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Supporting Elementary Teachers' Planning and Assessing of Mathematical Reasoning

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Abstract Attention to mathematical reasoning in curriculum standards is part of an international trend, but identifying and understanding reasoning continues to challenge teachers.We report on one component of an Australia-wide initiative supporting teachers to implement innovative pedagogies. This paper contains insights from design research that focused on trialling classroom materials to support elementary teachers in their planning and assessment of mathematical reasoning. Findings confirmed planning is a critical step to developing learning experiences that elicit student reasoning, including consideration to task modifications and teacher questioning. Teachers' capacity to assess their students' reasoning was explored using the purposefully designed Assessing Mathematical Reasoning Rubric. The results reveal the complexity involved in constructing accurate judgements of students' reasoning capabilities, particularly appreciating the non-linear nature of mathematical reasoning and the need to draw on multiple sources of evidence. Implications for supporting teachers in their planning for, and assessing of, mathematical reasoning are raised.

Keywords Assessment · Elementary teachers · Mathematical reasoning · Teacher planning

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Introduction

The focus on mathematical reasoning in curriculum standards is undergoing a resurgence globally: not only as a process which exemplifies mathematical thinking but as a strategy for learning mathematics (Australian Curriculum and Assessment Reporting Authority [ACARA], 2017; Ministry of Education Singapore, 2012; National Council of Teachers of Mathematics [NCTM], 2017a). For example, the Australian Curriculum: Mathematics (ACARA, 2017) described reasoning in the following way: students develop an increasingly sophisticated capacity for logical thought and actions, such as analysing, proving, evaluating, explaining, inferring, justifying and generalising (Key ideas, para. 5).

This attention on reasoning was inspired from the earlier report by Kilpatrick, Swafford and Findell (2001) which included adaptive reasoning as one of the five key aspects of mathematics, describing it as having ‘capacity for logical thought, reflection, explanation, and justification’ (p. 116). In cultivating reasoning, teachers are expected to provide ‘instructional programs from prekindergarten through grade 12 [that] enable all students to: recognize reasoning and proof as fundamental aspects of mathematics; make and investigate mathematical conjectures; develop and evaluate mathematical arguments and proofs; select and use various types of reasoning and methods of proof’. (NCTM, 2017b). In doing so, effective teachers need to pay careful attention to planning their lessons, which many teachers view as a core routine in their practice (Kilpatrick et al., 2001). Similarly, teachers in Australia are expected to make decisions about ‘how best to introduce concepts and processes, and how to progressively deepen understanding to maximise the engagement and learning of every student’ (ACARA, 2012, p. 19). These decisions, which are often contemplated during planning, include the consideration given to curriculum, tasks, pedagogy, assessment and differentiation (Sullivan, Clarke, Clarke, Farrell & Gerrard, 2012). Furthermore, these decisions have the power to directly impact student thinking and learning about mathematics (Kilpatrick et al., 2001). Yet, there is widespread concern that teachers experience difficulties in their understanding of what reasoning is and how to incorporate learning sequences and experiences that elicit reasoning in their students (Hunter, 2006; Rogers & Steele, 2016; Stacey, 2010). This issue is especially weighty given the links between teacher knowledge, effective teaching and student outcomes (Askew, Brown, Rhodes, Wiliam, & Johnson, 1997).

In recent years, interest has grown in reasoning and professional learning opportunities to support planning for its enactment in the classroom (see Clarke, Clarke, & Sullivan, 2012b; Herbert, Vale, Bragg, Loong & Widjaja 2015; Sullivan, Borcek, Walker, & Rennie, 2016; Stacey & Vincent, 2009). In this paper, we report on insights from design research of two year 3 and two year 4 teachers (students ages 8–10) and the ways they used the suite of reasoning resources to enhance their mathematical knowledge for teaching ([MKT], Ball, Thames & Phelps, 2008) and subsequently increase their awareness of ways to plan and assess reasoning. The research question addressed in this paper is ‘What elements of the reasoning resources did teachers consider useful in building their mathematical knowledge for teaching when planning and assessing reasoning learning experiences?’

Theoretical Background

The data reported below are informed by a framework that proposes teacher planning and subsequent classroom actions are a function of their disposition, their knowledge about mathematics and pedagogy, and the constraints teachers anticipate experiencing (see Sullivan et al., 2016). The node 'knowledge of mathematics and pedagogy' is the focus of the findings presented below, that is, the connection between teachers' MKT and their planning and assessment of reasoning. This node of this framework proposes that teachers' knowledge of mathematics and pedagogy impacts their ability to plan, teach and assess learning experiences that foster reasoning. Such knowledge is represented schematically by Ball et al. (2008) and includes two major categories: subject-matter knowledge (SMK) and pedagogical content knowledge (PCK), that is, teachers' knowledge of mathematics and their knowledge of ways of teaching mathematics—collectively, these are referred to as MKT. In terms of mathematical reasoning, teachers' MKT includes teachers' knowledge about reasoning generally, including an awareness that reasoning involves more than just explaining (Loong, Vale, Herbert, Bragg & Widjaja 2017) and knowledge about teaching approaches that facilitate students' development of reasoning, such as identifying reasoning potential in tasks; anticipating student responses; eliciting reasoning through effective prompting; and how to notice, nurture and assess students' reasoning when it occurs (Clarke et al., 2012b; Lannin, Ellis & Elliot, 2011). Therefore, it is reasonable to assume that the role of teachers' MKT becomes influential in shaping their competency to plan and assess reasoning accurately and effectively. The inference is that teachers will be more likely to incorporate reasoning into their mathematics lessons if they understand the opportunities for creating student reasoning and how to assess the reasoning actions (Clarke et al., 2012b; Loong et al., 2017; Sullivan & Davidson, 2014).

