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# Graphical Forms: The Adaptation of Sherin's Symbolic Forms for the Analysis of Graphical Reasoning Across Disciplines

Jon-Marc G. Rodriguez[1] · Kinsey Bain[2] · Marcy H. Towns[1]

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## Abstract
In this paper, we introduce and discuss a construct called *graphical forms*, an extension of Sherin's symbolic forms. In its original conceptualization, symbolic forms characterize the ideas students associate with patterns in a mathematical expression. To expand symbolic forms beyond only characterizing mathematical equations, we use the general term *registration* to describe structural features attended to by individuals (parts of an equation or regions in a graph). When mathematical ideas are assigned to registrations in a graph, we characterize this as reasoning using graphical forms. As an analytic framework, graphical forms provide the language to discuss intuitive mathematical ideas associated with features in a graph, but we are also interested in engagement in modeling. Our approach to investigating graphical reasoning involves conceptualizing modeling as discussing mathematical narratives. This affords the language to describe reasoning about the process (or "story") that could give rise to a graph; in practice, this occurs when mathematical reasoning (i.e. reasoning using graphical forms) is integrated with context-specific ideas. In this work we describe graphical forms as an extension of symbolic forms and emphasize its utility for analyzing graphical reasoning. In order to illustrate how the framework could be applied, we provide examples of interpretations of graphs across disciplines, using graphs selected from introductory biology, calculus, chemistry, and physics textbooks.

**Keywords** Framework · Inter-disciplinary · Mathematical/graphical reasoning · Modeling

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✉ Marcy H. Towns
mtowns@purdue.edu

[1] Department of Chemistry, Purdue University, West Lafayette, IN, USA

[2] Department of Chemistry, Michigan State University, East Lansing, MI, USA

## Overview

The primary goal of this paper is to discuss a construct called *graphical forms.* Reasoning characterized by the use of graphical forms reflects the assignment of meaning to specific structural features in a graph. The idea of graphical forms is a logical extension of Sherin's (2001) symbolic forms, which describes mathematical reasoning involving associating ideas with specific patterns in equations. In recent work, the authors have used symbolic forms and illustrated how symbolic forms could be expanded to focus on graphical reasoning in a chemistry context (Rodriguez, Bain, Elmgren, Towns, & Ho, 2019; Rodriguez, Bain, & Towns, 2019; Rodriguez, Santos-Diaz, Bain, & Towns, 2018). Here, graphical forms and its relation to symbolic forms are discussed in detail, and a case is made for the utility of using graphical forms as an analytic framework to characterize graphical reasoning across different contexts. In order to illustrate how the framework could be applied, graphs in textbooks were selected and analyzed by the authors, focusing on the reasoning illustrated in the presentation of graphs and the associated text. To provide context, the discussion begins with background literature related to graphical reasoning.

### Mathematical Reasoning Involving Graphs

Modeling, or the "coordination of quantities with other types of knowledge" (Izak, 2004), plays a critical role across science, technology, engineering, and mathematics (STEM), warranting further analysis to provide insight regarding how instruction can help students successfully integrate equations and graphs with contextual information (Bain, Rodriguez, Moon & Towns, 2018; Becker & Towns, 2012; Lunsford, Melear, Roth, Perkins, & Hickok, 2007). A review of the literature indicates reasoning involving interpreting graphs is multifaceted and complex, with many factors potentially increasing the difficulty of graphical comprehension. For example, some factors contributing to the challenge associated with interpreting graphs include the complexity of the graph (e.g. number of variables/functions represented, relationship/interaction between variables, type of function, etc.), students' algebraic/mathematical proficiency, and the domain knowledge required for interpretation (Carpenter & Shah, 1998; Glazer, 2011; Ivanjeck, Susac, Planinic, Andrasevic & Milin-Sipus, 2016; Kozma & Russell, 1997; Phage, Lemmer, & Hitage, 2017; Planinic, Ivanjeck, Susac & Millin-Sipus, 2013; Potgieter, Harding, & Engelbrecht, 2007). The extent to which these factors interact and influence graphical reasoning and problem solving in the physical sciences has been investigated by comparing student reasoning associated with discipline-situated graphical problems and parallel mathematics-only graphical problems (where the discipline-specific context has been removed), in which some research indicates a lack of mathematical proficiency hinders success in interpreting graphs (e.g. Potgieter et al., 2007); other research asserts that the issue lacks necessary content knowledge (e.g. Phage et al., 2017; Planinic et al., 2013). Taken together, these results indicate that the role and interaction of content knowledge and mathematical ability varies depending on the population and topic being investigated, and even if students have all the necessary pieces of knowledge, they must be able to access and blend the appropriate discipline-specific ideas with mathematical reasoning (Bain et al., 2018).

