International Journal of Science and Mathematics Education (2020) 18:123–143
https://doi.org/10.1007/s10763-018-09940-5

Four Fundamental Modes of Participation in Mathematics Group Activities

Chiara Andrà[1] · Domenico Brunetto[1] · Nicola Parolini[1] · Marco Verani[1]

Received: 28 October 2017 / Accepted: 6 November 2018 / Published online: 28 November 2018

Abstract
This paper aims at contributing to the debate in Mathematics Education about the understanding of the dynamics of students' group interactions by proposing an interpretative lens, which defines four modes of participating in a group on the basis of different kinds of utterances, gestures, postures, and glances that each student makes. We apply this lens to two selected cases of students working in a small group, and, by comparing and contrasting similarities and differences observed through our interpretative lens, we attempt to understand how, and under which circumstances, the students reach mathematical understanding as a group, or not.

Keywords Embodiment · Emotions · Group dynamics · Self-confidence

Introduction

Group interaction has gained more and more attention in mathematical curricula in many countries and instruction that builds on students' mathematical thinking has been endorsed in many reform documents (e.g., Clarke & Ziebel, 2017, for China and Australia; National Council of Teachers in Mathematics [NCTM], 2000, for US; Radford & Demers, 2004, for Ontario; Unione Matematica Italiana — Commissione Italiana per l'Insegnamento della Matematica [UMI-CIIM], 2001, 2003, for Italy). A feature of the

✉ Chiara Andrà
chiara.andra@gmail.com

Domenico Brunetto
domenico.brunetto@polimi.it

Nicola Parolini
nicola.parolini@polimi.it

Marco Verani
marco.verani@polimi.it

[1] Dipartimento di Matematica, Politecnico di Milano, Via Bonardi, 9, 20133 Milan, Italy

reform-based classroom since the early 1990s has been promoting talk in the classroom (Williams & Baxter, 1996), and more recently, many studies in Mathematics Education around the world (e.g., Arzarello, 2006; Bartolini Bussi & Mariotti, 2008; Bikner-Ahsbahs & Halverscheid, 2014; Goos, 2004; Nilsson & Ryve, 2010; Radford, 2011; Sfard, 2008; Sfard & Kieran, 2001) show how—and discuss to what extent—both classroom discussion and group interaction promote mathematical understanding. Some research conceptualises interaction as empowering acts in which belonging, agency, and identity are jointly constituted (e.g. Radford, 2011), while others examine how opportunities to learn the content are established in interactions (e.g. Nilsson & Ryve, 2010). The former strand of research leads researchers to focus on how identity, self-efficacy, and becoming a skilful participant in the discussion co-evolve and are determined by each other, while the latter one yields questioning about what counts as good communication and when it is effective. In the present paper, we aim at taking a unified view of these two strands, proposing an interpretative lens that takes into account both the sense of self-efficacy at individual and group level, and the interactional cues that allow the researcher to infer whether a student is communicating and listening to her peers. This choice allows us to contribute to the ongoing debate about the benefits and the limits of group interaction. With Sfard, Nesher, Streefland, Cobb, and Mason (1998), we acknowledge that "there are many ways to turn classroom discussion or group work into a great supplier of learning opportunities; there are even more ways to turn them into a waste of time, or worse than that — into a barrier to learning" (p. 50). In the sequel, we firstly recall the main results in mathematics education on group work, and then we propose an interpretative lens that builds on these results and their theoretical premises and identifies four modes of participation. We subsequently present data showing how the four modes of participation can be inferred in group work and we discuss what do they add to our knowledge of students' interactions.

Fundamental Elements in Group Interactions

Theoretical Background on the Value of Interactions

Interactional research states that the learning of mathematics takes place in a social context through interactions. Ernest (1998) defines mathematics as "inescapably conversational" (p. 169) and he places conversation at the centre of a cyclic process through which collective mathematical knowledge and personal knowledge of mathematics re-create each other. Sfard (2008) operationalises thinking as "the individualised version of interpersonal communication" (p. 85), and she sees meaning as an aspect of human discursive activity. Research in Mathematics Education has provided evidence that not only *cognitive*, but also *social* and *affective* aspects of students' interaction play a role in mathematical understanding as it emerges in group work: for example, Lave (1988) maintains that "developing an identity as a member of a community and becoming knowledgeably skilful are part of the same process" (p. 65), Davis (1996) further argues that for a true dialogue to take place, the interlocutors need to be willing to engage in the conversation, and Goos (2004) observes that community is essential to both the development of a sense of belonging and to the students' active participation. Classroom discussion and group work allow the growth of mutual understanding and coordination between the individual and the rest of the community (Sfard, 2001). To see knowledge as participation has been taken further by embodied views about knowledge, which recognise that bodily experiences ground the abstractions that are the basis of mathematical thought (Lakoff & Núñez, 2000). Embodied cognition entails a continuum rather than a clear cut between rational evaluation, feelings, and underlying psychological processes in the body. This continuum is put forward also by Roth and Radford (2011), who consider the emotional connection to cognition as a key element in any conversation. While Sfard (2008) suggests that a student's understanding can be gleaned from her talk, Roth and Radford (2011) suggest that talk is inextricably tied into the emotional state of the student in working with the problem. Furthermore, Roth and Radford (2011) show how the students' various and varying emotional states can be inferred from their postures, gestures, and tone of voice.

One can further argue that not only the students' fleeting feelings about themselves and their peers play a role in group activities, but also the more stable beliefs about themselves which can be difficult to change (for a general overview, see the seminal work in Leder, Pehkonen, and Toerner (2002)). Hannula (2011) suggests to identify a "trait" feature of emotions and beliefs that is more stable, and a "state" feature of them, which may change quickly. In the students' group work, it is necessary to focus on the "belief state", which plays a central role for the development of the activity itself since; following Roth and Radford (2011), the individual consciousness is produced by the movement of the activity. Similarly, the emotions of a student fluctuate and change rapidly during problem-solving and shape the mathematical activity itself (Roth & Radford, 2011).

We dwell for a moment on a particular belief, namely the one in being "good at something". Bandura (1977) argues that a student's self-efficacy is a major factor in whether she will attempt a given task, how much effort she will put on it, and how resilient she will be when difficulties arise. Self-efficacy can have a trait and a state feature, like any belief, but in group work, it takes the form of a "belief state" and may either propel the individual to persevere and develop the understanding as she goes on further into the problem, or—if low—provide the individual with a sense of being lost (see also Roth & Radford, 2011). Thus, three main arguments concerning mathematics group work are emerging, namely (a) thinking is communicating, (b) there's no thought without emotion, and (c) self-efficacy is the engine for mathematical thinking and doing. The way these arguments are intertwined is all but simple, as we discuss in what follows. Baxter, Woodward, and Olson (2001) have revealed that high-achieving and well-acknowledged students tend to dominate the discussion as well as to give valuable insights to the mathematical conversations, while low achievers remain passive and their ideas are sometimes muddled. Not only does there seem to be a link among high self-confidence, acknowledged ability, and mathematical outcomes, but Baxter et al. (2001) also report that the exposure to a wide range of ideas, strategies, and solution pathways from the more able peers resulted for the weaker students in richer socio-emotional and cognitive outcomes. Personal identity develops during group work jointly with becoming knowledgeably skilful (Lave, 1988). However, not always does group work result in positive outcomes for all the students: Barnes (2005) reports the case of two students who offer to the group many mathematical insights during the interaction, but they are frequently interrupted and their efforts are ignored by their peers. Barnes reports that these students learn less and they tend to lose confidence in their mathematical ability. Sfard and Kieran (2001) make the point that interaction with others may be counterproductive and show how articulating mathematical thinking to oneself can have beneficial effects for the individual. Different modes of participation into the group activity, thus, can be beneficial or not to mathematical understanding and to person's identity depending on several factors. Is it possible to characterise in-the-moment different modes for each student in a group to participate in a conversation, in order to delineate types of conversations that either allow or impede the fruition of the learning potential of students' conversations? To answer this question, we present our interpretative lens.