Connecting the Research Framework and Planning and Assessing Reasoning

When planning learning experiences promoting reasoning, Kilpatrick et al. (2001) offered advice recommending careful attention should be given to the following: (a) writing and adapting tasks that promote conjecturing and generalising, (b) teacher actions that support conjecturing, generalising and justifying and (c) classroom norms promoting reasoning. Building on Kilpatrick et al.'s work, Lannin et al. (2011) listed two fundamental reasons why we assess students' reasoning: to understand *how* students form their generalisations and *why* they think their mathematical statements are true. Their descriptions of teachers assessing student reasoning emphasised assessment as occurring 'in the moment', that is, teachers noticing instances of student reasoning when it takes place during a lesson and knowing specific teacher actions to further facilitate students' reasoning. This view of assessment resonates with more recent calls for authentic assessment practices (Clarke, 2011) providing teachers with information for planning.

In adopting assessment practices that promote effective teaching, researchers have long advocated the use of open-ended tasks (see Askew et al., 1997; Boaler, 2002, Stein, Grover & Henningsen, 1996). One advantage of such tasks are the opportunities they provide for teacher professional learning as teachers analyse and interpret work produced by students and consider questions they might ask students to further advance their understandings (Clarke, 2011; Stein, Smith, Henningsen & Silver, 2009). In this regard, the use of open-ended tasks are becoming more widely used in Australian schools (Clarke et al., 2012b) and are seen as a legitimate vehicle for instruction and assessment (Clarke, 2011). It is therefore reasonable to assume the planning and assessment of open-ended tasks are complex, specifically those that focus on reasoning, and shaped in part by teachers' MKT.

One aspect of teachers' MKT is the planning decisions teachers make in selecting the tasks they will teach (Ball et al., 2008). While tasks are often viewed as critical in creating potential for student learning (Anthony & Walshaw, 2009), it appears the parameters for selecting such tasks and teachers' capacity to decide on the relevance of tasks is varied (Sullivan, Clarke & Clarke, 2012). It has also been found that some teachers experience difficulties in articulating the 'big ideas' which inform their teaching (Clarke, Clarke, & Sullivan, 2012a) and knowing the questions to ask to elicit reasoning in their students (Martino & Maher, 1999). So, this will impact on teachers' selection and use of appropriate tasks and subsequent teacher professional judgements of student performance (Moon, 1997; Watson, 2000). Connected to teacher judgements are the concerns raised in the literature about the degree of reliability and consistency when assessing such tasks (Morgan, Tsatsaroni & Lerman, 2002). To overcome such difficulties in assessment, Wilson, Mojica and Confrey (2013) found the use of a learning trajectory supported teachers to make informed decisions based on theoretical underpinnings and their actual observations of students' work. We propose such a learning trajectory in the form a rubric for assessing reasoning.

Furthermore, in the last few years, more information on the way lesson planning documentation supports teachers in anticipating possible directions a lesson might take has become available (Mutton, Hagger & Burn, 2011; Sullivan et al., 2016; Davidson, 2017). Sullivan et al. (2016) provided written lesson suggestions to teachers including curriculum links, pedagogical considerations and task modifications, with teachers reporting that such advice was helpful to their planning, teaching and assessment of student understandings. Furthermore, recent evidence reveals teachers learn from the process of teaching and planning lessons. Roche, Clarke, Clarke and Chan (2016) drew attention to teachers' professional experimentation when they taught a lesson that was provided to them in the form of a lesson plan. This experience prompted their teachers to reflect on their students' thinking and plan a follow-up lesson to address common misconceptions, which teachers indicated was challenging. Roche et al. (2016) concluded that teachers' MKT, specifically about the task they are teaching, determines 'both the teacher's instructional effectiveness and their capacity to engage in further learning with respect to the teaching of the relevant content' (p. 566) and that teachers experience challenges in planning of subsequent lessons, especially when engaging with unfamiliar mathematical content. In this case, the mathematical content referred to is reasoning and the associated challenges teachers experience in planning for and assessing it.

Methods

This study was conducted as part of an Australia-wide programme supporting teachers to implement innovative pedagogies by providing free resources to all Australian teachers (see Australian Government Department of Education and Training, 2017). A suite of resources for elementary teachers was developed to support their planning, teaching and assessment of reasoning. While 32 elementary teachers across four schools of varying socioeconomic backgrounds participated in the study, the following paper contains findings from four experienced teachers with between 5 to 15 years of classroom teaching experience from a large metropolitan government school servicing a diverse middle to upper class population. These four teachers taught students of ages 8–10, that is, two year 3 teachers (Emery[1] and Jackie) and two year 4 teachers (Vivian and Morgan) chosen because their school was the first to participate, experiencing each iteration in a short time frame of approximately 2 weeks, enabling fewer interruptions and supporting teachers to remember the materials more clearly. In addition, despite their years of experience, each teacher perceived having had limited exposure and professional development in reasoning. While reporting on only four teachers from one school may risk overgeneralising, the intention is to provide key insights into the factors influencing planning and assessment of reasoning.

The collaboration between the researchers and the teacher participants is an example of design research because it 'attempts to support arguments constructed around the results of active innovation and intervention in classrooms' (Kelly, 2003, p. 3). The key elements of design research are an *intervention* refined through an *iterative* process to address issues of *practice* (Kelly, 2003). So, this study is design research because the refined interventions were the lesson exemplars and a purposefully designed rubric. The iterative approach included opportunities over a 2-week period for the teacher participants and the researchers to evaluate and refine the usefulness of the resources to support teachers in their planning, teaching and assessment of reasoning.