From a purely mathematical perspective, there is a large body of literature that characterizes different perspectives associated with interpreting the information presented in a graph (Even, 1990; Moschkovich, Schoenfeld, & Arcavi, 1993; Schwartz & Yerushalmy, 1992; Sfard, 1992). Building on this body of work, Moore and Thompson (2015) describe *shape thinking*, where students were characterized as engaging in emergent reasoning or static reasoning. The distinction made by Moore and Thompson (2015) is that emergent reasoning involves viewing the function as a mapping of all input and output values with a more explicit emphasis on covariation, whereas static reasoning involves viewing the function as an entity that can be translated and manipulated as a whole. Emergent reasoning emphasizes the covariational relationship and the process communicated by the graph, and when students are unable to engage in emergent reasoning, they tend to have an incomplete understanding of the graphical representation. As a construct, viewing a graph using static reasoning is not necessarily unproductive, because different perspectives of a function may be useful or sufficient to answer questions in a given context (Even, 1990; Schwartz & Yerushalmy, 1992; Sfard, 1992; Rodriguez, Bain, Elmgren, Towns, & Ho, 2019). Furthermore, the added complexity of conceptualizing a graphical representation from different perspectives is worth discussing because it can influence how students interpret graphs in discipline-specific contexts (Potgieter et al., 2007).

Although the importance and challenges associated with graphical reasoning in STEM fields has been the focus of this discussion, the ability to engage in the critical analysis of graphs and related representations of data is important for people pursuing careers outside of STEM. Being able to interpret graphs and understand what they communicate—as well as having a firm understanding of how the data was collected and the associated limitations of claims that can be made with the data—is critical to interact with modern society and make informed decisions about current global problems (Driver, Asoko, Leach, Mortimer & Scott, 1994; Driver, Leach, Millar & Scott, 1996; Grassian et al., 2007; Mahaffy et al., 2017; Matlin, Mehta, Hopf & Krief, 2016; National Research Council, 2012). Thus, a working understanding of graphical reasoning and related analytical skills are relevant for society across STEM and non-STEM careers, suggesting the importance of focusing on how experts describe processes across STEM fields to elucidate patterns in reasoning that will help guide and scaffold students with their reasoning (Glazer, 2011; Ivanjeck et al., 2016, Kozma & Russell, 1997).

## An Overview of Symbolic Forms

Most studies that utilize the symbolic forms framework theoretically base their work using resource-based model of cognition (Sherin, 2001). Within the resources framework, knowledge is described as a complex and dynamic network of cognitive units of varying sizes and degrees of connectivity (Hammer & Elby, 2002, 2003; Hammer, Elby, Scherr, and Redish, 2005). The resource perspective builds on diSessa's (1993) discussion of phenomenological primitives (p-prims)—simple and intuitive ideas individuals develop based on observations and experiences—and the conceptualization of cognition as knowledge-in-pieces, where fine-grained pieces of knowledge (resources, some of which may be symbolic forms, for example) are activated in specific contexts. Based on the utility of the resource perspective in previous studies, the approach that we have outlined for the analysis of graphical reasoning will follow the example set by the literature and the resource perspective will serve as the theoretical foundation for our approach to investigating mathematical reasoning.

Symbolic forms involve a combination of a symbol template and a conceptual schema, where the symbol template is a recognizable pattern in an equation, and the conceptual schema involves the ideas or knowledge elements associated with the pattern of symbols. Sherin (2001) developed the symbolic forms framework based on his analysis of students working through physics problems, during which he compiled an initial list of symbolic forms and commented on the need for more work to identify additional symbolic forms. In a supplemental file, we have provided a comprehensive list of symbolic forms identified by other researchers and discussed in the literature (Dreyfus, Elby, Gupta, & Sohr, 2017; Dorko & Speer, 2015; Hu & Rubello, 2013; Izak, 2004; Jones, 2013, 2015a, 2015b; Rodriguez et al., 2018; Sherin, 2001; Schermerhorn & Thompson, 2016; Von Korff & Rubello, 2014); some symbolic forms have been selected and are provided in Table 1. Although work utilizing the symbolic forms framework has focused primarily on the intuitive ideas that students associate with equations, it is asserted that experts have access to a variety of mathematical resources and engage in reasoning using symbolic forms (Dreyfus et al., 2017).

