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Chinese Students' Hierarchical Understanding of Part-whole and Measure Subconstructs

Zheng Jiang[1] · **Ida Ah Chee Mok**[1] · **Jianhua Li**[2]

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Abstract
Fractions are a very important but difficult topic in elementary and secondary mathematics curricula around the world. The difficulty in teaching and learning fractions is usually caused by their various subconstructs. To facilitate learning of fractions, a test consisting of 20 items was developed to examine 310 sixth graders' hierarchical understanding of the part-whole and measure subconstructs of fractions. Rasch analysis and cluster analysis were used to analyze the data. The results showed six distinguishable levels of understanding of the two subconstructs. Briefly, constructing fractions alone was the easiest to develop, followed by repartitioning/reconstructing and regarding fractions as numbers. The abilities involved at the highest level were successively partitioning, defining a unit interval freely and placing fractions on number lines. The results also indicated that most of the students were able to deal with part-whole subconstruct problems but still had difficulty handling measure subconstruct problems by the end of their elementary education. They could construct fractions/the whole, recognize the number notion of fractions and fractional unit and measure distances, but lacked enough ability to partition successively and place fractions on number lines. This study provides a clearer picture of hierarchy of understanding the two subconstructs and contributes to the corresponding development of instructional resources and the teaching/learning of fractions.

Keywords Fractions · Hierarchical understanding · Measure subconstruct · Part-whole subconstruct · Rasch model

✉ Zheng Jiang
jiangzheng20081022@126.com

Ida Ah Chee Mok
iacmok@hku.hk

Jianhua Li
lijianhua@bnu.edu.cn

Extended author information available on the last page of the article

Introduction

Fractions are the basis of rational number learning and predictors of secondary school students' mathematics achievement (Siegler et al., 2012; Torbeyns, Schneider, Xin, & Siegler, 2015). However, they are also a rather complicated and difficult concept in primary and secondary mathematics (Behr, Harel, Post, & Lesh, 1992; Behr, Lesh, Post, & Silver, 1983; Gabriel et al., 2013). Some of the most important factors contributing to the complexities and difficulties of teaching and learning fractions are their various subconstructs: part-whole, measure, ratio, quotient and operator (e.g. Charalambous & Pitta-Pantazi, 2007; Gabriel et al., 2013; Kieren, 1976; Lamon, 2012). For example, 2/3 can be conceived as a part of a whole (two parts of a whole having three equal parts), as a measure (as a sum of two measure units 1/3), as a ratio (two to three), as a quotient (two divided by three) and as an operator (two thirds of a quantity). Among the five subconstructs, the part-whole subconstruct has often been treated as fundamental to the development of the other four (Behr et al., 1983). It has prevailed in most mathematics textbooks and has played a predominant role in fraction instruction (Charalambous & Pitta-Pantazi, 2007; Lamon, 2012; Pantziara & Philippou, 2012). The measure subconstruct is also important to the development of children's knowledge of fractions (Fuchs et al., 2013) and a dominant factor in building children's sense of rational numbers (Lamon, 2012). However, children have been found to be the least competent in the measure subconstruct, compared with other subconstructs (Charalambous & Pitta-Pantazi, 2007; Ni, 1999, 2001). Therefore, this study focuses on children's understanding of part-whole and measure subconstructs so that the results can provide insights into facilitating their learning of fractions. The part-whole subconstruct is included in this study to determine how far away from the measure subconstruct it is, with the hope of building connections between them.

Mathematical understanding, the major goal of mathematics teaching and learning, is hierarchical (e.g. Hart, 1981b; Hart et al., 1981; Hiebert & Carpenter, 1992; Sierpinska, 1990). The hierarchical approach to teaching mathematics can facilitate children's learning. Nonetheless, although some studies have examined students' hierarchical understanding of fraction-related knowledge (Hart, 1981a; Nicolaou & Pitta-Pantazi, 2016; Pantziara & Philippou, 2012; Pirie & Kieren, 1989), there has been a lack of clear quantitative evidence supporting the understanding path of the part-whole and measure subconstructs (Nicolaou & Pitta-Pantazi, 2016; Zhang & Xu, 2016).

The Mathematics Curriculum Standards for Compulsory Education (MCSCE) in mainland China proposed understanding the meanings of fractions as a requirement in primary mathematics (Ministry of Education, 2001, 2012). However, the logical progression involved in the part-whole and measure subconstructs was not mentioned. Therefore, to provide a clear picture of the sequence of learning the two subconstructs and to facilitate learning of fractions, this study investigates the Chinese students' hierarchical understanding of the part-whole and measure subconstructs of fractions. Specifically, this study intends to answer the following two research questions:

1. What are the levels of understanding of the part-whole and measure subconstructs?
2. What levels of the part-whole and measure subconstructs of fractions can Chinese students achieve by the end of their elementary education?

Theoretical Framework and Literature Review

Fractions: the Part-Whole and the Measure Subconstructs

Fractions have five subconstructs: part-whole, ratio, operator, quotient and measure (e.g. Kieren, 1976; Lamon, 2012). Because this study focuses only on the part-whole and measure subconstructs, the following paragraphs mainly review the studies in these two areas.

The part-whole subconstruct is defined as "a situation in which a continuous quantity or a set of discrete objects are partitioned into parts of equal size" (Charalambous & Pitta-Pantazi, 2007, p. 296). It is fundamental to the development of the other four subconstructs (Behr et al., 1983; Charalambous & Pitta-Pantazi, 2007; Kieren, 1976), and it has been used as a model to introduce fractions in many countries (e.g. Lamon, 2012; Pantziara & Philippou, 2012). To master the part-whole subconstruct of fractions, students need several abilities, including the ability to discern whether a continuous quantity or a set of discrete objects is divided into parts of equal size and then to proceed with partitioning and unitizing to construct a fraction and conceive the whole (Hansen & Leeming, 2014; Lamon, 2012; Pantziara & Philippou, 2012). During these activities, also necessary are a number of ideas: (1) the parts must exhaust the whole; (2) the more parts the whole is divided into, the smaller the produced parts become; and (3) the size, shape, colour and arrangement of the equal parts cannot affect the relationship between the parts and the whole (Charalambous & Pitta-Pantazi, 2007; Hansen & Leeming, 2014; Lamon, 2012; Pantziara & Philippou, 2012). In addition, students are able to repartition already equipartitioned wholes and reconstruct the unit (Baturo, 2004; Charalambous & Pitta-Pantazi, 2007; Pantziara & Philippou, 2012).

