based on a resemblance that she observed between kites and rhombuses, which she regarded as similar to the resemblance between rectangles and squares. She then explained that since the rectangles and the squares were sorted together (both as parallelograms), the kites and the rhombuses should also be sorted together (both as not parallelograms). At that meeting, we observed a somewhat complementary state to the *mis-in* one, namely, a *missing concept image* situation, in which an example of the concept (in Tal's case, the rhombus) is erroneously excluded from a set (in this case, the set of parallelograms).

Discussion

At the beginning of this paper, we stated that the theoretical framework that we adopted for this study is that of concept definition–concept image and that a main aim of this paper is to introduce additional constructs to this framework. We also noted that another aim of this paper is to describe possible sources of students' decisions regarding their ways of sorting figures into examples and non-examples of parallelograms. In this discussion, we share with the readers three main observations that emerge from the findings of our present study. The first, theoretical in nature, suggests an extension of the concept definition–concept image framework. The second addresses various, possible sources of decisions regarding ways of sorting geometrical figures. The third describes and discusses two aspects of the methodology that we employed in this study.

Regarding the concept definition–concept image framework, the interviews that we conducted led us to suggest two additional constructs: *missing* concept images and *mis-in* concept images. We observed that Tal erroneously excluded the rhombus, an example of parallelograms, from the set of parallelograms. As a result, the rhombus, which should have been included in the set of parallelograms, was *missing* from this set. Accordingly, we propose to use the notion *missing concept image* to define a situation in which an example of a concept is erroneously categorized as a non-example of the concept. We also perceived that she mistakenly included the isosceles trapezoids, non-examples of parallelograms, in the set of parallelograms. *Mis-in concept image* is the somewhat complementary case to that of *missing concept image*, in which a non-example of the concept is *mis(takenly) in(cluded) in* the set of examples of the concept and consequently this non-example is mistakenly identified as an example of the concept.

Obviously, erroneously including non-examples and not including examples of concepts when sorting given figures is not unique to Tal and to sorting examples and non-examples of parallelograms. However, differentiating between *missing* and *mis-in* concept images provides a way to identify, describe, and study issues related to each of these situations. One set of research questions that comes to mind focuses on identifying *missing* and *mis-in* concept images, including questions such as: What are possible ways of detecting *missing* concept images? What are promising ways of identifying *mis-in* concept images? Are there any specific, differential ways for tracing each of these two types of undesirable concept images? Are there specific concepts that are more apt to generate *missing* concept images and others that will more likely produce *mis-in* concept images? Another set of questions concerns instructional decisions. What are beneficial ways of addressing, in instruction, issues related to *missing* concept images? What are useful ways of dealing with pitfalls regarding *mis-in* concept images? Are there any substantial differences between productive manners of addressing *missing* and *mis-in* concept images in instruction? Is one of them more rigid and consequently more resistance to change than the other?

Concerning instructional decisions, one could argue that *missing* concept images and *mis-in* concept images should be treated in the same manner, namely, by straightening the formal knowledge of the students. Such argument could, for instance, sounds like "Attending to the formal definition of a concept is the mathematical way to differentiate between examples and non-examples of a concept. Students should be encouraged to use the definition, and only the definition, to make such decisions and more generally, to consult the definition of the concept when solving related tasks." We cannot agree more with such a statement. However, various studies suggest that students often provide seemingly adequate definitions of concepts but their ways of sorting given figures into examples and non-examples or performing other, related tasks reveal the substantial discrepancies between their personal concept definition and their evoked concept images of the concept (e.g. Tall & Vinner, 1981; Vinner, 1990; Marmur et al., 2020). In fact, a similar situation was detected in this study. Tal provided personal concept definitions of parallel lines and of parallelograms that seemed to match the formal concept definitions of these concepts, but her way of sorting the figures and her related explanations uncovered the significant differences between her actual definitions and the formal definitions of these concepts. Furthermore, the differentiation between *missing* and *mis-in* concept images could potentially assist in the attempts to tailor the instructional sequences in a manner that would encourage the learners to re-build personal concept definitions that not only seemingly but also precisely matches the accepted, formal definitions and to adjust the mental pictures and the associated properties comprising their concept images of the concepts accordingly.

