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Epistemological Matrix of Rational Number: a Look at the Different Meanings of Rational Numbers

Henrique Rizek Elias[1] · Alessandro Jacques Ribeiro[2] · Angela Marta Pereira das Dores Savioli[3]

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Abstract
Our goal in this article is to organise an epistemological matrix for the concept of rational number that contemplates its different meanings. Aiming at understanding the various ways to signify the rational numbers and based on the theoretical-methodological approach of conceptual profiles, we produce and analyse data from various sources, including the sociocultural and ontogenetic domains, namely data extracted from secondary sources on the history of mathematics, mathematical analysis of subconstructs of rational numbers, analysis of textbooks for the middle and high school, as well as textbooks for higher education, research data on students' alternative conceptions, interviews with 3 teacher educators and 4 in-service teachers from middle and high school. Such data allowed us, by means of an active interpretation of the researchers, to list 11 themes from which the rational numbers can be signified and to organise them into an epistemological matrix. Finally, we conduct a theoretical discussion that compares our epistemological matrix with an already quite widespread perspective in mathematics education, which is Kieren's subconstructs.

Keywords Conceptual profile · Epistemological matrix · Mathematics education · Rational numbers · Significance

The concept of rational number and the concepts connected to it are central for the learning of mathematics throughout all levels of elementary and secondary education (ages 6 to 17) and for the understanding of more advanced mathematical ideas. However, this concept challenges those who want to teach it and those who seek to

✉ Henrique Rizek Elias
henriqueelias@utfpr.edu.br

1 Universidade Tecnológica Federal do Paraná (UTFPR), Londrina, Paraná, Brazil

2 Universidade Federal do ABC (UFABC), Santo André, Brazil

3 Universidade Estadual de Londrina (UEL), Londrina, Brazil

understand it. Behr, Lesh, Post and Silver (1983) note that the concepts associated with the rational numbers are among the most complex and important ideas that students face along the first few years of schooling. This complexity can be debated under several perspectives: teaching, learning, students' thoughts, a proper initial teacher education for the teaching of the rational numbers and the role of the rational numbers in the future success of the students in mathematics, among others. Therefore, the rational numbers arouse the interest of many researchers in mathematics education, in seeking to set out the nuances that involve this mathematical concept and understand its complexities.

The *Rational Number Project* (RNP), for example, brings together a group of researchers who produced several projects and papers between 1979 and 2013, investigating both students' learning and teachers' improvement. Another example is the research summarised in the book *Rational numbers: an integration of research*, edited by Carpenter, Fennema, and Romberg (2009), which brings an important debate on the outcomes of studies about the rational numbers, including one of Thomas Kieren's research works. Our research takes special interest in Kieren's work, since many mathematical educators of diverse countries based their researches on Kieren's approach the concept of rational number (for example, Shahbari & Peled, 2017).

In Brazil, in particular, Kieren's works (1976, 1980, 1988) are much used and cited and have motivated several studies[1] that propose to investigate the rational numbers. Those Brazilian researchers focus, basically, on the five subconstructs of Kieren (1980), namely: part-whole, measure, operator, quotient and ratio. However, Kieren's research goes far beyond subconstructs, even providing a model for the construction of mathematical knowledge referring specifically to the rational numbers (Kieren, 1988, 2009).

Since Kieren's first work (1976)—when he presented seven interpretations (already called subconstructs) for rational numbers (fractions, decimal fractions, ordered pairs, element of a quotient body, measures, operators and ratio)—this mathematician opened doors for other researchers to propose variations for the subconstructs and to broaden the understanding of these numbers. Behr et al. (1983), for example, redefined some of Kieren's subconstructs (1980) and subdivided others, considering seven, namely, measure, ratio, rate, quotient, linear coordinate, decimal and operator.

Although accepted by many, Kieren's subconstructs perspective of (Kieren, 1976, 1980) presents limitations, especially when thinking about the teaching of rational numbers. For Thompson and Saldanha (2003), for example, Kieren's attempt "to map systems of complementary meanings into the formal mathematical system of rational numbers will necessarily be unsatisfactory in regard to designing instruction for an integrative understanding of fractions" (p. 13).

We agree with the critiques of Thompson and Saldanha (2003), especially when they say that "to focus on subconstructs or meanings of the mathematical system of rational numbers ultimately runs the risk of asking students to develop meanings for a big idea that they do not have" (p. 13). For us, Kieren's subconstructs organise rational numbers in a way with low attention to the different ways of thinking the rational numbers in varied contexts, privileging a single way of thinking, that of the mathematician. As

[1] In literature reviews conducted, Fávero and Neves (2012) and Angeli (2017) show that many Brazilian researches on rational numbers take Kieren's works as theoretical framework.

Thompson and Saldanha (2003), who consider the fraction as the synthesis of a scheme of conceptual operations (measurement, multiplication, division and fraction), different researchers have written about other possibilities to organise the ways of thinking about the rational numbers. Steffe and Olive (2010), for example, take into account social interaction in the elaboration of their model on children's fractional knowledge. This model is, as the authors consider, a model that observers can construct, from the observed person, and in this sense, mathematical knowledge is assumed as a kind of knowledge socially constructed.

As Norton and Wilkins (2010) point out, Kieren's subconstructs and Steffe's fractional schemes can be considered two prominent models of students' fractions knowledge. These two models present relevant differences between them, but they have common constructivist assumptions, both based on Piaget. In our study, we present another approach to model the different meanings for rational numbers, a socio-interactionist approach, based on Vygotsky, called the theory of conceptual profiles. This theory seeks to organise the heterogeneity of modes of thinking and ways of speaking about a polysemic concept, as is the case of the concept of rational number.