The initial iteration incorporated a 1-h whole school professional learning session delivered by one of the authors. The session included defining the three key reasoning actions, exploring key reasoning ideas and the actions associated with them, and ways to support teachers to notice, elicit and assess reasoning. Teachers engaged in tasks that we deemed as having high potential for reasoning: teachers solved the task for themselves, discussed various solutions and proposed task modifications. The teachers' participation in this session addressed the key aspects of the research framework (Sullivan et al., 2016), specifically teachers' knowledge of mathematics and pedagogy, the intention being to enhance teachers' understanding of reasoning before attempting to plan, teach and assess it.

At the conclusion of the professional development session, a researcher met with the teacher participants to share and discuss the reasoning resources that teachers would trial in the subsequent iteration (Lesson 1). Teachers were provided with materials including definitions of the three focus reasoning actions (analysing, generalising and justifying); five lesson exemplars; reasoning prompts aligned to the three reasoning actions to elicit student reasoning; and a purposefully designed rubric to assess reasoning.

Following the professional development session, pairs of teachers from the same year level worked together firstly to select one of the exemplars and then teach the lesson (approximately 1 h) with peer and researcher observation. Researchers and teachers then engaged in an audio-recorded post-lesson discussion (approximately

[1] Pseudonyms used throughout

45 min) to discuss the usefulness of the resources and assess their students' reasoning against the rubric. This process was repeated 2 weeks later except teachers planned their own lesson that focused on reasoning. The following provides details about the lesson exemplars, the purposefully designed rubric and reasoning prompts that were the main focus of the suite of resources provided to the teachers.

The Lesson Exemplars

We employ the term 'lesson exemplar' rather than 'lesson plan' as these exemplars were designed to provide a sense of the reasoning learning experiences might resemble from years 3–6 (ages 8–11) to promote anticipatory thinking (Mutton et al., 2011), rather than a step-by-step guide for lesson delivering. The exemplars offered ways to *plan* tasks that embedded the reasoning actions. It should be noted that while suggested year levels were provided which aligned these with relevant curriculum standards, ways to adapt the tasks for various year levels were offered.

The following lesson exemplar was selected by Emery (year 3) to teach in the first set of classroom observations and formed part of the focus of the post-lesson discussion. The 'Is it True?' task (see Fig. 1) was designed to offer students' opportunities to notice place value misconceptions including grouping, regrouping and renaming for two-digit addition and prompt them to form conjectures and generalisations that lead to justifications. This task allows students opportunities to determine their own approaches, to identify and describe the place value misconceptions and to justify their reasoning about those misconceptions. Students can represent their ideas in their own way and communicate their ideas to other students. The task has the advantage of offering a 'low floor-high ceiling' (Sullivan et al., 2016), in that all students can readily develop a truth statement but offers opportunities for these students to extend their conjectures to generalisations.

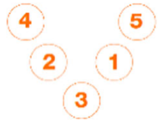
The exemplar described the reasoning purpose for teachers of this task as to explain and justify true statements, to explore and notice relationships between numerical structures and to form conjectures and generalisations. Determining the truth or otherwise of a conjecture is vital in reasoning and is a key aspect of generalising (Lannin et al., 2011). Furthermore, Lannin et al. emphasised that investigating the truth of a statement is an essential, yet 'often-overlooked part of mathematical reasoning' (p. 30). Table 1 presents title and reasoning tasks taught by the other teachers in Lesson 1 observations that comprise the impetus for the results reported on below.

The exemplar included a lesson abstract, the relevant extract from the curriculum standards, advice for introducing the reasoning inquiry and the reasoning focus. Enabling prompts for students experiencing difficulties and extending prompts for

27 + 34 = 511
Is this true?
Why or why not?

Fig. 1 'Is it True?' task

Table 1 Reasoning tasks taught by the teachers in Lesson 1 observations

Teacher	Year level	Exemplar title	Reasoning task
Jackie	3	*What else belongs?*	I wonder could these numbers 30, 12, 18 belong together? What other numbers do you think could belong with these numbers? How do you know that all these numbers fit with your reason? Use words, numbers or drawings to explain.
Vivian and Morgan	4	*Magic V*	A 'V' is said to be a 'Magic V' when the total of each arm on the V is the same. Sam said 'It is impossible to make a Magic V with an even number at the bottom'. Is Sam right? Explain why or why not? 

those who required further challenge (Sullivan, Mousley & Jorgensen, 2009) were included in the exemplar. The exemplar also contained examples of applying the rubric, including likely student reasoning actions and how these anticipated responses matched the relevant reasoning actions and levels in the rubric. Complementing each anticipated response was sample teacher prompts to support and/or extend students' thinking. Teachers were exposed to the exemplar tasks during the professional learning session.

Prompts to Elicit Mathematical Reasoning

It is well known that skillful questioning can support teachers to gain insights into students' mathematical thinking (Martino & Maher, 1999; Mason, 2003; Reid & Zack, 2009). Therefore, a second key feature of the suite of resources included prompts to elicit reasoning aligned against the three key reasoning actions to support teachers to find out how students analyse, generalise and justify (Table 2). The intention is that teachers would consider such prompts in their planning for use when teaching and reflect on such prompts after the lesson that might be useful in a follow-up lesson. Sample prompts for each key reasoning action are presented in the table below.

The Assessing Mathematical Reasoning Rubric: an Overview

Considerable ambiguity exists in how the actions of reasoning are defined and assessed (see Brodie, 2010; Jeannotte & Kieran, 2017; Kilpatrick et al., 2001; Lannin et al.,

Table 2 Sample prompts to elicit mathematical reasoning against the three key reasoning actions

Analysing	Generalising	Justifying
What is the same and different about …?	How can you describe the pattern?	Is it [the conjecture] just sometimes true, or is it always true?
What stays the same and what changes?	If …then …	How can we be sure?
What do you notice?	Are there other examples that fit the rule?	Convince me.

