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Using and Understanding Algorithms to Solve Double and Multiple Proportionality Problems

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Abstract
We analyzed the written answers and explanations by 26 Brazilian high-school students who attempted to solve double and multiple proportionality problems using the cross-product algorithm. We focused on students' awareness of the scalar and functional relations among the quantities described in verbal problems, an aspect that has been part of their school instruction. Our results show that, in their written work, the students included data tables or pairs of values, usually with their referents and connected by arrows indicating the scalar or the functional relations described in the problems. In most cases, during individual interviews, they explained their solutions in terms of these relations. Our findings suggest that students' understanding of proportion relationships between quantities in verbal problems constitutes a basis for their understanding and correct use of algorithms.

Keywords Cross-product algorithm · Double proportionality problems · Multiple proportionality problems · Scalar and functional relations

Introduction

This study explores how high-school students, while using and explaining their use of the cross-product algorithm, represent, solve, and understand the relationships between quantities described in double and multiple proportionality verbal problems.

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Participants were enrolled in a Brazilian school where adopted textbooks and instructors placed emphasis, before the cross-product algorithm was introduced in the classrooms, upon the functional and scalar relationships (Vergnaud, 1983, 1988, 1994) among quantities described in proportionality problems.

Proportionality, a core concept in cognitive development and in the characterization of adolescent reasoning (Inhelder & Piaget, 1958), is also a central topic in the middle-school curriculum (Lesh, Post, & Behr, 1988) and an entry point to algebraic reasoning and to algebra and function representations in middle and high school (Post, Behr, & Lesh, 1988). It is part of the conceptual field of multiplicative structures that includes fractions, ratios, rational numbers, linear and *n*-linear functions with dimensional analysis, and vector spaces (Vergnaud, 1983, 1988, 1994). As Vergnaud cautions, “The main problem for students is that rational numbers are numbers and that entities involved in multiplicative structures are not pure numbers but measures and relationships (p. 161). Therefore, teaching for understanding of concepts and procedures within the multiplicative structures requires attention to how measured quantities are interrelated.

Over the years, researchers of cognitive development and of mathematics education have produced a wealth of analyses of how students attempt to solve simple proportionality problems (see Tourniaire & Pulos, 1985, early review of studies and Hart’s, 1984, extensive description of secondary school students’ strategies and errors in solving ratio and proportion problems).

Vergnaud (1983, 1988, 1994), in his analyses of the conceptual field of multiplicative structures, describes the scalar and functional relations in proportionality problems. His work led to a number of studies demonstrating that school children and unschooled adults favor strategies based on scalar and/or functional relations when they solve simple proportionality problems (e.g., Nunes, Schliemann, & Carraher, 1993; Ricco, 1982; Schliemann & Carraher, 1992; Schliemann & Nunes, 1990). More recently, Silvestre and Ponte (2012) described, in a qualitative study, the use of scalar and functional strategies by four 11-year-olds who, before receiving formal instruction on proportionality, attempted to solve simple proportionality problems.

Even though the above studies show that people can solve proportionality problems using scalar and functional strategies, instead of formal algorithms, research on the understanding of proportionality in schools reveals students’ difficulties that persist beyond the elementary school years. For example, Hart (1984) found a predominance of erroneous additive strategies in paper-and-pencil tests of ratio and proportion among 12- to 16-year-old students in England. Van Dooren, De Bock, Hessels, Janssens, and Verschaffel (2005) analyzed, in Flemish schools, the responses to proportional and non-proportional problems by 1062 students in grades 2 through 8. They found that, although correct answers increase across the grades, with proportionality strategies replacing erroneous additive strategies, still more than one-fifth of the eighth graders’ answers were incorrect, with many students wrongly applying proportionality strategies to non-proportional situations. Van Dooren et al.’s (2005); see also Van Dooren, De Bock, Janssens, & Verschaffel, 2008 and Van Dooren, De Bock, Evers, & Verschaffel, 2008) analysis of the overuse of proportionality strategies suggests that students’ difficulties may result from instruction that emphasizes procedural routines, without the required analysis and understanding of how the quantities in the problems are interrelated. Fernández, Llinares, Modestou, and Gagatsis (2010) also analyzed the development of strategies to solve proportional and non-proportional problems by 755 students in Spain, from primary (from grade 4) to secondary school (to grade 10). They found that most primary school students used the additive strategy in proportional and in additive situations. On the other hand, secondary school presented a wider variety of strategies, such as use of ratios and of the cross-product algorithm but, in most cases, used proportional strategies in proportional problems as well as in additive problems and also used additive strategies while trying to solve both types of problems. The authors conclude that exclusive use of one strategy without a meaningful understanding of multiplicative reasoning constitutes a procedurally oriented approach obstructing students' initial use of proportional reasoning. In view of their results, Fernández et al. (2010) recommend that, to prevent students' use of strategies without understanding, instruction needs to include attention to the mathematical relationships between quantities.

Other studies show that understanding of ratio and proportion may be challenging for pre-service teachers. For example, Livy and Vale (2011) analyzed the responses of Australian pre-service teachers to a 49-item test of mathematical knowledge for instruction in years 5 to 8. The two items with the smallest proportion of correct responses asked participants to identify the missing part of a proportion and interpret a whole-whole scale ratio problem. Ekawati, Lin, and Yang (2015), in an analysis of another instrument to measure elementary (grades 1–6) in-service teachers' content knowledge of ratio and proportion in Indonesia, found that participant teachers had difficulties with problem situations involving stretching and shrinking and in ratio comparison problems focused on conceptual understanding. They performed better in missing value problems, applying formal algorithms to solve them, but applied the same formal procedures to non-proportional situations. And Buforn, Llinares, and Fernández (2018) found that pre-service teachers from a university preparation program for teachers of 6 to 12-year-old students in Spain, although producing a large percentage of correct answers to missing value proportionality problems using the cross-product algorithm, had difficulties recognizing a non-proportional situation.