The theory of conceptual profiles, developed by a Brazilian researcher called Eduardo Mortimer et al. (Mortimer, 1995, 2000; Mortimer, Scott, & El-Hani, 2009), takes as a premise that people have different ways of conceiving and representing the world, exposing different modes of thinking, used in different contexts. Thus, polysemic scientific concepts admit the construction of a conceptual profile, which has the role of modelling that heterogeneity of modes of thinking and ways to talk about such concepts. Given the polysemy of a concept, according to the theory of conceptual profiles, the use that an individual makes of one or another meaning in a given context is tied to the diverse and multifaceted social experience that the individual has had throughout his life. According to Mortimer, Scott, Amaral, and El-Hani (2014),

> [...] we have at our disposal a diversity of stabilized meanings in different social languages, the weight each of them in our personal way of thinking depending on the extent to which we had opportunities to fruitfully use them throughout our development, in order to face challenges posed by our experiences. (p. 14)

Originating from Mortimer's doctoral thesis (1994), the approach of conceptual profiles has been widespread in the area of teaching of sciences. For example, in biology, several concepts have already been profiled, as the concepts of life, death and breathing; in physics, the concepts of mass and strength; in chemistry, the concepts of heat and substance.

In mathematics education, this theoretical model remains little explored, but it can be foundational for some studies to discuss polysemic mathematical concepts. In this aspect, Ribeiro's research (2013) identified and categorised some zones of a conceptual profile of equation; Machado (1998) delineated zones of a conceptual profile of function; and Angeli (2014) aimed to investigate zones of a conceptual profile for the concept of variable.

Elias (2017) started the construction of a conceptual profile of rational number. The process of construction of a conceptual profile, which we describe in the next section, is long and requires certain methodological rigour. In this work, we bring into discussion one stage of this construction, configured in the organisation of a methodological tool, called epistemological matrix (Sepulveda, Mortimer, & El-Hani, 2014), based on themes from which the rational numbers can be signified (called hereafter epistemological themes).

Therefore, the aim of this article is to arrange an epistemological matrix for the concept of rational number (MENR[2]) covering different meanings (epistemological themes) of that concept. As an implication of the organisation of the MENR, we present and discuss here the extent to which our approach departs from Kieren's subconstructs and claim new possibilities to use the MENR in the teaching and learning processes of that mathematical concept.

We start this article with a section that discusses the theoretical model of conceptual profiles, mainly its methodological principles. In the following section, we bring our methodological choices to build the epistemological matrix, explaining the contexts and the procedures of data collection. As a result, we delimited the epistemological themes and organised the MENR. In the penultimate section, we discuss the differences between the approach of conceptual profiles used for the rational numbers and that considered in Kieren's subconstructs (1976, 1980, 1988). We finish the paper with some final considerations and possibilities for future research that will enable to actualize, from the MENR, a conceptual profile of rational number.

Theoretical Framework: the Model of Conceptual Profiles

In 1990, Mortimer (1994, 1995) introduced conceptual profiles as a way of modelling the heterogeneity of modes of thinking and ways of talking about scientific concepts that are present and interacting in science classes. This approach was inspired by Bachelard (1984) epistemological profile, but, in order to investigate the teaching and learning in science, other elements have been added to Bachelard's ideas, such as the characterisation of conceptual profiles, taking into account, in a balanced way, the ontological and epistemological aspects, and not only epistemological as in the original idea. In later years, however, the philosophical foundations of the theory of conceptual profiles withdrew even more from Bachelard's ideas (Mortimer et al., 2014), keeping only a few similarities with the epistemological profile, such as the notions of "profile" and of "zones".

The theoretical model of conceptual profiles is based on the distinction between *sense* and *meaning* given in Vygotsky's work. While sense is, for Vygotsky (2000), dependent on the context, meaning is much more stable and repeatable. Meaning thus offers the possibility of intersubjectivity, i.e., a situation in which two or more persons may share the meaning of a word, although they may differ in the senses attributed to it.

Therefore, "to learn a concept is to learn its meaning, generalize, go from personal senses to socially accepted meanings" (Mortimer et al., 2009, p. 4, our translation). Yet, the production of sense is an entirely personal process, in which each individual produces different *senses* for the same word and the same individual can also vary in the senses produced in different contexts. However, the authors state, "When

[2] We choose to use and preserve the acronym of Epistemological Matrix of Rational Number as MENR, because it is referring to *Matriz Epistemológica de Número Racional*, in Portuguese.

conceptual thinking is fully formed, the production of sense is restricted by the socially accepted meanings" (p. 4, our translation).

In view of this, we highlight two central aspects in our work: (i) to learn is to internalise meanings produced within a certain social practice; and (ii) people have different ways of conceptualising the world and, as it was said, the approach of the conceptual profiles aims to model this heterogeneity of modes of thinking and ways of speaking in complex social places, such as the classroom. According to Mortimer et al. (2014), the conceptual profile theory is grounded in the idea that people exhibit different ways of seeing and conceptualising the world and, thus, different modes of thinking that are used in different contexts. Modes of thinking are stable manners of conceptualising a given kind of experience, by ascribing to it a socially constructed meaning attributed to a certain concept.

The variety of meanings assigned to a particular concept constitutes what is called the polysemic concept. For Mortimer (1994), polysemic concepts, such as atom and mass, enable the elaboration of conceptual profiles. The profiles are composed of different *zones*, that correspond to different ways of seeing, representing and signifying the world. The different zones of a conceptual profile can coexist simultaneously in the same individual and each zone is more or less powerful in different contexts.

According to Mortimer et al. (2009), besides the acquisition of new zones of a conceptual profile, it is "a crucial goal of teaching and learning to promote a clear vision, among students, of the demarcation between the modes of thinking and meanings, as well as between their contexts and application" (p. 7, our translation). This awareness of demarcation of zones of a conceptual profile of a given concept implies that the student is able to use a scientific idea in appropriate contexts, including in daily life, preserving, at the same time, modes of thinking and ways of speaking other than the scientific ones, in situations where they show to be pragmatically appropriate (Mortimer et al., 2009).

For the construction of a conceptual profile, it is necessary to consider a diversity of ideas and contexts of production of meanings, including at least three of the four genetic domains pointed out by Vygotsky in his investigations on the relationship between thought, language and concepts formation, namely, the socio-cultural, onto-genetic and micro-genetic domains. This is a first methodological principle to profile a concept: to contemplate a variety of ideas and contexts of production of meanings.

