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First-Year Non-STEM Majors' Use of Definitions to Solve Calculus Tasks: Benefits of Using Concept Image over Concept Definition?

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Abstract Six US first-year university students in humanities or social science degree programmes were interviewed while solving 4 tasks on continuity and asymptotes in a required mathematics course. The focus was on how the students referred to the definitions or to the concept images when solving the tasks and if partial understandings appeared. Partial understanding denotes when the definition or the concept image has correct parts but essential parts are missing or incoherencies occur. Some students confused continuity with differentiability or perceived it as a graph not having holes, which is also reported by other studies. Another misconception that emerged is to perceive points on the *x*-axis as non-continuous. The partial understandings of asymptotes were related to the vertical asymptote as it was confused with a function property. Some students referred to the definitions when correctly solving some of the tasks and some students with coherent concept images were successful solving some of the tasks even when they did not refer to the definitions.

Keywords Asymptote · Calculus · Concept definition and concept image · Continuity · Non-STEM students

Introduction

This paper focuses on university students who are required to study mathematics, for instance calculus, even though they are not in any science, technology, engineering and mathematics (STEM) degree programme. Many students who enrol at a university do not wish to have mathematics as a compulsory part of their degree programme (Guzman, Hodgson, Robert & Villani, 1998) and first-year engineering students often

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complain about having to study calculus (Harterich, Kiss, Rooch, Monnigmann, Darup & Span, 2012). However in many countries, students within non-STEM degree programmes such as history, literature, education and language, are required to study mathematics, just as students in STEM degree programmes are required to study subjects in the humanities and social sciences. For instance, in the USA, such courses satisfy the universities' disciplinary breadth requirements whose purposes are to make the students acquainted with various traditions and thinking modes. The required mathematics is different from "mathematics as a service subject" (Howson, Kahane, Lauginie & de Turckheim, 1988, p. 3) seen in subjects like physics, chemistry, engineering, biology and economics, where mathematics is a useful tool. Although STEM university students' learning of calculus has been studied, not much research has yet been done on the large number of non-STEM university students who are required to study mathematics to satisfy such breadth requirements. Their understandings of mathematics may, or may not, be similar to those of the STEM students, but one might anticipate that their level of motivation and understanding is lower and that they approach mathematics differently from the STEM students. One might furthermore argue that they also deserve quality teaching aimed at their learning needs, and more research is therefore needed about how they learn mathematics.

Background Literature

Previous Studies on University Students' Difficulties in Calculus

Students in different STEM degree programmes do not have similar ways of learning calculus. Maull and Berry (2000); Bingolbali, Monaghan and Roper (2007) and Bingolbali and Monaghan (2008) argue that mathematics and engineering students' mathematical concepts develop differently and have different meanings to the students. Engineering students see mathematics as a tool and wish to see applications as part of a course. Rensaa (2014) studies an engineering student whose main strategy is instrumental, i.e. does not aim at relational understanding. Engineering students often lack fundamental understanding of difficult concepts due to an inability to perform deductive reasoning (Morgan, 1990). Macbean (2004) compares physics with biochemistry students and finds significant differences in how they approach mathematics and in their conceptions of mathematics. The student groups also have different learning outcomes even when they take the same course. Jukić (2011) compares mathematics, science and engineering students in a joint first-year calculus course. On all the tasks with a significant difference in the groups' performance, the mathematics students outperform the non-mathematics students. Jukić and Dahl (2012) study the long-term retention of core calculus concepts and find that the science and engineering students mostly have limited knowledge even though they encountered calculus in other courses later in their degree programme.

In terms of non-STEM students' learning of mathematics, Jóelsdóttir, Lindenskov, Misfeldt and Rattleff (2012) investigated economics students in an introductory mathematics course integrated with macro-economics and statistics. They argued that a transdisciplinary approach might help the students apply mathematics, but if mathematics is not taught on its own, it might be harder for the students to activate it later in new situations. Another study by Abramovich and Grinshpan (2008) argued for special teaching of mathematics to engineering, business and life sciences students using real-life projects.

Hence, from this overview of some of the literature, it appears that STEM students are not *one* group but they differ in how they learn calculus. It also appears that *non-*STEM students are different from any of the STEM students. Therefore, one cannot *know* how the students in non-STEM degree programmes learn and perceive mathematics when they take a required mathematics course like calculus. One might have expectations, but research is needed in order to verify revise, or reject such expectations, which is a goal of this study.