Analysis of Group Interactions

We exploit an idea introduced by Sfard and Kieran (2001) to capture "two types of speaker's meta-discursive intentions: the wish to react to a previous contribution of a partner or the wish to evoke a response in another interlocutor" (p. 58): reactive and proactive utterances, respectively. Reactive utterances are made in response to a previous utterance; namely, they are statements made in order to reply or react to an interlocutor. Proactive utterances usually point towards the person or people from whom a reaction is expected and appear as "new", namely as if new proposals are made to the group. We claim that in group interactions, there are modes of participation that are rather "reactive", while other modes are rather "proactive", but a student in a certain segment of the conversation can be both reactive and proactive, as well as neither proactive nor reactive (being silent or detaching from the activity). We can consider a fictitious example and imagine that a group of five students is working on a task about some arithmetic facts. Let us name them Ann, David, John, William, and Zoe. They read the task and Ann proposes a strategy, say to divide a number by another one. David says "I agree with you. I think you are right". John says "I am not sure I have understood what's your solution. Can you explain it to me?"—and William adds "Yes, I think we should do a multiplication, not a division". Zoe stays silent. Ann is proactive, David is reactive and Zoe is neither proactive nor reactive. John and William are clearly reactive, but they are also proactive, since the nature of their interventions is also to provoke a reaction from Ann (actually, two different kinds of reaction: to give an explanation and to discuss her proposal).

Following Sfard and Kieran (2001), but also extending their work, students' utterances are taken into account by our interpretative lens and distinguished among proactive, reactive, proactive and reactive, and neither proactive nor reactive. Moreover, if communication is an act of thinking, then "thinking can take any form, including gesturing" (Sfard, 2009, p. 195): gestures *with* utterances are the building blocks of commognitive processes in Sfard's view. There is more to gestures in interaction than just being part of thinking: simulation theories (e.g. Goldman, 2006) refer to mirror neuronal circuits to suggest that in order to recognise an interlocutor's actions, the perceived action is simulated in one's own motor system. The idea extends to emotions (e.g. Gallese, Eagle, & Migone, 2007), so that in understanding how others feel, we experience that emotion ourselves. By extension, then, when we are engaged with others in social interaction, it seems that one aspect of the interaction should be such simulation to the point that interlocutors mimic each other's actions, including gestures, and share similar emotions.

Vertegaal, Van der Veer, and Vons (2000) make a strong link between the amount of eye contact people give and receive, and their degree of participation in group communications. The occurrence of mimicry, or echoing, in co-speech gesturing has been examined by Kimbara (2008) and in a face-to-face communication by Holler and Wilkin (2011), who concluded that "mimicked gestures play an important role in creating mutually shared understanding" (p. 148). Non-verbal gestures were also found to be important in signalling incremental understanding, something the authors paraphrased as "I am following what you are saying" (p. 145). Roth (2000) notes that "the human body maintains an essential rationality and provides others with the interpretative resources they need for building common ground and mutual intelligibility" (p. 1685). He adds that such gesturing provides resources to collaborative thinking-through-processes by affording the production of public accounts. Hence, Goos' (2004) sense of belonging and active participation of the students in a group can be further characterised by exchange of glances, mirroring gestures, and echoing emotions. On the one side, we focus on the proactive and reactive nature of utterances; on the other side, we focus on propositional gestures and on how (if so) they are mimicked within the group, and on non-verbal gestures and postures that reveal the extent to which a student is with her peers (or not) in the conversation. A student can, in fact, make many proactive utterances but at the same time be not engaged: Kotsopoulos (2010), for example, provides examples of students undertaking the role of the "foreman" in group work. The foreman gives many directive orders to her peers (proactive) but engages in the task in very limited ways, and the mathematical quality of group work is poor. Liljedahl and Andrà (2014) apply Sfard's & Kieran's (2001) tool to analyse the dynamics of four students where two leaders are present: both make many proactive statements, one of them suggested the correct solution but the group adopts the solution of the other leader, which was not correct. To explain why, the researchers pay attention to the eye contacts among the students and they noticed that the second leader received a lot of glances from his peers, while the first leader received glances only from the other leader. Not always, however, does having a (proactive) leader in the group result in poor mathematical outcomes: Armstrong (2008) reports the case of a group where a student takes the lead and presents a solution that all his peers quickly accepted. Reading Armstrong's paper with Sfard's & Kieran's eyes, one can notice many proactive and reactive utterances. However, Armstrong argues that "there was nothing in their behaviour that suggested that any of them felt especially connected to any other member of the group" (p. 113). In other words, the students do not demonstrate a sense of togetherness (Condon, 1986), namely a synchronisation of physical and verbal behaviours. Proactive and reactive utterances, intertwined with gestures, postures, and glances, allow the researcher to infer the extent to which a student is participating in group work and his sense of efficacy. It is the intertwining of utterances, gestures, glances, and postures that allows us to identify four modes of participation in group interactions.

The Interpretative Lens

Our interpretative lens proposes four modes of participation in group mathematical activities: we assign them labels, even if we are aware that labels help identify some features of the concept they refer to, while hindering others. We summarise the four modes as follows:

- Mode 1: cooperation, when the student makes both proactive and reactive statements (not necessarily at the same time), i.e. she shares her mathematical ideas with her peers and is open to the ideas coming from the others. An interesting example comes from literature, and specifically it is an episode analysed in Roth and Radford (2011): three students, Aurelie, Mario, and Therese are working on a task. Mario and Therese are interacting, since Therese expresses doubts about how Mario proposes to work. An intense flow of glances among Mario, Therese, and the worksheet goes on. The two students stare at each other, and Therese follows at Mario's gestures, intensively and repeatedly. In the segment, Mario is talking and "his intonation—based on the correlates between prosody and emotion identified in psychological research—expresses firmness and confidence. During his explanation, his gestures make an embodied link between goblet-chip model (left hand index finger) and the worksheet in front of him (right hand pencil)" (Roth & Radford, 2011, p. 35). Mario is our example of a student in a cooperative mode
- Mode 2: obstination, when the student has mathematical ideas, which can be right or wrong, but tends not to listen to her peers' ones. This state is inferred from many proactive utterances but absence of reactive ones. A student who does not glance at her classmates, but looks at the paper, or elsewhere, may not recognise a competence in them. The student would propose her solution, without listening to the other members of the group, she can be so much self-confident to the point that she does not want to mediate. For example, she would ignore her peers' questions, she may interrupt another student's speech, or she would impose her view by increasing the tone of her voice and changing her posture. The students in Armstrong's (2008) example can be considered as students in obstinate mode
- Mode 3: isolation, when the student does not talk and does not listen to her peers. Absence of both proactive and reactive statements is observed. The student may glance at the paper, or elsewhere from her peers. Her posture reveals that she is not in the conversation, like Aurelie in Roth and Radford's (2011) example, who "throws herself back against the backrest" (p. 35). This mode has been proven to be particularly interesting also in Sfard and Kieran (2001) and it is well recognised in literature that isolation can be important even for the most famous mathematicians (see e.g. Liljedahl, 2012): before an illumination, many thinkers experience a feeling of being lost, a sense of "darkness", which can very well resemble what students experience in group activities when they do not know how to sort the problem out
- Mode 4: follower, when the student does not have mathematical new ideas, but follows her peers' ones. The student does not make proactive utterances, but she can either make reactive ones or stay silent. Her posture reveals that she is listening to her peers, and she looks at them (rather than looking at the paper, or isolating herself from the rest of the group, for example). It can be the case that the student asks many questions, since she needs to go slow in order to grasp the sense of what another student is saying. She can also echo her peers' gestures, and/or rephrase their utterances. Therese, in the aforementioned example from Roth and Radford (2011), can be interpreted as being in a follower mode. The student who is in this

Table 1 The coding scheme for the interpretative lens of participation modes

	Utterances	Glances	Gestures	Posture	Intonation
Cooperation	Proactive and reactive	Mostly to the peers, some to the worksheet	Many, co-speech, connected to the meaning-making process	Pounding the desk, being open to her peers	Firmness, confidence
Obstination	Proactive	Mostly to the worksheet (ignores her peers)	Many, co-speech; some can be interactional and tell a peer to stop talking	Pounding the desk, turning her back towards a peer to exclude her	Firmness, confidence
Isolation	Neither proactive nor reactive (silence)	To the worksheet or nowhere	Few; if present, they accompany thought and are non-communicative	Detached, throwing the back against the backrest, or placing the head on her hands	
Follower	Silence or reactive	To the peers, to worksheet, or nowhere (may depend on where who is talking is glancing at)	Few; if present, they can echo another peer's gesture	Pounding the desk, may scratch her head	May express frustration, or sense of being lost, or need for help

mode contributes to deepen the understanding of the whole group—as Therese does in the sequel of the episode in Roth and Radford (2011), but see also Liljedahl and Andrà (2014)

Table 1 summarises the coding that informs our lens of analysis. Following Sfard and Kieran (2001), we pay attention to verbal utterances and classify them into proactive, reactive, proactive and reactive, and neither proactive nor reactive. Following Roth and Radford (2011), we look at students' postures and tone of voice that reveal their emotional statuses while undertaking the activity at each moment. Following Liljedahl & Andrà, we also look at glances of the student that is talking and of the students that are not talking, taking into account whether they stare at a classmate or at the paper. And following the embodied cognition paradigm, we take also gestures into account.

There are no "positive" or "negative" modes of participation in group interactions in our interpretative lens: all them contribute to the shape the students' understanding in different moments of the activity and all of them reveal to be fruitful or not during the sharing of knowledge. The modes have a dynamic nature, since they change, and it is possible to depict the movements of each student throughout the four modes during the activity. For instance, a student who is in an obstinate mode at the beginning of the activity would impose her ideas on the rest of the group, until someone questions them. At this point, she can reach a cooperative mode if she recognises a competence in one of their peers (likely, the one who questioned her ideas), or transit to isolated mode if she has a feeling that her peer was right in questioning but she has no insight. If an obstinate student comes to isolation, it can be the opportunity for the rest of the group to express their own ideas. A completely different case is if the student in isolated mode was the one that was not self-confident and was mostly silent in the previous moments of the interaction: the risk is that she will stay in this state for the rest of the activity.

Aim and Research Questions

Literature review tells that it is possible to infer a student's mode of participation in a group by looking at verbal utterances, postures, gestures, glances, and intonation of voice. In this paper, we aim at applying an interpretative lens, which has been built on the basis of previous, relevant findings in Mathematics Education (and which is summarised in Table 1), in order to answer the following research question: What does the focus on the four modes of participation add to the understanding of group work activities? More precisely, how can this lens of interpretation help us understand better the conditions under which the students as a group learn from the mathematical activity, and those under which learning is somehow impeded? In order to answer this question, we selected two episodes from a set of around 80 h of video-recorded group work classroom activities. The episodes show interesting dynamics that mirror two typical, and different, group dynamics. In both groups, for example, there is a student who reaches an obstinate mode of participation and in one case, the mathematical quality in the group work is poor (like in Kotsopoulos' (2010) study), but in the other case, it is richer. Furthermore, in both episodes, there is at least one student who reaches isolation, but in one case, this results in an opportunity for the entire group to enhance their understanding (confirming Sfard and Kieran's (2001) claim), but in the other case it is detrimental to the entire group outcome. We noticed that it was not only the particular mode of participation of a single student that counts in order to understand these differences, but the modes of participation of all the peers, in that moment, that shape the outcome of the mathematical activity. The two episodes were chosen because they allow us to focus on these differences, and on the similarities, that we noticed. Case selection, then, was purposive (Yin, 1994).

Material, Context, and Methods

The Context, the Participants, and the Tasks

The research presented in this paper has been developed within the project BetOnMath (http://betonmath.polimi.it), which promotes learning activities that aim at preventing gambling abuse. One of the main goals of the project is to engage students in mathematics by offering them challenging opportunities to apply mathematical ideas in the context of betting games, rather than simply memorise and execute routine procedures. Boesen, Lithner, and Palm (2010) argue that the kind of task assigned to the students affects their learning: tasks with low levels of cognitive demand lead to rote-learning by students and, consequently, their inability to solve problems that are unfamiliar to them (for instance, the ones that require conceptual understanding). Breen, O'Shea, and Pfeiffer (2013) define an "unfamiliar task" as a task "for which students have no algorithm, well-rehearsed procedure or previously demonstrated process to follow" (p. 2318) and provide evidence that this kind of task raises an awareness about the need for more than procedural understanding of mathematics. Hence, the tasks of the BetOnMath project were designed so that the mathematics that is needed to understand gambling situations is new for the students, or requires them to connect different ideas in a new situation.