Algorithms and Scalar and Function Relations in Solving Proportionality Problems

In schools, students are mainly taught to solve simple proportionality problems using the cross-product algorithm, also known as the rule of three, for computing the fourth (missing) proportional value. Surprisingly, this algorithm is rarely used by school children and adolescents (Hart, 1984; Levain, 1992; Schliemann & Carraher, 1992; Vergnaud, 1983), perhaps due to lack of understanding of why the algorithmic rules relate to how the quantities in a problem are inter-related.

Consider the following problem:

If 3 notebooks cost 9 dollars, how much do I have to pay for 12 notebooks?

Using the *scalar strategy*, one would perform successive additions of 3 notebooks until reaching 12, or reason that 3 notebooks times 4 gives 12 notebooks, and perform corresponding operations on the price (9 dollars) of 3 notebooks. The *functional* *strategy* would focus on the relation between the two variables, number of notebooks and price in dollars, a ratio of 1 to 3, and multiply the number of notebooks (12) by 3, to get the number of dollars. Using the *cross-product algorithm*, one would set the two ratios as equal, that is, $3/9 = 12/x$, which leads to the equation $x = 9 \times \frac{12}{3}$ and to $x = 36$.

Levain and Vergnaud (1994–1995) propose that a focus on scalar and functional relations in proportionality problems may constitute a path towards the appropriation of more advanced understandings, representations, and procedures. This would require a basic understanding of functional and scalar relations before learning the cross-product algorithm. The analyses of how students relate algorithmic procedures to the quantities in a problem would help the development of teaching approaches that could lead to a better understanding of proportionality and of algorithmic procedures.

Vergnaud (1983, 1988, 2011) describes three main types of proportionality problems: (a) simple proportion, characterized by a constant relation between two variables; (b) double proportion, where two or more independent proportional relations are linked by a common variable; and (c) multiple proportion, where two simple proportional relations are interconnected (see also Levain & Vergnaud, 1994–1995).

Empirical studies on students' use of scalar and functional relations and of the cross-product algorithm have been mostly restricted to simple proportionality problems, similar to the one exemplified above. Exceptions are the studies by Levain (1992), Gitirana, Campos, Magina, and Spinillo (2014), Lautert, Schliemann, and Leite (2017), and Leite, Lautert, and Schliemann (submitted). These studies show that choice of scalar or functional strategies, as was the case in studies of simple proportionality problems (Levain, 1992; Schliemann & Carraher, 1992), seems to be related to the numerical relations between the elements in the problem.

Leite, Lautert, and Schliemann (submitted) analyzed 90 Brazilian middle-school students' (grades 7 to 9; 10- to 14-year-olds) strategies and representations to solve two double and two multiple proportionality problems. Textbooks adopted by their school emphasize the analysis of scalar and functional strategies from grade 7, before introduction of the cross-product algorithm. They found that the two types of problems were equally accessible for the group of students, with percentages of correct answers ranging from 67 to 97%, per problem and per school year, with no significant differences between types of problems or school years. Only one grade 9 student (out of all students) used the cross-product algorithm to solve one of the problems. He explained his strategy through description of scalar relations.

Lautert, Schliemann, and Leite (2017) analyzed the responses of 16 (out of 82 participants) high-school Brazilian students (aged 15 to 17 years of age) who had used the cross-product algorithm to solve one multiple proportion program (out of the four problems they were given: two multiple proportion and two double proportion problems). In that study, we briefly described how these students, besides the cross-product algorithm, represented in writing the relationships among quantities in the problem, considering scalar and functional relationships among variables, and whether or not, during an individual interview that took place a week later, they explained their solution strategies, displaying, or not displaying, an understanding of scalar and functional relationships in the problem. Their written work and interview data showed that the written representations were set up after considerations about the scalar and/or functional relations among the quantities in the problem. These were depicted as data tables, similar to those in Vergnaud's (1988) work, or as pairs of values with arrows to indicate the scalar or the functional relations between two elements in a pair. They also explained their solutions in terms of these relations. In addition, most of their written work included numbers as well as their referents and a few participants also represented the unknowns as letters. Only a few students represented the relevant ratios for use of the cross-product algorithm as an equality between ratios and, perhaps because the numbers were small and computations could be easily done mentally, only a few showed the computational steps to reach the solution to the problem.

Study Goals

Building upon Lautert, Schliemann, and Leite's (2017) initial findings, in the present study, we conducted a more extensive and deeper analysis of the written answers and explanations given by 26 high-school students to solve one double proportionality problems, one multiple proportionality problem, or both of these problems. These 26 students were the only ones (out of the 82 students) who used the cross-product algorithm. Here, we compare the students' written solutions and verbal explanations during an interview, about each of their solutions. We focus upon (a) how the students used the cross-product algorithm to fully or partially represent and solve each problem, (b) considered and represented, in writing, the scalar and functional relations between quantities in the problem, and (c) displayed an understanding of scalar and functional relations while explaining their solution strategies.

Method

Participants

Participants were 26 Brazilian high schoolers who had used the cross-product algorithm to fully or partially solve a double or a multiple proportionality problem. Eight students used the algorithm for both problems, 10 did so for the double proportionality problem, and eight for the multiple proportionality problem.

The students, in years 1 to 3 of high school, were from a public school in Recife (Brazil) serving a middle- and upper-class population and considered to be one of the best in the northeast region of the country. Students enter the middle- and high-school institution from the sixth year of schooling (first year of middle school which goes from grades 6 to 9) through public lottery selection. In grade 7, different from what may happen in most textbooks and traditional school instruction, the textbooks adopted by the school appropriately place emphasis upon the relations among quantities in proportionality problems. After that, the cross-product algorithm is introduced as the sequence of computation steps to find the unknown values in a problem. Ratio and proportions are then part of the remaining middle-school years.

