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The Use of a Length and Measurement HLT by Pre-Service Kindergarten Teachers' to Notice Children's Mathematical Thinking

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Abstract
The objective of the study was to characterise how pre-service kindergarten teachers used a Hypothetical Learning Trajectory on length and its measurement to notice children's mathematical thinking. A total of 64 pre-service kindergarten teachers enrolled in an Early Years Education mathematics teaching course were asked to notice teaching situations focusing on kindergarten children's learning of length. On the one hand, three profiles of pre-service kindergarten teachers were found according to the use they made of the Hypothetical Learning Trajectory. These profiles were characterised by three ways of learning to use the Hypothetical Learning Trajectory based on the type of mathematical elements they identified: only mathematical elements related to the magnitude; only mathematical elements related to the measurement of length; or both magnitude and measurement elements. On the other hand, when considering the three skills of professional noticing, by identifying the mathematical elements required to solve the proposed task, a group of pre-service kindergarten teachers within each profile were able to notice the thinking of these elements by children and to suggest activities. Our findings provide learning opportunities to pre-service kindergarten teachers who use a Hypothetical Learning Trajectory. It provides them with 'check-points' to answer the proposed questions, in order to learn the specialised knowledge for teaching length and its measurement as well as to develop the skill of noticing student's mathematical thinking.

Keywords Hypothetical Learning Trajectory · Length and measurement · Pre-service kindergarten teachers · Professional noticing

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Introduction

In mathematics education, the teaching competence of professional noticing significantly contributes to the quality of teaching-learning processes. For this reason, the subject is relevant today and constitutes a recent and important research area in mathematics education (Sherin, Jacobs, & Philipp, 2011). Research has focused on initial and ongoing training of teachers at all educational levels. Findings show that pre-service teachers tend to have difficulties in perceiving or interpreting students' work and that their mathematical knowledge seems to influence their skill at professional noticing (Stahnke, Schueler, & Roesken-Winter, 2016).

Noticing mobilises specialised content knowledge for teaching when pre-service teachers have to attend to students' answers to interpret their understanding (Walkoe, 2015). In addition, studies on pre-service kindergarten teachers' (PKTs) pedagogical content knowledge have indicated in some cases that this knowledge is unsuitable (Lee, 2010). Teacher educators thus face the new challenge of providing learning opportunities to PKTs so they develop noticing skills.

To address this issue, some studies have begun to provide information on the use of Hypothetical Learning Trajectories (HLTs) as a way of organising the knowledge to be learnt and helping pre-service teachers notice students' mathematical thinking (Ivars, Fernandez, & Llinares, 2017; Wilson, Mojica, & Confrey, 2013; Wilson, Sztajn, Edgington, & Confrey, 2014). Although these studies are producing findings for pre-service primary teachers, less is known on the subject in the case of PKTs (Stahnke et al., 2016). Our work contributes to this latter line of research.

In terms of contents, most of the research on teachers' professional development in early childhood education has focused on fields such as counting—which includes cardinality, one-to-one correspondence and subitising—and geometry—identifying, analysing, comparing and creating forms (Parks & Wager, 2015) though less on the learning of measurement (Smith III, van den Heuvel-Panhuizen, & Teppo, 2011). There is, however, a growing body of research indicating that measurement is inadequately learnt at school in many countries (Baturo & Naso, 1996; Chappell & Thompson, 1999; Clements & Bright, 2003; Congdon, Kwon, & Levine, 2018; Gómezescobar, Fernández-Cézar, & Guerrero, 2018; Irwin, Ell, & Vistro-Yu, 2004; Kamii, 1995; Kotsopoulos, Makosz, Zambrzycka, & McCarthy, 2015; Solomon, Vasilyeva, Huttenlocher, & Levine, 2015; Zacharos, 2006). There is also evidence that for kindergarten and elementary teachers who are responsible for guiding students' learning of measurement, it is a challenge to overcome their own shallow understanding of the subject (Menon, 1998; Simon & Blume, 1994). Thus, classroom instruction seems to typically focus on standard measurement procedures without considering the conceptual principles that underlie and justify those procedures (Castle & Needham, 2007; Lehrer, 2003; Stephan & Clements, 2003).

Taking all this into account, our research goal consisted in characterising pre-service kindergarten teachers' use of a HLT of length and measurement to notice children's thinking.

Theoretical Framework

In this section, we present the notion of Hypothetical Learning Trajectories and that of professional noticing of children's mathematical thinking.

Hypothetical Learning Trajectories

In recent years, HLTs have been used as content in professional development programs (Panorkou & Kobrin, 2017). Different approaches to HLTs (Lobato & Walkers, 2017) have been identified. For example, Simon (1995) in a constructivist approach introduced the term HLTs to capture the result of a process in which teachers posit a conjecture regarding their students' current understanding of a specific concept and then develop learning activities in order to construct more sophisticated ways of reasoning toward a particular learning goal. On the other hand, Clements and Sarama (2004) conceptualise learning trajectories (LTs) according to Simon and define LTs as:

> Descriptions of children's thinking and learning in a specific mathematical domain and a related, conjectured route through a set of instructional tasks designed to engender those mental processes or actions hypothesized to move children through a developmental progression of levels of thinking, created with the intend of supporting children's achievement of specific goals in that mathematical domain (p. 83).