The measure subconstruct can predict the confluence of several other subconstructs of fractions (Hannula, 2003). It is an important factor in developing students' fraction knowledge and building rational number sense (Fuchs et al., 2013; Lamon, 2012). In the measure interpretation, a fraction is associated with two notions: numbers and the measure assigned to some intervals (Charalambous & Pitta-Pantazi, 2007; Lamon, 2012). Specifically, students' understanding of the measure subconstruct relies on four abilities: (1) to identify a fraction as a number, namely, how big the fraction is; (2) to comprehend the density property of fractions, that is, that there are an infinite number of fractions between any two given fractions; (3) to perform successive partitioning other than halving; and (4) to define a unit fraction (i.e. 1/a) as a unit of measure and to use it repeatedly to measure distances from a zero point on a number line (Charalambous & Pitta-Pantazi, 2007; Hannula, 2003; Hansen & Leeming, 2014; Lamon, 2012; Pantziara & Philippou, 2012). Students should be able to identify a distance from the zero point as a fraction, locate a fraction on a number line and conversely identify a fraction represented by a certain point on the number line (Hansen & Leeming, 2014). In this study, the abilities or skills required by part-whole and measure subconstructs were treated as the criteria in developing the test.

In the past three decades, a wealth of research has been devoted to the teaching and learning of fractions (Čadež & Kolar, 2018; Lamon, 2012; Pitkethly & Hunting, 1996). It has generally shown that students have a better understanding of the part-whole subconstruct than the measure subconstruct. Ni (1999, 2001) tested 413 fifth and sixth grade Chinese students to determine whether they could understand the concept of fractions, compute fractions or apply them to solve problems. Ni found that both the fifth and sixth-grade Chinese students had developed a full understanding of the part-whole meaning of fractions. However, they did not perform well on the measure subconstruct. Hannula (2003) examined 3067 fifth and seventh grade Finnish students' understanding of two interpretations of 3/4: as part of a whole and as a location on a number line. He found that the part-whole comparison played a dominant role in the students' thinking. In contrast, only 20% of the fifth graders could locate the fraction 3/4 on a number line marked with 0 and 1, and even the seventh-grade students had difficulty identifying fractions as numbers on a number line. Charalambous and Pitta-Pantazi (2007) investigated 646 fifth and sixth grade Cypriot students' understanding of five interpretations of fractions and the equivalence and operations of fractions. They found that the mean of the students' performance on the part-whole subconstruct was significantly higher than it was on the measure subconstruct. Kurt and Cakiroglu (2009) investigated 1456 Turkish middle grade students' performance in translating various fractional representations and showed that students performed best on tasks related to area and discrete objects and worse on tasks involving number lines. These students were incapable of conceptualizing a fraction as a point on a number line. Tunç-Pekkan (2015) examined the performance of 656 fourth and fifth grade US students on parallel fractional knowledge problems that used different external graphical representations (circle, rectangle and number line) and found that the students' performance on the items with number lines was significantly lower than it was on the circle and rectangle items. Van Hoof, Degrande, Ceulemans, Verschaffel, and Van Dooren (2018) conducted a 2-year longitudinal study following 201 fourth ($n = 113$) and fifth grade ($n = 88$) students in Belgium, investigating their understanding of the comparison, operation and density of fractions. Their results showed that only a limited number of students could fully understand the density property of fractions by the end of sixth grade.

The Hierarchical Understanding of Fractions

Numerous studies have focused on the nature and development of mathematical understanding (e.g. Carpenter & Lehrer, 2009; Hart, 1981b; Hiebert & Carpenter, 1992). An individual's mathematical understanding is hierarchical (Carpenter & Lehrer, 2009; Hart, 1981b; Sierpinska, 1994). This hierarchy has been described as a learning sequence or sequence of understanding that implies a string of skills, levels or stages that are ordered from simple to complex (Hart, 1981b). Mathematical understanding is not an all-or-nothing phenomenon, and all complex mathematics concepts can be understood on a number of levels (Carpenter & Lehrer, 2009). For example, Herscovics and Bergeron (1983) put forward four levels of understanding: intuitive, procedural, abstract and formal. The process of understanding has also been viewed as a dialectic game between two ways of grasping the object of understanding (Sierpinska, 1994). For instance, Sfard (1991) proposed that mathematical concepts can be understood in two ways: operationally as processes and structurally as objects. Operational understanding usually precedes structural understanding in the process of concept acquisition. In addition, representational fluency is an important part of mathematical understanding (Cramer, 2003). A person's understanding of a mathematical concept is reflected in his or her ability to represent it in different modes and by making connections within and between these modes of representation (Cramer, 2003). For example, for a student to understand fractions, he or she must be able to express fraction ideas using an area model, a set model, a line model or written symbols (Lesh, Landau, & Hamilton, 1983). Therefore, these models have often been presented in mathematics textbooks and test instruments to assess students' understanding of fractions.

Based on theories of mathematical understanding, a small number of studies have partly addressed the levels of understanding related to the subconstructs of fractions. Two studies are introduced in detail below to illustrate the existing level of children's hierarchical understanding of the part-whole and measure subconstructs of fractions.