2011); hence, one aspect of this larger project sought to add clarity to the discussion through an analysis of the relevant literature to create the purposefully designed rubric that centred on the following three reasoning actions: analysing, generalising and justifying and enhance teachers' abilities to reliably assess their students' reasoning. The rubric includes five levels (not evident, beginning, developing, consolidating and extending) for assessing the three reasoning actions: (a) analysing, (b) generalising and (c) justifying. For example, a student working at the developing level for the reasoning action of analysing notices a common numerical or spatial property; recalls, repeats and extends patterns using numerical structure or spatial structure; sorts and classifies cases according to a common property; orders cases to show what is the same or stays the same and what is different or changes; describes the case or pattern by labelling the category or sequence. Table 3 presents an example of the ways the reasoning actions were aligned against anticipated student responses for the exemplar 'Is it True?' and suggested teacher prompts to support or extend students' reasoning.

Teachers assessed their students' work samples against the five reasoning levels and make judgements about their students' capacities to reason against the three reasoning actions. Based on the outcomes of the first post-lesson discussion, the teaching pair were required to select, plan, teach and assess a follow-up lesson that focused on reasoning.

Data Analysis

As the focus of this paper is on the ways teachers perceived the usefulness of the reasoning resources to support their planning and assessment of reasoning, the data reported below are sourced from the post-lesson discussion audio recordings which 'offer[s] one of the best opportunities to document the learning process of the research team' (Gravemeijer & Cobb, 2006). We also draw on student work samples and teachers' use of the rubric during each post-lesson discussion.

Post-lesson discussion recordings were professionally transcribed and checked with minimal editing. Data analysis proceeded through three cycles. The first cycle comprised thematic coding, using the research framework to guide the analysis in which interesting comments made by teachers about planning and assessing reasoning over the two post-lesson discussions were noted. For example, the following comment met the criteria for MKT as the teacher was analysing the students' reasoning, yet the comment also alluded to the limitation of making a judgement based on a work sample alone:

Table 3 An example of aligning anticipated student responses with the rubric for 'Is it True?'

Reasoning level	Anticipated student reasoning	Rubric action	Additional notes	Sample reasoning prompt
Developing	No, because $20 + 30 = 50$, $7 + 4 = 11$, $50 + 11 = 61$ Split 11 into 10 and 1	*Justifying* verifies truth of statements by using a common property, rule or known facts that confirm each case. It may also use materials and informal methods.	Displays knowledge of place value and partitioning.	Convince your partner you are correct.

> Well, I think that she is consolidating because she's got quite a few, she's got some developing, some in consolidating, but she doesn't quite have extending, she hasn't articulated that other than in the algorithm, but do I know what she's thinking there or is it just a process? If she explained that to me like Sylvia and Evie did then I'd know, but I can't just assume that she knows what she's doing.

Where an item was coded in multiple categories, that item was included in each category. The second cycle involved a priori coding where categories were established from the theoretical framework of Ball et al. (2008). For example, we sought examples of teacher's discussing planning and assessing reasoning that reflected, for example, specialised content knowledge. The third cycle involved revisiting quotes coded in the second cycle and determined whether they were connected to planning or assessment and the related resource/s, that is, the definitions, exemplars, prompts or rubric. Discrepancies in coding were reconciled through discussions between the authors. It should be noted the intention was not to examine individual teachers' responses, but rather identify recurring themes across the four teachers' comments during the post-lesson discussions to provide rich insights into the contributions of the resources to support teachers in their planning and assessment of reasoning.

Results

The results are presented in two sections: the first post-lesson discussion followed by the second post-lesson discussion. The initial iteration involved the participants' engagement in the professional learning session; data was not collected during this phase and is therefore not relevant to the results section. The results focus on what the teachers noticed in the lesson observations and described about their planning and assessment of reasoning during each post-lesson discussion.

Lesson 1: Teachers' *Planning* for Mathematical Reasoning

Teachers made generally positive comments regarding the suite of resources. They appreciated the flexibility of the exemplars in providing sufficient information for the teachers to know and understand the reasoning potential of the task, ways to modify the task and anticipated student responses, yet the resources offered scope for teachers to modify the exemplar for their class' needs. For example, the teachers commented that the openness of the exemplar was 'more helpful' than a full lesson plan with 'lesson ideas not necessarily fully scripted lessons'.

In terms of *specialised content knowledge* (Ball et al., 2008), teachers commented retrospectively that the definitions of reasoning actions would be helpful to future planning:

> Vivian: I think it would be good having that [reasoning definitions] when planning a lesson because then you can go 'all right I want to focus on …' Have you looked at your rubrics and you have seen that there's not that much justifying [in the task] and to actually have a look at the definition of justifying and logical argument and then you can plan your lessons from there.

In unpacking the reasoning potential of a task, that is, *knowledge of content and teaching* (Ball et al., 2008) after teaching the Magic V, Vivian described the task as focusing on:

> forming conjectures and generalizing. Actually it had a little bit of everything in it. It had a bit of analyzing because they had to recall and repeat a pattern, and had forming conjectures and generalizing because they had to explain the meaning of the rule, and justifying logical argument because they had to say why and because.