Clements and Sarama (2009) described an interrelated web of 10 LTs from pre-K to grade 5 across a variety of domains, including length and its measurement. Each LT consists of three components: (1) a learning goal; (2) levels of thinking; and (3) instructional tasks. More recently, Barrett, Sarama, and Clements (2017) used the LT on length and its measurement to clarify how children learn and apply measurement knowledge in pre-K through grade 5.

When Clements and Sarama (2009) described levels of thinking about length and its measurement, they used relevant elements of length as magnitude (recognition of length, conservation and transitivity) and length measurement (equal partitioning, unit of measurement, uniqueness, iteration, accumulation, measurement unit universality, and the relationship between the number and the unit of measurement). They highlighted the inclusive nature of thinking levels and provided instructional tasks to support the transition between levels of thinking. This approach emphasises the fact that what children learn is sensitive to the instructional tasks they engage in.

In this paper, we use the term Hypothetical Learning Trajectory to refer to the synthesis of previous research results leading to a student learning progress model. This model is used in professional development programs.

HLTs are used as a reference to identify learning progression in specific domains. In this sense, Mojica and Confrey (2009) indicated that HLTs can help identify and describe observable aspects to interpret students' understandings and make teaching decisions.

Professional Noticing

To improve the teaching and learning of a mathematical topic, we need to make sense of complex situations. Van Es and Sherin (2008) consider three characteristics of classroom situations: (a) identifying noteworthy aspects; (b) using knowledge about the context to reason about the classroom interactions; and (c) making connections between the specific classroom events and broader principles of teaching and learning.

A particular aspect of this competence is that of noticing children's mathematical thinking. Jacobs, Lamb, and Philipp (2010) conceptualise this expertise as a set of three interrelated skills: (a) attending to children's strategies; (b) interpreting children's mathematical understandings; and (c) deciding how to respond on the basis of children's understandings. Some studies have demonstrated that such teaching competence can be learned, but it does not come naturally (Fernandez, Llinares & Valls, 2012; Stockero, 2014).

Being able to understand and analyse students' mathematical reasoning involves to deduce of the students' understanding from what the student writes, says or does. Noticing students' mathematical thinking demands more than just pointing out what is correct or incorrect in their answers, but requires determining in what way the students' answers are or are not meaningful from a mathematical learning standpoint (Hines & McMahon, 2005; Wilson et al., 2013).

The development of noticing of students' mathematical thinking supported by an HLT implies generating a structured approach to identifying the relevant elements in the students' answers, interpreting pupils' understanding from the perspective of a progression in learning and making instructional decisions in accordance with the inferred understanding (Jacobs et al., 2010). These skills are interrelated since teachers can only decide their teaching approach taking students' mathematical thinking into account if they are able to interpret their understanding (Barnhart & van Es, 2015; Sherin et al., 2011). In this context, HLTs can be used as guidelines to help pre-service teachers focus on childrens' thinking.

The importance of HLTs to develop noticing skills is being emphasised under different approaches, but few studies have focused on the case of PKTs (Stahnke et al., 2016). In our research, we adapted HLTs from Sarama and Clements (2009) on length and its measurement and we analysed how PKTs appropriated it.

Our research question can thus be formulated as follows: How do pre-service kindergarten teachers make use of an HLT on length and its measurement as a framework to develop their noticing of children's mathematical thinking?

Method

The study was based on a qualitative methodological design, in which the population of reference was students in the sixth term of their 'Early Years Education Teacher Training Degree'. This means that they were completing their mathematical training in the degree. Intentional sampling was applied since we only included the students who were in the classroom when the instrument was implemented.

Participants and Context

The participants were 64 PKTs, in the sixth term of their 'Early Years Education Teacher Training Degree'. These PKTs came from different study backgrounds before enrolling in the degree: upper secondary education, professional training or University access for people over 25 years of age. Some of them combined their studies with work (canteen supervisors, play supervisors, etc.) but they did not work in formal Early Childhood Education schools (since to do so, the degree they were enrolled in would have been required). The only contact they had with these schools was their Practicum I, in which they learnt about the schools' organisation.

However, they had not observed aspects related to the teaching-learning of mathematics (Practicum 2 and 3). When the study was carried out, these students were taking their last mathematics subject in the degree, which included the module 'Length and its measurement'. This module was designed by a group of experts in mathematics education and it is structured around an HLT adapted from Sarama and Clements (2009) as a framework for supporting the development of professional noticing.

The Teaching Module

The teaching module was made up of 5 sessions: the first four lasted 100 min and the fifth was a 60-min evaluation session.

