Int J of Sci and Math Educ (2019) 17:523–544
https://doi.org/10.1007/s10763-017-9876-5

Students' Application of Concavity and Inflection Points to Real-World Contexts

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Received: 13 June 2017 / Accepted: 23 December 2017 / Published online: 23 January 2018

Abstract The calculus concepts of concavity and inflection points are critical for a complete understanding of quantities' behavior, making them important topics of research for those interested in the intersections of STEM disciplines. This study seeks to provide insight into this area by reporting on trends in students' concept projections of these concepts in a range of real-world contexts, including temperature, economics, human height, and the universe's expansion. While the students mostly thought of concavity and inflection points in pure mathematics as shapes associated with graphs, the projections with the real-world contexts were based much more on changing rates of change. Pedagogically important student confusions were also evident in their attempts at applying concavity and inflection points to the real-world contexts, and this paper uses the concept projection perspective to identify possible sources of these confusions.

Keywords Calculus · Concavity · Inflection points · Real-world contexts

Introduction: Calculus Concepts in Real-World Contexts

In recent years, there has been much more attention given to students' application of mathematics to real-world contexts (National Council of Teachers of Mathematics [NCTM], 2014; National Governors Association Center for Best Practices [NGACBP] & Council of Chief State School Officers [CCSSO], 2010). In this paper, I use "real-world" simply to refer to any context pertaining to the physical world we live in, such as physics, chemistry, biology, economics, geography, or cosmology. Calculus education research in particular has begun to more deeply investigate the connections

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between calculus concepts and real-world contexts, including the topics of function (e.g. Carlson, Jacobs, Coe, Larsen, & Hsu, 2002), the derivative (e.g. Jones, 2017; Marrongelle, 2004; Roorda, Vos, & Goedhart, 2010), and the integral (e.g. Blomhøj & Kjeldsen, 2007; Hu & Rebello, 2013; Jones, 2015). Yet, it has only been recently that there has been an implicit recognition of how important the concavity and inflection point concepts in and of themselves might be for a complete understanding of real-world quantities (Carlson et al., 2002). In analyzing real-world phenomena, it is important for students to be able to make sense of contexts like decreasing rates of inflation (NCTM, 2000) or accumulated interest in a savings account (NGACBP & CCSSO, 2010), both of which are closely related to the concept of concavity. In addition, concavity and inflection points are used to relate scientific information such as the expansion of our universe (e.g. Enqvist, Mazumdar, & Stephens, 2010). Thus, to help students be successful in applying these concepts beyond pure mathematics, educators must better understand how students might think about them in real-world contexts.

While no study has explicitly examined students' interpretations of concavity and inflection point in a range of real-world contexts, the important work of Carlson and colleagues (Carlson, 1998; Carlson et al., 2002; Oehrtman, Carlson, & Thompson, 2008) has examined how students reasoned about changing rates of change in real-world contexts. Changing rates of change can be considered to be at the heart of the concavity and inflection point concepts, and these studies have established how crucial this type of reasoning is for a complete understanding of quantities' behaviors. Without it, students may struggle to interpret whether a quantity would be increasing at an increasing rate, increasing at a decreasing rate, or decreasing at an increasing/decreasing rate (Carlson et al., 2002). However, the studies by Carlson and colleagues were focused on the concept of function, and the concepts of concavity and inflection points were fairly implicit to their studies. Consequently, there has been no investigation into how students actually apply the mathematical concepts "concavity" and "inflection points," in and of themselves, to real-world contexts. To explicitly take up this approach, this study was designed to present students with the ideas of "concavity" and "inflection point" in connection with various real-world contexts and to examine how students made sense of these concepts in the given contexts. In summary, the research questions for this study are as follows: (1) What trends exist in the nature of students' concept projections (defined subsequently) of concavity and inflection points in various real-world contexts? (2) What common confusions might students encounter in applying concavity and inflection points to real-world contexts?

Research on Student Understanding of Concavity and Inflection Points

I first briefly outline what is known about student understanding of the concavity and inflection point concepts from the quite small set of studies that have *explicitly* examined them (Baker, Cooley, & Trigueros, 2000; Gómez & Carulla, 2001; Tsamir & Ovodenko, 2013). I note that this research has mostly been focused on pure-mathematics contexts, like graphs, rather than on real-world contexts. In a study focused on inflection points, Tsamir and Ovodenko (2013) found that many students may believe in incorrect properties for inflection points, such as believing that the first derivative *must* be zero in order for an inflection point to occur, or that the first derivative must *not* be zero in order for an inflection point to occur. Also, some students seemed to believe that inflection points would be where a graph was about to change from shallow slopes to steeper slopes. In another study, focused on graphing, Baker et al. (2000) found difficulties students had in coordinating information from the first derivative (monotonicity) and second derivative (concavity) in creating graphs. In fact, some students treated concave up and concave down at times as though they meant the graph should be increasing or decreasing (pp. 567–568). The study reported that the second derivative, which provides information about concavity, was actually an obstacle for many students in creating an accurate graph. Lastly, Gómez and Carulla (2001) added the brief result that inflection points sometimes serve for students as a "center-point" for graphs, such as the graphs of cubic functions. Such interpretations mean that inflection points may have additional visual meanings for students beyond that given by their typical mathematical definitions. These studies show that students have difficulties even in pure-mathematics contexts with concavity and inflection points, making it likely that students would also struggle to apply these concepts to real-world contexts.

Concept Projection

This study uses the construct of *concept projection* (diSessa, 2004; diSessa & Wagner, 2005). This construct is built on the premise that certain types of "concepts," including concepts like concavity, do not consist of a single cognitive entity, but are rather made up of a collection of individual knowledge elements. Thus, when a person applies or uses a concept in a given situation, they may not activate *all* knowledge pertaining to that concept, but rather may activate a *subset* of that knowledge. diSessa (2004) called "the functioning of the concept in one context" the *concept projection* for that concept in that context (p. 139). Concept projections for the same concept in two different contexts may make use of different subsets of knowledge, even though they are both projections of the same "concept" (diSessa, 2004). For example, it is possible to interpret the derivative, dy/dx, in one context as the slope of a tangent line and in another context as the rate of change of y as x changes (Bingolbali, Monaghan, & Roper, 2007; Zandieh, 2000). While there are certainly *connections* between slope and rate of change, they nevertheless consist of separate subsets of knowledge pertaining to the derivative. Similarly, it is possible to interpret the integral, $\int_a^b f(x)dx$, in one context as an anti-derivative, in another context as the area under a curve, and in yet another as the accumulation of little bits of a quantity (Jones, 2015). In this same way, it is entirely possible that concavity and inflection points may also be applied to different contexts by using distinct subsets of knowledge.