As shown in Table 1, different conceptual schemas can be attributed to the same symbol template (e.g. the *balancing* and *same amount* symbolic forms have the same pattern of terms, but reflect different ideas) and symbolic forms can involve assigning ideas to expressions with varying levels of sophistication; see supplemental file for more examples, such as the simple and more nuanced forms of the *measurement* symbolic form (Dorko & Speer, 2015). In practice, some symbolic forms are more of

**Table 1** Abridged list of symbolic forms, from Sherin (2001)

| Symbolic form | Symbol template | Brief description of conceptual schema |
|---|---|---|
| Balancing | □ = □ | Two influences, each associated with a side of the equation, in balance so the system is in equilibrium |
| Same amount | □ = □ | Two amounts, each associated with a side, are the same. |
| Dependence | [...$x$...] | A whole depends on a quantity associated with an individual symbol. |
| Prop+ | $\left[\frac{\ldots x \ldots}{\ldots}\right]$ | Directly proportional to a quantity, $x$, which appears as an individual symbol in the numerator |
| Prop− | $\left[\frac{\ldots}{\ldots x \ldots}\right]$ | Indirectly proportional to a quantity, $x$, appears as an individual symbol in the denominator. |

Template Key

[...] Expression in brackets corresponds to an entity in the schema

$x$, $y$, $n$ Individual symbols in an expression

□ A term or group of terms

a restatement of a simple mathematical relationship, such as the *prop−* symbolic form describing how changing a value in the denominator influences the term as a whole (Sherin, 2001). Other symbolic forms, such as those identified by Jones (2013), are associated with defining advanced mathematical operations (e.g. the information an integral communicates), and some symbolic forms have been identified in which non-normative ideas are assigned to symbol templates (Jones, 2013). Furthermore, as noted by Dreyfus et al. (2017), the *eigenvector-eigenvalue* symbolic form is a "compound symbolic form", which is built using multiple "primitive symbolic forms". For students, symbolic forms can be particularly productive, because upon seeing an equation, even if they are not familiar with the relevant context, they can access useful resources to build an understanding. In a similar way, it is asserted that intuitive mathematical ideas associated with features in a graph are fundamental for interpreting graphs across various contexts.

## Investigating Graphical Reasoning

### Rethinking Symbolic Forms Using Registrations

Borrowing from Roschelle (1991), the term *registration* is used to refer to ideas that individuals attend to in representations. Roschelle (1991), as well as Sengupta and Wilensky (2009), used registrations to describe aspects of computer simulations that students focused on, in which students selected specific parts of the representation and subsequently *registered* (attributed) specific ideas to these pieces. According to Lee and Sherin (2006), redefining symbolic forms in terms of registrations can broaden the applicability of the framework. By conceptualizing registrations as structural features that students focus on in representations, in cases where they assign ideas to registrations, students are reasoning using symbolic forms (Lee & Sherin, 2006). This definition of symbolic forms allows for its general use to multiple types of representations, in which students are focusing on structural features that could be part of an expression (i.e. symbolic forms), but they could also be regions of a graph (i.e. graphical forms). Although the potential for the adaptation of symbolic forms to graphical representations has been suggested (e.g. Izak, 2000; Lee & Sherin, 2006; Sherin, 2001); in practice, this has not yet been taken up in the literature.

Within their conceptualization of registrations, Lee and Sherin (2006) noted that registrations can be of varying granularity, encompassing regions of a graph or an entire graph. This brings to mind Moore and Thompson's (2015) *shape thinking*, where students view a graph using static reasoning or emergent reasoning, where static reasoning involves viewing the whole graph as an entity to which ideas are associated, and emergent reasoning involves using covariational reasoning to coordinate changes in the variables. One way to interpret these types of reasoning is that students who engage in static reasoning are focusing on a particularly large registration—the graph as a whole—as opposed to students that engage in emergent reasoning that are able to think about the graph as being made up of smaller pieces—smaller registrations that form a process with many discrete steps. In this sense, the static/emergent distinction is a way to further characterize graphical forms based on the extent in which covariational reasoning is evoked.

### Mathematical Narratives

The integration of mathematical reasoning and discipline-specific content are critical instructional targets (Cooper, 2013; Ivanjeck et al., 2016; National Research Council, 2012; Reed & Holme, 2014; Underwood, Posey, Herrington, Carmel & Cooper, 2018). Literature on problem solving in the physical sciences emphasizes the importance of conceptual reasoning, asserting that when working through problems, experts can successfully integrate mathematical reasoning with the appropriate discipline-specific content and focus on details that are relevant for reaching a solution; however, students need support making these types of connections (Bain et al., 2018; Chi, Feltovich, & Glaser, 1981; Chi, Glaser, & Rees, 1982; Kuo, Hull, Gupta & Elby, 2013). In a study that involved using graphical representations to depict the changes associated with velocity, distance, and time, Nemirovsky (1996) focused on students' understanding of the process or "story" that could give rise to a particular graph. Central to how graphical reasoning is framed in this paper is the idea of *mathematical narratives*. According to Nemirovsky (1996) "A mathematical narrative fuses events and situations with properties of symbols and notations", or more tersely stated, discussing a mathematical narrative is modeling. The storytelling perspective of modeling is favored because of the way it emphasizes the process represented in a graph, which is particularly productive when thinking about graphical reasoning, affording the necessary language to describe how students think about the events, or a series of discrete events represented in a graph.