The Work of Hart et al. in the UK. Rooted in the Piagetian framework, Hart (1981a) developed four levels of understanding for fraction problems and computations, using a sample of five hundred fifty-five 12- to 13-year-olds and five hundred twenty-three 14- to 15-year-olds from UK secondary schools. The fraction problems mainly involved a portion of the part-whole subconstruct, the equivalence and the addition and subtraction of fractions for ages 12–13. This extended to multiplication and division for ages 14–15. Item facility (i.e. correct percentage) was used to obtain the four levels of understanding for these problems. With respect to interpreting fractions, for the 12- to 13-year-olds, naming shaded equal parts and representing fractions in area models were categorized as Level 1, identifying the meaning of a fraction as a subset of a set or naming a given configuration of pieces were at Level 2 and reconstructing fractions was at Level 3. For the 14- to 15-year-olds, representing parts of a whole was at Level 1, and reconstructing fractions was at Level 2. For all groups of students, understanding the differences in the wholes of fractions was the most difficult, but the level of this was not mentioned. Although Hart's work partially involved British students' understanding of the part-whole subconstruct, the full understanding sequence of the part-whole and measure subconstructs was not elaborated, and the hierarchy was also strongly dependent on the items included in the test (Hart, Brown, Kerslake, Küchemann, & Ruddock, 1985). Therefore, it is necessary to elaborately examine the hierarchical understanding of the two subconstructs using a more rigorous method.

The Work of Pantziara and Philippou in Cyprus. Sfard (1991) theorized that understanding a concept needs to be experienced in three stages: interiorization, condensation and reification. To examine this, Pantziara and Philippou (2012) measured 321 Cypriot sixth graders' hierarchical understanding of fraction-related content. Their test consisted of 21 items that examined five aspects: a portion of the part-whole and measure subconstructs, fraction equivalence, comparison and addition. Six items were related to the part-whole subconstruct, including constructing a fraction and constructing a whole. Four items were associated with the measure subconstruct, including locating a fraction on a number line and identifying the density property of fractions. Rasch analysis and cluster analysis were used to classify the students' understanding of fraction-related content on six levels. Constructing, representing a fraction and locating the fraction 3/5 on a number line with given unit intervals were at Level 3. Constructing a fraction in different models and constructing a whole were at Level 4. Locating fractions on a complex number line was at Level 5, and identifying the density property of fractions was at Level 6. Pantziara and Philippou's work provided useful information for the items and methods used in this study. However, some of the abilities of the two subconstructs, such as successive partitioning, were not examined.

Fractions Included in the Chinese Mathematics Curriculum and Textbooks

In mainland China, the MCSCE completely allocates fractions to primary mathematics (Ministry of Education, 2001, 2012). Pursuant to the MCSCE, most primary mathematics textbook series introduce the concept of fractions and the addition/subtraction of fractions using the same denominator in the third grade and then introduce the meaning of fractions and the four operations of fractions in the fifth and sixth grades (Zhang, 2014).

In addition, the MCSCE requires students to understand the interpretations of fractions in a variety of specific situations (Ministry of Education, 2001, 2012). However, the logical progression to be followed in the part-whole and measure subconstructs is not mentioned. Three popular textbook series used in Beijing were examined for this study. Among the three series of textbooks, we applied a content analysis with 3 steps to the most popular ones published by People's Education Press (PEP) and used them as a reference. Firstly, we examined arrangement sequences of PEP textbooks related to fractions in detail. Secondly, we coded the type of items in two fraction-related chapters (i.e. one chapter regarding the fraction concept in textbook 3A and one chapter about the meanings of fractions in textbook 5B). Thirdly, based on the theoretical framework of this study, the items were coded from the perspective of possible abilities related to the part-whole and measure subconstructs. The analysis showed that the PEP textbooks assign relatively parallel but unbalanced arrangements to the part-whole and measure subconstructs (People's Education Press, 2007, 2014; Wang & Peng, 2016). The part-whole and measure subconstructs are presented formally and simultaneously in a section of the Grade 5 textbook (People's Education Press, 2007, 2014). In contrast, the part-whole subconstruct dominates the textbooks, whereas the measure subconstruct appears less frequently. The part-whole subconstruct is used to introduce the fraction concept as the starting point in the third grade textbook and plays an important role in the comparison of fractions, improper fractions and fraction operations in the fifth grade textbooks (Zhang, 2014; Zhang & Xu, 2016). In contrast, most of the content on the measure subconstruct is scattered in limited exercises in a chapter of the Grade 5 textbook. This includes locating fractions on a number line with a predetermined unit, successive partitioning and the density property of fractions (People's Education Press, 2007, 2014).

In mainland China, mathematics textbooks are mediators between the intended curriculum and the implemented curriculum (van den Ham & Heinze, 2018), and Chinese mathematics teachers design their teaching closely based on mathematics textbooks (Xu, 2013). Given that textbooks often drive instruction, this study could also offer some insights as to whether the coverage of certain topics in the textbooks and their sequencing might reflect in students' understanding and performance on the test.

Method

Participants

This study was conducted in five primary schools in Beijing, China. The participants were selected through convenient sampling. Sixth graders were chosen because by this grade, students in China have already learned the part-whole and measure subconstructs of fractions. In total, 310 sixth graders voluntarily participated in the test, which was administered by their class mathematics teachers and took about 25 min to complete. The average age of the participants was 11.5 years. There were 164 boys and 144 girls (2 unspecified). At the time of the test, all of the schools in mainland China followed the MCSCE (2012). With respect to the students' learning experiences with fractions, the participants had learned the concept of fractions and the addition and subtraction of fractions using same denominator in the third grade. In the fifth grade, they had learned the meanings of fractions, the properties of fractions and the addition and subtraction of fractions. Finally, in the sixth grade, they learned how to multiply and divide fractions. Among the participants, 25.5% were from the first band of schools, 37.7% were from the second band of schools, and 36.8% were from the third band of schools. These schools had different levels of school quality (e.g. educational resources, teaching quality, reputation), with the first band of schools being the best. Therefore, it was assumed that the participants had a distribution of different mathematical abilities.