Next, the concept projection perspective interprets student difficulties in an important way. Rather than necessarily being the product of an inherently faulty "concept," a difficulty often stems simply from making an incorrect inference between otherwise correct knowledge elements. For example, suppose a student is considering the function, $x(t)$, that gives the position of a car traveling along a road at a given time. Suppose the student is confused about how the function can have "positive" derivative values (i.e. velocity), and claims that because the car is traveling *horizontally* the derivative should be zero. Here, the student may simply be making an inference between "derivative" and "graphical slope." Graphical slope is not an *incorrect* way to interpret the derivative, and it would not be desirable to remove that knowledge element. Rather, the issue would be in helping the student see that, in this specific context, an inference between “derivative” and “rate of change” may be more helpful for understanding the derivative value.

Brief Conceptual Analysis of the Concavity and Inflection Point Concepts in Mathematics

Having described concept projections, it is useful at this point to articulate what mathematical knowledge elements might be within the concavity and inflection point concepts, in order to have a baseline of what might be used in a concept projection of them for a given context. Concavity is often discussed, both in calculus textbooks (e.g. Hughes-Hallett et al., 2012; Stewart, 2015) and in educational research (e.g. Baker et al., 2000; Carlson et al., 2002; Tsamir & Ovodenko, 2013), as (a) a graphical shape, (b) the sign of the second derivative, (c) increasing or decreasing first derivative values or slopes, or (d) changing rates of change (see Fig. 1). First, as a graphical shape, concavity is often thought of in terms of the *iconic appearance* of a particular section of a graph. This includes the familiar U-shape for concave up and the ∩-shape for concave down, as in Fig. 1a. Of course, concave up or down may only contain a portion of one of these shapes, such as only the left or right half of the U or ∩. Second, as the sign of the second derivative, concavity is seen as an attribute of the second derivative. Concave up is defined to be where the second derivative is positive, $f''(x) > 0$, and concave down is defined to be where the second derivative is negative, $f''(x) < 0$, which may be identified either symbolically or graphically (Fig. 1b).

Third, as changing derivative values or slopes, concave up is defined as increasing f' and concave down is defined as decreasing f', which correspond to whether the slopes

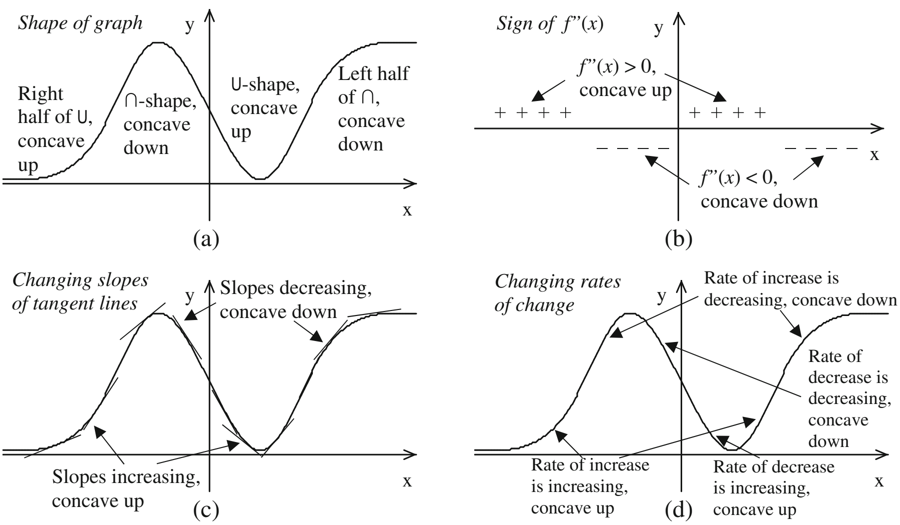

Fig. 1 Concavity interpreted as **a** the shape of a graph, **b** the sign of second derivative, **c** changing values of f' or slopes, and **d** changing rates of change

of tangent lines of a graph are increasing or decreasing (Fig. 1c). That is, if the tangent lines, moving from left to right, have increasing slopes, it represents concave up, and if they have decreasing slopes, it represents concave down. Note that for this conception, I have combined *derivative values* with *slopes*, even though some (including myself) advocate that the derivative's meaning be rooted in concepts other than only "slope." I have combined them simply because in the current state of affairs, these are essentially synonymous in typical calculus classrooms and textbooks (e.g. see Hughes-Hallett et al., 2012; Stewart, 2015). Thus, for the sake of examining students from typical classrooms, this classification lent itself best to the analysis.

Fourth, as changing rates of change, concave up deals with increasing rates of increase and increasing rates of decrease, and concave down deals with decreasing rates of increase and decreasing rates of decrease (Fig. 1d). It is important to note that when I speak of, for example, "increasing rates of increase," I mean *increasing* to be an adjective to *rates of increase*. In other words, the phrase should be read that the object "rate of increase" is itself increasing in value. Also, I clarify that throughout this paper, the words "increasing" and "decreasing" are defined in terms of left-to-right movement along the number line, where "decreasing" negatives mean that the negative numbers are becoming *larger* in absolute value. Thus, "decreasing rates of decrease" means that the rate is negative and is becoming larger in magnitude.

While I combined derivative values and slopes for the previous conception, I have separated *changing rates of change* as a distinct conceptualization. This is because research has shown that, even for typical students, "slope" and "rate of change" are conceptually different in important ways (see Bingolbali et al., 2007; Stump, 2001). Similarly, textbooks sometimes treat "rate of change" as a concept separate from derivative values as slopes (e.g. Stewart, 2015).