In summary, mathematical reasoning can be characterized using graphical forms which can be described as static or emergent (shape thinking). Engaging in discussing mathematical narratives involves an additional layer of understanding that build on the graphical forms by incorporating discipline-specific content or contextual ideas. Thus, by using Sherin's (2001) symbolic forms framework, different forms of mathematical reasoning can be characterized, reasoning that can be productive for a variety of contexts; Moore and Thompson's (2015) shape thinking helps further characterize students' mathematical reasoning; and Nemirovsky's (1996) mathematical narratives provide a way to connect students mathematical reasoning with the overall story represented by the graph.

## Examples From Across Disciplines: Textbook Analysis

### Graph Selection

In the sections that follow, examples are provided of analysis guided by the graphical forms framework, focusing on textbook descriptions of graphs. Although there are limitations associated with applying this framework to textbook examples, the textbooks are used as a readily available illustrative tool to exemplify the type of reasoning that is of interest. Moreover, examples of students' reasoning are not provided in this contribution, but we have previously utilized the graphical forms framework to investigate students' understanding of graphs in chemical kinetics (Rodriguez, Bain, Elmgren, Towns, & Ho, 2019; Rodriguez, Bain, & Towns, 2019; Rodriguez et al., 2018), and direct the readers to these studies for additional examples of reasoning characterized by graphical forms.

Graphs were selected from introductory biology, calculus, chemistry, and physics textbooks, focusing on the explanation provided that accompanied each graph and how experts conveyed ideas in the textbooks. The textbooks, listed in Table 2, were chosen based on the assigned textbooks for the introductory biology, calculus, chemistry, and physics courses that are required each year for the thousands of first-year engineering students at Purdue University, a large research-intensive university in the Midwestern USA. In order to narrow the scope of the analysis, a single graph was selected from each textbook, where each graph had (1) time as the independent variable, (2) a rich description accompanying the graph, and (3) the incorporation of both mathematical and domain-specific ideas. Each discipline's distinct rate-related graph (e.g. logistic population growth model for biology, chemical kinetics for chemistry, kinematics for physics, a model of water temperature for calculus) was modified and is presented in Table 2. The graphs were simplified by removing any annotations or additional associated text, leaving only the labeled axes and the curve/line.

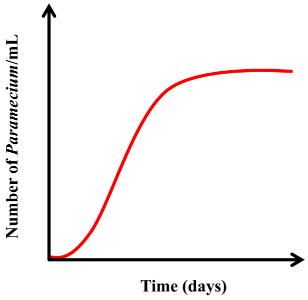
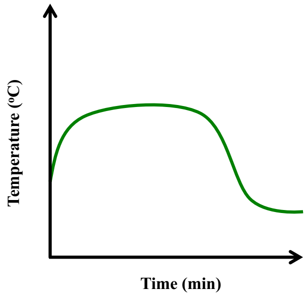
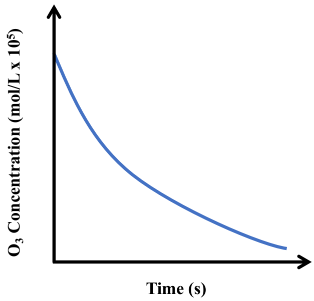
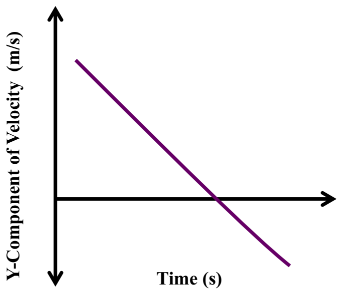
### Biology: Population vs. Time

The introductory biology textbook graph that was chosen was a representation of the population growth of the bacteria *Paramecium*, depicting the number of *Paramecium*/mL as a function of time. Urry, Cain, Wasserman, Minorsky and Reece (2017) presented the population growth of the bacteria as an example of the logistic population growth model, stating that the *Paramecium* graph has a characteristic sigmoidal curve. In their description, Urry et al. (2017) focused on distinct regions of the graph, incorporating mathematical reasoning with a description of what is physically happening at different points along the graph. When describing the rate of population growth within the logistic model, they stated, "New individuals are added to the population most rapidly at intermediate populations sizes, when there is not only a breeding population of substantial size, but also lots of available space and other resources in the environment" (Urry et al., 2017, pp. 859-860). In terms of graphical forms, we characterize this as the intuitive mathematical idea *steepness as rate*, in which an individual is attending to a region of the graph (registration) and assigning ideas related to the relative magnitude of the rate. We can see that the mathematical reasoning employed by Urry et al. (2017) serves as a basis to describe the phenomena being modeled. In this case, they attended to the region of the graph with the steeper slope and subsequently provided a discipline-specific explanation for this observation, namely, why the population can grow at such a fast rate. Completing the story communicated by the graph, Urry et al. (2017) used the same mathematical reasoning to discuss how the model reflects that "the number of individuals added to the population decreases dramatically" due to factors such as limited resources (Urry et al., 2017, p. 860). As before, the intuitive mathematical ideas about the graph serve as an anchor to assign physical meaning. A summary of the graphical forms discussed in relation to the biology graph is presented in Fig. 1.