The content of the teaching module was an HLT on length and its measurement relating to children aged 3 to 6 years, which consisted of a learning objective, a model of progression of the understanding of length and its measurement (Table 1), and examples of instructional activities (Szilagyi, Clements, & Sarama, 2013; Clements & Sarama, 2004; Sarama & Clements, 2009; van den Heuvel-Panhuizen & Buys, 2008). Learning progression included the understanding of the following mathematical elements: recognition of length, conservation of length, transitivity, uniqueness and iteration of the unit of measure, equipartition, accumulation, universality of the unit of measurement, relationship between number and measurement unit (Table 1). The three first mathematical elements are related only to length as a magnitude and the other mathematical elements are related only to length measurement.

A number of studies have found that children's progression does not always follow this sequence, because they sometimes acquire ideas about measurement before developing the notion of length conservation (Clements, 2010; Hiebert, 1981).

Table 1 Levels of understanding of an HLT on length and its measurement

Levels	Development of progression
1	Recognise the concept of *length as a magnitude*: • Identify the qualities of length. • Make direct comparisons, considering length intuitively.
2	Recognise *the conservation of length*: • Make direct comparisons through the displacement of objects.
3	Use the *transitive* property to: • Make indirect comparisons. • Order objects. • Measure lengths.
4	Recognise the *uniqueness* of the quantity taken as a unit. Identify a unit and make *iterations* of it. Recognise the property of *accumulation*.
5	Recognise the *universality* of the unit of measurement. Recognise the relationship between *number and unit of measurement*. Start to make *estimations*.

Adapted from Sarama and Clements (2009)

On the other hand, children have been found to have difficulties in understanding the notion of accumulation (Gómezescobar, Guerrero, & Fernández-Cézar, 2020; Stephan, Bowers, Cobb, & Gravemeijer, 2003), the uniqueness of the length unit (Ellis, Siegler, & Van Voorhis, 2003), in inferring the length of objects regardless of the unit size (Nunes & Bryant, 1996), and in relating a number to a length and indicating its meaning (Skoumpourdi, 2015).

The PKTs were asked to analyse five teaching situations focusing on children's mathematical thinking (Table 2) using the framework of a HLT. This HLT was worked on in class. The learning progression levels contained in it were explained and different examples of activities that could help children move from one level of understanding to the next were given. After working on all these aspects in class, the PKTs were asked to analyse the 5 situations described in Table 2. The analyses performed by the PKTs were discussed in a large group with the teacher's feedback.

In this article, we focus on the task proposed in the last session (session 5) to characterise what may have been learned after participating in the module (Fig. 1). In this session, the PKTs did not dispose of the teaching module's theoretical content.

As indicated, the research data was constituted by the responses of the PKTs to the questions proposed in session 5 (Fig. 1). The teaching scenario consisted in a teacher proposing that children aged 5 years make necklaces using strings of different lengths and shapes (rolled, stretched and folded), as well as pasta tubes of different shapes and sizes (macaroni, stars, etc.). According to the LTs used, we understand that when children fill the string with a series of units in this task ('tiling', Lehrer, 2003), they are dividing it into sections in order to perform an iteration which allows defining its length.

The different strategies used by the children were observed: Mario used pasta tubes of different sizes and answered incorrectly; Almudena chose pasta tubes of the same size, inserted them leaving gaps between them and answered incorrectly; Elena and Luis used pasta tubes of the same size, inserted them without gaps and answered correctly.

Once the necklaces were made, a dialogue took place between the teacher and the children. This dialogue, together with the children's strategies, revealed different

Table 2 Teaching situations proposed in the module

Session	Teaching situation
1	Teaching situation of length where the teacher asks the children to cut a strip of paper as long as its height; children compare the strips and the teacher arranges them in a certain order (video clip) (Van den Heuvel-Panhuizen & Buys, 2005)
2	Teaching situation of initiation to measurement where children measure the length of their desks in a classroom using anthropomorphic measures (created ad hoc).
3	Teaching situation of measurement where children have to measure the contour of trees in a park (adaptation of Alsina, 2011)
4	Planning situation of a classroom session: search, select and analyse a task on length and its measurement, extracted from books, projects, web pages, etc.
5	Teaching situation of length and its measurement in which a teacher proposes making necklaces with different string lengths and different types of beads (written transcript, created ad hoc).

Alicia is an Early Years teacher in a state school. Her pupils are between 5 and 6 years of age. A week ago, she began working with them on the concept of length as a magnitude and is measurement. Today, in art class, she invites the children to make necklaces using different materials (coloured beads and different types of pasta tubes) and different lengths of string (A, B and C):

Once she has explained the task, the children choose their pieces of string and accessories and start making necklaces. When all the necklaces are finished, Alicia asks the children:

Teacher: *Who's made the longest necklace?*
Mario: *I've made my necklace with the piece of string that looks like a stick* [string C] *and I've used 13 pasta tubes* [he has used different types of pasta tubes]
Almudena: *Miss, I've made a necklace with the pink string* [string A] *and I've used 15 stars* [the stares are very spaced out]
Luis: *Mine has 12 pasta tubes* [he has used all the same type] *and I chose the string that looks like a spiral* [string B], *but it's longer than Mario's because the piece of string is longer.*
Elena: *I also used the pink string [string A] and I used 20 stars* [the stars are close together],
Almudena: *So Elena's necklace is the longest one of them all.*

Based on these responses, Alicia asks the children:

Teacher: *Do you agree?*
Mario: *No miss, I don't agree whit Luis, because mine has more pasta tubes*

Fig. 1 Teaching situation presented in session 5

characteristics in the understanding of length and its measurement (Table 3). Mario and Almudena showed difficulties in understanding the conservation of length, so they were considered to be at level 1 of understanding, while Luis and Elena used a single unit of measure (uniqueness) and iterated that unit to perform the measurement, so they were considered to be at level 4 of understanding.