The Role of Definitions and Concept Images in Learning and Problem Solving

Vinner (1991) argues that definitions create problems for learning mathematics since they represent a difference between mathematics as a structured body of knowledge and the cognitive processes of learning mathematics. In mathematics contexts, as opposed to everyday contexts, "you should consult definitions, otherwise mistakes might occur" (Vinner, 1991, p. 67). He further argues that whenever one hears a concept's name, what is usually evoked in one's memory is not its definition, but what he calls one's *concept image*. A concept image is a non-verbal association in the mind. It might include visual representations and other impressions and experiences, and it is entirely individual. All these mental associations form a cognitive structure and they might at some point be translated into verbal forms, but this is not what is first evoked in a person's memory. A part of the process of acquiring a concept is the development of one's own concept image because knowing a definition by heart does not necessarily entail an understanding of the concept. The term 'concept definition' denotes the definition accepted by the mathematical community (Tall & Vinner, 1981). According to Vinner (1991), in many everyday situations when a concept is learnt by definition, e.g. a forest, a concept image is formed and "the moment the image is formed, the definition becomes dispensable" (p. 69). This is different from mathematics where definitions do not only help form the concept images but they are also essential when solving tasks. Zandieh and Rasmussen (2010) argue that a main role of mathematical definitions is to describe a characteristic and single out a concept with certainty. This is not contradictory to Vinner (1991) but has another focus than its role in solving tasks. A concept image might not include all aspects of a concept, so just relying on a concept image when solving tasks can lead to mistakes. The relationship between the concept image and the definition is two-way: if a concept is introduced by a definition, the concept image will be empty at first but it will be gradually 'filled' after the person sees some examples. On the other hand, a person might first see some examples and later get introduced to the definition, which then alters the concept image. According to Vinner (1991, p. 71 – 72), a solution should only be attempted after consulting the definition; he describes three ways for this:

1. Consult the definition, then via interplay with one's concept image, solve the task.
2. Solve the task solely using the definition.
3. Via one's concept image, a proper definition is chosen to solve the task.

Vinner (1991) argues that due to habits from everyday life, usually, only the concept image is consulted when solving tasks. This is only adequate when the tasks are routine. In concurrence with Vinner (1991), several studies have shown that students often do not use the definitions in building the concept image (Vinner & Dreyfus, 1989) and they do not rely on the definition but on the concept image when solving problems (Rasslan & Tall, 2002). Sometimes, undergraduate mathematics students do not use definitions even when they are able to state and explain them correctly. If there is a conflict between the definition and the concept image, the latter wins (Edwards & Ward, 2004). Rösken and Rolka (2007) report that definitions play a marginal role for 24 German final-year secondary school students' conceptual knowledge of integral calculus. The students mainly lean on their concept images, which cause difficulties in their reasoning and problem solving. Kidron (2011) describes a student who experiences a conflict between her concept image of the horizontal asymptote and the definition relating to the dual nature of the notion of limit as a process and as a number. Williams (1991) describes some college students' models of limit and show that they vary widely. Juter (2005) also shows that although students claim that a function cannot attain its limit values, they still consider it possible in problem solving. Hence, their concept image of limit is not good enough to solve the problems. Jukić and Dahl (2011) investigate science students in a calculus course and show that no student knows the definitions, which causes problems solving the tasks. Wawro, Sweeney and Rabin (2011) argue that university students' descriptions of subspace in linear algebra often differ substantially from the definition. Randahl (2012) finds that first-year engineering students starting a calculus course have major difficulties making sense of and use textbook definitions given by formal definitions.

These studies are all done on STEM students and document that definitions are often either not known or not used when students build up their concept images or when they solve tasks. This impacts the STEM students' ability to solve tasks negatively. The present study investigates whether six non-STEM students have a similar marginal use of definitions and lack of success solving tasks using the concept images. This includes investigating which of Vinner's (1991) three ways of using a definition is chosen, if any, when they solve tasks.

Partial Understandings

Tall and Vinner (1981) argue that a definition can be learnt in either a rote fashion or more meaningfully by being related to the concept. Viholainen (2008) defines a concept image as *coherent* if the conception about the concept is clear; all conceptions concerning the concept are connected to each other; there are no internal contradictions, and the concept image does not contain conceptions contradictory to the formal system of mathematics. Otherwise, the concept image is *in*coherent. Tall and Vinner (1981) argue that as the concept image develops, it does not need to be coherent at all times but can include contradictions to the definition, which students may not be aware of. Also, Vinner and Dreyfus (1989) argue that learning a concept does not happen in one step: "In these intermediate stages, some peculiar behaviours are likely to occur. Several cognitive schemes, some even conflicting with each other, may act in the same person in different situations that are closely related in time" (p. 365). This indicates that the route to having understood a concept is a process with ups and downs, not a jump from no understanding to full understanding. Ron, Dreyfus and Hershkowitz (2010), in this context, define 'partially correct' to describe situations where the match between a student's construct and the corresponding formal mathematics is partial. One might further argue that while the term 'incoherent' emphasises the *inadequacies* of the students' understanding, the term 'partially correct' emphasises the *qualities* of the students' incoherent understanding. In this study, the term 'partial understanding' is used when students' expressed definition has correct parts but essential parts are missing, or when a concept image is incoherent but also shows some coherence with the formal definition. An example of what this paper terms 'partial understanding' is seen in Tall and Vinner (1981) where students call a function discontinuous as it is not in one piece. This is correct if we deal with real intervals, but if students are not aware of this, they have a partial understanding.