Procedure

The students were asked, during a math class, to individually solve, in writing, four problems, two on double proportion situations and two on multiple proportions. In this study, we analyzed the answers from students who used the cross-product algorithm to answer two problems, one of each type. Six to 15 days later, they were individually interviewed and asked to explain how they had solved the two problems we analyze here. The problems were as follows:

(a) A double proportionality problem:

A group of 6 students collected 20 kilograms of food in 5 days. How many kilograms would they have collected if the group was made by 18 students, working for 10 days?

(b) A multiple proportionality problem:

Marina is preparing a chocolate cake. The recipe states that, for each cup of milk, one should use 2 eggs. For each egg, one uses 3 cups of flour. How many cups of flour does Marina need to make a cake with 3 cups of milk?

Figures 1 and 2 show schematic representations of the relations among quantities in each of these problems

Presentation of problems was randomized, with half of the students solving the multiple proportion problem first and the other half the double proportion problem first. All students first explained the procedure for the multiple proportionality problem, followed by the other problem. All interviews were audio-recorded and transcribed in full.

Results

Students' written materials and answers to the interviews on how they had attempted to solve each problem were analyzed by two independent judges. Disagreements were discussed by the judges until they reached a consensus.

Two distinct but complementary aspects were analyzed: (a) how each student represented each problem in writing, considering relevant data, display of known and

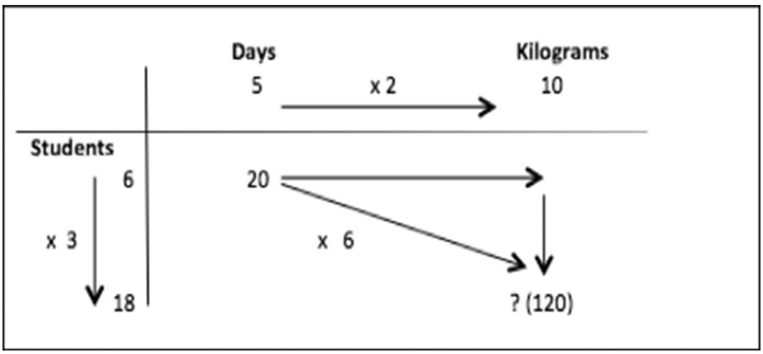

Fig. 1 Representation of the function (horizontal arrows) and scalar (vertical arrows) relations among quantities in the double proportionality problem (adapted from Vergnaud, 1988)

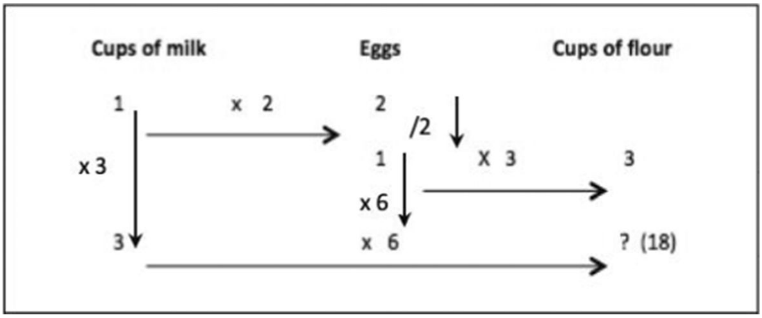

Fig. 2 Representation of the function (horizontal arrows) and scalar (vertical arrows) relations among quantities in the multiple proportionality problem (adapted from Vergnaud, 1988)

unknown values in cross-product arrangements, and computation steps towards solution and (2) whether they mentioned in the interview, implicitly or explicitly, their use of the cross-product algorithm and the scalar and/or functional relations between quantities in the problem.

The quantitative analysis that follows is complemented by a qualitative analysis of examples of the written work and corresponding interview answers by five students on the double proportionality problem and by four students on the multiple proportionality problem.

Quantitative Analysis

Overall, the 26 students' written work showed only two wrong answers to the double proportionality problem. In solving the multiple proportionality problem, one of the answers was correct but included mistakes in the cross-multiplication display of relations.

Table 1 shows that students more often used a table to organize data in the double proportionality problem. For the multiple proportionality problem, they more often listed the relevant data as pairs. Only one student used both forms of representation.

We analyzed students' representations in terms of numbers only, representation of the missing value as a letter (in general *x*), and/or references to quantities (e.g., 5 days, 19 students, 2 eggs, 3 cups of milk, etc.).

Table 2 shows that, for both problems, all students included numbers and letters in their written representation of the relevant data and most of them considered quantities by adding referents to the numbers.

Table 1 Frequency and percentage (in parentheses) of students by type of written representation of data for each problem

Problem				Total number of responses by the students
	Pairs	Table	Pairs and table	
Double proportionality	2 (11.00)	16 (89.00)	0 (0.00)	18 (100)
Multiple proportionality	12 (75,00)	3 (18.75)	1 (6.25)	16 (100)

Table 2 Frequency and percentage (in parentheses) of students who used numbers, letters, and/or quantities to represent each problem

Problem	Numbers and letters	Numbers, letters, and quantities	Total number of responses by students
Double proportionality	6 (33.0)	12 (77.0)	18 (100)
Multiple proportionality	2 (12.5)	14 (87.5)	16 (100)

Table 3 shows that, for the double proportionality problem, half of the students chose to work in two separate steps, using the cross-product representation for one or for both steps, therefore dealing, in sequence, with two simple proportionality problems. The first step determined how many kilograms would be collected by 18 students in 5 days (20 kg); the second step led to how many kilograms would be collected by the 18 students in 10 days (120 kg). The other half chose to use the algorithm for the compound cross-products, which considers the three ratios in the problem, setting up an equation that isolates the fraction with the unknown value as equal to the product of the other two. The two wrong answers to this problem were given by students using this algorithm: one of them mistakenly considered inverse proportions and the other, while solving the problem, isolated a known ratio instead of the ratio with the unknown quantity.