Instrument

Based on the abilities required by part-whole and measure subconstructs, a test consisting of 20 items was developed to assess the participants' understanding of the two subconstructs (see Appendix). Half of the items measured the part-whole subconstruct, and the other half focused on the measure subconstruct. Except for items M2, M4 and M5, all of the items were drawn from published studies (e.g. Charalambous & Pitta-Pantazi, 2007; Hart, 1981a; Pantziara & Philippou, 2012).

Item P1 (Wong, 2010) assessed naming of equally partitioned parts represented by an area model. Items P2 and P3 (Pantziara & Philippou, 2012; Saxe, Taylor, McIntosh, & Gearhart, 2005) required the students to construct fractions by discerning unequal sizes, partitioning a continuous quantity into equal-sized parts and understanding that the parts must exhaust the whole. Items P4 (Charalambous & Pitta-Pantazi, 2007) and P5 (Pantziara & Philippou, 2012) asked the students to represent the same fraction in area or set models. To do this, they had to understand that the relationship between the parts and the whole is preserved regardless of the size, shape or arrangement of the equivalent parts and then build relationships between the different representations and equally partition the discrete objects. Item P6 (Lamon, 2012; Pantziara & Philippou, 2012) asked the students to represent a given fraction by partitioning a set of discrete objects into equal sets. Items P7 (Charalambous & Pitta-Pantazi, 2007) and P8 (Cramer, Post, & del Mas, 2002) asked the students to construct the whole when a fraction and fractional parts were given in a set model and a length model, respectively. Item P9 (Hart et al., 1981) required the students to repartition shaded areas into six equal parts and reconstruct a fraction. Item P10 (Hart et al., 1981) assessed the students' ability to conceive the wholes when two different fractions were given in a real-life problem.

Item M1 (Behr & Post, 1992) measured the students' ability to perceive the quantitative notion of a fraction and how big the fraction is. Item M2 was designed to examine the students' understanding of a unit of measure (i.e. 1/a), which is a fundamental concept of the measure subconstruct (Charalambous & Pitta-Pantazi, 2007; Hansen & Leeming, 2014; Lamon, 2012). Item M3 (Pantziara & Philippou, 2012) required the students to place a given fraction at a point that was three intervals from zero in terms of the given unit distance. Items M4 and M5 assessed the students' ability to choose a given subinterval as a unit of measure and then use it to measure given distances from zero as a fraction (Hansen & Leeming, 2014). Here, the number line is regarded as a ruler. Item M6 (Pantziara & Philippou, 2012) asked students to identify a fraction represented by an area model, to choose a subunit from given subintervals and then to use it to locate the same fraction on a number line from 0 to 1. Item M7 (Lamon, 2012; Pantziara & Philippou, 2012) examined the students' ability to determine the size of subunits and to identify a fraction represented by a point on a number line from 0 to 1/2. Item M8 (Charalambous & Pitta-Pantazi, 2007) only gave the points 0 and 5/9 on a number line and asked the students to successively partition and define the size of the unit interval and then to choose a point to represent the number on the number line. Item M9 (Charalambous & Pitta-Pantazi, 2007; Pantziara & Philippou, 2012) examined the students' understanding of the density property of fractions. Item M10 (Charalambous & Pitta-Pantazi, 2007) measured the students' ability to identify a fraction as a number.

The test was translated into Chinese, and an experienced primary mathematics teacher was invited to check the wording, the arrangement of the items and whether there were any items that were too difficult for the participants. A pilot test was carried out to examine the reliability and wording. After this, the wording of some items was revised.

Data Analysis Methods

The participants' responses were scored on a 0–1 scale, with 1 point given for a correct response and 0 for an incorrect one. The dichotomous Rasch model and a cluster analysis developed by Marcoulides and Drezner (2000) (MD method) were used for data analysis. The dichotomous Rasch model is a rather simple but powerful item response theory (IRT) model used to analyze dichotomous data (Andrich, 2004; Fischer & Molenaar, 2012). It assumes that the probability of a person succeeding at answering an item is a function of the difference between the person's ability and the item's difficulty. As an IRT model, the Rasch model addresses some deficiencies inherent in traditional approaches to fraction assessment. It could provide student ability estimates independent of particular item groups and item difficulty estimates independent of particular sample (Embretson & Reise, 2000; Fischer & Molenaar, 2012). The Rasch model also enabled us to predict the probability of how a given student would answer other items and how other students would answer the designated item without actually observing the response to such item-person pairs (Bond & Fox, 2015; Fischer & Molenaar, 2012). Furthermore, using Rasch model enabled us to acquire some desirable properties which other IRT models (e.g. two-parameter IRT model) do not have (Bond & Fox, 2015). Apart from its simplicity and small sample size, the Rasch model provided this study with interval measures and invariant comparisons between item difficulties and between student abilities (Andrich, 2004; Bond & Fox, 2015; Embretson & Reise, 2000). It also ordered the items and students on a logit scale and provided this study with a clear item-person map related to the two subconstructs (Bond & Fox, 2015). The MD method, a procedure used to detect pattern clustering in data, segments the observed measurements into constituent groups (or clusters) so that the members of each group are similar based on the selected criteria (Marcoulides & Drezner, 2000). Based on the item's difficulty derived from the Rasch analysis, this method enabled us to classify the items into different groups and to determine the pattern at each level.

Model Fit of Items and Reliability

Infit and outfit mean squares (MNSQs) of items were used to examine the model fit of the items. Usually, if the infit and outfit MNSQs are in the range of 0.75–1.30, the items fit the Rasch model (Bond & Fox, 2015). Because the outfit MNSQs were too large, three items (P1, M3 and M10) were removed from the test. All of the remaining 17 items had acceptable infit and outfit MNSQs. The mean of the infit (M = 0.99, SD = 0.08) and outfit (M = 1.0, SD = 0.15) MNSQs were also very close to the Rasch-modelled expectations of 1 (Bond & Fox, 2015). In addition, except for M5 with slightly low infit *t* value (i.e. −2.1), all the values of infit and outfit *t* for other 16 items were greater than −2 and smaller than 2. The infit *t* (M = −0.10, SD = 1.00) and outfit *t* (M = 0, SD = .80) values had a mean near 0. These indicated that the data basically fit the Rasch model (Bond & Fox, 2015).