According to Sepulveda, Mortimer, and El-Hani (2013), the development of a concept in sociocultural domain has been investigated on the basis of ideas related to the concept in question found in the history of science, in epistemological reviews of the concept and in the context of the production of school knowledge, by the way this concept is discussed in textbooks. With respect to the ontogenetic domain, these authors affirm that there is a diversity of research already produced about how central concepts from different areas of knowledge are learned by students of different ages, and how these ideas evolve throughout the history of life of individuals. The empirical data produced in original studies on the construction of meanings for the concept in a school context, through interviews, questionnaires or footage of discursive interactions in the classroom, have enabled the study of the genesis of concepts in short periods of time, covering the micro-genetic domain.

From the point of view of the analyses, as stated by Mortimer et al. (2014), what is sought in the data set are the so-called epistemological and ontological commitments.

These commitments stabilise modes of thinking and ways of speaking about the concepts that allow for the identification of zones of a profile and for profiling a concept.

The epistemological commitment refers to the nature of the concept, to its genesis and to its stabilisation as such, whether in academy, whether in school or in other contexts. The ontological commitment refers to the individual member of a culture and to the way they appropriate the socially constructed meanings. It is in this aspect that the zones of a conceptual profile differ, for example, from Kieren's subconstructs, inasmuch as each zone is not composed of mathematical relations only (epistemological commitment), but considers the learning subjects (ontological commitment) and their different modes of thinking inside and outside classrooms (including the so-called non-scientific forms). The modes of thinking of the subjects of a given culture determine the zones, not the other way around.

The process of identifying the zones of a conceptual profile is not merely a categorisation of meanings, although it includes this procedure. This becomes obvious when one considers that individuation of the zones is made through the epistemological and ontological commitments that structure different modes of thinking and ways of speaking about a concept. These commitments are not given explicitly in statements, it is necessary to interpret the statements of the subjects in terms of a repertoire of ontological and epistemological commitments prepared as hypotheses and constantly reformulated by the investigator, in the light of his/her data sources. The individuation of zones of a conceptual profile is, therefore, the result of an active interpretation of the researcher, who bases his/her investigation on hypotheses formulated from a dialogue between their data sources in a way that he/she can work such commitments. Here is a second methodological principle, important for the construction of a conceptual profile: the data set must be examined in a non-sequential, dialogic way, suggesting that the contents obtained in each domain are, at all times, articulated with each other (Sepulveda, 2010).

A third methodological principle, according to Sepulveda (2010), considers that when examining data on the development of the concept in focus in different genetic domains, we should not draw parallels between the contents characteristic to each one of them. We should not, for example, draw parallels between students' ideas and those found in the history of science, but rather articulate them, in order to have a broader view of the genesis of the concepts.

An important methodological step to organise the polysemy around the concept investigated, to generate categories that will be derived from the zones of a profile, is called epistemological matrix, proposed by Sepulveda (2010) and further discussed in Sepulveda et al. (2014). The epistemological matrix is a stage that precedes the individuation of zones of a conceptual profile and consists in organise different epistemological themes from which the concept in focus can be signified.

An *epistemological theme* is a matter that can give rise to the concept, i.e., a theme from which the concept can be signified. From its origin, different modes of thinking the concept can arise and be *categorised* based on the *ontological and epistemological commitments* that structure the interpretation of the concept in question.

According to Sepulveda (2010), the zones of the profile model derive from the combination of various ontological and epistemological commitments referring to the epistemological themes identified. The process of individuation of zones is not a focus of discussion of this article. First, because the contexts in which we investigate the concept of rational number do not include all the expected genetic domains (for example, they do not embrace the micro-genetic domain). Second, because, in our analyses, we have not executed the movements of reformulation expected to individuate the zones of a conceptual profile. Therefore, we propose to present the construction process of the MENR, which was based on certain epistemological themes that will be discussed in the following sections.

Research Methodology

Research Context

This article is the result of broader research,[3] in which we sought to understand meaning of rational numbers as elements of an ordered field ($\mathbb{Q}$, $+$, $*$) can contribute to the development of the professional knowledge of the middle and high school prospective teachers. To this end, we had established four specific objectives. One of them was to organise, in MENR, different ways to signify the concept of rational number. In this article, we present in detail that stage of the original research.

The methodological approach we assume here follows the principles of qualitative research (Esteban, 2010), in an interpretative theoretical perspective (Crotty, 1998). This article was developed from the perspective of establishing a parallel between the modes of thinking and ways of speaking about the rational numbers in a school context and the modes of thinking and ways of speaking about the rational numbers in the context of the pre-service mathematics teacher education. Thus, for all data source provided for the rational numbers in basic school, an equivalent source in higher education was also considered. To contemplate these two contexts for the construction of the MENR, data were produced from literature review, historical review about the concept of rational number, analysis of textbooks and interviews with mathematics teachers. This will be better explored in the next section.

Data Sources and Analysis Method

The academic research gathered in the literature review was obtained from searches both in journals and in congress, both from Brazil and from abroad, and dissertations and theses of graduate programs in mathematics education in Brazil. This study has prioritised research on the students' alternative conceptions, as well as research involving conceptual analyses on the rational numbers and processes of teaching and learning of those numbers. In the case of textbooks, for the school context, we chose a collection of mathematics textbooks for the middle education (students from 11 to 14 years), and for the context of pre-service teacher education, we chose books that often appear as basic references in disciplines that are present in the curriculum matrices of undergraduate courses in mathematics in Brazil. In the case of interviews with teachers, we interviewed four middle and high school teachers who had extensive experience in teaching (nearly 20 years of experience) and three teacher educators who worked in

[3] A doctoral thesis (Angeli 1, 2017), held in Brazil, which had as main objective to investigate and propose theoretical-methodological foundations for teaching the field of rational numbers in Mathematics Teacher Education undergraduate courses.

various public institutions of higher education in Brazil and that taught the discipline of algebra for pre-service teacher education.