Each of these four conceptual orientations toward concavity have corresponding orientations for inflection points. When concavity is viewed as the shape of a graph, the inflection point is the location of a *switch* in the shape type. For the sign of the second derivative, an inflection point happens at a point in the domain where a *switch* occurs in sign of the second derivative from negative to positive, or positive to negative. When concavity is viewed as increasing or decreasing f' or slopes, an inflection point is a location for which the slopes *switch* from increasing to decreasing or vice versa. Finally, for concavity as changing rates of change, an inflection point is as a *switch* from increasing rates of change to decreasing rates of change, or vice versa. I highlight that in all four cases, inflection points essentially denote a "*switch*" from one type of concavity to the other; however, concavity is defined.

Methods

Interview Protocol and Interview Participants

For this study, a semi-structured interview protocol was used that asked the students to consider the mathematical concepts of concavity and inflection points in real-world contexts (see Table 1). In addition, the interview also contained a "pure mathematics" prompt, which was used to gather background information about how the students interpreted concavity and inflection points within pure-mathematics contexts. The pure-mathematics prompt used was similar to the graph-sketching prompt from Baker et al. (2000) and to graph-sketching exercises commonly seen in calculus textbooks, though adapted to focus primarily on concavity and inflection points. Note that the point of this graphing prompt was *not* to assess their graphing skills, as done in Baker et al. (2000), but only to provide background information on which conceptualizations of concavity and inflection points the students used within pure mathematics.

The main part of the interview protocol consisted of four prompts (see Table 1), which were designed to draw from a range of real-world contexts, and not only from one specific area. These contexts included temperature (physics/engineering), housing prices (economics), human height (biology), and the expansion of the universe (cosmology). In each context, the students were asked to describe how concavity or inflection points related, in order to elicit a concept projection into that context. While the first two prompts began with questions about inflection points and the second two began with questions about concavity, both concepts were brought up throughout the interview. Follow-up questions were used to explore student thinking, to elaborate on student statements, and to clarify student responses. If a student desired clarification on a context (such as what a "recession" was), that clarification was given by the interviewer.

The participants for this study consisted of eight students randomly selected from a pool of willing participants from two different first-semester calculus classes (about 200–250 students in each class). The instructors for both classes had reputations as being good instructors, both by students and colleagues, though both taught in what were essentially typical, "traditional" formats and styles. The students were recruited at the end of their calculus courses in order to ensure that they had experienced classroom learning regarding concavity and inflection points. There were three female and five male students in the sample, and in this paper, they are given the pseudonyms: Camille, Doug, Gavin, Jose, Kylie, Mary, Nathan, and Ryan.

Table 1 Interview prompts

Background pure math prompt	Suppose the following is true for a function $f(x)$. Sketch the graph of $f(x)$. $f(x)$ is increasing to the left of $x = 0$, $f(x)$ decreasing to the right of $x = 0$, $f''(x) > 0$ to the left of $x = -1$, $f''(x) < 0$ in between $x = -1$ and $x = 1$, $f''(x) > 0$ to the right of $x = 1$
Real-world prompt #1	A meteorologist is studying the weather at [name of city] and is keeping track of the temperature throughout the day. The meteorologist reports an "inflection point" in the temperature at 12:00 pm (noon). What might that mean?
Real-world prompt #2	You might be familiar with the "economic recession" of the late 2000's. In particular, housing prices began to decline in 2007. An economist claims that there was an "inflection point" in housing prices in 2009. What might that mean?
Real-world prompt #3	Suppose I recorded the height of a person during their *entire* life from birth to adulthood. Describe the *concavity* of the person's height over time.
Real-world prompt #4	Astronomers believe the universe is expanding. Let $S(t)$ denote the overall "size" of the universe over time. Data has led astronomers to conclude that during the earlier period of expansion, $S''(t) < 0$, but that for the current period of expansion $S''(t) > 0$. Describe the way in which the universe has been expanding over time.

Analysis of the Background Pure-Mathematics Data

As a precursor to the main analysis, I first examined how each student interpreted concavity and inflection points in the pure-mathematics prompt. For this, I identified which of the four types of concavity or inflection point conceptualizations—graph shape, sign of f'', increasing/decreasing f' or slope, changing rates of change—each student seemed to use in describing either concavity or inflection point. In addition to using the pure-mathematics prompt for the background data, I make here an important note about how certain other portions of the interview were included with the "pure math" prompt. Several students, during an applied interview prompt, would leave the real-world context and revert to the pure-mathematics context in order to express ideas regarding concavity or inflection points. The interviewer allowed the students the space to do so, letting them discuss the pure-mathematics context before returning to the real-world context. In these cases, the students were no longer mapping concavity and inflection points into the real-world context, and their statements had much more to do with describing purely mathematical meanings of the concepts. When this occurred, these portions of their responses were grouped with the pure-mathematics prompt as part of the "pure-mathematics" data, even though they originated as deviations from the real-world contexts.

Analysis of Concept Projections

The main analysis consisted of examining the concept projections of concavity and inflection points in each of the real-world contexts, based on what the students said, wrote, and gestured. Concept projections were identified in terms of the knowledge elements pertaining to concavity or inflection points evidenced as being connected to the given real-world context, and how that knowledge was used to describe or explain the context. The descriptions of concavity and inflection points given earlier in this paper formed the basis for the knowledge of these concepts that I looked for during analysis. For each real-world interview prompt, any instance was marked in which a student appeared to make a connection between concavity or inflection points and the real-world context in the prompt. For example, suppose a student said, "Concave up means that the person is growing by larger and larger amounts." This instance would be marked as part of the concept projection of concavity in the height context, and would be coded as having used the changing rates of change conception. Further, it would be recorded that changing rates of change were used to describe that the height quantity was increasing by larger and larger increments. Of course, students were not always so direct as to explicitly say, "Concave up means…," but places were only marked in the data where it was clear that the *students themselves* were making a connection between concavity or inflection point and the context. Note that students often made more than one connection between concavity or inflection points and a given real-world context. These connections were all considered part of the concept projection in that context, even if a student later changed their mind or later said something that contradicted an earlier statement.

Trends were then observed for the students' concept projections across all of the real-world contexts. Of course, the context-sensitivity of the concept projection lens means that each real-world context could be reported on individually and separately from the others. However, this exploratory study is intended to focus on the broad contexts of "real-world" versus "pure-mathematics," and how concept projections may differ between them in general. This limitation means that the next step would be, in fact, to do the work of shifting the contextual focus from "real-world" generally to examining each of the specific real-world contexts individually.