Instrument

Our data collection instrument consisted in three professional questions the PKTs had to answer. These professional questions were validated by a group of researchers in Mathematics Education specialised in professional noticing. Los aspectos trabajados en las distintas sesiones del módulo conforman los conocimientos previous que los PKTs poseen para responderlas.

Table 3 Characteristics of the children's responses and learning goals to suggest instructional tasks

Children	Level	Characteristics	Learning goals
Mario	1	There is evidence that he: • DOES NOT understand the *conservation* of length (magnitude). • DOES NOT consider the *uniqueness* of the quantity taken as a unit	Appreciate the conservation of length
Almudena		There is evidence that she: • DOES NOT understand the *conservation* of length (magnitude). • DOES consider the *uniqueness* of the quantity taken as a unit • DOES NOT consider the *iteration* of the unit of measurement	
Luis	4	There is evidence that he: • DOES understand the *conservation* of length (magnitude) • DOES consider the *uniqueness* of the quantity taken as a unit, along with *iteration* and *accumulation*. There is no evidence that he establishes a relationship between number and measurement • DOES NOT make use of the inverse relationship between number and measurement.	Acquire the universality of the unit of measurement Establish the relationship between the number and the unit of measurement
Elena		There is evidence that she: • DOES consider the *uniqueness* of the quantity taken as a unit, along with *iteration* and *accumulation*. There is no evidence of establishing a relationship between number and measurement.	

- Question 1. Indicate the mathematical elements needed to complete the task.
- Question 2. What is each child's comprehension level? Reason your answer based on the characteristics of the children's interventions.
- Question 3. Assuming you are Alicia, propose an activity for the child that you consider to be at the lowest level of comprehension, and for the child at the highest level, so that they continue to progress in their understanding of length and its measurement.

In this last prompt, the PKTs had to define learning goal to each child to properly justify the proposed activity.

Data Analysis

In our analysis, we used a grounded theory approach (Strauss & Corbin, 1994) to consider the three interrelated skills of professional noticing based on pre-service kindergarten teachers' answers. We started by analysing a data sample over an iterative process to generate characteristics on how PKTs identified relevant mathematical elements in the situation, interpreting the children's understanding and proposing new activities.

To ensure the validity and reliability of the analysis, a group of five researchers first analysed a small sample and discussed the encodings linking them to the evidence. Once a consensus was reached, new data were included to check the system of categories that was beginning to emerge from the data.

PKTs' responses to the three questions were analysed over three stages:

- In the first stage, we analysed PKT's answers to questions 1 and 2 to characterise how they identified the relevant elements of length and its measurement needed to complete the activity (prompt 1) and that were present in the children's dialogue.
- In the second stage. we focused on how PKTs interpreted children's understandings of these elements (prompt 2) providing evidence to support their inferences.
- In the third stage, we focused on how PKTs proposed suitable activities in relation to children's understanding (prompt 3).

Table 4 shows examples of the different categories.

Table 4 Example of analysis of a PKT's answers (Lara)

Skills	Example	Researcher's analytical inference
Identifying the mathematical elements	'You need to recognise length and you need to know *that there is conservation when manipulating the strings and comparing them.*	Identifies the elements of conservation, uniqueness and iteration and provides evidence.
Interpreting the understanding of mathematical elements	Mario is not able to recognise that string C is clearly the shortest, *he does not possess the conservation element so he is at level 1.* Almudena is at level 1 like Mario because she *does not make direct comparisons to see which necklace is longer.* Luis and Elena are at *level 4, they use the same types of units and place them without any gaps and without overlapping them*; there are no signs of transitivity.'	Provides evidence that Almudena and Mario are at level 1 and that Luis and Elena are at level 4, according to the learning progression indicated in the HLT.
Proposing activities	'In both cases, because none of the four children makes direct comparisons (level 2), I would suggest the same activity. *Using the same necklaces composed in the same way, I would ask them to compare them and say which is longest.* This will force them to experiment, compare, and put them together to complete the task. If we observe that Luis and Elena perform the task easily, we would go up a level and ask them to arrange the necklaces (level 3, transitivity) to acquire new knowledge and strengthen previous knowledge.'	This PKT proposes suitable activities for low level children (Mario and Almudena). But the task this PKT suggests for the highest level (Luis and Elena) is not appropriate because these children are at level 4 of understanding and the PKT has proposed a task from level 3.