Thereby, our analyses made from sources that cover the sociocultural and ontogenetic domains proposed by Vygotsky. As for the sociocultural domain, the sources were information extracted from secondary sources on the history of the rational numbers (Ifrah, 1996; Katz, 2010; Roque, 2012); mathematical analyses of subconstructs proposed by Kieren (1976, 1980); analysis of textbooks for middle and high school (Chavante, 2015a, 2015b, 2015c) and for higher education (Caraça, 1951; Carvalho, Lopes, & Souza, 1984; Domingues & Iezzi, 2003; Niven, 1984).

As for the ontogenetic domain, the sources were information obtained in the literature (Behr et al., 1983; Damico, 2007; Pinto & Tall, 1996; Wasserman, 2014); interviews with three teacher educators and four middle and high school's teachers to understand some of their ways of thinking rational numbers.

Data collection and its organisation occurred in the following order: study of Kieren's subconstructs, historical reconstruction of rational numbers, interviews with teachers, analysis of textbooks and, finally, research on alternative conceptions. During this first data organisation, preliminary analyses were conducted, seeking ways that would indicate us the origin of the concept of rational number. Later, the analyses that we carried out followed in a dialogical perspective, in the sense that we would promote comings and goings in each interpretation held from the data, allowing a "conversation" between them. For now, our active interpretation allowed us to list 11 epistemological themes and the categories that comprise the MENR, which we present henceforward.

Data Analysis

Because it is a part of the broader research, and for reasons of space limitations, we had to make some choices about what to consider in this paper. We chose to focus more attention on the presentation and organisation of the MENR than in the process of its construction. From now on, we present each of the 11 epistemological themes, characterising and explaining them. As a result, we present the MENR.

The Epistemological Theme 1: Relationship between Dimensions. A theme around which the rational number can be signified is the relationship between dimensions (of the same or different species), in which the rational number $\frac{a}{b}$ is connected to the idea of quantitative comparison between two quantities a and b.

The idea of a relationship between two quantities, that appears since the Greeks, as Katz (2010) presents, indicates a genesis of the concept of rational number other than the idea of fraction. During Euclidean times, the concept of ratio as a multiplicative comparative between quantities was not equivalent to a fraction between numbers (Roque, 2012). One reason was taken as a size ratio between two quantities of the same type (two lines, two surfaces or two solids) and not as a particular point on the number line and on which standard arithmetic operations could be performed (Katz, 2010).

The very notion of ratio presents different ways of being signified. For example, for Gottfried Leibniz, different from the fraction, a ratio is a relation that is independent of the terms that compose it. This leads us to an idea of *rate*, associated with a variation of a magnitude in relation to other quantities (of different kinds).[4] Kieren (1976, 1980) has an approach to ratio in the sense of a comparison between numbers or quantities of similar nature. This mode leads us to another meaning of ratio.

The Epistemological Theme 2: Division between Two Integer Numbers. A theme around which the rational number may be signified is the division between two numbers, in which a rational number is directly related with the idea of division between two integer numbers.

According to Roque (2012), for Leibniz a fraction meant a division of two numbers, "therefore it was a quantity obtained by dividing two quantities" (p. 361, our translation). In addition to evidence presented in excerpts from the history of mathematics, research in mathematics education and the middle and high school textbooks lead us to the following modes of signifying the rational numbers: (i) the symbol $\frac{a}{b}$ can also be used to represent an operation, i.e., $\frac{a}{b}$ is used as a form of writing $a \div b$ (Behr et al., 1983). More than a number, $\frac{a}{b}$ indicates a process, a "something to be done", that we call indicated division; (ii) as stated by Katz (2010), for the Egyptians of the so-called ancient mathematics, fractions appeared when they needed to divide two numbers and the result of this division was not exact. In this case, the representation $\frac{a}{b}$ ($a \in \mathbb{Z}$, $b \in \mathbb{Z}^*$) indicates a quotient. This mode of signifying the rational numbers as a quotient is not restricted to a non-exact division and can extend to exact divisions.

The Epistemological Theme 3: the Inverse of an Integer Number. A theme around which the rational number can be signified is the inverse of a integer number, in which a rational number $\frac{1}{b}$ is conceived as the inverse of the multiplicative of the integer number b.

As pointed out by Roque (2012), the Egyptians attributed this meaning to rational numbers. In the case of the Babylonians and their *tables of reciprocals*, the inverses were used to perform divisions: dividing by a is the same as multiplying by the inverse of a.

This way of signifying the rational numbers has a natural connection with the integer numbers. To some extent, this has some connection with the discourse of teacher Victor, one of the teacher educators interviewed, when he explains that, in order to maintain algebraic consistency, the passage from $\mathbb{Z}$ to $\mathbb{Q}$ adds not just the multiplicative inverses of the integer numbers, but also a lot of elements that are neither integer numbers nor multiplicative inverse of integer numbers. That is, the extension of integer numbers to rational numbers involves the inclusion of two "types" of elements: the multiplicative inverses of integers and those that are neither integer nor inverse of integer numbers (for example, $\frac{2}{3}$). The multiplicative inverses of integer numbers have some specificity in signifying the rationals.

The Epistemological Theme 4: Process of Double Counting. A theme around which a rational number can be signified is the process of double counting (the count of the parts of the whole (n) and the count of the parts taken (x)), in which a whole is

[4] For a more discussion about the concept of rate, we indicate Thompson (1994).

partitioned in n equal parts (each part may be represented by $\frac{1}{n}$) and we take a number x of parts (we will have $\frac{x}{n}$ of the whole).

Katz (2010) asserts that both the Greeks in Euclidean times and the ones before him did not use fraction of a whole in their formal work, since the unit could not be divided. Along the historical development of the rational numbers we showed earlier, we did not identify, explicitly, the relationship part-whole as a process of double counting. The notion part-whole is a recent guidance of education, being mobilised in types of tasks that did not appear in the early days of the construction of the fractional numbers field.