Lastly, during the documentation of concept projections, certain confusions were observed that were common to several students as they discussed concavity and inflection point concepts in the different contexts. I kept track of these confusions, with three main ones having been recorded by the end of the main analysis phase. Subsequently, I took these three confusions and passed through the entire interview data again to record any instance of any of them. I then grouped together all instances of similar confusions across the students. Based on the concept projection interpretation of student difficulties, I also attempted to deduce a plausible underlying student inference that may have led to each type of confusion.

Supplemental Survey Data to Verify Confusions

When I observed the common confusions in the interview data, a possible explanation was that I simply had a particularly weak group of calculus students for my interviews. In order to determine if the three common confusions were unique to these students, or if they were relevant to the larger calculus student population, I collected additional survey data to see if the confusions would be observed in a somewhat larger sample. The surveys were given to students in two smaller calculus classes (~ 30 students each), different from those the interview students were recruited from. I created four versions of the survey, each with only one of the real-world prompts from the interview. The four versions were alternately handed out in an effort to receive roughly equal numbers of responses to each prompt. Of course, as students could choose to participate or not, the numbers of responses per prompt were not equal. There were 34 surveys included in the data set, with 13 responses for the temperature prompt, six responses for the economics prompt, eight for the height prompt, and seven for the universe prompt. Since the purpose of the survey was only to test whether the confusions I had observed in the interviews (described in the results section) were observed in a somewhat larger sample, the surveys were only analyzed by whether each survey contained evidence for one of these confusions.

Background Results: Concavity and Inflection Points in Pure Mathematics

In this section, I describe the background results from the pure-mathematics portions of the interviews. Of the four conceptualizations described previously, the students relied most heavily on the "shape of a graph" interpretation in the pure-mathematics context (see Table 2). All eight students regularly described concave up in pure mathematics as some kind of picture of a "U" shape and concave down as some kind of picture of a "∩" shape (for more on graphs as "pictures," see Beichner, 1996; Elby, 2000; McDermott, Rosenquist, & van Zee, 1987). Furthermore, each student seemed to have their own particular type of shapes that they used to identify or describe concavity. Camille, Doug, and Ryan each thought of concave up/down as represented by upward

Table 2 Conceptions of concavity or inflection points used in the pure-mathematics context, where x means used by student, (x) means only rarely used by student, and – means not used by student

	Camille	Doug	Gavin	Jose	Kylie	Mary	Nathan	Ryan
Graph shape	x	x	x	x	x	x	x	x
Sign of f''	x	(x)	x	x	x	x	x	(x)
Inc/dec f' or slope	(x)	(x)	x	x	x	–	–	x
Changing rates	–	–	–	(x)	–	(x)	–	–

and downward facing "bowls" (cf. Baker et al., 2000). Kylie and Mary both used the visual cues of smiles and frowns; Nathan described them as hills and valleys; Gavin used the images of a lowercase "u" or "n;" and Jose thought of portions of the graph as "upward facing" or "downward facing." I name these visual cues for concavity *image pairs*, whether upward/downward bowls, smiles/frowns, or hills/valleys, since they always consisted of two complementary images for concave up and concave down, as in the following two excerpts.

Ryan: It's like a bowl of soup, whether it keeps the soup in. So like, [concave up] you have, like, soup here, it's being held in, and [concave down], all the soup falls out.

Gavin: I know that kind of makes a u shape, and then this one does the same thing. I just kind of look for the u shape, and the n shape.

In tandem with their shape-based interpretations of concavity, all eight students interpreted inflection point in terms of *shape switching* from one element of the image pair to the other. For example, Mary described finding an inflection point as "looking for when it was from a smiley face to a frowny face." Overall, the shape of graph conception was by far the most frequently used by all eight students for both concavity and inflection point.

For the next conception, all eight students identified the relationship between the sign of the second derivative and concavity (see Table 2). Yet, the students did not seem to rely on this conceptualization as much in their thinking, and most of the statements related to this conception were fairly brief, such as Jose stating, "Concavity usually reflects the second derivative." While they frequently mentioned information regarding the second derivative, they mostly seemed to use it to quickly infer other conceptions, such as the graph-shape conception of concavity. Sometimes this happened so quickly it only consisted of a single utterance, such as Mary stating, "Second derivative is positive, so smiley face." For inflection points, all eight students interpreted them as when the sign of the second derivative *switched* from positive to negative or vice versa. Some students overgeneralized this to incorrectly assert that inflection points occurred *whenever* $f''(x) = 0$ (cf. Tsamir & Ovodenko, 2013).

For the third conceptualization, six of the students (all except Mary and Nathan) at one point or another invoked the increasing or decreasing slope conception (see

Table 2). However, the changing slope conception was also much less frequently observed than the graph shape conception. When using this conception for concavity, the students made statements such as “The slope… is going to be a lot steeper, or a lot higher” (Doug) for concave up, or “The slope would be, it would still be positive, but it would be decreasing” (Gavin) for concave down.

Only Gavin and Kylie described inflection points using this conception, by explaining what would be happening to the slopes before and after the inflection points, as in Kylie’s excerpt.

> *Kylie*: It starts off with less of a slope, then it gets more of a slope [i.e. steeper slope], and there’s the inflection point, and then it has more a slope to less of a slope.

Finally, for the last conceptualization, only Jose and Mary made any statements at all during the pure-mathematics portions of the interviews that reflected an interpretation of concavity or inflection points as changing rates of change (see Table 2). They each only made brief statements according to this conception, and their statements only pertained to attempts at explaining inflection points. Jose explained that for an inflection point, “It’s basically the point where it changes the rate that it’s changing at.” Mary stated that for x^3, the inflection point would be “where the rate is the least, it’s where the rate is changing from being sharp [i.e. large] on either, on both ends.” Thus, the changing rate conception was very *infrequently* used by this group of students in the pure-mathematics contexts.

Results: Concavity and Inflection Points in Real-World Contexts

I now move to the main focus of this paper and describe the students’ concept projections in the real-world contexts. I first describe projections involving changing rates of change, then increasing or decreasing f' values or slopes, then the sign of the second derivative, and, finally, the shape of the graph. This reverses the order from the previous sections, because, interestingly, it turned out that changing rates of change were used much more frequently in the concept projections, and the sign of the second derivative and graph shape were used much less. Next, there were common confusions seen across the students when projecting each conception to the real-world contexts. Thus, as I describe how each conception was used in the projections, I also include a subsection describing the corresponding confusion associated with that conception. In these subsections, I include a brief discussion on plausible inferences in the concept projection that may have led to them. I end with supporting evidence from the surveys that these confusions were not unique to the interviewed students, but may be present in the larger calculus student population.