### Calculus: Temperature vs. Time

The graph chosen from the calculus textbook was from the first chapter, "Functions and Models", of the Stewart (2016) textbook. In this chapter, Stewart (2016) discussed the

**Table 2** Textbooks and graphs used in this study

| Discipline | Graph |
|---|---|
| **Biology**<br>*Campbell Biology in Focus* (2nd Ed.)<br>Urry et al., 2017 |  |
| **Calculus**<br>*Calculus: Early Transcendentals* (8th Ed.)<br>Stewart, 2016 |  |
| **Chemistry**<br>*Chemistry: The Molecular Nature of Matter and Change* (8th Ed.)<br>Silberberg & Amateis, 2018 |  |
| **Physics**<br>*Matter & Interactions: Modern Mechanics* (4th Ed.)<br>Chabay & Sherwood, 2015 |  |

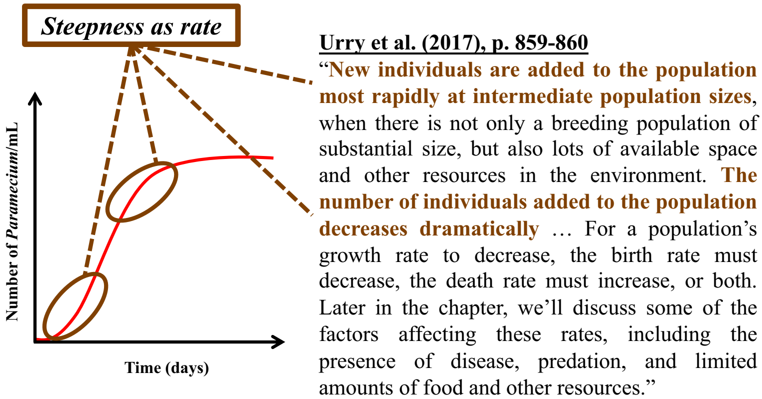



**Fig. 1** Summary of graphical forms used in Urry et al.'s (2017) discussion of the logistic model of population; features attended to (registrations) are circled in the graph and the ideas associated with these features (conceptual schema) are emphasized in the accompanying text

nature of functions and how they can be used to model phenomena that we may encounter in our daily lives. The graph selected describes how the temperature of water coming out of a hot-water faucet changes as a function of time. Although there are not explicit "discipline-specific" ideas, the graph has a clear narrative—a context that is informed by everyday experiences. Stewart (2016) described the initial region of the graph using the previously mentioned *steepness as rate* graphical form, stating that temperature of the water begins at room temperature, but "when water from the hot-water tank starts flowing from the faucet, $T$ [temperature] increases quickly" (Stewart, 2016, p. 14). Stewart (2016) cued into the relative steepness of the graph, translating this to mean a large rate of change for the temperature. When describing the next portion of the graph, Stewart (2016) reasoned using the *straight means constant* graphical form (associating constant rate with a straight line), stating that the temperature is constant because the water flowing from the faucet is the same temperature as the water from the hot-water tank. For the final region of the graph, Stewart (2016) reasoned that the temperature of the water would eventually decrease to the temperature of the water supply as the water from the hot-water tank is drained, illustrating the use of contextual information to support his graphical reasoning. In this example, Stewart (2016) attended to different registrations and attributed different intuitive mathematical ideas, to which he assigned a particular aspect of the story modeled. In Fig. 2, we have provided a summary of the graphical forms discussed in relation to this graph.

### Chemistry: Concentration vs. Time

The chemistry graph was taken from a chapter on chemical kinetics, which is concerned with how the concentration of reactants and products in a reaction change over time. The graph shows how the concentration of ozone ($O_3$) changes over time during a reaction involving ozone and ethylene ($C_2H_2$). When reasoning about the graph, one of the first comments Silberberg and Amateis (2018) made is about the overall shape of the graph and its implications for rate. Silberberg and Amateis stated that because the graph is a curve, the rate would change over time, whereas "a straight line would mean