We collected 80 h of video-recorded group work activities in 20 classrooms from eight high schools (grades 10–13, age 14–18 years), of different types and with different background in mathematics. On average, each class has been video-recorded for 6 h, and data presented here do not come from the first lesson videotaped, which has been considered a warm up for the students to become used to both our presence and the presence of the camera. We present and analyse data from two video excerpts, which have been selected because they allow us to discuss

(a)

Sum	Probability	Prize
2	1/36	
3	2/36	
4	3/36	
5	4/36	
6	5/36	
7	6/36	
8	5/36	
9	4/36	
10	3/36	
11	2/36	
12	1/36	

(b)

Probability of 'gold bar-other-other': 64/729 → 9%

Probability of getting 1 gold bar: 192/729

Outcome	Prize	Probability
1 gold bar	1 euro	
2 gold bars	10 euros	
3 gold bars	100 euros	

Figure 1 The two tasks partially solved by **a** EFGM and **b** ABCD

the complexity of group dynamics as well as to contrast different modes of participation that result in different mathematical outcomes. The first excerpt comes from a group activity of four grade-12 (17 years old) students: Enrico (E), Federico (F), Giovanni (G), and Michele (M). They are attending a technical high school program in a school located in the city of Milan and are asked to invent a fair game using two dice (possible outcomes are all the sums from 2 to 12). In a fair game, the expected winning equals the ticket price. They have computed the probability of each sum, reporting it on the paper (Fig. 1a). The excerpt begins when they have to assign the prize to each sum from 2 to 12. The second excerpt comes from a group activity of four grade-10 (15 years old) students: Alice (A), Barbara (B), Carola (C), and Dora (D). They are attending a scientific high school program in a school located in the suburbs of Milan, and they are analysing a slot machine, which has three rolls with nine different symbols each. The number of different possible sequences is $9^3 = 729$. There is only one winning symbol on each roll: the gold bar. The students computed the probabilities of one, two, and three gold bars and reported them on the paper (Fig. 1b). The excerpt begins when they compute the expected winning. They have to report the probabilities in the table, multiply each of them by the corresponding prize, and sum up the results (weighted prizes).

Each group received one worksheet and one pen, so that they had to interact in order to decide which answer needs to be written on the paper. In case isolation mode would have been reached, the possibility that a student would start working independently on her own worksheet is avoided.

Method of Data Collection and Video Analysis

During group activities, the students were seated in a semi-circle. A camera was put in front of them, so that it was possible to catch all the participants, their postures as well as their utterances. A researcher took also field notes on each student's glances, moving around during the group work.

Once data had been collected, we transcribed the excerpts and added side notes on each student's posture, intonation, and glances. We, thus, assigned a mode of participation to each student in the group in subsequent moments of the activity. We refer to Table 1 for assigning a student to a mode. When a change in a student's mode occurred, we considered a transition to another mode.

Excerpt 1 and Its Analysis

In this excerpt, two students (F and G) propose different solutions. G's one is grounded on previous in-classroom mathematical activities, while F's one is linked to betting practices.

1. G: (*looks at the paper*) We should start with 7, which has the highest prize. … No, the lowest prize (*looks at F*).
2. F: (*staring at the paper*) No, let's do, let's bet 1 euro (*his right hand is on the paper as if he wants to write on it*).
3. M: Easy.
4. F: (*keeps looking at the paper*) If e.g. you bet on 12, it comes out, you win…

Table 2 The interpretative lens applied to excerpt 1

Turns	Modes				Indicators (utterances, glances, gestures, posture, intonation)
1–6	Obstination	Cooperation			G looks down, then turns his eyes to look at F, and makes a proactive utterance. Intonation shows self-confidence. G pounds the desk.
	F	G	M		
	Isolation	Follower			M makes a reactive and proactive utterance (in 3 and 5). Intonation reveals self-confidence and pounds the desk. For the most of time, he looks at G and F.
		E			
					F makes three proactive statements looking at the paper. He detaches from the group with his posture, lying with his back to the back of the chair. F seems to act as in self-talk rather than to involve his peers. F speaks over his classmates contradicting G without any justification.
					E is silent and glances to F. In 6 he takes the paper and brings it closer to him, starting reading the task silently. E's gesture pointing on the paper echoes F's one.
7–11	Obstination	Cooperation			G is reactive and looks at the paper mirroring and staring at F.
	F	G			F pounds the desk and talks looking at the paper. E and M are silent, but their posture tells they follow F and G.
	Isolation	Follower			
		G	M		
12–15	Obstination	Cooperation			G looks at the paper and remains silent.
	G	F	E	M	E and M question F with higher self-confidence, looking at F.
					F is proactive and makes his first (and unique) reactive statement to E, looking at him.
	Isolation	Follower			
16–18	Obstination	Cooperation			F makes another proactive statement, looking at the paper, but he adds a detail, looking at E. Intonation expresses self-confidence.
		F	E		
	G	M			E looks at the paper and reacts to F. M is looking intensely at F.
	Isolation	Follower			G has still a questioning facial expression and looks "nowhere".
19–20	Obstination	Cooperation			G makes proactive statement looking intensely at F.
	F	G	E	M	F does not reply to G, but goes on with his reasoning, detaches from the table, looks at the paper, remains silent, and does not write.
	Isolation	Follower			E and M also look at the paper, mirroring F's glance.

5. M: But let's bet 2 euros.
6. F: …you win 36 euros (*detaches from the paper with his back*). Let's do 36 to 1 (*looks at G*).
7. G: (*looks at the paper*) But, wait: 7, how much is it? We should compute the mean of the prizes and…
8. F: It's enough to do this (*points to 36, the denominator*) divided by this (*points to 6, the numerator of the probability to get a sum of 7*). If you do 36 divided by 7, what do you get? (*makes computations with the calculator*) 5. If you bet 1 euro on 7, you win 5 euros.
9. G: Hence, the minimum you can win is 5 euros (*keeps looking at the paper*).
10. E: (*inaudible, looks at F*)
11. G: It's too much (*looks at F, then looks at the paper with a questioning facial expression*).
12. F: The highest prize is 36 euros (*keeps looking at the paper, takes a pen*).
13. E: (*looks at F*) Otherwise, let's bet 50 cents.
14. F: (*looks M*) It's the same, then, finally. If we bet 1 at least we have (*inaudible*)
15. M: (*inaudible, keeps looking at F*)
16. F: (*looks at the paper*) For example, 12 is given … like the SNAI[1] (*all laugh*)
17. F: (*looks at E*) 12 is given 36 to 1. If I bet 1 euro I win 36 euros. (*stays silent for some seconds, looking at E*)
18. E: (*nods, looking at the paper*)
19. G: mmm it's too much, because, then, the 7…? (*looks nowhere, with a questioning facial expression*)
20. F: Eh, no, because… (*looks at the paper, remains silent, stops writing, detaches from the table. G stares intensely at him, E and M look at the paper*)

Table 2 provides both a graphical representation of EFGM's modes of participation in a 4-cell squared table (second column), and the interactional clues, namely utterances, gestures, posture, and intonation, upon which we based our interpretation (third column). We provided many details for the turns 1 to 6, then for brevity we pinpointed only the main facts.

In the turns 1–6, G and F propose two different approaches to the problem: either to start with the most probable event (i.e. 7), or to start with the least probable one (12), which means to decide the highest prize to assign. After his first utterance, G remains silent, while F makes many other proactive statements.

In 7–11, there emerges a conflict between two different ways of approaching the problem: G's one is resorting to previous classroom experiences ("we should compute the means …" recalls the previous mathematical activity), while F's one is resorting to betting practices. F's point of view is taken over by the group.