Concept Projections Involving Changing Rates of Change

The changing rates of change conception were the most commonly used in the projections of concavity and inflection points to the real-world contexts. All eight students gave evidence of concept projections based on changing rates of change. Note that this is in stark contrast to how little this particular conceptualization was used by the students in the pure-mathematics context. The student explanations based on this conception involved describing a changing amount of the quantity defined by the prompt, where the change itself was also changing, as in the following representative examples.

> *Gavin* [concavity + universe]: [Concave down] would mean that the rate at which the universe was expanding was decreasing. And then [concave up], the rate which it is expanding is increasing… The rate at which it's expanding is increasing sometimes and decreasing sometimes.

> *Kylie* [inflection point + temperature]: I would summarize [an inflection point] something like, I would say, so, the temperature change went from changing gradually, to changing hastily, to changing hastily, to gradually… It's going from less change to more change, changing to more change to less change.

> *Jose* [inflection point + height]: From here to here [early in life] it's less increase of height, whereas from here to there [closer to teenage years]… it's a greater increase of height… And whatever point that they start to not grow as quickly… it'll still be increasing, it'll be increasing at a decreasing rate, which means that, that would have to be an inflection point for that rate at which their height is changing.

In these examples, "rate" has been connected to variations in the quantities' amounts over time, and "changing rates" has been connected to *changes* in these variations, such as smaller or larger changes in the quantities' amounts at different points in time.

In this concept projection, the students were at times able to draw on knowledge of rates of change to successfully identify the behavior of the quantity. That is, they were able to speak about more than the amount or size of the quantity, but also how that quantity changes or evolves over time. The fact that all of the students used changing rates in their concept projections, and some of the students extensively, suggests that the students found rates of change knowledge useful in projecting concavity and inflection points to these contexts.

Confusion with Changing Rates of Change: Concavity as Fast or Slow Change Yet, there was a common confusion exhibited by several students when using changing rates of change. This confusion essentially cast concave up as meaning a *fast* rate of change and concave down as meaning a *slow* rate of change. The students used the word "fast" to mean that the dependent variable changed by a lot as the independent variable increased, and the word "slow" to mean that the dependent variable changed only by a little as the independent variable increased. Of course, thinking of concave up or down in this way does not have to be true, since rates of change could be increasing, and hence concave up, while still being "slow" (think $y = e^x$ over $(-\infty,0]$). Or, rates of change could be decreasing, and hence concave down, while still being "fast" (think $y = \ln(x)$ over $(0,1]$). Four of the students gave evidence of this confusion at least once (Kylie, Mary, Nathan, and Ryan), as seen in the following representative excerpts.

Ryan [concavity + height]: [Concave up] is growing faster, as opposed to once they level off… and then I guess switches back to concave down once they hit adulthood because they're not growing anymore.

Nathan [concavity + universe]: [Concave down is] the slowest of its expansion… Well, I would say that [concave up] is when it's probably fastest and then [concave down] is where it's probably the slowest.

To describe what might account for this confusion, I briefly discuss a plausible inference in the concept projection that may have led to it. The students' classes often used kinematics examples, which associates concavity with acceleration. Increasing rates of change in this case can at times be associated with *speeding up*, and decreasing rates of change with *slowing down*. However, *speeding up* then may have been linked to *fast*, likely from the common experience that speeding up results in going fast. On the other hand, *slowing down* may have been linked to *slow*, from the common experience that slowing down results in going slow. Thus, as advocated by the concept projection perspective, this confusion may not be the result of a deep misconception about concavity, but rather the use of an inference that is true in some cases but not others. In fact, I note that some of these same students did, at other times, accurately describe concave up as increasing rates of change and concave down as decreasing rates of change, without making assumptions as to whether the rates were "fast" or "slow."

Concept Projections Involving Increasing/Decreasing f' Values or Slopes

The next most common types of concept projection were those involving increasing/decreasing f' values or slopes. This conception was also used by all eight students. In these projections, the students connected the idea of slope directly with changes in the quantities. The students often began with graphical images meant to portray the real-world context, but rather than focusing on the *graph shape*, they instead focused on the nature of the slopes. This usually took the form of describing changes in the quantity through the notion of "*steepness*."

Kylie [concavity + temperature]: So the temperature was rising… but it's going to level off and stop rising so steeply.

Mary [inflection point + price]: So, at the inflection point, 2009… it would be the steepest jump.

Doug [concavity + height]: [After identifying concave up as a possible growth spurt], I guess, like, those growth spurts just represent, like, the slope becoming greater… so the slope, at least on the graph, is going to be a lot steeper.

I note that in many instances, the students used both the changing rates of change conception and the increasing or decreasing slope conception within the same concept projections for a single context. This shows the possible use of multiple knowledge elements in their concept projections for concavity and inflection points to the real-world contexts.

Confusion with Increasing/Decreasing f' Values or Slope: Inflection Points as Extrema There similarly was a common and recurrent confusion related to concept projections of this type, which seemed rooted in difficulty keeping track of exactly what was *changing* in concavity and, consequently, what *switched* at an inflection point. To preface it, recall that in this conception of concavity and inflection points, it is not the quantity itself that is increasing or decreasing, but rather the *derivative values* that are increasing or decreasing. An inflection point refers to a *switch* from increasing f' to decreasing f', or vice versa. Many of the students in this study instead described inflection points as where the *quantity itself* (i.e. temperature, price, height, or size) switched between increasing and decreasing, turning inflection points into maximums or minimums. Five of the eight interviewed students (Camille, Doug, Gavin, Nathan, and Ryan) made these types of statements, as seen in the following representative examples.

> *Ryan* [inflection point + temperature]: That'll be where it, um, it changes from, like, getting hot to getting cold. So if you're looking at, if you're looking at, like, the temperature, and it's getting hotter, hotter, hotter, it's not getting as hot anymore, and then it'll start getting colder, this would be the inflection point.