In Rasch model, all the information in the data is assumed to be explained by the Rasch measure, and the unexplained aspects of the data (i.e. the residuals) are random noise. To examine this unidimensionality, a principal component analysis of the Rasch residuals was used and aimed to identify no common variance among those residuals (Bond & Fox, 2015; Linacre, 1998). The result revealed that the Rasch dimension explained 40.2% of the raw variance in the data; the eigenvalues of the first contrast were 2, and it only involved 2 similar items (P2 and P3). These indicated that no second factor can be developed from the residuals; that is, the observed data satisfied the unidimensional assumption of the Rasch model. The reliability of the 17 items was quite high at 0.98, and the reliability of the personal ability estimates was acceptable at 0.78.

Results

Levels of Understanding of Part-Whole and Measure Subconstructs

Using Rasch Analysis to Estimate Item Difficulty. Each item's difficulty was estimated using Rasch analysis. As shown in Table 1, item difficulties had an average mean of 0 by default (SD = 1.41) and ranged from −2.64 to 2.71. P6 was the easiest, and M8 was the hardest.

Table 1 Rasch measure: difficulty of the 17 items

Item no.	P6	P2	P3	P4	P8	P7	M2	P10	P5
Difficulty	− 2.64	− 1.99	− 1.93	− 0.99	− 0.93	− 0.6	− 0.52	− 0.44	0.19
Item no.	P9	M1	M4	M5	M7	M6	M9	M8	
Difficulty	0.25	0.44	0.83	1.1	1.39	1.55	1.59	2.71	

As shown in the item-person map (see Fig. 1), on the right side of the logit scale, the items are ordered by item difficulty, with the easiest item at the bottom. On the left side of the scale, the students are ordered by their ability from the top down. From the right side of the scale, we can see that items focused on the part-whole subconstruct are located at the lower levels of the scale and the items emphasizing measure interpretation (except M2) are at the higher levels. The easiest item, P6, is at the bottom of the scale. It required the students to partition discrete objects equally and to represent the fraction 2/3. P5 is of medium difficulty. It asked the students to translate different representations of the fraction 5/6. The hardest, M8, is at the top of the scale. It asked the students to successively partition and determine a unit of measure and then place a number on a number line with only 0 and 5/9.

The pathway to understanding the measure subconstruct was rooted in or not far away from the part-whole subconstruct. As shown in Fig. 1, the items focused on the measure subconstruct are located at the upper level, adjoining the items emphasizing the part-whole subconstruct. Specifically, the two easiest items regarding the measure subconstruct are intertwined with some part-whole items. M2 is surrounded by part-whole items (P4, P7, P8 and P10), and M1 is close to P5 and P9.

Using MD Cluster Analysis to Specify Levels of Understanding of the Two Subconstructs. Based on their Rasch difficulty estimations, the 17 items related to the part-whole and measure subconstructs were clustered into 6 groups using the MD method (see Table 2).

These 6 clusters explained 69.7% of the total variance in the data. When the 17 items were clustered into 5 groups, the percentage of variance explained decreased by 7.3%. Therefore, the sixth graders' understanding of the part-whole and measure subconstructs was characterized by six levels or stages.

The task at the Level 1 needed to represent the fraction 2/3 by dividing discrete objects into three equal parts and shading two of them (P6). The problems at Level 2 required to construct the unit fractions 1/8 and 1/16 in area models with unequal partitions. During this process, necessary were the ability to identify whether the parts were of equal size and the idea that the sum of the parts equals the whole (P2 and P3). The tasks at Level 3 included the abilities to (1) construct the fraction 1/6 from an area model, build a relationship between an area model and a set model and then represent the same fraction in the set model (P4); (2) construct the whole in a length model and a set model when 3/4 or 2/3 and fractional parts were given, respectively (P7 and P8); (3) conceive the differences in the wholes (i.e. total money) of the fractions 1/4 and 1/2 so as to judge whether Mary could spend more money than John in a real-world word problem (P10); and (4) recognize the unit of measure of the fraction 3/8 (M2). The

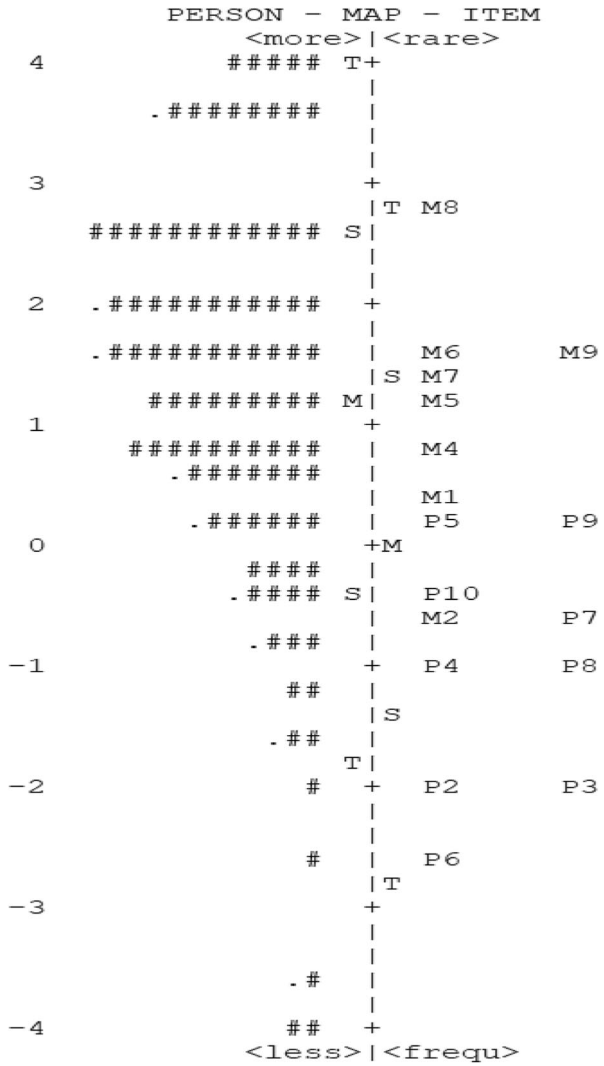


Fig. 1 Item-person map showing the student-participants' understanding of the part-whole and measure subconstructs. Each "#" is three students; each "." is one to two students