As regards the microgenetic domain, Steffe and Olive (2010), highlight that the idea of fractions as part-whole is a common meaning among children in the classroom. From the perspective of these authors, the "purpose or goal of a part-whole fraction scheme is to partition a segment into so many parts and establish a part-to-whole relation." (Steffe & Olive, 2010, p. 323).

This mode of signifying the rational numbers is related to the ability to partition a continuous quantity or a discrete set of objects in subparts of same size (Behr et al., 1983). Besides, the relationship part-whole has a strong geometric appeal.

The Epistemological Theme 5: Ways to Represent. A theme around which the rational number can be signified is the way to represent a number, in which a rational number is so strongly associated with its forms of representation that these are intertwined with the very concept.

The usual definitions for rationals favour this aspect: a rational number is a number that can be written in the form a/b with $a \in \mathbb{Z}$ and $b \in \mathbb{Z}^*$; or a rational number is a number whose decimal representation is either finite or a recurrent decimal. Chavante's textbooks (2015b, 2015c) assume these definitions for rational number.

Through research on the students' alternative conceptions, we found conceptions of rational number strongly based on forms of representation (fractional or decimal). In the case of the fractional form, a recurrent way of signifying the rational numbers, which is not consistent with the academic form or school, was a number that can be written in the form $\frac{a}{b}$ (without restricting a and b). Yet the definition that values the decimal representation can lead to a way of signifying the rational number from denying the irrationals (a rational number is one that is not irrational), as could be seen in the research by Pinto and Tall (1996).

The Epistemological Theme 6: Geometric. A theme around which the rational number may be signified is the geometric aspect associated with numbers, in which a rational number can be associated to a point on the straight line or the ratio between commensurable segments. The rational numbers as points on a straight line were already used by Dedekind, as stated by Roque (2012). In Chavante's book (2015b), the author presents the representation of a rational number on a number line, discussing its location through different representations (fraction, mixed, decimal number).

A second form of signifying the concept of rational number whose origin has a strong geometric bias is that presented by Caraça (1951), where the rational number is taken as the ratio between the lengths of two commensurable segments. This meaning, which is characterised as a comparison between lengths (*ratio*), establishes a close relationship between rational and irrational numbers.

The Epistemological Theme 7: Multiply and Divide. A theme around which the rational number may be signified is the multiplication and division of whole numbers, in which a rational number $\frac{a}{b}$ (with $a \in \mathbb{Z}$ and $b \in \mathbb{Z}^*$) is understood as dividing something into two b equal parts and multiplying the result by a (or in another order, multiplying something by a and dividing the result in b equal parts).

These cases are usually accompanied by the preposition "of", which carries with it a multiplicative sense. For example, $\frac{2}{3}$ of 60 is the same than $\frac{2}{3}.60 = \frac{2.60}{3} = \frac{120}{3} = 40$. In this example, we have that the fraction $\frac{2}{3}$ modified the number 60 (initial situation), turning it into 40 (final situation). We say that the rational numbers, from this perspective, have a characteristic of operator, that is, something that modifies a situation over which it operates. In addition, by their multiplicative sense, it is common to say that the rational number with this characteristic of operator defines a multiplicative structure of rational numbers (Damico, 2007; Kieren, 1976, 1980). In this aspect, there is a distinction, not always so well delimited, between this epistemological theme and the epistemological theme 2. In the epistemological theme 2, the number 40 is the result of division 120 by 3; here, the number 40 is obtained from the transformation of 60 by operator $\frac{2}{3}$.

In this meaning of a rational number $\frac{p}{q}$ as an operator, Behr et al. (1983) distinguish two cases: a transformation of a continuous object (such as the length of a segment of a straight line) or a transformation of a discrete object (a set with n elements).

The Epistemological Theme 8: Everyday Practices. A theme around which the rational number can be signified is associated with everyday practices, in which a rational number $\frac{a}{b}$ connects to numerous day-to-day activities without necessarily any connection with a particular meaning of those belonging to the school mathematics or mathematics education, but are derived from a mathematics of everyday life.

The historical study revealed that various habits and everyday practices, especially in situations of weights and measures, of different people and in different contexts, have brought about the creation of the Hindu Arabic positional system, which began to be used, in the west, only in the sixteenth century. Professor Roberto, one of middle and high school teachers, who had been interviewed, associates the rational numbers, in his representations of fraction and decimal numbers, to the use of the currency of the country, emphasising its relevance for the understanding of these numbers.

Those data sources indicate how the daily practices are crucial to signify the concepts. In some way, such everyday practices are brought to the interior of the school as a means to teach rational numbers from the reality of the student. However, these are situations in which, regardless of the school, the rational numbers appear and can be signified as *used numbers* to deal with money, weight, height, cooking recipes, time (fraction of seconds), discounts on trade, etc.

The Epistemological Theme 9: Equivalence Class of Ordered Pairs of Integer Numbers. A theme around which the rational number may be signified is the idea of equivalence class, in which a rational number is taken as equivalence classes of ordered pairs of integer numbers with a non-null second coordinate ($\mathbb{Q} = \mathbb{Z} \times \mathbb{Z}^*/\sim$). In this case, there is a certain strangeness in signifying the rational number as an equivalence class, i.e., as a set.

This meaning of rational numbers, typical of contemporary mathematics, appears in history in the eighteenth and nineteenth centuries. It is the result of a historical context favourable to mathematics as a logical-formal-deductive system and, in respect of the numbers, signified the construction of an abstract concept of number that is not subordinate to the idea of quantity (Roque, 2012).

The textbooks geared to higher education we analysed feature the construction of rational numbers as equivalence classes of ordered pairs of integer numbers with non-null second coordinate ($\mathbb{Q} = \mathbb{Z} \times \mathbb{Z}^{*}/\sim$), i.e., it is an extension of the set $\mathbb{Z}$ and whose properties are also deducted from those already known in $\mathbb{Z}$.

The Epistemological Theme 10: Element of a Field. A theme around which the rational number may be signified is the collection of the axioms of the field algebraic structure, in which a rational number is an element of the ordered field ($\mathbb{Q}$, +, .) and, therefore, is subject to the axioms of that structure.