Connected to *knowledge of content and students* (Ball et al., 2008) that would support their planning of reasoning, teachers focused on the centrality of the two particular aspects that supported their planning and subsequent teaching: the development of enabling and extending prompts and the inclusion of reasoning prompts, such as 'What is the same and what different about …?' (Mason, 2003, p. 24) to elicit students' thinking. Jackie suggested minimising the number of suggested enabling and extending prompts in the exemplar to one or two and including any remaining prompts as 'alternative or follow-up tasks' that teachers could use as a basis for planning subsequent lessons. These suggestions were met with agreement from the other teachers. When discussing reasoning prompts, Emery described how the generic prompts supported him to elicit students' reasoning by encouraging students to explain their thinking. Jackie described how the reasoning prompts helped her to encourage students to communicate their thinking, especially when they experienced difficulties in making a start on the task or to unpack their thinking further. The teachers suggested including targeted reasoning prompts within the lesson exemplar, rather than the larger selection of reasoning prompts provided which teachers must decide the most relevant prompts to use for a lesson. Lastly, based on her observation of Vivian, Jackie suggested including the reasoning prompts in the learning intention for students. Vivian trialled this approach in the following lesson that she planned.

> Maybe you could put that [reasoning prompt] in the reasoning purpose for the students … 'So in today's maths lesson when you are doing a task I want you to think about these things…' and then your prompts start—'How you are going to convince me? How are you going to justify …?'

These comments emphasised the valuable contribution prompts offer in supporting teachers to plan and implement lessons focused on eliciting reasoning. Following the first post-lesson discussions, teachers planned their own lesson that focused on reasoning.

Lesson 1: Teachers' *Assessment* of Mathematical Reasoning

When approaching assessment of their students' reasoning, the discussion commenced with a general reflection of what the teachers noticed about their students work and how to assess the key reasoning actions according to the rubric. Jackie and Emery approached their assessments by first summarising the general responses made by students in their classes and then proceeded to discuss specific students to make judgements about their reasoning levels:

> Emery: looking through [the work samples] … the highest level that any of my kids fit was perhaps consolidating and an example of that is … he used different language 'they're all in the six number pattern' but then down here [same work sample] 'I counted up by sixes and got all these numbers'

Vivian on the other hand commenced by matching the 'lowest' performing student's work sample to the examples of anticipated student reasoning in the annotated rubric from the exemplar, to make a judgement, then selected a student who she judged as working at consolidating and focused on the student's explanation to support her judgement. For example:

> So we will start with not evident—[reading student work sample] this is not a Magic V—he can't tell us anything why—he actually worked with someone and wrote down the Magic V and couldn't explain it. So not evident.
>
> she [another student] has talked about, 'because it equals the same as the other side'. I am ranking her at consolidating but I don't know if that's being a bit generous … when I look back at it [the work sample]—'because it equals the same amount'—she is actually kind of justifying by using the word 'because'.

Alternatively, Morgan commenced the assessment process by discussing the 'strong performers' in her class her recollections of the lesson to make a judgement based on the annotated rubric:

> she could explain it I would put her at working...Consolidating... she is able to explain her thinking fairly clearly and understands the relationship of odd and even… Consolidating—knowing her.

These quotes from Vivian and Morgan demonstrate their focus on the reasoning action of explaining. Additionally, Morgan's comment 'knowing her' indicates that assessing reasoning was based on the teacher's prior knowledge of the student performance and the reasoning the student demonstrated in the work sample and the student's utterances during the lesson; hence, the assessment of this child's reasoning was not just based solely on their work sample.

The post-lesson discussions generally continued in this vein with the teachers analysing particular student work samples and working with the researchers to understand what students were communicating. Further discussion also revealed complexities involved in identifying the relevant reasoning actions and levels. It became apparent to the teachers that in many cases, assigning students to a reasoning action and level was not as straightforward as it first appeared, particularly after discussing the range of strategies used by the students and recalling instances of students' reasoning actions during the lesson. Teachers referred to the constraints of making judgements based on the work sample alone. For example:

> Vivian: They're the hard ones to catch because … a few kids might've done that, but they don't record that, they just do that in their head...It's about catching it and hearing them to be able to go, 'Oh, good thinking'.... But it might be the difference between a beginning and a consolidating.
>
> Morgan: There is a lot more happening behind the scenes than what you actually think.

These comments suggest the limitation in making teacher judgements on the written work sample alone and that teachers need to supplement their judgements with further data, such as their knowledge of the student and recalling the student's comments and actions during the lesson. One way to circumvent this constraint was Vivian's suggestion of reformatting the rubric onto one page to monitor students' progress on the rubric over multiple lessons and including a comment box below the rubric for teachers to include additional comments, to supplement the work sample evidence and provide a more informed view of students' reasoning capabilities. Teachers also noticed that many of their students demonstrated different levels across the three reasoning actions (see Fig. 2).

Lesson 2: Teachers' *Planning* of Mathematical Reasoning

In planning the second lesson, teachers either modified the task they had previously taught or sourced and adapted a new task to teach (see Table 3). Resulting from the first post-lesson discussion, each teacher had identified a focus area by either addressing the mathematical content or the reasoning actions they wanted to develop in their students. For example, teachers Jackie and Emery adapted the exemplars *Is it True?* and *What else Belongs?* because of their focus on content. That is, after the first post-lesson discussion, Jackie noticed in her observations of 'What else belongs?' that students