Results

We grouped the findings taking into account the following: (1) the relevant mathematical elements identified by the PKT; (2) the interpretation of students' understanding; and (3) tasks proposed based on inferred comprehension. In this way, we first identified three PKT profiles:

- Profile 1: PKTs who make a partial use of the HLT, identifying mathematical elements relating only to length as a magnitude.
- Profile 2: PKTs who make a partial use of the HLT, identifying mathematical elements relating only to measurement.
- Profile 3: PKTs who make full use of the HLT, identifying mathematical elements relating to length as a magnitude and measurement.

Second, three groups of PKTs were identified within each profile:

- those who identified mathematical elements, but did not interpret children's understanding
- those who identified mathematical elements and interpreted understanding, but did not propose adequate tasks considering the inferred understanding; and
- those who identified mathematical elements, interpreted understanding and proposed appropriate tasks based on inferred understanding.

We now move on to describe each of these profiles and the groups they contain.

Profile 1. PKTs Who Make a Partial Use of the HL, Identifying the Mathematical Elements Relating Only to Length as a Magnitude

Within this profile, 12 PKTs identified mathematical elements relating to length as a magnitude only, and had difficulties with or did not recognise relevant measurement elements (uniqueness, iteration and accumulation) to describe children's answers.

These PKTs identified the 'conservation' element and some of them also identified length recognition as an object's property and/or transitivity. By identifying conservation, they would be able to interpret the understanding of low level children (Mario and/or Almudena) who did not understand conservation (Table 3); however, only 8 PKTs were able to while the remaining 4 PKTs were not.

For example, Miriam recognised the three length as a magnitude elements needed to solve the activity (length as an object's property, conservation and transitivity) but did not give evidence based on the children's answers to support the interpretation of Mario's and Almudena's understanding. She also made general comments regarding measurement elements—gaps between stars or relating to iteration:

> Miriam: The following length as a magnitude elements are necessary for this activity: recognition of magnitude, conservation and transitivity. Mario is at

level 2 because he gets confused and thinks that his necklace is longer because it has more macaroni than Luis's necklace. Almudena has the same problem as Mario since she is at level 2, and has not noticed that Elena's necklace has the same length as hers, the only difference consists in the gaps left between the stars.

Among the 8 PKTs to have correctly interpreted Mario's and/or Almudena's understanding, only 4 PKTs proposed tasks based on children's understanding of conservation and transitivity (Table 4). For example, Manuel interpreted Mario and Almudena's understanding of conservation and noted:

Manuel: I would classify Mario at level 1 as the child has not yet acquired conservation: he says that his necklace is longer than Luis's because he uses more macaroni than he does, regardless of the length of the string.
Almudena would also be at level 1, as she has not acquired conservation, and does not understand that her string and Elena's have the same length, regardless of which contains more or less macaroni.

But he could not identify Luis and Elena's mathematical elements, and consequently, he did not know how to interpret their understanding nor propose appropriate instructional tasks.

Manuel: Since Luis has acquired conservation, I would place him at the transition between levels 2 and 3. It is very difficult to classify Elena, she only mentioned the string she used and the number of stars.

This PKT proposed the following task to get Mario and Almudena to understand that length conservation was unrelated to the shape of the strings:

Manuel: I would suggest to Mario and Almudena that they remove the macaroni from the string [Mario's string C had a stick shape and Almudena's string A was elongated] and that they directly compare the longer and shorter string, in this way they should reach level 2 [when acquiring the length conservation element].

However, some PKTs were not able to suggest activities to help understand conservation. For example, Lucia considered that Mario and Almudena did not understand length conservation but proposed a task centred on measure unit iteration:

Lucia: I would place Mario at level 1, because based on what he says at the end of the dialogue, he does not differentiate the size of the string; instead, he takes into account the number of macaroni but without considering their size [referring to the conservation]. Almudena also seems to be at level 1. She focuses on the number of stars without comparing the size of the strings. I would suggest to them a measurement activity also using strings but to measure the tables in the classroom.

Profile 2: PKTs Who Make a Partial Use of the HLT, Identifying the Mathematical Elements Relating Only to Measurement

A total of 41 PKTs only identified elements relating to measurement, iteration, accumulation and universality, which allowed them to potentially interpret the understanding of high-level children (Luis and/or Elena, Table 3). However, only 29 of them actually did interpret the understanding. Furthermore, out of these 29 PKTs, only 16 proposed tasks based on inferred understanding (Table 4).

The PKTs belonging to this profile did not refer to length elements and only tried to infer children's understanding of measurement elements, so it was difficult for them to characterise the understanding of children who did not understand conservation (Mario and Almudena). For example, Alejandro refers to the understanding of iteration, universality and accumulation to interpret Mario and Almudena's comprehension and notes:

> Alejandro: Mario seems to be at level 3 because he did not complete the activity correctly, he gave a good justification and although he did not overlap the beads, he has not acquired universality so he is not at level 4. Almudena would be at level 3 because even though she knows how much she has used, she left a gap between each star.