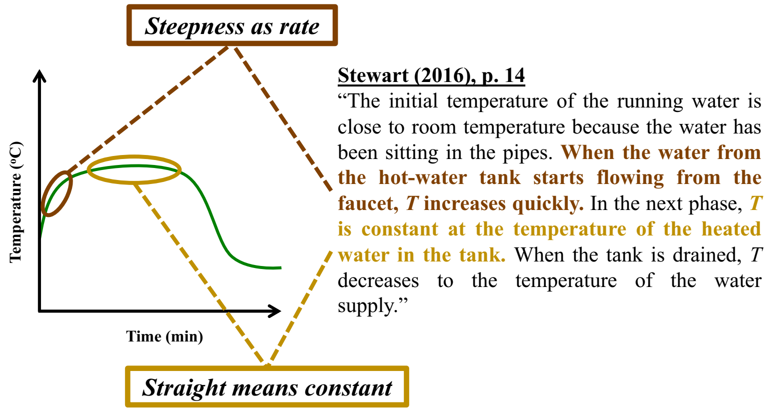



**Fig. 2** Summary of graphical forms used in Stewart's (2016) discussion of modeling changes in water temperature; features attended to (registrations) are circled in the graph and the ideas associated with these features (conceptual schema) are emphasized in the accompanying text

that the rate was constant" (Silberberg & Amateis, 2018, p. 694). In this section, Silberberg and Amateis (2018) described two graphical forms: *curve means change* and *straight means constant*. Reasoning using the *curve means change* and *straight means constant* graphical forms are closely related and represent two extremes, where one idea follows from the other (i.e. if a curve suggests a change in rate, then the lack of a curve—a straight line—indicates no change in rate). Both graphical forms involve focusing on a region of the graph or the graph as a whole (a registration of varying size) and assigning ideas regarding a general trend about the associated rate.

Similar inferences about the rate can be drawn when reasoning using the *steepness as rate* graphical form. Although this type of reasoning is similar to the previous two graphical forms, it is distinct. The previously discussed graphical forms, *curve means change* and *straight means constant*, allow us to think about how the rate changes, whereas *steepness as rate* involves considering the magnitude of rate, which is normally discussed relative to another region of the graph. Furthermore, the different ways that Silberberg and Amateis (2018) reasoned about the graph illustrates how we could further characterize *steepness as rate* using Moore and Thompson's (2015) considerations of shape reasoning. We view graphical reasoning that focuses holistically on the overall shape of the entire graph or focuses more on general shapes of a region of the graph as static reasoning. For our operationalization of emergent reasoning, we emphasize the importance of covariational reasoning, which involves thinking about both variables (such as thinking about slope as the relationship between two changing variables, bringing in ideas related to the derivative, and considering multiple points along the process modeled by the graph) (Saldanha & Thompson, 1998; Thompson & Carlson, 2017). This distinction is significant because of the role covariational reasoning serves in topics across mathematics, including graphical reasoning and modeling dynamic processes (Carlson, Jacobs, Coe, Larsen & Hsu, 2002; Confrey & Smith, 1995; Ellis, Ozgur, Kulow, Dogan, & Amidon, 2016; Habre 2012; Thompson, 1994). Using the example of *steepness as rate*, when Silberberg and Amateis worked through reasoning about the graph by comparing multiple points or intervals using considerations of average or instantaneous rate, the *steepness as rate* graphical form can be characterized as emergent from the perspective of shape thinking.

When discussing the relevant chemistry ideas pertaining to the graph, Silberberg and Amateis used mathematical reasoning as a starting point and modeled by providing a chemistry-based description that explains the observed trends in the graph. This is best illustrated when Silberberg and Amateis reasoned why the rate decreased over time, "… as $O_3$ molecules react, fewer are present to collide with $C_2H_4$ molecules, and the rate, the change in $[O_3]$ over time, therefore decreases" (Silberberg & Amateis, 2018, p. 695). Silberberg and Amateis provided a particulate-level description to describe the observed trends in the graph, which necessarily builds on the mathematical reasoning that allowed them to make the inference that the rate is changing and decreasing (*curve means change*, *steepness as rate*). For a summary of the graphical forms discussed in relation to the chemistry graph, see Fig. 3.

## Physics: Velocity vs. Time

For the physics textbook, a graph was chosen from a section that discussed kinematics, a field of study that focuses on modeling the motion of objects. The graph describes how the y-component of velocity changes over time for a soccer ball that was kicked into the air. When reasoning about the graph, Chabay and Sherwood (2015) attended to the fact that the graph "has a constant negative slope" (p. 64). Looking at the shape and directionality of the graph, Chabay and Sherwood (2015) were able to make inferences about the system, another example of the *straight means constant* graphical form. Figure 4 provides a summary of the graphical form identified for the physics graph.