In the multiple proportionality problem, eight students showed, in writing, the cross-product arrangement for two steps in the problem and eight did so for the final step only. The first step would determine the number of eggs one would need to make a cake with three glasses of milk (6 eggs); the second would lead to the number of cups of flour to make the cake with 6 eggs (18 cups of flour). One answer to this problem, although correct, showed an error in writing the relationships between the variables in the cross-product representation. No student used the compound cross-product algorithm to represent or to compute the result.

The equality of ratios was explicitly shown in the use of the equals sign by five students (31%) in the multiple proportionality problem ($1/3 = 2/x$ or $1/3 = 6/x$) and by 12 students (67%) in the double proportionality problem ($6/18 = 20/x$).

In the double proportionality problem, 12 students computed the final result as prescribed by the algorithm, and the other six students did not show the computation steps in writing. For the multiple proportionality problem, eight students used the cross-product algorithm to compute the problem result and eight did not show the computation steps in writing. For both problems, use of mental computation may have

Table 3 Frequency of students and percentage (in parentheses) considering one or two cross-product representations for each problem

Problem	One simple cross-product	Two simple cross-products	Compound cross-products	Total number of responses by students
Double proportionality	4 (2.2)	5 (27.8)	9 (50.0)	18 (100)
Multiple proportionality	8 (50.0)	8 (50.0)	0 (0.0)	16 (100)

resulted from the fact that the numbers in the problem were small and the relations between them easily memorized.

Twelve students used arrows or lines indicating the functional relations between the variables in the double proportionality problem, and 11 students did so in the multiple proportionality problem. One student used arrows or lines for the scalar relations for each variable in the multiple proportionality problem and six and seven students did so for both relations for the double and for the multiple proportionality problem, respectively.

Students' *explanations* were classified by the same two judges in terms of reference to function relations, a combination of scalar and function relations, multiplication (not clear if scalar or function relation since they lead to the same result), or the cross-product algorithm (see Table 4).

In the double proportionality problem, 55% of the students explained their solutions by combining scalar and function relations. In the multiple proportionality problem, students tended to mention just function relations (37.5%) or a combination of scalar and function relations (31%). No student mentioned only scalar relations.

The following analysis of the solutions and explanations by nine students will hopefully clarify how their understandings about the relations among quantities in double and multiple proportionality problems relate to their use of the cross-product algorithm.

Qualitative Analysis: Written Solutions and Explanations

The Double Proportionality Problem I

The problem (see diagram in Fig. 1) stated that

> *A group of 6 students collected 20 kilograms of food in 5 days. How many kilograms would they have collected if the group was made by 18 students, working for 10 days*?

The following are five examples of students' written work and corresponding interview answers to this problem:

Example DP 1. Two steps, correct, explanation on relations

Table 4 Frequency of students and percentage (in parentheses) of explanations that considered each kind of strategy in double and multiple proportionality problems

Type of problem	Multiplication*	Function relations	Scalar and function relations	Cross-product	Total number of responses
Double	3 (17.0)	5 (28.0)	10 (55.0)	0 (0.0)	18 (100)
Multiple	3 (19.0)	6 (37.5)	5 (31.0)	2 (12.5)	16 (100)

*Not clear if scalar or function factor since they lead to the same result

Figure 3 shows the written work by a girl in year 3 of high school. She wrote a functional relation as 6 students/5 days and listed all data as a three-column table with arrows linking the values across the three variables, thus focusing on the functional relations throughout. She then chose to set up the arrangement for solving the problem in two steps, one for each part of the problem, again indicating the functional relations with arrows. She correctly computed the missing proportional values, 60 for the first part and 120 for the second one, and concluded that 120 kg would be collected (in Portuguese, “seriam arrecadados”).

In the interview, when asked to explain her work, she mentioned the cross-product algorithm and described how the different values in the problem were inter-related, mentioning functional relations (number of students to kilograms and number of students to number of days) and, for the last step, a scalar relation (5 days to 10 days):

> *I know that this is a compound rule of three. But I divided into two smaller parts to allow me to solve it. Then, in the first one, I used the information that 6 students would gather 20 kg; then I considered how many kilograms 18 students would gather during the same 5 days and found that it would be 60 kg. But it is not only 5 days, it is 10 days. Then, if 60 kg are for 5 days, x are (collected) in 10 days and then the total is 120.*

Example DP 2. Two steps, correct, explanation on algorithm and relations

Figure 4 shows the written work by a boy in year 2 of high school. He displayed the data for each of the two parts of the problem in the arrangement for the cross-product algorithm for solving simple proportionality problems. The solution to the first part of the problem shows the relevant equation and the steps towards its correct solution (60). Using this result, he displayed the arrangement for the second part of the problem and arrived at the correct solution to the problem (120).