We could observe that Alejandro identified the fact that Mario and Almudena had difficulties with measurement elements and therefore knew that they were not at any level of understanding they should have acquired. So, he discarded level 4, knowing that they were at some lower level. But because Alejandro does not mention any length as a magnitude element nor checks whether Mario and Almudena have acquired conservation, which they have not, he is not able to place them at a level lower than 3.

Alejandro did identify characteristics of Luis and Elena's (high level) understanding of uniqueness and iteration, but the instructional activity proposed did not foster any learning progress:

> Alejandro: "Luis carries out equal partitioning [referring to uniqueness], he chooses the same pasta tubes, so he's at level 4"; "Elena always uses the same stars, so she knows how to carry out equal partitioning [referring to uniqueness] and also iterates correctly, without leaving any gaps."
> "For children with a higher level of understanding, which in this case would be Luis and Elena, I would work again with the beads, perhaps using a ruler so that they can see that not all the beads, lengths of string, and pasta shapes have the same length."

This PKT attempted to introduce the use of the ruler (characteristic of level 5) justifying that it would help Luis and Elena recognise the different lengths of the beads. We infer that this justification related to the fact that Alejandro believed the children needed to see this, since the units of measure differed according to the necklaces and their length could not be compared based on the number of macaroni or stars. This requirement would be related to length conservation, an element that Luis and Elena had already acquired. Therefore, the activity proposed by this PKT did not address characteristics that would help children to move up from comprehension level 4, in which they already were, to level 5 and therefore did not help them to progress in their learning.

An example of PKT of profile 2 that, in addition to interpreting the understanding of high-level children (Luis and Elena), proposed activities that helped them advance is Amelia. This student commented that Luis and Elena had acquired the uniqueness element, because they had respectively chosen the same macaroni and stars, and because Elena had iterated correctly, so she placed them at level 4; she also suggested the following task, which helped them move to the next level by working on measurement unit universality:

> Amelia: In order for Luis and Elena to advance, I would propose an activity in which they were offered different materials to measure the length of the class and thus be able to develop the notion of measurement unit universality. So that they use a single measurement instrument, they would have to agree on one and thus move up from level 4 to level 5.

Profile 3: PKTs Who Make Full Use of the HLT, Identifying the Mathematical Elements Relating to Length as a Magnitude and Measurement

This profile was formed by 11 PKTs who identified mathematical length as a magnitude and measurement elements, so they would be able to interpret the understanding of the four children; however, only 10 PKTs actually did. Furthermore, among those who interpreted the level of understanding of the four children, only 5 suggested suitable activities to progress in their learning (Table 4). Thus, 5 PKTs who interpreted the understanding of the four children did not propose a task that would help them to progress in their learning. This was the case of Lara, who interpreted the understanding of the four children by identifying the elements of recognition, conservation, uniqueness and iteration, as visible in her answer, but proposed general activities that did not take into account the inferences made regarding understanding:

> Lara: Mario is not able to see that string C [stick shape] is clearly the shortest, [so he does not understand] conservation and is at level 1. Almudena is at level 1 like Mario because she does not directly compare the necklaces to see which necklace is longer.
> Luis and Elena are both at level 4. They use the same type of units [referring to the uniqueness of the measuring unit: macaroni and stars respectively] and place them without overlapping or leaving gaps [referring to unit of measure iteration], there are no signs of transitivity. In both cases, because none of the four children makes direct comparisons (level 2), I would propose to do the same activity. I would ask the children to compare the same necklaces made up in the same way, and tell me which one is longest. This will force them to experiment, compare and put them together to perform the task. If Luis and Elena perform the task easily, I would go up one level and ask them to arrange the necklaces (level 3,

transitivity) to acquire new knowledge and strengthen previously acquired knowledge.

As we can observe in this PKT's answer, Lara has correctly interpreted the understanding of all children, placing Mario and Almudena at level 1 and Luis and Elena at level 4. Despite this, she proposes the same task for all of them to help them progress in characteristics of level 2 relating to the direct comparison between different necklaces. Moreover, this PKT specifies that if Luis and Elena find the task easy, she would suggest that they arrange the necklaces in order to work on transitivity, an element proper to understanding level 3.

Five PKTs in this group did propose appropriate activities. These activities were based on the HLT, since they used the learning progression contained in it to characterise each child's understanding and to propose a task to support the child's progress, taking the HLT's sequence into account. These 5 PKTs made a professional use of the HLT to professionally notice children's mathematical thinking, identifying the elements, both of magnitude and measurement, interpreting their understanding, and proposing activities that favour progression in learning.

An example is Ainhoa, who after characterising Mario's understanding by identifying that the child has not acquired conservation submits the following task to work on that element:

> Ainhoa: What I would do is give him two strings of different sizes and macaroni of the same size. So that he realizes that the strings do not share the same length. Or, I could stretch both strings so that he can thus observe why one is longer than the other even if the shorter string has more macaroni.