ReSolve RUBRIC for Mathematical Reasoning

	Analysing	Forming Conjectures and Generalising	Justifying and Logical argument
Not evident	• Does not notice numerical or spatial structure of examples or cases. • Attends to non-mathematical aspects of the examples or cases.	• Does not communicate a common property or rule for pattern. • Non-systematic recording of cases or pattern. • Random facts about cases, relationships or patterns.	• Does not justify. • Appeals to teacher or others.
Beginning	• Notices similarities across examples • Recalls random known facts related to the examples. • Attempts to sort cases based on a common property. • Recalls and repeats patterns displayed visually or through use of materials.	• Uses gesture and rhythm of gesture, drawing, counting and oral language to draw attention to and communicate a: ○ common property ○ repeated components in patterns • Adds extra cases visually or by using materials. • Extends patterns displayed visually using diagrams or through use of materials.	• Describing what they did and why it may or may not be correct. • Recognises what is correct or incorrect using materials, objects, or words. • Makes judgements based on simple criteria such as known facts. • The argument may not be coherent or include all steps in the reasoning process.
Developing	• Notices a common numerical or spatial property. • Sorts and classifies cases according to a common property. • Orders cases to show what is the same or stays the same and what is different or changes. • Recalls, repeats and extends patterns using numerical structure or spatial structure. • Describes the case or pattern by labelling the category or sequence.	• Communicates a rule about a ***property*** using words, diagrams or number sentences. • Communicate a rule about a ***pattern*** using words, diagrams to show recursion or number sentences to communicate the pattern as repeated addition. • Explain the meaning of the rule using one example.	• Checks the truth of statements using materials and informal methods. • Uses known facts to verify that the statement, common property, or rule for a pattern holds for each case. • Uses a counter example to refute a claim. • Starting statements in a logical argument is correct and accepted by the classroom • Detecting and correcting errors and inconsistencies using materials, diagrams and informal written methods.

Fig. 2 Example of assessment at different levels and actions in the second lesson

adhered to one strategy, such as fact families, and wanted to explore ways of stretching students' reasoning. She drew on the professional development workshop from the first iteration as inspiration behind the lesson she designed for the second lesson (Table 4).

Similarly, Emery discussed the misconception of applying commutativity to subtraction that his students experienced which led to her adapting this exemplar to focus on drawing out that misconception, whereas after teaching the Magic V, rather than focus on content, Vivian wanted to find a task that would promote the reasoning actions of forming conjectures and generalising, as well as justifying and logical argument. As an indication of Vivian's increased awareness of reasoning, she mentioned a useful resource she used in her planning for reasoning that supported her understanding of reasoning and how it differentiated from problem solving.

Vivian then selected a task about finding the length of a train that she sourced from a list of suggested freely available online resources provided during the first iteration. However, she described how, despite sourcing a task that had been identified as reasoning online, she felt the task required modifications to further draw out the reasoning potential and added reasoning prompts: 'What is the rule?'; 'What stays the same and changes?'; 'How can we be sure?'; and 'What is your reason?' to focus students' attention on the desired reasoning actions.

> Because when we were looking the question on [the website] unless you had some of these questions [reasoning prompts] on it I don't know how just the top bit [of question] could really fall under the reasoning and actions.

Furthermore, evident from the lesson observations and the teacher produced worksheets was the consideration given to the development of reasoning intentions to share with students and the use of enabling and extending prompts to support and/or extend their students reasoning. Jackie and Emery used the reasoning intentions for the initial exemplars and focused on explicitly developing extending prompts, 'Which other numbers do you think don't belong in this group? Why?' (Jackie) and 'Explain why someone might mistakenly think this problem to be true' (Emery), whereas Vivian planned an enabling prompt by simplifying the numbers and an extending prompt that offered an alternative task and encouraged students to think of a rule. She wrote the learning intention on the board 'We are learning to *explain* my thinking and *justify* my answers when problem solving' to indicate the reasoning focus of the lesson to her students. This evidence demonstrates teachers' increased attention towards planning mathematical experiences that focused on eliciting the key reasoning actions and suitable pedagogies.

Table 4 Reasoning tasks planned by year 4 teachers Jackie and Emery in Lesson 2

Teacher	Lesson title	Reasoning task
Jackie	*What Doesn't Belong?*	(a) Which number doesn't belong? Why? 60, 120, 123, 240 (b) What other numbers would not belong if this is the reason?
Emery	*Is it True?*	$511 - 34 = 27$ Is this true? Why or why not?

Lesson 2: Teachers' *Assessment* of Mathematical Reasoning

While each teacher approached the assessment of their students' reasoning in slightly different ways, there was a noticeable shift in the attention given to the three reasoning actions across the different levels. For example, Jackie came to the post-lesson discussion with clearly delineated strategies for 'three groups of children' in mind and shared work samples of each to assess against the rubric. Jackie initially used one student, together with her knowledge of the student to make a judgement, then used it to inform assessments of students' reasoning.

> Jackie: Alright so [this student] would probably be one of the higher end, rather than [other students] who listed some numbers ... So consolidating, forming conjectures and generalising [for this student] ... I think most other children were kind of sitting between beginning and developing.

Alternatively, Emery scrutinised two work samples, comparing and contrasting the strategies used by the two students and how those responses matched within and across the rubric:

> So what's standing out to me with both of these [work samples], especially [this student] is that she's finding within terms of her reasoning, multiple reasons and then explaining them in quite a detailed manner. So you're seeing some patterns and continuing them, and that mentions that in consolidating, under analysing ... So I'm thinking about maybe consolidating and some of these things don't match up exactly but I think it does fit.

As Emery continued to grapple with the assessment process, the conversation shifted to clarifying his understanding of the levels in the rubric. To do this, one approach was to compare the work sample to a conversation he had with another student during the lesson who used a similar strategy although experienced confusion in communicating her understandings. With the support of the researchers, Emery concluded that while the student communicated a rule about a property, the student needed to check the truth of statements and therefore assessed the student as *developing* for the reasoning action of analysing, although 'it's the justifying that she's perhaps might be a little lower'. One way of overcoming such assessment difficulties, as suggested by a researcher, was to follow-up with the student by asking, 'Can you tell me a bit about this? Can you show me something about this? or Show me how you can prove that?' This exchange led to the teachers' affirmation about the use of reasoning prompts to encourage students:

> Jackie: It's definitely worthwhile thinking about those [reasoning prompts] before going into the lesson.

> Vivian: it will be interesting when I have the discussion with her whether she was using this rule.