In the interview, he described the functional relations between the quantities and mentioned use of the algorithm:

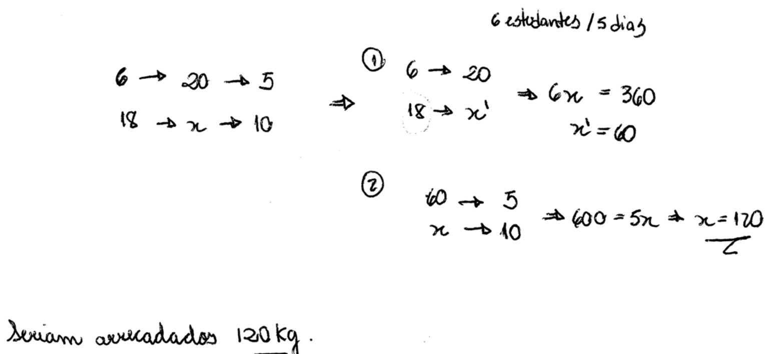

Fig. 3 Written work by a third-year student. The label (in Portuguese) refer to students (estudantes) and days (dias)

You have that 6 students gather 20 kilograms of food in 5 days. Then I used the cross-product: 6 students gather 20 kilograms in 5 days. So, 18 students will gather "x" kilograms, also in 5 days, to find out how many kilograms you will gather with 18 students in 5 days. So, I use the answer here, which is 60, but this 60 will be for 5 days and not for 10. So, I do another cross-product: In 5 days I gather 60 kilograms, then in 10 days I would gather "y". Then I solve the problem and get 120."

Example DP 3. Compound solution attempt, incorrect, fails to explain

Figure 5 shows the written work by a girl in year 2 of high school who tried to use the compound cross-product algorithm but fails in correctly applying it. She organized the data in the problem as a table, using horizontal arrows that suggest she was focusing on the functional relations. She tried to set up the information as compound cross-products, writing the ratios for number of students (6 to 18), kg (20 to x), and days (5 to 10), but failed to isolate the fraction with the unknown value as equal to the product of the other two. In doing so, she arrived at a wrong answer. The arrows in the written work suggest that she was considering scalar and function factors in representing the problem.

In the interview, she stated that she had used the compound rule of three (cross-product algorithm) but could not provide an explanation of why and how she did so and realized that her result was wrong.

Example DP 4. Compound solution correct, mistake in explanation followed by correction

The example in Fig. 6, by a boy in year 1 of high school, shows that he correctly applied the compound cross-product algorithm to solve the problem. He seems to have started with the computations on the upper right-hand side of the page. The

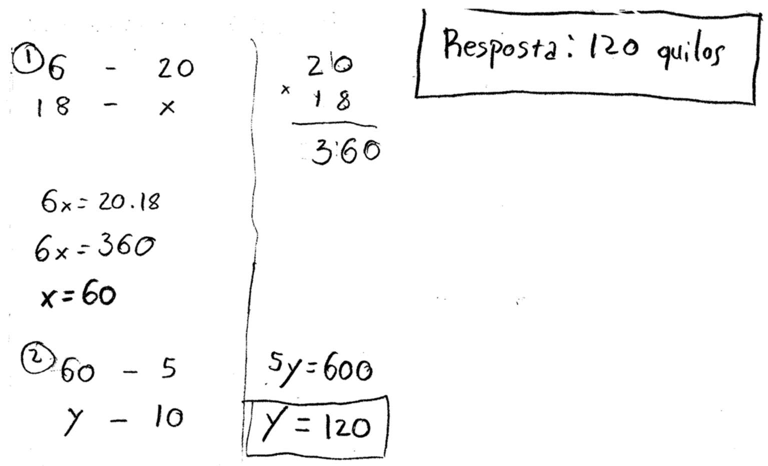

Fig. 4 Written work by a second-year student

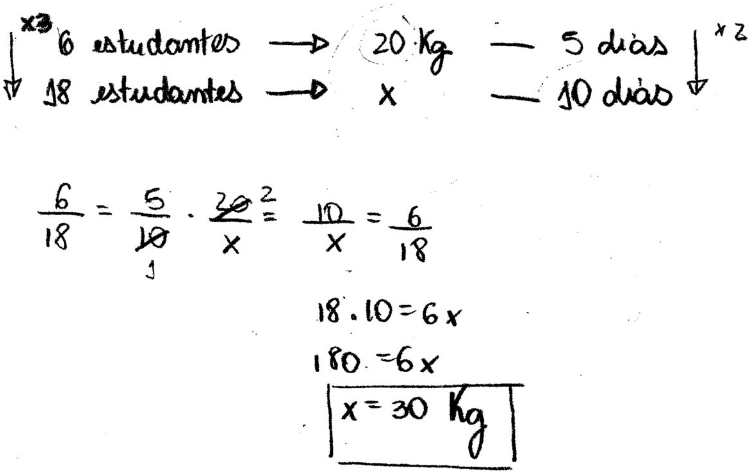

Fig. 5 Written work by a second-year student. The labels, in Portuguese, refer to students (estudantes) and days (dias)

computation of 20 (kg) divided by 5 (number of days) suggests that he was considering functional relations. He later displayed the data as a table and correctly set up the compound cross-product algorithm as the equality between the ratio with the unknown value (number of kilograms, that is, 20 to x) and the product of the ratios for the other two variables (number students, 6 to 18, and number of days, 5 to 10). He correctly applied the algorithm steps for solving the equation.