This PKT comments that Luis and Elena know how to iterate and have acquired accumulation; therefore, they are at understanding level 4. To help them progress, she proposes a transition task to level 5, working both the number-measure relationship and measure unit universality, as visible in her answer:

Table 5 Number of students per profile based on data analysis

Profiles	Identified elements	Identified mathematical elements only	Interpreted the understanding of the mathematical elements but did not suggest suitable tasks	Suggested suitable instructional tasks based on inferred understanding	Total
Profile 1	Length as a magnitude only	4	4	4	12
Profile 2	Measurement only	12	13	16	41
Profile 3	Length as a magnitude and measurement	1	5	5	11
	Total	17	22	25	64

> Ainhoa: I would ask Luis and Elena the following sort of question: if we inserted macaroni (which are larger than the stars), would the string measure the same length? Why? What if we put little beads (which are smaller than the stars)? Then I would invite them to check themselves so that they understand the relationship between unit and length, that is, the larger the object, the fewer objects will enter. They should also come to the conclusion that they need to use a universal measuring unit to achieve the same measurement.

To summarise (Table 5), the results showed that of the 12 PKTs with profile 1 (who make a partial use of the HLT, identifying the length elements as magnitude only), 4 of them only identified the mathematical elements (without interpreting the children's understanding and without proposing activities to help them to make progress); 4 interpreted the children's understanding based on identifying the elements, but they did not suggest the activities that were suitable to make progress in their learning; and 4 interpreted the children's understanding after having identified the mathematical elements, and based on that interpretation, proposed the suitable activities to help them to make progress.

Profile 2 (PKTs who made a partial use of the HLT, identifying measurement elements only) was the most common profile as it included 41 PKTs. Of those, 12 identified the mathematical elements but did not interpret the children's understanding based on that identification, nor did they propose activities; 13 PKTs did interpret children's understanding based on the identification of the elements, but they did not propose suitable activities; and 16 did identify the elements, built on them to interpret children's understanding, and proposed suitable activities.

Profile 3 included 11 PKTs who identified length elements such as magnitude and measurement and therefore made full use of the HLT. Of these, 1 identified the elements only; 5 interpreted children's understanding based on the identification of the elements but did not propose suitable tasks; and another 5 identified the elements, interpreted the children's understanding based on that interpretation and also proposed suitable tasks to help them progress in their learning.

Discussion and Conclusions

The research question in this study was as follows: how do pre-service kindergarten teachers use an HLT on length and its measurement as a framework to develop their noticing of children's mathematical thinking?

To answer this question, we first described three PKT profiles, each characterised by three different ways of using the HLT according to the type of mathematical elements they identified: the mathematical elements relating only to length as a magnitude (perception of length, conservation and transitivity); mathematical elements relating only to length measurement (equipartition, uniqueness, iteration, accumulation, universality of the unit of measure and relation between number and unit of measurement); or both length as a magnitude and measurement elements. In the first two cases, a partial use was made of the HLT, and in the third case, a full use was made of it.

Secondly, when considering the three skills of professional noticing, the fact of identifying the mathematical elements required to solve the proposed task allowed a group of PKTs within each profile to recognise children's understanding of these elements and suggest related activities.

The HLT, which the learning environment centred on, presented different types of 'check-points' to help PKTs answer the questions they were asked:

- From the conceptual viewpoint, the mathematical elements of length and measurement provided the PKTs with key concepts helping them to articulate a mathematical discourse that described the children's responses to the situation posed by the teacher.
- From the cognitive viewpoint, the fact that learning progression was divided into different levels allowed determining the degree of understanding of each child.
- From a didactic viewpoint, once the understanding of each student was interpreted, the HLT helped to formulate learning objectives and propose tasks to achieve them.

We discuss below the results based on these three perspectives.

Conceptual Perspective

A first characteristic of the use of the HLT related to recognising mathematical elements in the presented situation. Our results showed that of the 64 participants, 17 named one or more mathematical elements in a rhetorical way, without providing evidence, which did not lead them to interpret the children's understanding (Table 5). Consequently, they did not use the HLTs as a tool to structure their noticing of students' productions. Nevertheless, the rest of the PKTs (47) used the HLTs as a tool to interpret students' understanding with different levels of sophistication, which was due to their use of the information provided by the HLTs.

The mathematical elements were an important part of the teaching module, since progression in the learning of the HLT included the understanding of these elements, which we can consider as conceptual advances in the development of understanding by children (KDU, key development understanding, Simon, 2006). They also acted as an important reference to notice the mathematical thinking of children, focusing their attention on their understanding and being able to differentiate degrees of understanding.

Besides, as already indicated, most of the PKTs focused on the second part of the HLT relating to length measurement (41 out of 64). This can be explained by the fact that PKTs had perhaps not perceived the inclusive nature of the different levels of learning progression; in other words, these PKTs ignored that children who are, for example, at level 4, are able to perform object equipartitions, identify the unit of measurement and perform iterations, and had already acquired notions of length as a magnitude. In addition, a small group only identified length as a magnitude element (12 out of 64); therefore, they did not identify the mathematical elements involved in the process of length measurement.