Table 2 Six levels of the 17 items related to the part-whole and measure subconstructs

Clusters	Rasch estimates	Items
1	… to −2.64	P6
2	−1.99 to −1.93	P2, P3
3	−0.99 to −0.44	P4, P8, P7, M2, P10
4	0.19 to 0.44	P5, P9, M1
5	0.83 to 1.59	M4, M5, M7, M6, M9
6	2.71 to …	M8

tasks at Level 4 involved the following: (1) identifying the fraction 5/6 represented in a set model and making connections between different representations and then representing this fraction in another set model (P5); (2) repartitioning and reconstructing the fraction 1/8 of a whole based on 1/6 of the shaded parts of a pie graph (P9); and (3) identifying a fraction as a number rather than two different whole numbers and estimating the sum of the two fractions 12/13 and 7/8 (M1). The tasks at Level 5 required to (1) choose a unit interval to measure the length of objects from the origin (M4, M5); (2) identify a fraction represented by a point on a number line marked with 0, 1/2 and three subintervals (M7); (3) recognize the fraction 3/4 represented in an area model and then locate the fraction on a number line marked with 0, 1/2, 1 and subintervals (Item M6); and (4) identify the density property of fractions and explain why there are fractions between 1/8 and 1/9 (M9). The task at Level 6 involved successively partitioning rather than halving, defining a unit of measure and placing a fraction on a number line marked only with 0 and 5/9 (M8).

To summarize, in response to the first research question, the Rasch analysis and MD cluster analysis demonstrated that understanding part-whole and measure subconstructs could be characterized as six hierarchical levels: (1) representing fractions in a simple model is the easiest; (2) constructing fractions in models with unequal partitions is at Level 2 where equal partitioning and the relationship between the parts and the whole need to be emphasized; (3) the abilities involved in the two subconstructs start to be intertwined at Level 3, including construction of the whole and identification of the fractional unit; (4) in terms of the part-whole subconstruct, the most difficult abilities are at Level 4, involving translations among different representations, repartitioning and reconstructing. The number notion of fractions also starts to be involved; (5) the density property of fractions can be understood at Level 5; and (6) the measure assigned to some intervals is the highest demanding, including successively partitioning and locating fractions on number lines.

Levels Achieved by Sixth Graders

The mean of person ability estimations (M = 1.17, SD = 1.86) was higher than the mean of item difficulty (M = 0, SD = 1.41). From the item-person map in Fig. 1, we can see that the mean of the students' abilities was located at a point close to the difficulty level of item M4, which was higher than the difficulty levels of all part-whole items and lower than the difficulty levels of most measure items.

Based on six groups of tasks at different levels and their difficulty estimations, 310 sixth-grade students were categorized into 7 groups from Level 0 to Level 6. Specifically, in the Rasch model, one person has a 50% probability of success on an item with the same Rasch estimation on the logit scale. In this study, the greater the ability-difficulty differences in favour of a student's ability, the more likely the student was to succeed. The seventh group Level 0 was newly added. If a student's ability estimation was lower than or equal to the difficulty estimation of the items at Level 1, he or she was viewed as being at Level 0. Generally, if a student's ability estimation was higher than the difficulty estimations of all items at Level X and equal to/lower than all items at Level X + 1, he or she was treated as achieving the understanding of Level X.

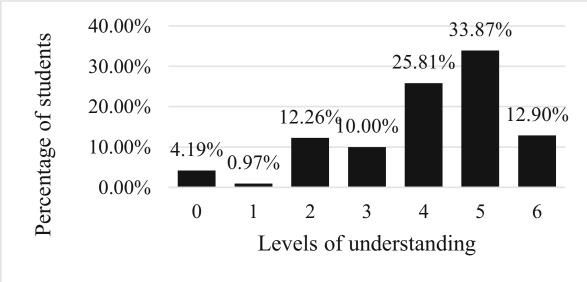


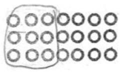
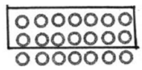
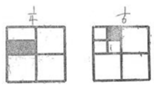
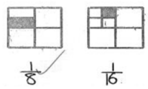
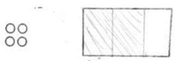
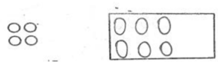
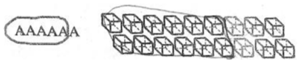
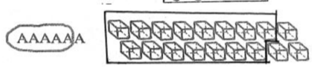
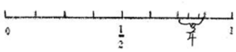
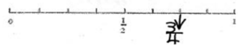
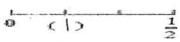
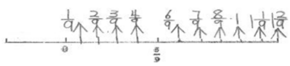
Fig. 2 Distribution of the Chinese student-participants' understanding

As shown in Fig. 2, around 60% of the students were located at Level 5 (33.87%) or Level 4 (25.81%). Only 12.90% had enough ability to solve all of the problems, whereas around 4.19% of them performed poorly, even on tasks at Level 1.

Two students are discussed below as examples: Lily (female) at Level 0 and Wilson (male) at Level 6. They were from the same class and the same school. Their responses to nine items at different levels are shown in Table 3.