Wasserman's work (2014) points out that valuing the collection of the axioms can be a fertile way to working algebraic structures in teacher's education courses. The path to this may be the polynomial equations of the first degree with an unknown factor. This means that rational numbers are regarded as a solution of an equation of the type $ax = b$, with $a \in \mathbb{Z}$ and $b \in \mathbb{Z}^{*}$.

In this context of a field element, the rational number is "anything" that satisfies those properties of the field of the rational numbers, regardless if that number means a quotient, a ratio or an equivalence class of ordered pairs of integer numbers. The structure overlaps the nature of the number.

The Epistemological Theme 11: the Field of Fractions of the Domain of Integrity $\mathbb{Z}$. A theme around which the rational number may be signified is the field of fractions of a domain of integrity, in that a rational number is an element of the field of fractions of the domain of integrity $\mathbb{Z}$.

This epistemological theme seems to us the furthest from the school mathematics. Some books devoted to higher education, as Domingues and Iezzi (2003), feature this mode of signifying $\mathbb{Q}$, a characteristic way of mathematics education and, from the point of view of history, is the most recent meaning attributed to rational numbers. From this recent perspective, the elements of the ratio set $K = (A \times A^{*})/\sim$, in that A it is a domain of integrity, are the *fractions* $\frac{a}{b}$ ($a \in A$ and $b \in A^{*}$). In this context, the meaning of fraction is coupled to the advanced concept of quotient set. When we take the domain $\mathbb{Z}$ as the domain of integrity A, then the field (K, +, $*$) is called the field of rational numbers and is denoted by ($\mathbb{Q} = (\mathbb{Z} \times \mathbb{Z}^{*})/\sim$, +, $*$).

In this case, the fact that Q is the body of fractions of the integral domain Z makes it "similar" to the set Q (X), since this is the body of fractions of the integral domain Z [X]. Note that the emphasis now is on the body of fractions and not on the nature of the elements (the same feature of epistemological theme 10, where the focus is on the algebraic structure, not the nature of the number). So, we say that the important thing here is to establish similarities to deal with different things in the same way.

The Epistemological Matrix of Rational Number

After having described the 11 epistemological themes, we will now present the main result of this article, the MENR. In MENR, for each epistemological theme (first column of the matrix), we highlight some categories (second column of the matrix), which are specific to each epistemological theme. In addition, we seek to make explicit ontological and epistemological commitments (third column of the matrix) that, in future research, will be combined to give rise to zones of the conceptual profile of rational number. The MENR, presented in Table 1, was built based on the epistemological matrix of the concept of adaptation by Sepulveda (2010) and Sepulveda et al. (2013).

The categories produced in this matrix expands different ontological and epistemological commitments for the same epistemological theme. These ontological and epistemological commitments will be combined to give rise to the zones that constitute the conceptual profile of rational numbers. When zones are individuated, they are likely to intersect with each other, since ontological and epistemological commitments do not (necessarily) present clear dividing lines. This means, for example, that the ontological and epistemological commitments of the categories of the epistemological theme 4, which has a strong geometrical appeal, may intersect with the ontological and epistemological commitments of the categories of the epistemological theme 6. The same holds true for the ontological and epistemological commitments of themes 1 and 8.

The fact that we have obtained 11 epistemological themes in our MENR does not mean that we will construct 11 zones of the conceptual profile of rational number. By combining the ontological and epistemological commitments to individuate the zones, we can obtain a smaller number of zones precisely because of the possibility of overlapping of the commitments.

It is also worth mentioning that the epistemological themes we bring do not exhaust all possible themes for rational numbers. Other researchers with data sources from other contexts may provide us with an organisation other than MENR. As presented in the second methodological principle, for the theory of conceptual profiles the active interpretation of the researcher is fundamental in the process of building the MENR. This does not mean, however, that each researcher must draw up their own MENR. The aim is that this MENR is taken to other contexts to be used and improved by other researchers, in order to increasingly encompass epistemological themes and refine the ontological and epistemological commitments, making the matrix as general as possible. The MENR is not static, it must always be in reformulation, since it has to be connected to the cultures.

Discussion

Despite presenting the construction of the MENR from data analysis—such as textbooks, surveys and interviews with teachers—the main contribution of this article is still in the theoretical field, proposing a way of modelling the heterogeneity of modes of thinking and ways of speaking about rational numbers that involve meanings stabilised in different contexts such as academic (universities), school (in the interaction between teacher and students in the classroom) and daily (from different realities).

This theoretical contribution will gain empirical status as we move forward, for example, in classroom discussions (microgenetic domain), involving students. This

Table 1 Epistemological matrix of rational number (MENR)

The epistemological theme	Categories	Ontological and epistemological commitments
1. Relationship between dimensions	Rate	Rational number is interpreted as a variation of a dimension in relation to another, of distinct natures
	Ratio	Rational number is interpreted as an indicator of comparison between two quantities of similar natures, not as a number
2. Division between two integers	Indicated division	Rational number is interpreted as a quantity obtained when two quantities are divided. $\frac{a}{b}$ is used as a way of writing $a \div b$
	Quotient	Rational number is interpreted as a number obtained when two integer numbers are divided
3. The inverse of an integer number	The inverse of the operation	Rational number is interpreted as the inverse of an integer: dividing by a is the same as multiplying by the inverse of a
	Multiple	Rational number $\frac{a}{b}$ is interpreted as a multiple of the inverse of the integer number b
4. Process of double counting	Partition the continuous whole	Rational number is interpreted as the partition of a continuum in subparts of the same size
	Partition the discrete whole	Rational number is interpreted as a partition of a discrete quantity in subparts of the same size, being the "part" the number of objects taken and the "whole" the total number of objects
5. Ways to represent	Fraction	Rational number is interpreted as a number that can be written in the form $\frac{a}{b}$, without considering the restrictions for the numbers a and b
	Fraction of integers	Rational number is interpreted as one that can be written in the form $\frac{a}{b}$, where a and b are integer numbers, with $b \neq 0$
	Decimal	Rational number is interpreted as one that can be represented in decimal form by a finite number of places or by a recurrent decimal
6. Geometric	Location on a straight line	Rational number is interpreted as a point on a number line
	Relationship between two commensurable segments	Rational number is interpreted as the ratio between two commensurable segments
7. Multiply and divide	Transformation of a continuous object	Rational number is interpreted as a stretcher-shrinker combination
	Transformation of a discrete object	Rational number is interpreted as a multiplier-divisor combination
	Functional	Rational number is interpreted as a function $f(x) = \frac{p}{q}.x$ that turns q into p or a machine that, when receiving q, returns p
8. Everyday practices	Association with the social usage	Rational number is interpreted as the numbers used in everyday practices