In terms of the mathematical narrative reflected by the graph, they explained that the linearity of the graph is result of a constant force acting against the vertical motion, "the y-component of the force has the constant value –*mg*, which makes the y-component of momentum [which is proportional to velocity] decrease at a constant rate," (Chabay & Sherwood, 2015, p. 64). From a purely mathematical perspective, since the graph is

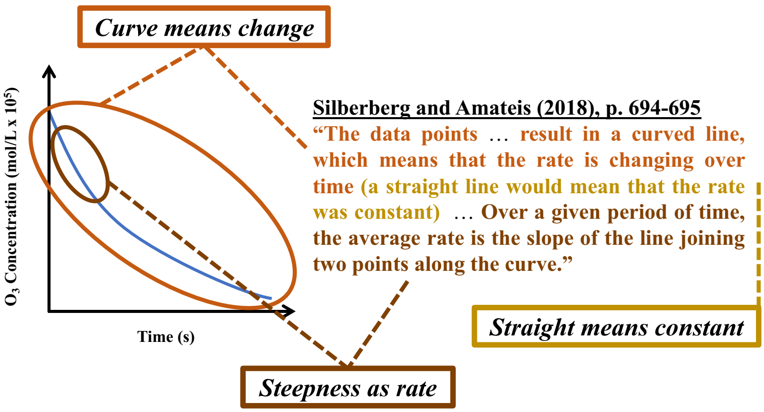



**Fig. 3** Summary of graphical forms used in Silberberg and Amateis' (2018) discussion of chemical kinetics; features attended to (registrations) are circled in the graph and the ideas associated with these features (conceptual schema) are emphasized in the accompanying text

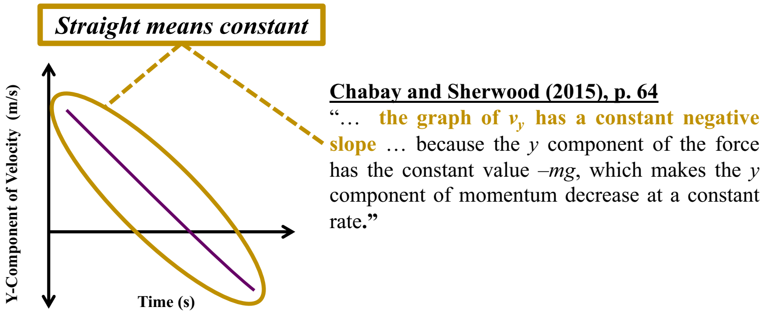



**Fig. 4** Summary of graphical forms used in Chabay and Sherwood's (2015) discussion of kinematics; features attended to (registrations) are circled in the graph and the ideas associated with these features (conceptual schema) are emphasized in the accompanying text

linear, it is relatively straightforward, having largely the same "story" for the duration of the graph. However, when considering the relevant physics concepts, the graph takes on some interesting nuances. Unlike the other graphs we have discussed so far, this graph has negative values for the y-axis, because velocity is a vector, having both magnitude and direction (in this case, upward or downward motion). In their description of the graph, Chabay and Sherwood (2015) described how the different regions of the graph correspond to different periods during the projectile's motion. According to authors, since the graph focuses only on the vertical motion of the ball, the positive velocity values correspond to when the ball is rising (after being kicked), the negative values correspond to when the ball is falling, and when the y-component of velocity is 0 (when the line crosses the x-axis), "… [the height of the ball] is at a maximum. At this instant the ball is neither rising nor falling" (Chabay & Sherwood, 2015, p. 64). Thus, by bringing in concepts from physics, Chabay and Sherwood (2015) described the mathematical narrative, explaining the observed features of the graph, such as why velocity takes on positive, negative, and zero values. As we observed with the previous experts, Chabay and Sherwood (2015) anchored their reasoning in mathematics (intuitive graphical reasoning) and attributed discipline-specific ideas to describe the system being studied.

## Conclusion and Implications

Graphical forms are useful for interpreting graphs because they are generalizable and less restricted to a particular context, which provides a foundation to consider the phenomena represented by any graph. In the case of the presentation of the content in the textbooks, we observed that although they were situated in different contexts, a common set of intuitive mathematical ideas was used in the description of the graphs. It is feasible to assume that even though a disciplinary expert may not have the relevant content knowledge outside their field of expertise, she could generate and interpret the process being modeled using the previously discussed intuitive mathematical ideas.

In the examples provided, mathematical reasoning served as an anchor to which discipline-specific ideas were attached; however, under a different set of circumstances—if a phenomenon was provided and an individual had to mathematize the process—we would observe a different trend. In this case, we would see a discussion that was anchored in the context (e.g. the rate of a zero-order reaction is constant), which would subsequently be translated to a graphical representation (e.g. the graph should have a straight line, the graphical form *straight means constant*). In our study on student problem solving in chemical kinetics, we considered this distinction to be a matter of *directionality*, that is, when modeling phenomena, differentiating between whether chemistry ideas were mapped onto mathematical reasoning or mathematical reasoning was mapped onto chemistry ideas (Bain, Rodriguez, Moon, & Towns, 2018). Regardless of the directionality of modeling, we argue that the ability to create mathematical narratives is highly dependent on an individual's ability to reason using graphical forms, such as those summarized in Table 3 and Fig. 5.