The need to triangulate the work sample with students' comments and actions during the lesson exposes the associated difficulties with making a judgement based on the work sample alone.

> Well I put her at developing but now when I think about it, I wrote here [in the box] she visually drew examples and she had doubled 15 to help her and in conversation she kind of knew the rule but she was just unsure how to just get it down on paper. So she is using her diagrams.

Figure 3 work sample demonstrates how Vivian assessed this student's reasoning actions across the levels, an approach that was common amongst other teachers' assessments of their students. Figure 3 shows the revised rubric based on teachers' feedback after the second lesson resulting in the reformatting of the rubric to one page, with comment box included below. As can be seen, the teacher has utilised the comment box to add supplementary information based on recollections of the students' actions during the lesson to strengthen the assessment of the student's reasoning.

Vivian drew attention to that fact that while some of her students were able to verbalise their reasoning, they experienced difficulties communicating it on paper. She emphasised the role of teacher in eliciting student reasoning:

> When she verbalised it ...—that's why I have written here [in the evidence box] 'in conversation she knew the rule' … but here [on the work sample] that could have also been from my prompts that she was understanding.

During the second post-lesson discussion, an increase in the teachers' attention to the three reasoning actions and levels was noted. In particular, teachers noticed the need to shift students between the levels and across the actions as evidenced in Jackie's comment:

> I'm starting to notice which children are sitting in which column. Your [high students] are really more here—generalising, the lower end of the students are sitting up here in the justifying, middle ones are just starting to analyse going a bit deeper. So I am starting to see them in columns … Every time we have looked at lower end students we have been looking at justifying and starting to notice some

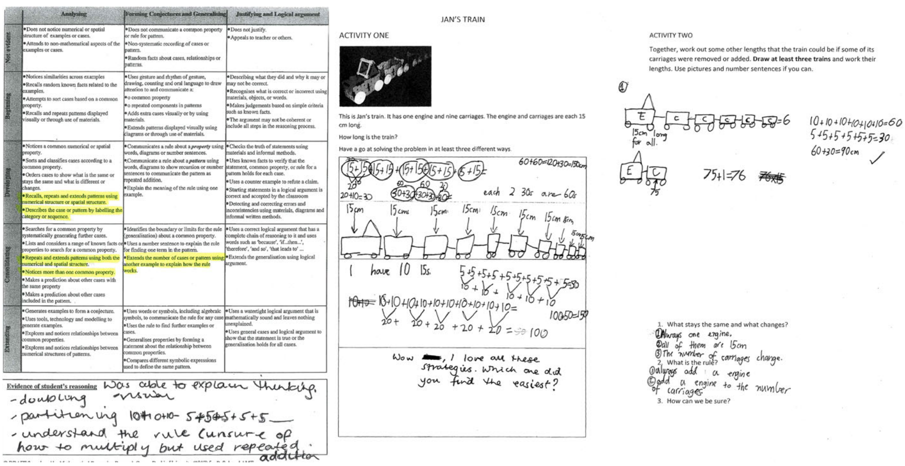


Fig. 3 Vivian's use of the rubric and accompanying work sample

common properties—similarities and differences but not really much of our conversation has been in this middle column and for our higher end students we have been talking a lot about these form of conjectures and generalising and justifying a logical argument having these watertight arguments, but haven't really been looking.

At the conclusion of the second post-lesson discussion, teachers expressed the benefits the rubric offered as an opportunity for ongoing formative assessment and monitoring students across multiple lessons for informed, professional judgements:

Jackie: The good thing about this rubric too is this is just one task. So let's just say we would stick [them] in the middle of both of them [reasoning actions] and give her some more experiences. And see if she's leaning more towards here or more towards here [teacher pointing to the reasoning actions on the rubric].

Overall, the post-lesson discussions revealed the important role MKT played in shaping teachers' decisions about planning and assessment of reasoning: teachers considered the usefulness and challenges associated with the lesson exemplars and the purposefully designed rubric and the ways these interventions supported them to plan and assess their own lesson that focused on reasoning.

Discussion and Implications

The research reported above intended to explore teachers' planning and assessment of mathematical reasoning, specifically the ways resources: the lesson exemplars, reasoning definitions, reasoning prompts and the purposefully designed rubric contributed to teachers' competency to design learning experiences that fostered reasoning and aided professional judgements about their students' reasoning actions. It appears the experience of being involved in the first post-lesson discussions aided the teachers' understandings of reasoning for their planning and assessment for the second lesson. The results revealed the influence of aspects of teachers' MKT (Ball et al., 2008) on shaping teachers' planning and assessment decisions (Sullivan, Clarke, Clarke, Farrell et al., 2012).

In responding to the research question, it was found that teachers reported the exemplars were useful in their planning as they provided sufficient information to understand the reasoning inherent in the task, yet provided flexibility in the pedagogical decisions to enact reasoning tailored to their class. Teachers identified the reasoning definitions as useful in unpacking the reasoning potential of tasks and were instrumental in the development of their understanding of the nature of reasoning. After teaching the first lesson, teachers discussed the role of the reasoning prompts in eliciting students' reasoning which had implications for the ways teachers offered such prompts (Sullivan et al., 2009): one teacher suggested sharing reasoning prompts with the students at the commencement of the lesson consistent with using an inquiry-based approach and setting norms where students are expected to engage in the desired reasoning actions (Reid & Zack, 2009).