> *Doug* [inflection point + temperature]: And, I guess what it's saying, an inflection point in the temperature maybe just means that it's just changing from going up, up [gestures upward] to down [gestures downward], or maybe it's going from down to up [gestures downward then upward]. Maybe it just means that, um, the temperature has reached, you know, a certain low point or a certain high point before changing direction.

> *Camille* [inflection point + price]: Maybe it just keeps going down until 2009, it can't go down anymore, then that [i.e. the inflection point], so it starts to go back up.

As before, to discuss what may account for this confusion, I briefly describe a possible inference in the concept projection that may have caused it. In the students own words, they recognized inflection points as where a *switch* in something happens. For this conception, it should represent a switch from increasing derivative values to decreasing, or vice versa. However, research has shown that it can be very difficult for students to reason about *changing* derivative values (Baker et al., 2000; Carlson et al., 2002). Thus, in attempting to make an inference between an inflection point and a *switch* in something, the students appeared to have incorrectly inferred a switch in when the *quantity* itself was increasing or decreasing. Again, this confusion may not necessarily be the result of a general misconception of inflection points, but rather an in-the-moment incorrect inference. Many of these same students at other points in the interviews described inflection points in ways that did *not* cast them as extrema. Also, no student made this kind of interpretation of inflection points in the pure-mathematics contexts, making it unlikely to be due to inherently incorrect ideas about inflection points from pure mathematics.

Concept Projections Involving the Sign of the Second Derivative

There was actually very little evidence of concept projections that used the sign of the second derivative in the real-world contexts, even in the universe prompt. The students seemed to think of the sign of S'' only in purely mathematical terms. In other words, the students took the information $S''(t) > 0$ and $S''(t) < 0$ and inferred a different conceptualization (changing rates, increasing/decreasing f', or graph shape) *before* making explicit connections to the real-world context. In fact, Gavin was the only student to provide any evidence of this type of projection, and even his was not directly done with the quantities defined in the interview prompts. Rather, Gavin seemed to have used an inference that the second derivative is equivalent to the real-world concept of *acceleration* (similar to Marrongelle's (2004) "contextualizers"). His concept projection then involved discussing acceleration in the real-world contexts, as in this example.

> *Gavin* [sign of 2nd derivative + acceleration + temperature]: So, it's concave up and concave down, that's keeping track of, it would be found in the second derivative. And so that would mean that acceleration, and second derivative's talking about acceleration… it would go from decreasing to increasing in temperature, or in accelerating in temperature, um, whether it be going down faster, or going up faster.

With so few instances of concept projections based on the second derivative, I saw no empirically based confusions corresponding to them. Of course, using acceleration to project concavity might bring in other inferences that could conflict with the contexts, but I saw no direct evidence of it. In fact, thinking of *acceleration* seemed to help organize Gavin's projection, though I suspect that he was using some sort of "generalized acceleration" that no longer dealt explicitly with, and only with, the physical movement of a static object.

Concept Projections Involving Graph Shapes

Lastly, I describe concept projections that used the graph shape conceptualization. In contrast to how dominant the graph shape conception was in the student thinking in the pure mathematics contexts, this conception was not used very frequently by the students to project concavity and inflection points to the real-world contexts. Only four students (Doug, Kylie, Mary, and Nathan) showed any evidence of this, and Mary only rarely did so. In this projection, the familiar U-shape for concavity was mapped to the real-world quantity to identify the behavior of the quantity. Sometimes, this U-shaped image mapped well to the real-world context, such as seeing the temperature graph as a cyclical switch between U's and ∩'s (see Fig. 2).

However, using graph shape was often problematic because these students tended to project the *entire* U-shape or ∩-shape to the real-world contexts. While this behavior is true for temperature models, it needs not be true for other real-world contexts. As such, I proceed directly to discuss the confusion that arose for Doug, Nathan, and Kylie in using the graph shape in their concept projections.

Confusion with Graph Shapes: Concavity Requiring Extrema In this confusion, the students seemed to think that the full U or ∩ shape was required for a quantity to be associated with concave up or concave down. The problematic inference with this full U-shape is then that concave up requires a minimum and concave down requires a

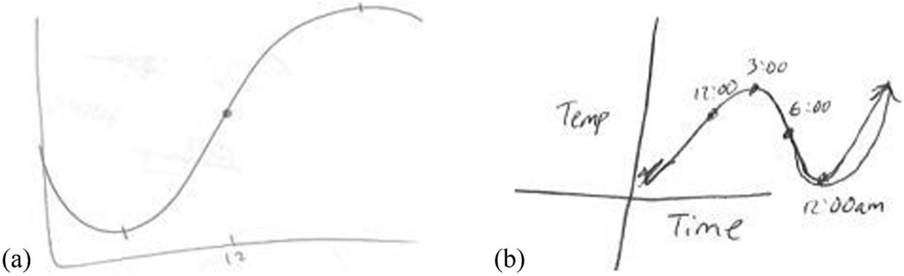

Fig. 2 Graphs relating concavity to temperature, from **a** Kylie and **b** Doug

maximum. To illustrate, first consider Nathan's initial description of concavity in relation to a person's height over time.

> *Nathan* [concavity + height]: You're steadily increasing [in height] over time, then you start to level off, and then possibly go down a little bit, so it would be concave down.

The interviewer then asked specifically about the younger part of a person's life, where there might be growth spurts, and the effect that that might have on concavity. Nathan's description suggested that concavity cannot exist without a minimum or a maximum present.

> *Nathan* [concavity + height]: You wouldn't necessarily change concavity… I'd say this [reaching adulthood and getting older] is definitely concave down. These ones [younger periods], I wouldn't be exactly sure about, I can't remember exactly… if there has to be the minimums or maximums. So definite, I guess, because there's not minimums or maximums here [during childhood or adolescence], I'd say there's not concavity there.

Doug also made statements that seemed to suggest the need for extrema, describing during the height prompt how people might grow and even *shrink* at different times in their lives.

> *Doug* [concavity + height]: I guess, in my mind, everybody either, like, increases over time and then they reach like a certain peak in their life, and then they go back down. I guess there are, like, certain people where maybe they, maybe they reach a growth spurt in high school, shrink a little bit in college, maybe a little bit, maybe grow back in their twenties or so. And I guess maybe that would change the concavity of it.

The interviewer then asked Doug whether concavity and inflection points could exist without necessarily having maximums and minimums associated with it.