Even though he had correctly set up the values and solved the problem as one on directly proportional relations, in his explanation, the student started by mistakenly claiming that the problem included an inverse proportional relation:

> *... inversely proportional to number of days and proportional to number of students; then I multiplied six times five... I used the cross-product.*

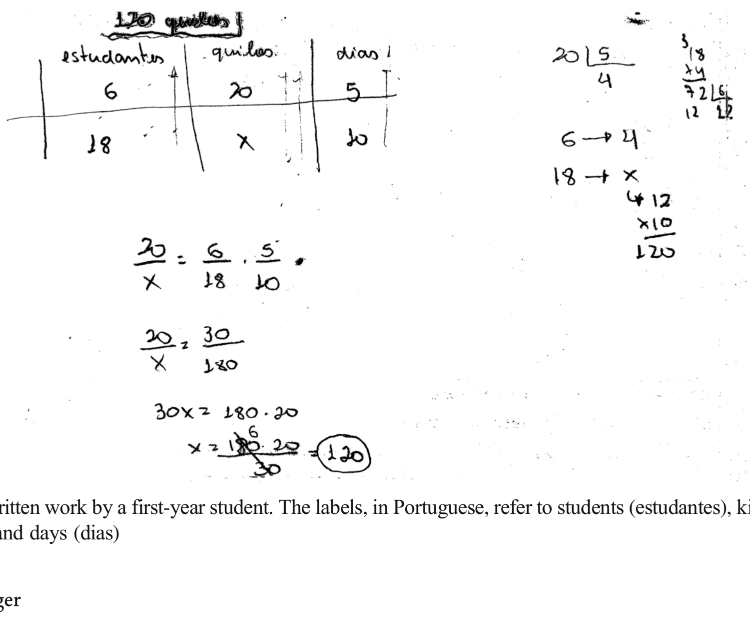

Fig. 6 Written work by a first-year student. The labels, in Portuguese, refer to students (estudantes), kilograms (quilos), and days (dias)

Upon the interviewer's question on what was inversely proportional, he recognized his mistake and stated, again mentioning function relations, that

> *It is directly proportional in both. If you increase the number of students you increase the number of kilos and increase the number of days.*

Example DP 5. Compound, correct, explanation on algorithm

Figure 7 shows the written work by a boy in year 3 of high school. He displayed all data as a three-column table. The vertical line and the words "diretamente proporcional" (directly proportional) suggest that he was considering the scalar relations. He correctly set up the elements of a compound cross-product algorithm as the equality between the ratio with the unknown value and the product of the other two ratios. He concludes that 120 kg of food would be collected (in Portuguese, "seriam arrecadados 120 kilos de alimentos").

In the interview, the student mentioned the functional relations in the problem to explain the steps for setting up the data and find the solution using the compound cross-product algorithm, as follows:

> *... if we have 6 students for 20 kg, in 5 days, you can relate them without inverting the fractions. This is what I did. I related, for example, 6 students to 5 days and to 20 and placed the x above, since this is what I want to discover.*

The Multiple Proportionality Problem

The problem (see diagram in Fig. 2) stated that

> *Marina is preparing a chocolate cake. The recipe states that, for each cup of milk, one should use 2 eggs. For each egg, one uses 3 cups of flour. How many cups of flour does Marina need to make a cake with 3 cups of milk?*

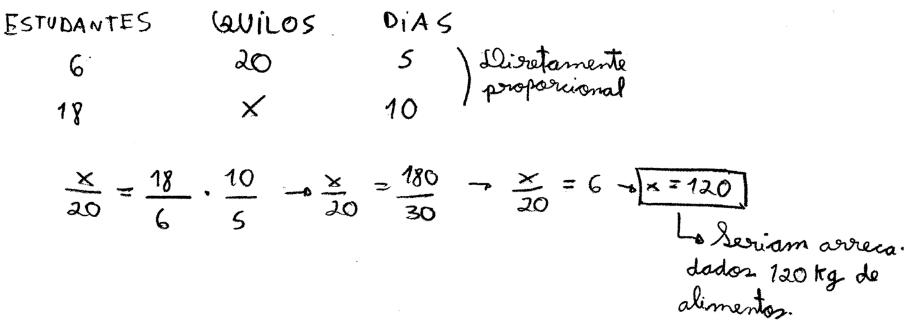

Fig. 7 Written work by a third-year student

Example MP 1. Two steps, correct, explanation on relations.

Figure 8 shows the written work by a boy in year 1 of high school. He organized the data in the problem as pairs and as ratios between two variables (the functional relations). The first ratio refers to the glasses of milk per number of eggs and the second to number of eggs per number of cups of flour (xt for cup of flour). He then displayed the information as the equality of two ratios to find the number of eggs (o) one would need to make a cake with three glasses of milk (3 cl). Using the result for this first step (6o), he did the same to compute the number of cups of flour (xt) one would need to make the cake with 6 eggs (6o). He arrived at 18xt as the result and wrote that Marina would need 18 cups of flour. In both cases, the correct result is shown without indication of the computational steps.

In the interview, the student claims that this is a proportionality problem but does not refer to the cross-product algorithm nor to the computations he may have performed. Instead, he repeats the problem's statements, expressing the functional relations among quantities. He also emphasized the scalar factor (three) to determine the number of cups of flour for three glasses of milk. This description matches the final computation he had to do to compute the value of *x*:

> *... with one cup of milk, after computing, I use six cups of flour. ... if for one cup of milk I use six cups of flower, for three cups (of milk) I use six (cups of flour) times three, which gives (...) eighteen.*

Example MP 2. Compound correct answer, mistake in procedure, mistake in explanation.