Lily only had the ability to procedurally represent/recognize simple fractions in simple area models, which had lower cognitive demands than the abilities required by items at Level 1. In contrast, Wilson showed competence in all of the abilities measured in this study. For item P6, Lily lacked the ability to divide discrete objects into three equal parts and to circle two out of the three parts. For items P2 and P3, Lily incorrectly treated the number of all white parts as the denominator and the number of shaded parts

Table 3 Two examples of the participants' responses to nine items

Item No.	An example of Lily at level 0	An example of Wilson at level 6
P6 (L1)		
P2 and P3 (L2)		
P7 (L3)		
P5 (L4)		
M6 (L5)		
M7 (L5)		
M9 (L5)	No response	 Answer: If they are enlarged, there are other fractions between them.
M8 (L6)	No response	

Note. L refers to understanding level of items as the numerator. In other words, when unequal parts were given in the area model, she did not understand that the parts should be equal and that the parts must exhaust the whole, and she could not partition the whole equally. When solving item P7, Lily mistakenly represented 2/3 of the area model rather than conceiving the fractional parts given in the set model and constructing the whole. These responses suggested that she could possibly partition a rectangle equally and represent a simple fraction. For M6, she could recognize the simple fraction 3/4 in a simple area model. However, after this, she ignored the labels on the number line and failed to identify the unit. Further, to our surprise, she partitioned a small segment into three parts, which she marked as 3/4. For item M7, Lily again ignored the number labels and placed the whole number one on the number line. This suggested that she did not acquire any measure-related abilities. In addition, she did not provide responses to items M9 and M8. Conversely, Wilson solved all problems. For example, he was able to construct fractions, conceptualize a fraction as a point on the number line and find a fraction between two fractions with consecutive denominators.

To summarize, in response to the second research question, the results showed that about 60% of the sixth-grade students in this study were located at Level 4 and Level 5 in terms of the part-whole and measure subconstructs. Two examples of actual student performances on the items were provided to show the different levels of understanding.

Discussion

This study aimed to unveil the hierarchical understanding of the part-whole and measure subconstructs of fractions and Chinese students' performance on them at the end of their elementary education. The Rasch model and cluster analysis were used for data analysis. The answers to the two research questions are summarized and discussed below.

Hierarchical Understanding of Part-Whole and Measure Subconstructs

The results demonstrated that understanding of the part-whole and measure subconstructs could be classified into six hierarchical levels. Starting from the lowest level to the highest, these are the construction of fractions; the construction of the whole; the identification of the unit of measure; the translation of fraction representations; repartition or reconstruction; the treatment of fractions as numbers; the placing of fractions on number lines; and the successive partitioning. The hierarchy identified in this study is basically consistent with the work of Pantziara and Philippou (2012) and Hart et al. (1981).

A powerful explanation of the hierarchy and the consistency between this study and the earlier studies is closely related to the various characteristics of mathematical understanding: (1) from a procedural understanding to a conceptual understanding (Sfard, 1991), (2) from a single representation to multiple representations (Cramer, 2003) and (3) from concrete actions to mathematical abstraction (Herscovics & Bergeron, 1983). In practice, at the lower level, a step-by-step procedure could be applied to construct fractions represented in a single area or set model, that is, counting all equal parts as a denominator and the number of shaded parts as a numerator. The procedures would then be gradually interiorized, and a lower level of conceptual understanding would be produced. Translating between different representations, conceiving the whole or identifying the unit of measure of fractions could also be developed. After that, learners could progressively detach from concrete objects and achieve mathematical abstraction. For example, they could reconstruct fractions and recognize density property. When subintervals are not marked, multiple thinking processes involving successive acts of partitioning and defining the unit of measure are integrated.

In addition, instruction following prescribed mathematics textbooks could help to accelerate students' understanding of the two subconstructs (van den Ham & Heinze, 2018). For example, adequate exercises in textbooks pertaining to the construction of fractions could be helpful to the development of this ability at lower levels, whereas a lack of exercises in textbooks on successive partitioning could be related to this ability at the highest level.

There were a few differences between the performance of the Chinese students and the Cypriot/English students, which could also be explained by different types of instruction following the textbooks. In terms of density property of fractions and the location of fractions on a number line with congruent intervals, the Chinese students presented the same level of understanding (Level 5). However, the Cypriot students found density property of fractions (Level 6) to be much more difficult than the location of fractions (Level 5) (Pantziara & Philippou, 2012). The Chinese students' better performance on the density of fractions may be due to the frequent practice of simplifying fractions to a common denominator. This is introduced in a whole section in Chinese textbooks, whereas locating fractions on a number line is only found in several scattered exercises. Chinese students also performed better on conceiving differences in wholes of fractions in word problems (Level 3) than on repartitioning and re-unitizing (Level 4). Hart et al. (1981) suggested that the former was more demanding than the latter (Level 2) for British students. The difference may be that textbooks in China contain items related to conceiving differences in wholes of fractions but have no items related to repartitioning and reconstructing.

Sixth Graders' Performance on Part-Whole and Measure Subconstructs

The results showed that by the end of primary education, most of the participants could deal with problems pertaining to the part-whole subconstruct but had not acquired enough ability to handle the measure subconstruct tasks. This is in line with previous studies (Charalambous & Pitta-Pantazi, 2007; Ni, 1999; Pantziara & Philippou, 2012).

In the case of China, there may be multiple factors contributing to the difference. The first possible explanation may be that the part-whole subconstruct is relatively easier or more widely accepted than the measure subconstruct (Charalambous & Pitta-Pantazi, 2007). Second, this situation could possibly be influenced by the domination of the part-whole subconstruct in corresponding textbooks in mainland China, where the measure subconstruct has appeared less frequently (van den Ham & Heinze, 2018). Third, the scarce use of number lines has possibly had a negative effect on students' performance on the measure subconstruct (Bass, 2018). Fourth, Chinese children's good performance on the part-whole subconstruct has probably benefitted from the relative simplicity and clarity of Chinese fraction-related language (Ngan Ng & Rao, 2010; Zhang, 2010). In the Chinese language, the mathematics terms for fractions are embedded in the concept of fractional parts and partitioning. For example, 1/3 is read as "三分之一" (literally 1 parts [attributive] 3).