In 12–15, F, E, and M are all in a cooperation mode, while G is in isolation as if he wants to think and find out why F's strategy works with the 12 but it seems to him that it does not work with the 7. G's isolation can also be seen as resulting from F's obstination: F and G have two contrasting starting points in 1–6, and they further contrasted different approaches in 7–11: this seems to provoke G's resignation. F comes

[1] SNAI is an Italian acronym: "National Consortium of horse-race Agencies".

to cooperation in 12–15 from obstination, while E and M from follower. It is as if F during the activity becomes able to recognise his peers' ability, while M and E become more confident in their own ability. It is as if the group as a whole perceives itself to be "more able" that it believed itself to be at the beginning of the activity. It is as if F's obstination depresses G's self-confidence and at the same time it prompts E and M to intervene in the activity: a reason for this can be that F's experience with betting games (e.g. SNAI) engages E and M (who follow), while G contrasts this approach.

In 16–18, G keeps his isolated mode, but in 19, he makes a question that is ignored by F, who in 20 goes on with his reasoning about 12 and concludes that something should be wrong. We notice that F and G get the same conclusion, following different strategies, but they are not able to recognise that it is so. In fact, when F was in cooperation mode, G was isolated, and when G got out from isolation and was following, F turned in isolation and no space for interaction was possible between the two students.

Also in the second excerpt, two students reach the same conclusion, like G and F in the first one, but with some struggle, they come to recognise so.

Excerpt 2 and its Analysis

A, B, C, and D are dealing with the task of computing the mean prize of the slot machine with three rolls. They have already computed the probability to get one gold bar (192/729), two gold bars (64/729), and three gold bars (1/729), which allow the player to get 1€, 10€, and 100€ respectively. In the first 15 s, B reads the task ("compute the mean prize of the slot machine"), then

1. B: We should use combinatorial mathematics (*looks at D, while A, C and D look at the paper*).
2. D: (*keeps looking at the paper*) No, we have computed that the probability of finding one gold bar and two other symbols (*takes the sheet of paper close to her, to be able to read it*) was…
3. B: …was it 192? (*reads the worksheet*)
4. D: No. No. This one maybe (*points to the probability of getting the sequence 'gold bar– other–other' on the worksheet*)
5. B: (*reads*) 'probability of gold bar-other-other' (*nods*). 9%
6. D: 9% [*The students make computations and report the results on the paper, then they ask the teacher if their work is correct, and the teacher confirms their doubt: it is wrong.*]
7. B: (*looks at the paper*) That is why we should use combinatorial mathematics. Otherwise, they would not have given it to us (*keeps writing*).
8. C: Let's use our ingenuity! (*looks at A*)
9. B: (*looks at D, her left hand makes a gesture that tells that she is pretty sure to be right*) For sure it's with combinatorics, then let's invent something. Well, in combinatorics one needs to multiply many numbers, so let's have a look… [*The teacher provides a further feedback about the mistake the group has made: the probability of getting one gold bar is not 9%.*]
10. A: This one (*points to 192/729*) is the probability to get one gold bar.

Table 3 The interpretative lens applied to excerpt 2

Turns	Modes			Indicators
1–6	Obstination	Cooperation		All the students look mostly at the paper. B glances at D for a while.
	D	B		B makes a proactive statement and reacts to D looking at her. B reads on the sheet of paper the correct answer.
		A	C	A and C are silent and look at the paper.
	Isolation	Follower		D contradicts B, and keeps looking at the paper and reacts to B.
7–9	Obstination	Cooperation		D is silent and keeps looking at the paper.
	B	C		C intervenes with self-confidence. C looks at A seemingly to encourage A to intervene.
	D	A		A does not intervene, but listens to her peers.
	Isolation	Follower		B looks at D, her left hand makes a gesture that tells that she is pretty sure to be right
10–13	Obstination	Cooperation		B and D look at each other, and B expresses a sense of revenge with respect to D. B's posture does not change.
	B	A	C	D's glance reveals that she is still thinking about the solution without following her peers.
	D			C's speech connects to B's speech, and A makes a proactive statement. A's and C's postures mirror each other.
	Isolation	Follower		
14–22	Obstination	Cooperation		D makes two proactive statements She looks at A and B, and at the paper.
		A	D	All her classmates look at D. A, B and C react to D proactive statements.
		B	C	B reacts and asks for explanation.
	Isolation	Follower		A intervenes, reacting to D but also adding what makes sense to her. A's posture reveals that she is in the conversation.
23–25	Obstination	Cooperation		A is react and proactive because she reformulates D's proposal
		A		D says "I do not know", and she looks down at nowhere.
	D	B	C	B and C remain silent looking at their peers.
	Isolation	Follower		C echoes D's admission "I do not know"
26–29	Obstination	Cooperation		D remains silent and stares at the paper.
		B	C	C reacts to B but looks at D
	D	A		B looks at D, but is proactive and reactive with C
	Isolation	Follower		A is silent and listens to her peers.

11. B: (*looks nowhere in front of herself*) Hence, I have said it correctly at the beginning! It is 192 divided by 729.
12. C: (*speaks over B, looking at A*) 192 divided by 729.
13. B: (*looking at D, pleased*) Ah! Ah! [*The students copy the (correct) probabilities into the table on the worksheet. Then, they stop and look at the empty cell (where the weighted sum should be computed and put).*]
14. C: (*looks at B*) But what should we compute?
15. B: (*looks at the paper, her left hand is lifted as if she asks for silence and time to think*) The total average prize. The… mm… namely…

16. D: (*looks at B*) All those prizes times the probability that you win them, namely that you win the prizes, divided by… all the cases? (*A looks at D*)
17. C: Eh? (*turns to D*)
18. D: 729? Because there are many cases in which you win nothing (*looks at A*).
19. A: (*looks at D, D looks at A*) Exactly. There are many cases where you get nothing, where you get 'other'.
20. B: (*looks at D*) So we should do the prize..? No, the cases in which you win, divided…
21. D: (*looks at A*) The cases in which you win, the total prize divided by all.
22. B: Why divided by all? (*looks at D*)
23. A: (*after few seconds of silence from the group, looks at D*) You are saying: to sum all the average prizes. Is it what you're saying?
24. D: I do not know (*smiles*).
25. C: Eh, (*echoing D*) 'I don't know what I am doing'! (*smiles*)
26. C: (*looks at B*) Wait. Is it asked (*A pulls the paper close to herself*) the mean of the prizes?
27. B: But, but I would have done this: I would have multiplied the mean prizes for the probability, then summing the probabilities, divided for all the cases (*looks at D*).
28. C: (*looks at B, and B looks at C*) Like the last time!
29. B: (*looks at D*) Yes, it is the weighted average.

Table 3 summarises our interpretations of ABCD's modes of participation.

In 1–6, the students are making sense of the task, but in 7–9, the glances are interesting: A and D look at the paper, while C looks at A, and B looks at D: it is as if C and B want their peers with them. They do not look at each other even if they talk—apparently—to each other. We notice that, in this move of the activity, each student has a different mode of engagement: A follower, B obstination, C cooperation, D isolation. Such a configuration is reached when the group is wondering what to do (and we can further notice that in the first excerpt it is not reached).