The second aim of this paper was to describe possible sources of students' decisions regarding their ways of sorting figures into examples and non-examples of parallelograms. In this paper, we uncovered two sources of erroneous decisions. One source is associated with an inadequate interpretation of parallel lines, allowing to *mis-in* non-examples in the set of parallelograms. The second resulted from a belief that two figures that resemble each other in the same, perceptual feature as two other figures, should be sorted in the same manner (i.e. similar figures should either belong or not belong to the set of parallelograms). This viewpoint resulted in *missing* an example of a parallelogram (i.e. rhombus) from the set of parallelograms. To the best of our knowledge, these two sources of students' decisions regarding their ways of sorting figures into examples and non-examples of parallelograms were not reported in the research literature. Identifying additional sources that impact on students' decisions when sorting geometrical figures into parallelograms and not parallelograms (sources that, possibly, impact students' classifications of other figures into examples and non-examples) is an important, ongoing task in mathematics education. The differentiation between *missing* and *mis-in* concept images could potentially assist in these attempts. Further studies could aim at identifying causes of *missing* concept images, detecting sources of *mis-in* concept images, and studying the differences between the sources of *missing* and *mis-in* concept images of parallelograms and other, mathematical concepts.

We would like to end this paper with two reflections on the methodology that we employed in this study: The set of tasks that we design for this study and the specific procedure that we implemented in the study. Regarding the tasks, we noted that we based our choices of the intuitive and unintuitive examples and non-examples of parallelograms on studies including information about students' conceptions of parallelograms. These studies led to both our choices of the figures and to our expectations regarding the manner of sorting them. However, quite a lot of these expectations were not fulfilled. Isosceles trapezoids, which were categorized as intuitive non-examples of parallelograms, were erroneously included in the set of parallelograms. Figures that were categorized as unintuitive examples of non-parallelograms were correctly identified as non-parallelograms. Finally, squares and rectangles, categorized as unintuitive examples of parallelograms, were correctly included in the set of parallelograms.

A question that arises in the light of these findings is as follows: What are possible explanations of the substantial gaps between our expectations regarding the sorting of the figures (which were based on existing research) and the actual manner in which they were classified? Several possible reasons for these discrepancies come to mind. The first relates to the two sources of decisions regarding the ways of sorting figures into examples and non-examples of parallelograms that were exhibited in this study. The specific view of lines bending towards each other "in the same amount" as parallel lines points to a need to attempt to analyze not only the concept images of the mathematics concept that is targeted in a study (in our case, parallelograms) but also the concept images of other, related concepts that might impact on the concept images that form the focus of the study (i.e. parallel lines and even lines).

The other source of decisions of ways of sorting figures was based on observing perceptual similarities in the relationship between kites and rectangles and those between rectangles and squares. The issue of perceptual similarity (and perceptual dissimilarity) between figures and their impact on students' decisions when classifying geometrical figures was discussed in various studies (e.g. Clements & Sarama, 2011; Fischbein & Nachlieli, 1998; Mariotti & Fischbein, 1997). Mariotti, & Fischbein (1997, p. 244) for instance, stated that in mathematics "very often theoretical classifications conflict with spontaneous ones… the definition dictates rigorously… sometimes in disagreement with the figural-perceptual constraints." Thus, one major task for researchers in mathematics education is to continue their search towards identifying spontaneous classifications of students that conflict with formal, mathematical classifications.

A second, possible cause of the gap between our expectations regarding the sorting of the figures and the actual manner in which they were classified relates to the specific context in which the study took place. We noted that in choosing the figures that were presented in the first interview we relied on existing research on students' ways of thinking about parallelograms. However, these studies were conducted in various grades and age groups, and in different countries. Our findings point to the importance of relating to the specific context in which a study takes place when diagnosing students' concept images.

Another facet of the methodology was the specific procedure that took place in this study. Both of us designed the tasks for the first meeting, were present in the first interview, conducted a conversation to reflect on the occurrences in this interview, formulized related hypotheses, and, accordingly, designed the tasks for the second interview. The second interview was also conducted in the presence of both of us and then, in our meeting, we reflected on the entire process and drew related conclusions. This sequence of steps is similar to the three phases of a design-based research, which often consists of one or more cycles of preliminary design phases, teaching experiment phase, and retrospective analysis phases (Cobb, Confrey, diSessa, Lehrer, & Schauble, 2003). Design-based research usually aimed at planning instructional sequences and not at identifying concept images, but could be adopted to studying students' ways of thinking. We whole-heartily recommend using the procedure that we implemented in this study in further studies that attempt to identify students' ways of thinking about mathematical concepts in general and students' *missing* and *mis-in* concept images in particular.

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