> *Doug* [concavity + height]: It would need to be based on, like, a maximum or a minimum [height]. At least, just to make sense in my mind, so that you can know,

like, something is concave up or concave down, because those always have a maximum or a minimum.

To emphasize this point, Doug drew a graph (see Fig. 3) of human height and explicitly inferred concave up and concave down portions by locating the maximums and minimums. This can be seen by the marks and labels (CU and CD) Doug made directly at the local extrema.

Lastly, during the universe prompt, Kylie drew a graph meant to portray the universe's size over time (Fig. 4). She recognized that the universe was always increasing and drew her corresponding graph to also be increasing. However, she was at first bothered by the fact that the prompt required a change in concavity even though the universe's size was always increasing.

Kylie [inflection point + universe]: This states that it's concave up now, and this [earlier] part's concave down… wait… I'm wondering if this is really an inflection point… It's definitely not this defined change from concave down to concave up… I guess I'm not sure where the max and min would be on this, because there should be a max somewhere over here [left portion of the graph]… I've just never looked at concavity that's not as defined, but it's required to have a max and a min if there's going to be concavity.

Eventually, Kylie did decide that it was acceptable to not have extrema in order to have concavity. Yet, overall, it is clear that the students' concept projections using graph shapes sometimes involved invoking the entire "U" or "∩" shape, leading them to sometimes believe that there was a problem with concavity existing in the absence of extrema.

Summary of Results

Here, I summarize the overall results of this study in Tables 3 and 4. Table 3 shows which of the students used each of the (a) changing rates of change, (b) increasing/decreasing f' or slope, (c) sign of f'', and (d) graph shape conceptions in their concept projections of concavity and inflection points with the real-world contexts. Table 4 then

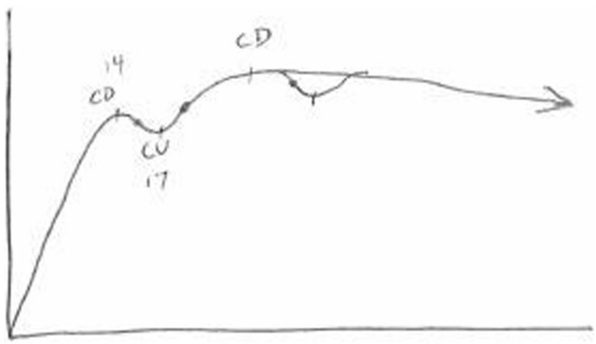

Fig. 3 Doug's usage of extrema to identify concave up (CU) and concave down (CD)

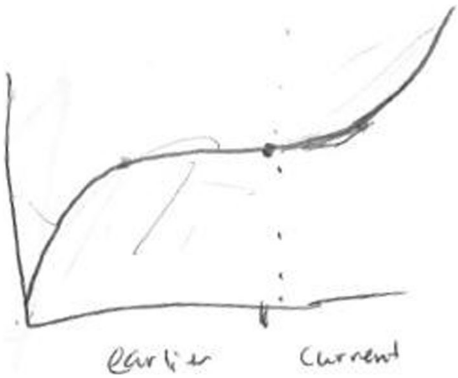

Fig. 4 Kylie's "not as defined" concavity due to lack of extrema

shows which students gave evidence during their interviews of each of the three main confusions described in this paper.

Survey Data in Support of the Three Confusions in the Larger Student Population

In this final subsection, I address the possibility that the three confusions described in the results were limited only to the small group of interviewed students in this study and do not have wider applicability to the calculus student population. The results from the surveys support that these three confusions are not limited to these students and may be shared more widely. Out of the 34 surveyed students, six (17.6%) made statements suggesting that concave up or concave down corresponded to fast or slow rates of change, 12 (35.3%) made statements suggesting a mix-up between inflection points and extrema, and six (17.6%) indicated in some way that concavity may require extrema. While these percentages do not represent majorities of students, I note that they also are not negligible in size, especially given the surveyed students' limited interaction with the survey. If all the surveyed students were interviewed more extensively, these same confusions might be more frequently observed. Here, I provide some examples from the surveys regarding each confusion. First, the following two excerpts are examples of survey responses that suggested that concave up and concave down represented fast or slow change:

> *Survey A* [height prompt]: The concavity will be positive during puberty because there is a level of growth there that is faster than the regular growth.

Table 3 Usage, by student, of each conception in concept projections, where x means used by student, (x) means only rarely used by student, and – means not used by student

	Camille	Doug	Gavin	Jose	Kylie	Mary	Nathan	Ryan
Changing rates	x	x	x	X	x	x	(x)	x
Inc/dec f' or slope	x	(x)	x	x	x	(x)	x	x
Sign of f''	–	–	x	–	–	–	–	–
Graph shape	–	x	–	–	x	(x)	x	–

Table 4 Confusions exhibited by each student at least once during their interview, indicated by an x

	Camille	Doug	Gavin	Jose	Kylie	Mary	Nathan	Ryan
Concavity as fast/slow change	–	–	–	–	x	x	x	x
Inflection points as extrema	x	x	x	–	–	–	x	x
Concavity requiring extrema	–	x	–	–	x	–	x	–

Survey B [economics prompt]: With the IP in 2009, the cost took a sharp declining slope (concave up) from a slower declining slope (concave down). [Note that the "concave" phrases in parenthesis are part of the student's response.]

Second, these next two excerpts are examples of survey responses that seemed to describe inflection points as either a maximum or a minimum:

Survey C [temperature prompt]: An inflection point is where the temperature changes from increasing in temperature to a decrease in temperature.
Survey D [economics prompt]: At the inflection point in 2009, the housing prices reached a minimum and started to increase again.

Finally, the following two are examples of responses that seemed to suggest the need for both increasing and decreasing parts for there to be concavity (i.e. requiring the full "U" shape).

Survey E [height prompt]: The concavity would be negative. The initial positive slope and the final very slight negative slope would make concavity negative.
Survey F [universe prompt]: $S''(t) < 0 \rightarrow$ concave down, $S''(t) > 0 \rightarrow$ concave up. [See accompanying graph in Fig. 5, which contradicts the always-increasing requirement.]

Of course, the brief statements written in these survey responses likely do not account for the entire concept projection these students might have for these contexts.