This discussion of graphical forms was informed in part by Nemirovsky's (1996) work related to mathematical storytelling and the "grammar" of graphs, in which he drew a distinction between a linear function, which tells the story of "growing steadily," and a curve that levels off asymptotically, which is "growing but slowing down." Similarly, in Jones' (2013) identification of symbolic forms related to student understanding of the integral he identifies what he calls "stable cognitive resources," in which students associated negative area with parts of a function that are oriented in a specific way. In the context of how graphical reasoning is discussed in this paper, these can be considered examples of graphical forms, with Nemirovsky's (1996) notion of a line "growing steadily" being consistent with the graphical form *straight means constant.*

By characterizing these ideas and drawing attention to their role in understanding graphical representations, the goal is to provide an avenue for future work in this area and suggest instructors across STEM fields emphasize graphical reasoning skills that have broad applicability for thinking about graphs in a variety of contexts. Although each textbook description did not necessarily use all the graphical forms discussed in this study, no claims are made regarding the ability of each of the textbook authors to engage in these types of reasoning, and no claims are made regarding identifying all the possible graphical forms. Cataloging a comprehensive list of graphical forms is beyond the scope of this work; instead, the intention herein is to introduce and describe an approach for analyzing graphical reasoning, which has been utilized in recent work by the authors (Rodriguez, Bain, Elmgren, Towns, & Ho, 2019; Rodriguez et al., 2019; Rodriguez et al., 2018). As shown in the examples provided, understanding ideas regarding rate are critical for being able to interpret graphs and given that previous research has indicated rate-related ideas are challenging for students (Orton, 1983;

**Table 3** Summary of graphical forms discussed in this work

| Graphical form[a] | Registration and conceptual schema |
|---|---|
| Steepness as rate | Varying levels of steepness in a graph correspond to different rates |
| Straight means constant | A straight line indicates a lack of change/constant rate |
| Curve means change | A curve indicates continuous change/changing rate |

[a] We anticipate more nuanced versions of these graphical forms exist where individuals focus on the overall shape of a curve or evoke ideas related to the mathematical definition of the derivative/slope as part of their reasoning (classifying graphical forms as static or emergent)

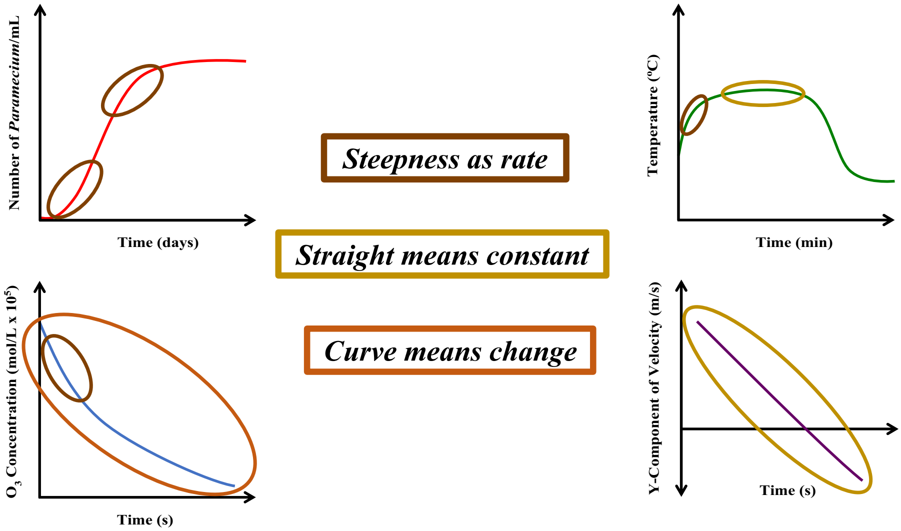



**Fig. 5** Summary of graphical forms identified for each of the graphs analyzed

Rasmussen, Marrongelle, & Borba, 2014; White & Mitchelmore, 1996), other facets of graphical reasoning should be explored, including thinking about examples of rate that do not have time as the independent variable (Jones, 2017). Moreover, the graphical forms discussed are specific to rate contexts in the Cartesian coordinate system, and it can be anticipated that other contexts and other coordinate systems would have unique graphical forms, some of which may have similar structural features (e.g. straight line) with different implications for the associated conceptual meaning. Finally, we note that Sherin's symbolic forms have been useful to researchers across disciplines in analyzing student reasoning. We hope that the idea of graphical forms will prove to be similarly fruitful to researchers.

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## Compliance with ethical standards

**Conflict of interest** The authors declare that they have no conflict of interest.

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