Table 1 (continued)

The epistemological theme	Categories	Ontological and epistemological commitments
9. Equivalence class of ordered pairs of integer numbers	Formal construction from the integers	Rational number is interpreted as an equivalence class of ordered pairs of integer numbers
	Abstraction	Rational number is interpreted as a set before becoming a number
10. Element of a field	Solution of an equation of the form $ax=b$ (with $b\in\mathbb{Z}$, $a\in\mathbb{Z}^*$)	Rational number is interpreted as an element of a quotient field
	The relevance of the structure overlaps the nature of the number	Rational number is interpreted as "anything" that satisfies those properties of the field of the rational number
11. The field of the fractions of the domain of integrity $\mathbb{Z}$	Establish similarities to deal with different things	Rational number is interpreted as an element of the field of the fractions of the domain of integrity $\mathbb{Z}$

should be done both on the basis of research conducted by ourselves, in our contexts, and by research carried out in classrooms in other contexts, which we did not do in the present study. In this sense, in the microgenetic domain, Steffe's works (e.g. Steffe, 2002; Steffe & Olive, 2010) will be central as they provide detailed descriptions of the meanings children have for the idea of fraction. Likewise, textbooks for the early years of elementary school will also need to be incorporated into research.

So far, as a theoretical contribution, we can debate this new perspective through the MENR in relation to other theoretical approaches that deal with rational numbers. Let us take the case of the already widespread Kieren's cognitive perspective (1976, 1980, 1988) and make some comparisons with the social interactionist perspective of learning of the rational number concept, based on the conceptual profiles, which we are assuming in this article when we propose the MENR.

We start by presenting briefly the intuitive model of mathematical knowledge building with special reference to rational numbers, proposed in Kieren (1988). This mode will be here called ring model, because it is represented by four concentric rings. The author suggests that this ring model of personal construction of mathematical knowledge is purposely organic in nature. The smaller ring, represented by E, is called by the author of ethnomathematical knowledge. It is a knowledge oriented in nature and is neither a "school-oriented" knowledge, nor identified as mathematics by the person who uses it. It consists of the "basic knowledge acquired as a result of living in a particular environment" (Damico, 2007, p. 66, our translation). For example, only half, a fourth and a third are known.

The second ring, which contains the first, is called the intuitive level, represented by I, and, according to Kieren (1988), it is usually a school-oriented or taught knowledge. Such knowledge can be built from everyday experience and relate to it. This level represents processes of abstractions from the experiences or that come from them.

The third ring from inside to outside represents the symbolic technical knowledge, represented by TS, which is constructed through the use of language pattern, notations and algorithms. So that this knowledge is stable and useful, Kieren (1988) asserts that it should be checked against some form of "reality" or represent a local logic sequence that can be assessed in terms of the axioms for rational numbers.

The last level and the one that encompasses all of the others, as shown in Fig. 1, is the axiomatic knowledge of rational numbers, called A (Kieren, 1988). Obviously, this implies the formal knowledge of rational numbers, seen as the field algebraic structure. Unlike the two previous rings (I and TS), which have a more direct connection with the school, ring A seems to be more related to formal university mathematics.

Despite bringing valuable contributions to our research, Kieren's approach diverges from the way we understand both a possible grouping of ways of thinking and ways of talking about the concept of rational number and its implications for teaching. As they are based on a mathematical analysis, the subconstructs gain, many times, an unnecessarily formal language. The approach of conceptual profiles, on the other hand, when considering the epistemological and ontological commitments necessary for the construction of a conceptual profile involve other aspects, including the social value assigned to the concept, which was not observed in Kieren's analyses.

Individuating zones of the conceptual profile of rational number involves identifying particular modes of thinking this concept (including students' alternative conceptions), considering that each mode of thinking may be related to a particular way of speaking. In other words, a formal mathematics language is not dominant in all zones of the conceptual profile of rational number, but can be characteristic of a specific zone.

From the point of view of the implications for the teaching of rational numbers, our view is that the construction of the conceptual profile involves the very social practice of the teacher and, therefore, is not outside it, as suggested by the Kieren's approach. For us, it does not make sense to think mathematics teaching from a purely mathematical analysis, since the concept, as it is socially constructed, is also configured inside the classroom.

Precisely because it is a fundamentally mathematical analysis, Kieren's works indicate certain "naturalness" in the perception that the algebraic structures of group and body relate to the construction of knowledge of rational numbers. In these cases, the analyses of the algebraic structures underlying the subconstruct operator, for example, are "working as

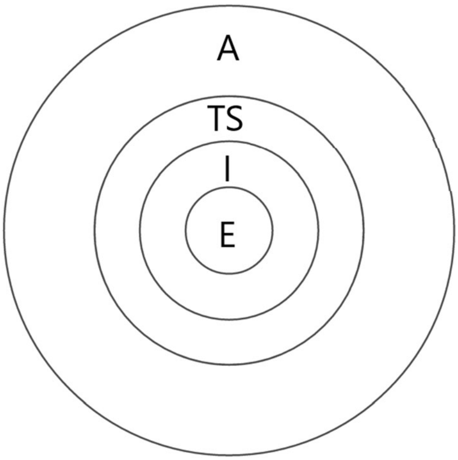

Fig. 1 The intuitive model for the construction of knowledge (prepared by the authors, adapted from Kieren, 1988, p. 170)

a teaching tool to promote the development of cognitive structures associated" (Moreira & Ferreira, 2008, p. 118, our translation), highlighting the following motion: "the 'mathematical' defining the 'cognitive' and the latter suggesting the 'didactic'" (Moreira & Ferreira, 2008, p. 118, our translation). Moreira and Ferreira (2008) emphasise that this logic can serve as a means to assign a role to academic mathematical knowledge in the development of school curriculum, without clarifying the superficial and simplistic argument that "to teach mathematics the teacher must know mathematics" or that "the teacher has to know more than what he/she going to teach", which, in a way, has been justifying the presence of those structures in teacher training courses.