The teachers did not develop a written lesson plan for their follow-up lesson, which is consistent with research describing that much of what is planned by teachers occurs mentally (McCutcheon, 1980; Roche, Clarke, Clarke & Sulivan, 2014; Yinger, 1980). However, it appeared the exemplars, together with the teachers' participation in the first post-lesson discussion, focused teachers' attention towards the three reasoning actions in their planning, in particular, task selection and task modifications. Together, the resources, lesson observations and the post-lesson discussions enhanced these teachers' awareness of reasoning (Roche et al., 2016). For example, based on the first post-lesson discussion, each teacher identified a follow-up focus to pursue. For the follow-up lesson, Jackie and Emery identified a content focus and modified tasks based on the original exemplars whereas Vivian identified a reasoning action focus and sourced a new task online. Vivian's increased awareness of reasoning was evident when she described the considerations she gave to modifying the task by embedding reasoning prompts to enhance the reasoning potential of the task. Vivian commented on being able better able to distinguish between problem solving and reasoning and the appropriate language associated with each to use with her students.

A recurring theme throughout the first post-lesson discussion was the importance of teachers' consideration to the reasoning prompts to elicit student thinking and the use of enabling and extending prompts. Jackie and Emery did not report developing or using any enabling prompts for their students, whereas Vivian had designed her own enabling prompts prior to the lesson. Jackie defended the choice to not design any enabling prompts as she did not perceive her students would require them. However, upon reflection, she indicated that such prompts would have been helpful to her teaching and should be given priority in future planning. Alternatively, Vivian, based on her observation of the first lesson, shared the reasoning prompts in her lesson introductions and in the students' worksheets and reflections she designed which appeared to focus students' attention on the reasoning expectations of the lesson (Reid & Zack, 2009).

In terms of the rubric to assess reasoning, the findings indicate the rubric provided a framework for teachers to notice and assess their students' reasoning actions and a common language to facilitate discussion of these actions in a similar fashion to the learning trajectory used by Wilson et al. (2013). The annotated rubrics in each exemplar, containing anticipated student responses and ways to utilise the rubric, supported teachers in approaching their initial assessments of their students. Generally speaking, the teachers in this study either commenced their assessments using two approaches; the first being to review a perceived 'strong' or 'weak' performer in their class and using this as a benchmark by which to assess subsequent students. The second approach witnessed teachers group students' strategies (Jacobs, Lamb & Philipp, 2010) and then form judgements about individual students. These two approaches to assessment are similar to those actions found by Wilson et al. (2013) of 'describing', 'comparing' and 'inferring' to make sense of students' work samples and mathematical thinking.

A salient feature amongst the post-lesson discussions was the need to compare subtle differences amongst the work samples to delineate between the reasoning actions and levels. Teachers relied on their memory of students' actions and vocalisations of reasoning during the lesson to make informed judgements of their reasoning capabilities: the revision of the rubric to incorporate an 'Evidence of student reasoning' comment box was one way teachers catered to this limitation. Teachers discussed following up their assessments with one-on-one student interviews to confirm their initial assessment (Clarke, 2011; Watson, 2000). Teachers' comments also reflected the non-linear nature of reasoning (Lannin et al., 2011). For example, it was evident that using the rubric increased teachers' attention on the three reasoning actions by noticing that students could be more proficient in one action over another in a similar to the way the learning trajectory used by Wilson et al. (2013) supported their teacher participants to more adequately interpret their students' thinking and draw on evidence from students' mathematics work.

Consistent with Loong et al., (2017), teachers' comments in the post-lesson discussions indicated they privileged explaining over other reasoning actions. Further work on supporting teachers in noticing beyond superficial reasoning actions is needed, such as justifying in order to develop teachers' knowledge of reasoning and how to teach and assess it (Clarke et al., 2012b; Schifter, 2009; Morris, 2009).

Overall, the process of teachers using the rubric appears to have engaged them in a cycle of planning and assessment. Teachers suggested that the descriptions of reasoning actions in the rubric could serve to inform their foci for planning follow-up lessons through auditing the reasoning potential in tasks and then identifying relevant reasoning prompts for that task.

This study confirms that teachers' MKT related to reasoning is vital in their planning to prepare learning experiences that elicit students' reasoning, that is, task selection, teacher questioning and assessment. Our study extends previous research by identifying specific resources that supported teachers to focus their attention on key reasoning actions when planning and assessing reasoning, that is, the reasoning definitions, exemplars, prompts and rubric.

While the data presented reports on only four teachers at one particular school, our findings are encouraging in that they provide practical ways to support teachers to become more aware of the nature of planning and assessing reasoning. Yet despite these teachers' years of experience and the provision of the reasoning resources, the planning and assessment of reasoning remains a complex process (Reid & Zack, 2009; Loong et al., 2017) which requires further investigation. Assessing reasoning requires an awareness of the possibility for students to demonstrate different levels of proficiency across the three reasoning actions and requires triangulation of various forms of evidence. In addition, teachers' MKT about reasoning influences their judgements (Roche et al., 2016). Teachers need to consider a variety of factors in seeking to assist students to develop their reasoning capacity: identifying the reasoning potential of the task; knowing teacher actions to facilitate reasoning; and awareness of the student's reasoning capabilities all of which are often considered during planning (Kilpatrick et al., 2001).

Conclusion

The findings of this study confirm that elementary teachers need assistance to build their knowledge about planning and assessing mathematical reasoning and this assistance may be in the form of readily available resources such as reSolve: Mathematics by Inquiry materials. Given the current interest in reasoning, particularly professional learning opportunities to support teachers in providing experiences that foster reasoning, further research into the ways materials can support teachers in the planning and assessment of reasoning is recommended. Caution is advocated in basing teacher professional judgements of reasoning solely on student work samples and that forming a comprehensive view of one student's reasoning capabilities occurs over time. Future studies should target the use of multiple sources of evidence, such as audio recordings of students' actions to aid in teacher recall for assessment, along with tasks that offer a variety of opportunities for students to reason mathematically and associated teacher actions to elicit reasoning.

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