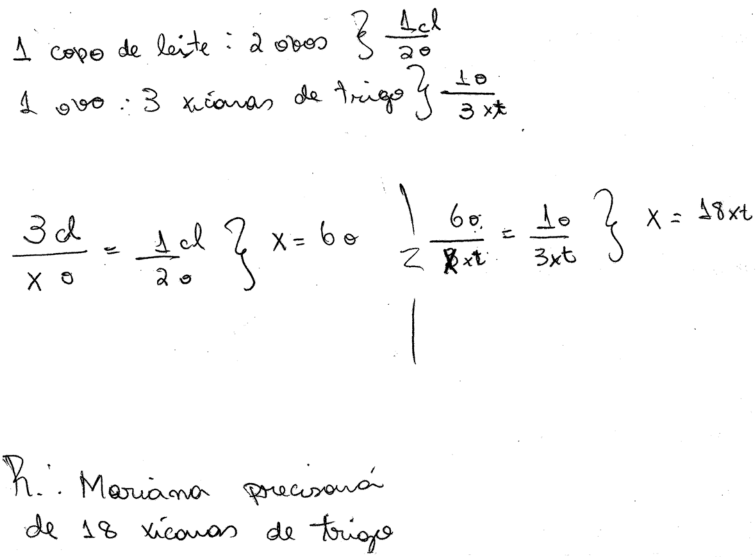

Fig. 8 Written work by a first-year student. The labels for values refer to, in Portuguese, glasses of milk (cl, copos de leite), egg (o, ovos), and cups of flour (xt, xícaras de trigo)

Figure 9 shows the written work by a girl in year 1 of high school. She listed two data pairs relating the values for different variables (functional relations), as described in the problem, and two pairs for the solutions to the first and second steps of the problem. The elements in the pairs showing the solutions were linked by arrows, indicating the functional relations. This was followed by the correct answer statement: 18 cups of flour. She also included the cross-product algorithm representation for computing the final result but, mistakenly, listed that 1 glass of milk (1c) would correspond to 6 eggs (6o) and 3 glasses of milk (3c) would correspond to x. Nevertheless, the expression she wrote for computing the result, $x = 3 \times 6 = 18$, led to the correct numerical result.

In the interview, she was at first uncertain and, after reading the sequence of pairs she had written, she stated, once more, the wrong relations shown in the algorithm representation, mistakenly referring to eggs instead of cups of flour. She then mentions the cross-product algorithm (rule of three) and explains her solution by referring to functional and scalar relations, applying the scalar factor, 3, to six eggs. She did not clarify if the result, 18, refers to eggs or to cups of flour:

> *... wait, let me remember, one glass of milk, we need six eggs, then for 3 glasses of milk we need x eggs. Then we just multiply, do a rule-of-three. Then x is equal to six times 3 which is eighteen.*

Example MP 3. Compound, correct, with explanation on scalar and functional relations.

Figure 10 shows the written work by a female student in year 2 of high school. She organized the data in a table and also set up two sets of data for computing the unknown values, as the equality of ratios. Focusing on the scalar relations, she writes the equality between the ratio for different number of eggs (2 to 1) and for different number of cups of flour (x to 3), and writes the solution as $x = 6$. After that, she writes

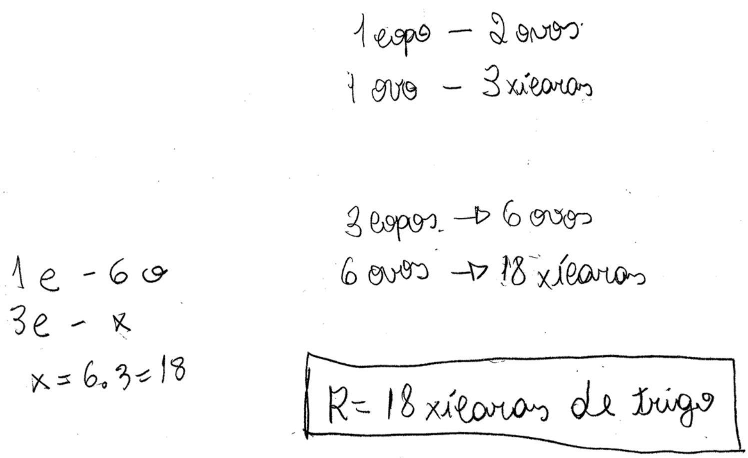

Fig. 9 Written work by a first-year high schooler. The labels for values refer to, in Portuguese, the words for glasses of milk (copo or c), eggs (ovos or o), and cups (xícaras)

the equality between the ratio for number of glasses of milk (1 to 3) and the ratio for number of cups of flour (6 to y), and the solution, $y = 18$.

She explains her solution for the first step mentioning the relation of one egg to three cups of flour, a functional relation:

> *If I have two eggs I will need six (cups of flour) because you just multiply by three. Then I will need six cups of flour.*

For the second step, she mentions functional and scalar ratios:

> *And I know that, for one glass of milk, I need two eggs (*a functional relation*) and it asks about when I have three cups of milk (*a scalar transformation*). Then, if I have three (glasses of milk) I will need six eggs. And because for two eggs I need six cups of flour (*a functional relation*), as I had discovered, then I just need to multiply by three (*the functional factor*). So, if I have six eggs I will need eighteen cups of flour.*

Example MP 4. Compound, correct, with explanation related to the algorithm rules.

Figure 11 shows the written work of a girl in year 3 of high school. She displayed all data in the problem as a rough table (first three written lines of her work). On the next two lines on the left side of the page, she repeated two relevant pairs of information: 1 cup of milk (copo de leite)–2 eggs (ovos) and 1 egg (ovo)–3 cups of flour (xícaras de trigo) (third and fourth line on the left side of the page). On the right side of the page, she showed the data arrangement and solution for the first step in the problem as, 2 eggs–x and 1 egg (ovo)–3 with, underneath, $6 = x$, and the conclusion 2 eggs (ovos)–6 cups of flour (xícaras de trigo). After that, she writes data on the question to be answered: 3 glasses of milk (copos de leite)–x cups of flour (xícaras de trigo), followed by the information that 3 glasses of milk (copos de leite) correspond to 6 eggs (ovos),