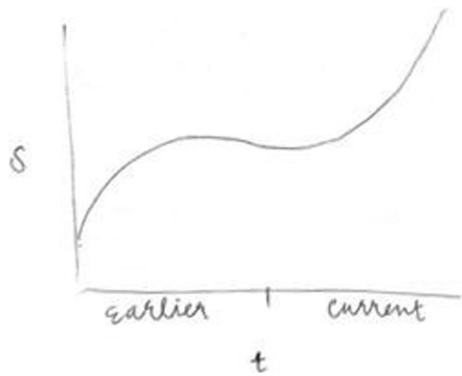

Fig. 5 Extrema present in concave up and concave down parts from Survey F

If asked for more detail, the students might correct the inferences that led to these confusions as the interviewed students sometimes did. Thus, in alignment with the concept projection perspective, I do not claim that these confusions indicate persistent misconceptions about concavity or inflection points. However, the fact that these confusions were evidenced by several students suggests that they are worthy of attention by calculus education instructors and researchers.

Discussion

This study shows an important contrast in how students tended to conceptualize concavity and inflection points in pure-mathematics contexts versus how they tended to project these concepts in real-world contexts. In the mathematics context, the students most frequently invoked the graph shape conception, in which graphs are essentially seen as image-shapes (similar to Beichner, 1994; Elby, 2000; McDermott et al., 1987). The students also rarely used the idea of changing rates of change in the mathematics context. On the other hand, when projecting these concepts in real-world contexts, the students most frequently drew on the changing rates and the increasing/decreasing f' conceptions, and used the graph shape conception much less. The frequent usage of the changing rates of change conception suggests that it was a useful part of their concavity and inflection point knowledge to project to the real-world contexts, allowing them to meaningfully infer about the quantities' behaviors (cf. Carlson et al., 2002). However, it is clear from the students' confusions that they also struggled to use appropriate aspects of covariational reasoning to make sense of concavity and inflection points (cf. Carlson et al., 2002). Several students mistakenly associated concave up with large rates of change and concave down with small rates of change. Many also confused concave up with an increasing quantity and concave down with a decreasing quantity (cf. Baker et al., 2000), leading them to incorrectly describe inflection points as local extrema.

In reflecting on the confusions reported on in this paper, I believe they are indicative of (a) difficulties with covariational reasoning in projecting the concavity and inflection points concepts and (b) issues with iconic shape-based thinking in projecting these concepts. To explain what I mean, consider the most common confusion, *inflection-points-as-extrema*, evidenced by five of the students. Recall that an "inflection point" is defined, both in typical curriculum (e.g. Stewart, 2015), and by each of the interviewed students in their own words, as a *switch* from concave up to down, or vice versa. In the real-world contexts, the students seemed to be attempting to identify some sort of phenomenon that would fit the required *switch* for an inflection point. But thinking in terms of *changing rates of change* is a difficult cognitive activity for students, corresponding to higher levels of covariational reasoning (Carlson et al., 2002). Thus, at times, the students seemed to slip into an *easier* type of a "switch": a switch directly in the whether the *quantity itself* was increasing or decreasing. Using a *switch* in the quantity's behavior rather than a *switch* in the type of rate of change seemed to be a natural inference to use in trying to identify an inflection point. Thus, the problem is not with the idea of a *switch*, but rather in the inference of *what* switches. The reason may simply be the complex cognitive activities required in tracking the covariation between a *rate of change* of a quantity and the independent variable. This is far more complicated than tracking the covariation between the *amount* of a quantity and the independent variable. Interestingly, as a sidenote, I observe that the *inflection-point-as-extrema* confusion actually resonates with ways that business contexts sometimes use the term "inflection point" (e.g. Comaford, 2012).

Next, to explain what I mean about the iconic *shape-based* thinking being potentially problematic, I describe the relationship I saw between the *concavity-requires-extrema* confusion and the students' tendency to use U-shaped, image-based conception of concavity in pure mathematics. All of the students relied heavily on U-shaped images for concavity in the pure-mathematics context. However, it appears that viewing concavity as a static image might have led a few students (Doug, Nathan, and Kylie) to link concave up mostly with the full "U" image and concave down mostly with the full "∩" image, which inherently have a minimum or a maximum. I hypothesize that this static shape-based thinking may be responsible for these students' inferences that a quantity must attain a maximum to be concave down and a minimum to be concave up. In fact, I see the *concavity-requires-extrema* confusion as connected to what Aspinwall, Shaw, and Presmeg (1997) have termed *uncontrollable imagery*, especially for Doug and Nathan. An uncontrollable image is one that "rise[s] unbidden in an individual's thought" and that "persist[s] even in the face of contrary evidence" (p. 303). In Aspinwall et al.'s study, a student was convinced that a parabola had vertical asymptotes because of how the graph of a parabola looks, which caused deep confusion for the student. Similarly, in this study, Doug, Nathan, and Kylie made problematic conclusions, presumably to fit their image of a full "U" shape. Nathan, in particular, struggled quite a bit both in the height context and in the universe context because of this strong "U" image. He fully recognized the conflict, even stating once that he "wished" he could be "wrong with the minimums and maximums." As such, I feel it appropriate to label the "U" image of concavity as a potentially *uncontrollable image*.

Overall, the results suggest that some knowledge elements may be more helpful in projecting concavity and inflection points into real-world contexts than others. For example, the typical graph-based idea of concavity in mathematics may not always translate well into an understanding of what concavity and inflection points might mean in real-world contexts. This is not to say that it is an *incorrect* conception, but it may lead to problematic inferences. Since graphical depictions of concavity are the norm in calculus curriculum, students may not be learning these concepts in a way that will help them meet the challenge of using them in real-world situations (see NCTM, 2000). Rather, it may be that in mathematics classes, we need to help students develop meanings for concavity as representing, explicitly, changing rates of change (see Jones, 2016). In addition to a basic understanding of two-quantity covariation, students may need time to understand what it means to have covariation between the *rate of change* itself as one part of the covariation and the independent variable as the other part. I believe this study supports conclusions and recommendations made by Carlson et al. (2002) about the importance of covariational reasoning abilities for understanding real-world quantities. Explicitly bringing up the covariation of real-world quantities with calculus students in connection with concavity and inflection points, while staying true to mathematical definitions of first and second derivatives, may help students develop knowledge of these important concepts in ways that can help students apply them to real-world contexts.

Acknowledgements There are no grants nor funding to acknowledge for this article.

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