Implications

The hierarchical understanding of the meanings of fractions has not been fully explored in previous research (Nicolaou & Pitta-Pantazi, 2016). This study presents a clear and nuanced picture of the hierarchical understanding of the part-whole and measure subconstructs. It adds quantitative evidence to enrich this topic and fills in the gaps left by previous research in three aspects.

First, this study builds on and advances Hart (1981a) in two main aspects. Drawing on the curriculum, textbooks and updated literature on the meanings of fractions produced since the 1980s, it not only expands our knowledge on measuring the part-whole subconstruct but also incorporates the measure subconstruct of fractions, which Hart (1981a) did not address. In addition, instead of depending on item facility, this study relies on a rigorous and influential method (i.e. the Rasch model) to develop a scale and reveal a clear picture of understanding of the two subconstructs. Item difficulty and each student's ability are presented simultaneously on a logit scale. Within the context of a curriculum unified with textbooks, the hierarchy of understanding of the two subconstructs is data-driven and has invariant measurement properties.

Second, this study builds on Pantziara and Philippou (2012) and enriches the literature on the two subconstructs by assessing four additional abilities and revealing their levels: (1) conceiving different wholes at Level 3, (2) repartitioning and reconstructing at Level 4, (3) identifying a fraction as a number at Level 4 and (4) successively partitioning at Level 6. These abilities play an important role in the development of the part-whole and measure subconstructs (Lamon, 2012), but they have not received enough attention in mathematics textbooks in mainland China.

Third, this study not only makes up for the deficiency in the relevant curriculum documents in mainland China but also could allow researchers and educators to learn about Chinese students' performance and apply the hierarchy to non-Chinese students. The hierarchy of understanding applied to the part-whole and measure subconstructs in this study is basically consistent with the findings of Pantziara and Philippou (2012) and Hart et al. (1981). The hierarchy of understanding of the two subconstructs informs the instructional design and the development of teaching resources. Based on the hierarchy, teachers could design their teaching, select mathematical tasks and diagnose the extent of students' understanding in their classroom practices. Textbook developers could design corresponding textbooks.

Limitations of the Study

There are three limitations that need to be considered when interpreting the findings. First, there were not enough items included in this study, and some students did not have enough items to assess their abilities. Items assessing the other three subconstructs could make up for this gap in the future. Second, because the items M4 and M5 were self-designed for this study to assess the measure subconstruct, the students might have responded to the item M4 based on their understanding of the part-whole subconstruct, and the infit t value of M5 was slightly lower than -2. The two items need to be further validated. Third, this study suggested some possible factors without further examination. Future research could explore the effect of the factors or conduct experiments to improve students' learning of the measure subconstruct based on the hierarchy.

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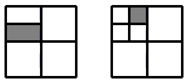
Compliance with Ethical Standards

Conflict of Interest The authors declare that they have no conflict of interest.

Appendix: Test Items

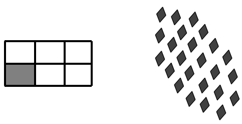
P1. What fraction is shaded in the following figure?

P2. and P3. Write as a fraction the shaded part in each shape.

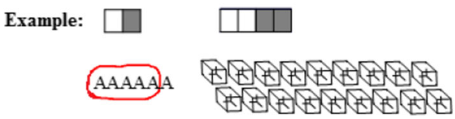


(1) (2)

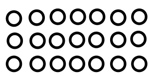
P4. Use the diagram on the right to represent a fraction that is also represented by the left diagram.



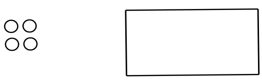
P5. The picture on the left represents a fraction. Use the picture on the right to represent the same fraction as the one in the left picture. The example below will help you.



P6. Please circle 2/3 of the whole.

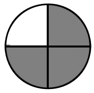


P7. If the four circles below represent 2/3 of the whole, draw the whole in the box below.



P8. There is a string. John cuts 3/4 of the length of the string as below. Please draw the whole string exactly.

P9. Shade in 1/6 of the dotted section of the circle. What fraction of the whole circle have you shaded



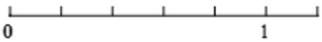
P10. Mary and John both have pocket money. Mary spends $1/4$ of hers, and John spends $1/2$ of his. Is it possible for Mary to have spent more than John? Why do you think this?

M1. Don't calculate. Estimate: $\frac{12}{13}+\frac{7}{8}$ is about equal to ______.

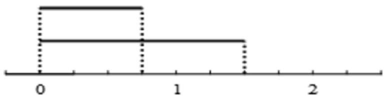
A 1 B 2 C19 D 21 E don't know

M2. The fractional unit of 3/8 is _______.

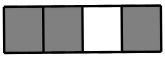
M3. Please locate the fraction $3/5$ on the given number line.

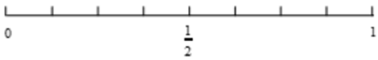


M4. and M5. Please use fractions to represent the length of the two sticks.

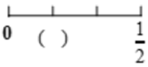


M6. Please locate the fraction that represents the shaded part of the rectangle on the number line.

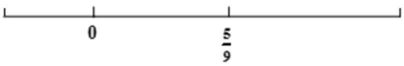




M7. Write the correct number in the bracket below on the number line.



M8. Please locate a number on the following number line.



M9. Is there any fraction that appears between $\frac{1}{8}$ and $\frac{1}{9}$? Please explain your answer.

M10. Which of the following are numbers? Put a circle around them.

A 4 * 1.7 16 0.006 $\frac{2}{5}$ 47.5 $\frac{1}{2}$ $ $1\frac{4}{5}$

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Affiliations

Zheng Jiang[1] · Ida Ah Chee Mok[1] · Jianhua Li[2]

[1] Faculty of Education, The University of Hong Kong, Pokfulam Road, Hong Kong 999077, China

[2] School of Mathematical Sciences, Beijing Normal University, No. 19, XinJieKouWai Street, HaiDian District, Beijing 100875, China