In 10–13, A's proactive statement provokes a change in the group at the level of their understanding. A was listening and following, but at this point, she recognises the correct number and B expresses a sense of revenge with respect to D. Like F and G, also B and D have contrasting ideas about the start of the activity, but in the second case, a student in follower mode takes the point of view of the student who was not in obstination mode. The first moves across the four modes of participation bring A and D to be in cooperation mode, as D comes here from isolation in 14–22. B and D are following, but participating with many questions. Then, in 26–29, B and C switch to cooperation mode and they together find the correct solution. They find it after A in 23, in cooperation mode, had reformulated D's proposal and D remains silent. The exchange of glances is intense among A, B, and C.

Before discussing the relevance of these data for the issues raised at the beginning of the paper, we would like to dwell, at the end of data analysis, on some details that we found handy on a methodological level, in case a reader is interested in applying our lens of analysis to her data. We noticed that both E and M are silent in 7–8 and both of them talk in 9–15: their mode is *follower* in 7–8 and change to cooperation in 9–15. If we look at utterances alone, however, it is clearer that E moves to *cooperation* since there is evidence that he interacts with F: F reacts to E's utterance. Furthermore, E and F glance at each other. In order to understand M's mode, it is necessary to go beyond words and consider his posture: in 9–15, he is clearly in the conversation with his body, bending his back towards his peers and staring at them intensely, in a will to intervene (and finally he intervenes in 15). Glances, furthermore, are really helpful when the recognition of the peers' capability needs to be inferred, and glances of different kinds allow us to understand different modes. Let us consider, for example, B's glances. In 1, she looks at D, as if B recognises her competence and, even if D is not looking at her, D reacts to her while A and C remain silent for a long while. In 8, the same happens between A and C, namely C looks at A and, even if A does not look at her, she will speak after that moment. Such glances cannot be considered as an invitation to speak, since the person towards whom are directed does not see them, but rather as expressing a sense of confidence towards a peer. A completely different glance is made by B towards D in 9, when B is sure to be right: in that moment, B is not expecting D's intervention, but she is stressing her self-confidence.

A limitation of this model is that is it based on inferences and one can question about the reliability of such interpretations. In the section dedicated to our theoretical framework, we have, however, reported several studies in Mathematics Education that show that it is possible to infer the students' participation in a group work (Barnes, 2005; Goos, 2004; Kotsopoulos, 2010; Sfard & Kieran, 2001), their ways of interacting with their peers and their degree of valuing the others' ideas from their utterances, postures, glances, and gestures (Armstrong, 2008; Holler & Wilkin, 2011; Kimbara, 2008; Liljedahl & Andrà, 2014; Roth & Radford, 2011; Sfard, 2008). We have also provided theoretical foundations of how this is possible (Gallese et al., 2007; Goldman, 2006; Roth, 2000; Vertegaal et al., 2000), and the analysis of the two excerpts we made is a further example that inferences of this sort can be reliable.

Discussion

In this paper we have focused on both self-confidence and confidence-in-peers as two key elements in group dynamics and in mathematical understanding: we can say that the four modes of participation can be further characterised by different degrees of "I can" and "you can". We may think of "I can" as self-confidence or perceived competence, while we may conceptualise "you can" as competence recognised to one's peers. In obstination, the student's "I can" is high, while the "you can" is low, since she has a high self-confidence but does not trust her peers as if she does not recognise them as competent. Also, cooperation belongs to the "I can" area, while follower and isolation seem to belong to "I cannot" one. In follower, the "you can" is high as well as in cooperation, while obstination and isolation seem to belong to the "you cannot" area (for further details, we refer also to Andrà, Brunetto, Parolini, & Verani, 2015a).

In our research, to pay attention to every student's "I can"/"I cannot" and "you can"/ "you cannot" is not considered important per se, but it is so because it allows us also to investigate whether a "we can" is reached. In our examples, ABCD reach a "we can", while EFGM do not. If we compare the initial situation of both groups, we can see that there are not so many differences: in EFGM case, there are two students in cooperation mode (G and M), one student (F) in obstination, and one in follower (E); in ABCD case there are: one student (B) in cooperation mode, one in obstination (D), and two in follower (A and C). There are also other similarities: G and B make the first utterance and they both are in cooperation mode at the beginning of the activity. F and D, who are in *obstination* mode, make the second utterance, which in both cases begins with a "no", namely F and D respectively contradict G and B's proposals. Similar incipits give rise to very different interactions: the third utterance in EFGM is made by M, the other student in cooperation mode, and G remains silent for some time; in ABCD, there's no other student in cooperation mode and B makes the third utterance. In EFGM, an interaction between F and M takes place and G is silent, while in ABCD, the interaction is between B and D while A and C seem to be warming up. This time of warming up for A and C seems to have a positive effect for their interventions in the rest of the excerpt, since they really contribute to the development of their activity, while E and M would propose different prices for the ticket but would not make any intervention regarding the strategy. This confirms and extends Sfard and Kieran's (2001) observation that isolation, understood also as silence and/or time for "warming up", is essential especially for those students who are low achieving, or not well acknowledged, or who tend not to dominate the discussion. Back to Baxter et al. (2001), it is true that these students tend to remain passive, but it is true that specific group dynamics may lead to a different scenario.

F and D, the students who in respective groups are the most proactive, both start in obstination mode. However, in the second excerpt, D, who tends to lead the conversation, is said to be wrong earlier than F, the obstinate student in the first excerpt. This allows B, a cooperative student at the beginning of the activity (like G), to feel a sense of revenge, instead of being isolated like G. G's case recalls Barnes's (2005) case of students who were ignored by their peers and a specific focus on "I can" and "you can" may help us to understand why. D's trajectory navigates the modes in this order: obstination → isolation → cooperation → isolation. F's trajectory makes obstination → cooperation → isolation. The two trajectories are very different: while F's trajectory lies for the most of time in the "I can" area, D's one goes to "I cannot" two times and in these occasions, her peers are allowed space to say their ideas. F's permanence in the "I can" area causes a shift towards the "I cannot" area to G and this turns out to be disadvantageous also for F, since also F ends up in isolation (and the group does not solve the task). High self-confidence from students who are high achievers is important for the development of the activity, but this example shows that it is also important for a "dominant" student to leave room for her peers: too much "I can" from a student can depress the other's "I can". Group leaders take advantage from their peers' self-confidence and the entire group may reach a sense of "we can", which mirrors better learning outcomes.

Finally, there is another interesting difference between ABCD and EFGM, a detail that deserves attention: at a first sight, A's trajectory is similar to E's one (and A catches some of D's glances like E does with F), and M's trajectory looks similar to C's one, even if M does cooperation → follower → cooperation → follower, and C does follower → cooperation → isolation → follower → cooperation. These trajectories lie in the "you can" area (except the short isolation for C). But what happens when C is in cooperation mode (the second time)? B is also in cooperation and they cooperate to find a correct solution. When M is in cooperation (the second time), G (who is the analogous of B) has already reached isolation and ignores M's proposal. For low achievers, it is vital that their peers feel a sense of "you can" towards them, so that they can contribute to the activity (if not at the beginning of it, later on). This confirms and extends Baxter et al.'s (2001) findings that weaker students benefit from the exposure to a wider range of ideas, strategies, and solution pathways provided by their more able peers, *if* also the weaker peers can offer insights on the solution to task posed. The conditions upon which a less-acknowledged student is allowed to offer such an insight, or not, depend on the different modes of participation of all the students in the group, in the very moment such an insight is offered.