Cognitive Perspective

Our results reveal that identifying mathematical elements in students' answers does not necessarily lead to using them to interpret students' understanding: out of PKTs, only

47 interpreted some children's understanding (Table 5). There is therefore a gap. Authors (2017) interpret this gap taking into account the cognitive demands of the tasks. From a cognitive perspective, the task of characterising a student's understanding of a mathematical task involves greater cognitive demands than the task of identifying mathematical elements in students' answers. In this sense, the HLTs would help to build bridges between identifying mathematical elements and interpreting students' understanding, since they provide a model of learning progression in which the descriptors of the levels of comprehension correspond to the mathematical elements identified in children's answers. As Mojica and Confrey (2009) point out, student understanding is not directly observable, but HLTs can help teachers identify and describe observable behaviours and thus make students' mathematical thinking more visible to the teacher.

On the other hand, according to Mason (2002), noticing involves two processes. The first consists in the observation of, or 'accounting of', the phenomena: the objective is to inform of the phenomena as directly as possible, avoiding interpretation, judgement or evaluation. The second process, 'accounting for', aims at explaining and interpreting what is perceived. While describing the students' answers 'taking account of', explaining the understanding of length and its measurement involves the act of interpretation based on a number of answers. This cognitive activity requires specific discourse that utilises mathematical elements deemed significant and links between facts and causes or consequences. Our research provides empirical evidence of the differing demands on teachers of these two cognitive acts.

Didactic Perspective

Our results also indicate that the ability to interpret students' understanding is more developed than that of proposing instructional tasks. Of the 47 PKTs who had interpreted children's understanding based on evidence, only 25 were able to propose suitable tasks for some children or for all children, depending on the mathematical elements identified, in order to help them progress in their learning. This shows how difficult it is to make instructional decisions, despite having inferred children's understanding.

To do this, the PKTs, after interpreting the children's understanding, had to propose a learning objective and design a relevant task or tasks to achieve it. The HLTs provided would have the potential to help them set objectives based on the understanding identified in the framework of the progression of learning. As in other studies (Gupta, Soto, Dick, Broderick, & Appelgate, 2018), among the PKTs who proposed appropriate tasks, some reproduced the instructional tasks given as examples to move from one level of learning to the next, and others designed their own tasks. The PKTs in our research had received theoretical-practical training relating to the HLT, and had had contact with the kindergarten classes to know their organisation (Practicum 1); they had not, however, observed aspects of teaching-learning of mathematics (Practicum 2 and 3). This fact may have undoubtedly influenced the results on suitable instructional task proposals.

Implications for Teacher Educators

In this paper, we described the professional development of pre-service kindergarten teachers. There has been little research on professional noticing (Stahnke et al., 2016) relating to this group. We studied the professional noticing of this group relating to one subject, measurement, for which there is evidence that it is a challenge for kindergarten and elementary teachers (responsible for guiding students' measurement) to overcome their own shallow understanding themselves (Menon, 1998; Simon & Blume, 1994).

Our results confirm and add to existing research findings that HLTs help pre-service teachers make inferences about students' understanding and enable them to propose instructional tasks to progress in their learning (Ivars et al., 2017; Panorkou & Kobrin, 2017; Wilson et al., 2013; Wilson et al., 2014).

We provided learning opportunities to the PKTs using an HLT in order to learn specialised knowledge for teaching length and its measurement and develop the skill of noticing students' mathematical thinking. Our findings provide information for teacher educators on how to use an HLT's 'check-points' as a way of organising the knowledge to be learnt and help PKTs to notice students' mathematical thinking.

Therefore, we consider that our work is replicable and that other teachers will be able to benefit from it. However, it should be noted that we encountered a number of specific challenges, particularly in the interpretation phase of the answers. This challenge was overcome by performing a triangulation of all the researchers' interpretations. When reproducing this work in the future, we recommend completing the study by interviewing the students whose answers were not supported by evidence.

Concerning the reproducibility of our study, we believe that the instrument designed in our research can be a starting point for devising teaching materials for teacher education programs aiming at developing the noticing of students' mathematical thinking. Our work could be useful for supporting the development of PTKs' expertise in identifying and interpreting students' answers. Moreover, the profiles we identified help to describe how PKTs develop the competence of noticing students' mathematical thinking within the context of length and its measurement.

Besides, the PKTs found it easier to identify certain mathematical elements (conservation, uniqueness, iteration and accumulation), probably due to the characteristics of the task. For this reason, it is important to propose that PKTs analyse children's responses to different types of activities that show the diversity of elements involved in the understanding of length as a magnitude and its measurement.

It is also necessary to emphasise the type of instructional decisions associated with each level of understanding, given that the PKTs found it difficult to make appropriate decisions based on inferred understanding, particularly in the case of pupils with higher levels of understanding.

The results of our study revealed that most students made a partial use of the HLT; that is, they focused their attention on only one part of it (length as magnitude or length as measurement). This reveals their difficulties in perceiving the sequentiality and learning progression levels nested in the proposed HLT. In our study, we focused on a type of spatial magnitude and length, and we approached it based on the conceptual and procedural aspects that make up a 'measurement theory' (nature of the units, equipartition, iteration of the unit, etc.) and that underlie the measurement of different types of magnitude. Our results revealed, however, that the relationships between these general aspects need to be emphasised further.

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