Another distinction we make concerns the hierarchy of the different meanings. The notion of conceptual profile seeks to break with the commitment of an epistemological hierarchy of zones of a profile to adopt the concept of verbal heterogeneity, grounding the organisation of the profile according to the principle of complementarity (Sepulveda, 2010). By complementarity, Sepulveda (2010) explains, we understand that different interpretations of certain phenomena provide information of reality that are not excluding, but, on the contrary, they complement each other. Yet the model of knowledge construction of Kieren establishes a hierarchy other than that considered in the conceptual profiles. Kieren's ring model for rational numbers suggests that more "advanced" levels contain others and, therefore, the individual who deals with a concept in its academic sense seems to necessarily dominate the other levels. However, this domain seems to be static, because, since it is growing, if the person has reached the highest level, he/she has reached his/her full development. On the other hand, the idea of the conceptual profile seems more dynamic in the sense that it is modified by the social experiences of the person.

If we are to establish differences between the approach of the conceptual profiles adopted here and the perspective of the subconstructs of Kieren in this article, this was with the sole intention of presenting to the academic community another way to think about the teaching and learning of rational numbers, as well as for the mathematics teacher education. We might have sought to establish a dialogue with other perspectives, such as Steffe's fractional schemes, a constructivist approach that takes into account classroom social interactions and students' conceptions to understand children's knowledge about fractions. However, we believe that these two prominent models (Norton & Wilkins, 2010) have a strong contribution to the microgenetic and ontogenetic domains, and it is necessary, according to the theory of conceptual profiles, to advance in the socio-cultural domain. It is in this sense that we propose the construction of a conceptual profile of rational numbers, a model that, in the long term, can lead to different socially stable meanings of an even broader concept that is the concept of number.

Final Remarks

In this article, we present the MENR, an alternative to look at rational numbers, which brings new possibilities to the processes of teaching and learning this mathematical concept and to mathematics teacher education. As to incorporating an approach based on the model of conceptual profiles in mathematics classes, Ribeiro (2013) suggests that if it seems to be possible, with such an approach, to enable teachers to understand the diversity of meanings which can compose the knowledge of their students. Ribeiro (2013) also suggests that there is an urgent need to understand whether and how an approach to teaching based on the model of conceptual profiles contributes to a better learning of mathematics in classrooms of basic education and training of teachers.

We also consider that the MENR, as a construction that precedes the development of a conceptual profile of rational number, can be a useful tool for be discussed in initial teacher education courses. In the next paragraphs, we indicate some paths for future research.

We believe that an approach to the rational numbers via MENR enables future teachers to know and work with different modes of thinking and ways of speaking about the rational numbers, including those that are not produced in the school context, as, for instance, the epistemological theme 8. School is a social place that strongly influences the development of the subject but it is not the only one. Each student brings into the classroom a "cultural background" (Mortimer, 2000) that should not be ignored by the teacher. In this sense, the work with the MENR can promote an understanding of the teaching of rational numbers in schools that do not require to replace the students' previous ideas about rational numbers by new scientific ideas. On the contrary, the work with the MENR seeks to highlight the heterogeneity of modes of thinking in the classroom, allowing the teacher to provide opportunities to his/her students to internalise other socially produced meanings for the rational numbers, beyond those already internalised in previous social interactions.

In the model of conceptual profiles, the modes of thinking make up the zones, not the other way around. That is why, for example, we include the category "fraction" on the epistemological theme 5 because the literature of alternative conceptions indicates that this may be a way of thinking the rational numbers, even if "incorrectly" from the point of view of school mathematics. In a purely mathematical categorisation of rational numbers, which disregards the students' modes of thinking, a way of thinking that disagrees with the scientifically accepted way is rejected; in the approach of conceptual profiles, it may be taken as a zone of the profile, given its pragmatic value.

Another potential refers to the meanings of the concept of rational number that are characteristic of mathematics education, such as those from epistemological themes 9, 10 and 11, and their domains of application. To recognise that these epistemological issues are more characteristic of scientific mathematics can lead the future teacher to an awareness of the contexts in which their uses are more or less fertile to deal with various problems, moving away from a common trend in mathematics education of compressing the different meanings of the concept of rational number into a single and formal mode of thinking. In the teacher education, an epistemological theme can be approached in a way to support teacher's mathematical knowledge for teaching as a compressed form to understand the rational numbers. The researches from Wasserman (2014) are examples of possibilities to approach, for example, the epistemological theme 10—elements of a field—in the teacher education through the collective valorization of the axioms of the algebraic structure of field.

We finish this article stating our intention to go on working with our MENR, moving to the next step to build effectively a conceptual profile of rational number. To that end, we will continue to do research involving and covering a larger and more varied set of empirical data, making them engage in dialogue with the epistemological themes, aiming at individuating the zones of a conceptual profile of rational number.

We know that the theme rational numbers, because it is a complex subject (Behr et al., 1983), arouses the interest of several mathematical educators and therefore already has a large number of researches on different aspects of these numbers. Aware that we have not yet explored all of these researches, we will follow up on our work to increasingly embrace the research already produced. Both the individuation of zones of conceptual profile and the investigations about the potentialities of the MENR in training teachers constitute a research purpose that we intend to develop in our upcoming investigations.

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