What is relevant, and original in our study, is that our interpretative lens allows us to have a *unified view of the group* at any moment of the interaction: a group that has cognitive, affective, and social dynamics that intertwine. Even when students work in groups, many researches focus on individual students' gestures, utterances, signs, as if the peers' role is to provide an interlocutor for the student-having-the-idea to express it. Chamberlin (2010) has provided evidence that researches on group work activities focus mostly on the individual factors (such as self-competence, enjoyment, or goal orientations). We take a different viewpoint and when F proactively proposes his strategy to the group, for example, our focus is not only on F's posture, gestures, glances, but they are considered together with his glances, postures, utterances, since F's idea is emerging in a specific group, F is sharing his idea with those particular students who react in a certain way to it and not only the way they react to it, but also the way F feels they would react to it has an impact on F's idea and on how it emerges. Or, let us consider D's gestures and tentative speech when she intuits to use the weighted average in 14–22: it is D who has an idea, but it is A who rephrases it and it is C who finds the words to make it understandable for herself and for B. D's idea is no longer D's idea, but it has become ABCD's idea as a group.

A focus on the group's "we can" and on in-the-moment individual "I can" and "you can" allows us also to reach some interesting conclusions. For example, we have commented that to temporarily reach the "I cannot" area for a student can be beneficial for the entire group, either because she allows herself time to think/warm up, or because she leaves space to her peers. Hence, *it is not necessary that a student lies in the* "*I can*" *area for the most of time for the group to reach a* "*we can*". Or, *too much* "*I can*" *from only one student might depress the others'* "*I can*" *and it might result into a* "*we cannot*". The fact that a certain degree of "I cannot" can be good has been observed also by Zaslavsky (2005), who argues that tasks that elicit *learners' uncertainty* in the mathematical validity of a claim, problem-solving method, conclusion, or outcome can facilitate learning of mathematics. The researcher roots her arguments in the conflict theory, variations of which were acknowledged by many scholars (e.g. Dewey, 1933; Festinger, 1957; Piaget, 1985). Generally speaking, scholars agree that when an individual is experiencing uncertainty (cognitive conflict or disequilibrium, in terms of Piaget; dissonance, in terms of Festinger; perplexity, confusion or doubt, in terms of Dewey), she will be motivated to modify something in her ways of acting and thinking in attempt to escape from this situation. Employing this kind of task in their research, Kontorovich and Zazkis (in press) further argue that the uncertainty that such tasks elicit among learners can expose some of the classroom socio-mathematical norms, which become tangible when there is a deviation from them. The idea was supported also by Morselli and Sabena (2015): it was seen that having emotional ups and downs (e.g. being inspired—getting confused—succeeding) was an effective way to improve cognitive understanding.

Conclusions and Further Development

In this paper, we show two examples of group activities that at the certain point get stuck but in one case, the students are able to go on and surprisingly a weak student provides the key input for this. In order to understand why this happens in one case and not in the other, we resorted to an interpretative lens of analysis that draws on previous research findings about utterances, glances, postures, gestures, and intonations as indicators of different modes of participation in a group. One of the strengths of this lens of analysis is that it allows us to focus also on the group as a whole and examine why in some cases, a group of students reaches a “we can” and positive learning outcomes while in other cases, it does not.

Our interpretative lens is descriptive, not prescriptive. How to provoke or stimulate group dynamics that lead to more engaging and productive learning outcome is outside the reach of our research. Teachers’ professional development is challenging; however, our wish is that our interpretative lens can became a tool in the hands of a teacher to (i) recognise the students’ modes of participation, (ii) identify those which can lead to either bogging down or going on, and (iii) decide whether and how to intervene in a group of her students, not to give them mathematical inputs, but rather to increase a student’s “I can” (or, “you can”), and/or lower another student’s “I can”. The first attempt towards this direction has been made in Andrà, Brunetto, Parolini, and Verani (2015b).

Finally, we may raise the question whether this lens is unique to mathematical learning, or it can be used for any subject. An argument in favour of the former case is that limitations and potentialities for group interactions that constitute the background of our study have emerged in researches in Mathematics Education, and in our study, we had in mind mathematics teachers in mathematics classrooms. At the same time, the interested reader may find this lens suitable for other subjects—with the necessary adaptations.

References

Andrà, C., Brunetto, D., Parolini, N., & Verani, M. (2015a). ‘I can – You can’: Cooperation in group activities. In K. Krainer & N. Vondrová (Eds.), *Proceedings of the Ninth Congress of the European society for Research in Mathematics Education* (pp. 1109–1115). Prague, Czech: CERME.

Andrà, C., Brunetto, D., Parolini, N., & Verani, M. (2015b), Teachers’ perspectives on group dynamics. In K. Krainer & N. Vondrová (Eds.), *Proceedings of the Ninth Congress of the European Society for Research in Mathematics Education* (pp. 1309–1310). Prague, Czech: CERME.

Armstrong, A. (2008). The fragility of group flow: The experiences of two small groups in a middle school mathematics classroom. *The Journal of Mathematical Behaviour, 27*, 101–115.

Arzarello, F. (2006). Semiosis as a multimodal process. *Revista Latinoamericana de Investigación en Matemática Educativa,* Special Issue on Semiotics, Culture and Mathematical Thinking, 267–299.

Bandura, A. (1977). Self-efficacy: Toward a unifying theory of behavioural change. *Psychological Review, 84*, 191–215.

Barnes, M. (2005). ‘Outsiders’ in a collaborative learning classroom. In M. Goos, C. Kanes, & R. Brown (Eds.), *Mathematics education and society* (pp. 58–68). Brisbane, Australia: Griffith University.

Bartolini Bussi, M. G. & Mariotti, M. A. (2008). Semiotic mediation in the mathematics classroom: Artifacts and signs after a Vygotskian perspective. In L. English (Ed.), *Handbook of international research in mathematics education (2nd edition)*. New York, NY: Routledge.

Baxter, J., Woodward, J., & Olson, D. (2001). Effects of reform-based mathematics instruction in five third grade classrooms. *Eelmentary School Journal, 101*(5), 529–548.

Bikner-Ahsbahs, A., & Halverscheid, S. (2014). Introduction to the theory of interest-dense situations. In A. Bikner-Ahsbahs, S. Prediger, & the Networking Theories Group (Eds.), *Networking of theories as a research practice in mathematics education* (pp. 88–102). New York, NY: Springer.

Boesen, J., Lithner, J., & Palm, T. (2010). The relation between types of assessment tasks and the mathematical reasoning students use. *Educational Studies in Mathematics, 75*, 89–105.

Breen, S., O’Shea, A., & Pfeiffer, K. (2013). The use of unfamiliar tasks in first year calculus courses to aid the transition from school to university mathematics. In B. Ubuz, C. Haser, & M. A. Mariotti (Eds.), *Proceedings of the 8th Congress of the European society for Research of Mathematics Education* (pp. 2316–2325). Antalya, TR: ERME.

Chamberlin, S. A. (2010). A review of instruments created to assess affect in mathematics. *Journal of Mathematics Education, 3*(1), 167–182.