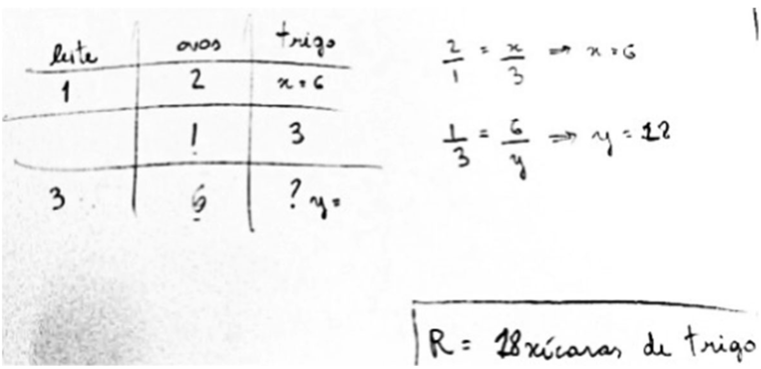

Fig. 10 Work of a second-year high schooler. The column labels in Portuguese correspond to glasses of milk (leite), eggs (ovos), and cups of flour (trigo)

and the answer to the problem: 3 glasses of milk (copos de leite)–18 cups of flour (xícaras de trigo). These results may have been obtained through mental computation, after she had listed, on the right side of the table, the pairs 1 egg (ovo)–3 and 6 eggs (ovos)–x, followed by the cross-product arrangement and the equality of $(1/6 = 3/x)$ leading to the correct result, $x = 18$.

In the interview, she emphasized her use of the cross-product algorithm to solve the problem:

> *It says that, for one egg it needs three cups of flour. And three cups of flour it means ... It uses six eggs...*

She seems unsure at this point. In fact, she had correctly written that 3 cups of milk, not 3 cups of flour, corresponds to 6 eggs. She then proceeds and simply explains:

> *Then it asks for how many cups of flour for six eggs. Then I did the rule-of-three and found eighteen.*

Discussion

We examined the written solutions and explanations by 26 high schoolers who used the cross-product algorithm to represent and solve double and multiple proportionality problems. We focused on their awareness of the scalar and functional relations among the quantities in the problem.

We found that their written representations of the problems while they attempted to solve them are set up after considerations about the scalar and/or functional relations

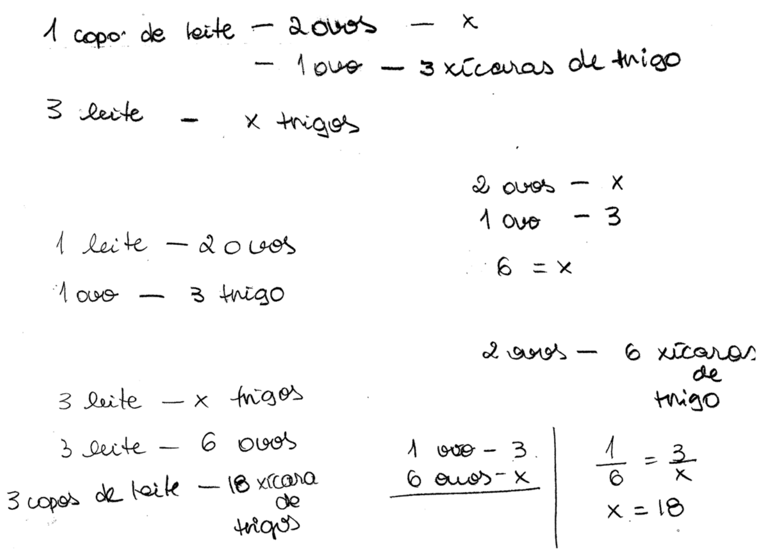

Fig. 11 Written work by a third-year high schooler. The words in Portuguese correspond to glasses of milk (copo de leite), eggs (ovos), and cups of flour (xícaras de trigo)

among the quantities in the problem. These are depicted as data tables, similar to the diagrams proposed by Vergnaud (1988), or as pairs of values, with arrows to indicate the scalar or the functional relation between two elements in a pair. Most of them also explain their solutions in terms of these relations.

The majority of students included numbers as well as their referents in their written work. A few also represented the unknowns as letters. Only a few students represented the problem data as an equality of two ratios and, perhaps because the numbers were small and computations could be easily performed mentally, only a few showed the computational steps to reach the solution to the problem.

Instruction in the school attended by this study's participants placed emphasis upon the relations among quantities (numerical values and their referents) in proportionality problems before the cross-product algorithm was introduced. This may explain students' use, awareness, and understanding of scalar and functional relations in our data. This tentative conclusion, however, needs to be evaluated by studies of solutions by students from schools where introduction of the cross-product algorithm is not preceded by instruction on relations among quantities.

Participants' use of number with referents is in keeping with Nunes' (1997) findings that students prefer to do so. Although reasoning about pure number relations is an important part of mathematics, reference to numbers allows the students to more easily consider the problem situation and follow the steps in the process of solving verbal problems.

As emphasized by Vergnaud (1994, p. 59),

> *... we must devote all our attention to conceptual aspects of schemes and to the conceptual analysis of the situations for which students develop their schemas, in school or in real life.*

Our data suggest that students' understanding of relations among quantities in verbal problems can in fact constitute a basis for their understanding of mathematical concepts, representations, and algorithms. Thus, it may be more helpful for students to associate pairs of physical quantities, or of numbers with their referents, rather than pairs of pure numbers, as they usually appear in the cross-product representation.

Our study was limited to students' responses to problems which, without exception, refer to situations involving proportional relations among quantities. As highlighted by others (e.g., Buforn et al., 2018; Fernández et al., 2010; Van Dooren et al., 2005; Van Dooren, De Bock, Evers, & Verschaffel, 2008; Van Dooren, De Bock, Janssens, & Verschaffel, 2008), understanding proportionality requires not only solving proportionality problems, but also discriminating between proportional and non-proportional relationships. Future studies in this area would benefit from examining students' answers and strategies while they deal with proportional and non-proportional situations. This would further clarify how they represent and understand relations between quantities.

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