Clarke, D., & Ziebel, N. (2017). Aligning curriculum, instruction and assessment. In T. Bentley & G. Savage (Eds.), *Educating Australia: Challenges for the decade ahead* (pp. 257–276). Melbourne, Australia: Melbourne University Press.

Condon, W. (1986). Communication: Rhythm and structure. In J. Evans & M. Clyne (Eds.), *Rhythm in psychological, linguistic and musical processes* (pp. 55–78). Springfield, IL: Charles C. Thomas.

Davis, B. (1996). *Teaching mathematics: Toward a sound alternative*. New York & London: Garland Publishing, INC.

Dewey, J. (1933). *How we think: A restatement of the relation of reflective thinking to the educational process*. Lexington, MA: Heath.

Ernest, P. (1998). *Social constructivism as a philosophy of mathematics*. New York, NY: State University of New York Press.

Festinger, L. (1957). *A theory of cognitive dissonance*. Stanford, CA: Stanford University Press.

Gallese, V., Eagle, M. E., & Migone, P. (2007). Intentional attunement: Mirror neurons and the neural underpinnings of interpersonal relations. *Journal of the American Psychoanalytic Association, 55*, 131–176.

Goldman, S. L. (2006). Science wars: What scientists know and how they know it. Chantilly, VA: The Teaching Company.

Goos, M. (2004). Learning mathematics in a classroom community of inquiry. *Journal for Research in Mathematics Education, 35*(4), 258–291.

Hannula, M. S. (2011). The structure and dynamics of affect in mathematical thinking and learning. In M. Pytlak, T. Rowland, & E. Swoboda (Eds.), *Proceedings of the Seventh Congress of the European society for Research of Mathematics Education* (pp. 34–60). Rzeszow, PL: ERME.

Holler, J., & Wilkin, K. (2011). Co-speech gesture mimicry in the process of collaborative referring during face-to-face dialogue. *Journal of Nonverbal Behavior, 35*, 133–153.

Kimbara, I. (2008). Gesture form convergence in joint description. *Journal of Nonverbal Behavior, 32*(2), 123–131.

Kontorovich, I. & Zazkis, R. (in press). Evoking the feeling of uncertainty for enhancing conceptual knowledge. In C. Andrà, D. Brunetto, E. Levenson & P. Liljedahl (Eds.), *Teaching and learning in math classrooms — Emerging themes in affect-related research: Teachers’ beliefs, students’ engagement and social interaction* (pp. 175–184). New York, NY: Springer.

Kotsopoulos, D. (2010). When collaborative is not collaborative: Supporting student learning through self-surveillance. *International Journal of Educational Research, 49*, 129–140.

Lakoff, G., & Núñez, R. (2000). *Where mathematics comes from: How the embodied mind brings mathematics into being*. New York, NY: Basic books.

Lave, J. (1988). *Cognition in practice: Mind, mathematics and culture in everyday life*. Cambridge, UK: Cambridge University Press.

Leder, G., Pehkonen, E. & Toerner, G. (2002). Beliefs: A hidden variable in mathematics education? Dordrecht, NL: Kluwer Academic Publishers.

Liljedahl, P. (2012). Illumination: Cognitive or affective? In M. S. Hannula, P. Portaankorva-Koivisto, A. Laine, & L. Näveri (Eds.), *Proceedings of the 18th international conference on Mathematical Views* (pp. 187–199). Helsinki, FI: University of Helsinki.

Liljedahl, P. & Andrà, C. (2014). Students' Gazes: New Insights into Student Interactions. In C. Bernack-Schüler, R. Erens, T. Leuders, & A. Eichler (Eds.), *Views and beliefs in mathematics education* (pp. 213–226). New York, NY: Springer.
Morselli, F. & Sabena, C. (2015). "Let's play! Let's try with numbers!": Preservice teachers' affective pathways in problem solving. In K. Krainer & N. Vondrová (Eds.), *Proceedings of the 9th Conference of the Society for the European Research in Mathematics Education* (pp. 1231–1237). Prague, Czech: ERME.
National Council of Teachers in Mathematics [NCTM]. (2000). *Principles and standards for school mathematics*. Reston, VA: NCTM.
Nilsson, P., & Ryve, A. (2010). Focal event, contextualization and effective communication in the mathematics classroom. *Educational Studies in Mathematics, 74*, 241–258.
Piaget, J. (1985). *The equilibration of cognitive structures: The central problem of intellectual development*. Chicago, IL: University of Chicago Press.
Radford, L. (2011). Classroom interaction: Why is it good, really? *Educational Studies in Mathematics, 76*, 101–115.
Radford, L., & Demers, S. (2004). *Communication et apprentissage. Repères conceptuels et pratiques pour la salle de classe de mathématiques* [Communication and learning. Conceptual and practical benchmarks for the mathematics classroom]. Ottawa, Canada: Centre franco-ontarien des ressources pédagogiques.
Roth, W.-M. (2000). From gesture to scientific language. *Journal of Pragmatics, 32*, 1683–1714.
Roth, W.-M., & Radford, L. (2011). *A cultural historical perspective on teaching and learning*. Rotterdam, NL: Sense Publishers.
Sfard, A. (2001). There is more to discourse than meets the ears: Looking at thinking as communicating to learn more about mathematical learning. *Educational Studies in Mathematics, 46*(1), 13–57.
Sfard, A. (2008). *Thinking as communicating: Human development, the growth of discourses, and mathematizing*. New York, NY: Cambridge University Press.
Sfard, A. (2009). What's all the fuss about gestures? *Educational Studies in Mathematics, 70*, 191–200.
Sfard, A., & Kieran, C. (2001). Cognition as communication: Rethinking learning-by-talking through multi-faceted analysis of students' mathematical interactions. *Mind, Culture and Activity, 8*(1), 42–76.
Sfard, A., Nesher, P., Streefland, L., Cobb, P., & Mason, J. (1998). Learning mathematics through conversation: Is it as good as they say? *For the Learning of Mathematics, 18*(1), 41–51.
Unione Matematica Italiana — Commissione Italiana per l'Insegnamento della Matematica [UMI-CIIM] (2001). Matematica 2001. Retrieved on http://www.umi-ciim.it/materiali-umi-ciim.
Unione Matematica Italiana — Commissione Italiana per l'Insegnamento della Matematica [UMI-CIIM] (2003). Matematica 2003. Retrieved on http://www.umi-ciim.it/materiali-umi-ciim.
Vertegaal, R., Van der Veer, G. C., & Vons, H. (2000). Effects of gaze on multiparty mediated communication. In D. Mould & S. Noel (Eds.), *Proceedings of Graphic Interface 2000* (pp. 95–102). Montreal, CA: Morgan Kaufmann Publishers.
Williams, S. R., & Baxter, J. A. (1996). Dilemmas of discourse-oriented teaching in one middle school mathematics classroom. *The Elementary School Journal, 97*, 21–38.
Yin, R. K. (1994). *Case study research: Design and methods*. Thousand Oaks, CA: Sage Publications.
Zaslavsky, O. (2005). Seizing the opportunity to create uncertainty in learning mathematics. *Educational Studies in Mathematics, 60*(3), 297–232.