Research Questions

It is not yet known if *non*-STEM students have the same perception of calculus as various STEM students, and it is also not known how they use definitions or apply their concept images when solving tasks. The learning of asymptotes has also not been studied extensively. The paper addresses the following research questions:

1. How do the six students apply the definitions or their concept images when solving the given tasks in continuity and asymptotes?
2. Which correct, wrong or partially correct understandings occur and what are their qualities?

Methodology

The Students, Lessons and Textbook Definitions

The study took place in the autumn of 2011 at a private US university in the top 10 of the Times Higher Education ranking of North American universities 2011 – 2012 (Times, 2012). The university offered three sequences for calculus. One was an Honours sequence for students intending to study mathematics and physics and the two others were standard single variable courses. These two options covered the same material but at a different pace. Students were encouraged to take the slower one if they only needed calculus to satisfy the university's disciplinary breadth requirement or basic medical school requirements. The faster course was for students who needed calculus for engineering, science and economics studies. The students reported in this paper were from the slower course, which consisted of 191 first-year students. The author was informed that the majority of the students had either not taken the Advanced Placement (AP) calculus exam (US high school level) or they did not get a good score on it. To find participants for the study, the lecturers encouraged the students, and the author stood up in class and asked for volunteers. Hence, the sampling was voluntary, which means that it was unlikely that the students represented the whole student body. The students in this sample came from different non-STEM programmes such as history, literature, education and languages. The students' midterm or final grade was not known to the author due to confidentiality, but grades were also not relevant in determining how they used the definitions or concept images.

The course was taught by two lecturers from the mathematics department. The students were divided into three sub-groups. One lecturer taught two sub-groups and the other taught the third sub-group. Each lecturer taught two or three times during the week, serving each of the three sub-groups, but the students were free to attend any lecture they wanted. The teaching styles of the lecturers were quite similar (one being under guidance of the other). Both lecturers used the blackboard and mixed their lectures with questions to the students as well as minor tasks the students should discuss with each other. The atmosphere was informal and questions were welcomed. Each lecture was around 85 min.

The textbook (Stewart 2009, p. 113 – 121) defined a function f to be continuous at $x = a$ if $\lim_{x\to a} f(x) = f(a)$ with three things implied: (1) $f(a)$ is defined ($x = a$ is in the domain), (2) $\lim_{x\to a} f(x)$ exists (i.e. $\lim_{x\to a-} f(x) = \lim_{x\to a+} f(x)$, and (3) $\lim_{x\to a} f(x) = f(a)$. During class, the lecturers stated, 'Intuitively, the graph of a continuous function has no gaps', and, 'You can sketch the graph without lifting your pen' (author's notes). The textbook used a similar description: "Geometrically, you can think of a function that is continuous at every number in an interval as a function whose graph has no break in it. The graph can be drawn without removing your pen from the paper" (Stewart, 2009, p. 113). Concerning asymptotes, Stewart (2009, p. 125 – 129) stated that the line $y = L$ is a horizontal asymptote of $y = f(x)$ if $\lim_{x\to +/-\infty} f(x) = L$. A function has a vertical asymptote at $x = a$ if any of these six conditions hold: $\lim_{x\to a+/-} x\to af(x) = +/-\infty$. Limit was defined as $\lim_{x\to a} f(x) = L$ and "the limit of $f(x)$, as x approaches a, equals L … the values of $f(x)$ tend to get closer and closer to the number L as x gets closer and closer to the number a" (Stewart, 2009, p. 95). These were therefore given without the precise epsilon-delta definitions. Similar definitions were given in class.

Qualitative Study Design

Given the small sample of six students, one cannot reach general conclusions about how non-STEM students, as a whole, behave. Furthermore, the students are from different degree programmes, which may or may not have an impact. What they *do* have in common is a greater interest in studying non-STEM topics, and that they did not have good results from calculus in high school (as stated above); but given that they now study at a top university, one can assume that they generally get good grades and know how to study. Since one cannot reach general conclusions in qualitative studies, Schofield (1990) argues for replacing the notion of generalisability by 'fittingness', "the degree to which the situation matches other situations in which we are interested" (p. 207). Ergo, detailed descriptions are important to determine the applicability of a study to other situations. Therefore, even though the sampling was voluntarily, the results might be of interest beyond the six students.

The data collection was pair or single student interviews to achieve details and personal accounts (Morgan, 1998b). The students were challenged to elaborate their answers and thoughts about the solutions to the given tasks (Kvale, 1996). During a pair discussion, the participants did a lot of the exploration for the interviewer (Morgan, 1998a), since when two work together, there is more verbalisation, and the reasons for the decisions are open. Furthermore, when working in pairs, the students discuss the tasks more with each other than the author, limiting the need for the author to prompt, hence influencing the students' thinking and choices. The students were paired according to when they had time to meet the author, but due to practicalities such as students asking to reschedule, two were interviewed alone. Since all interviews were qualitative and aimed for details and personal accounts, the methodological consequence of four being done in pairs and two alone is not significant, but the author needed to prompt more when interviewing students alone. The interviews lasted 30–45 min each. To compensate them for their trouble and to make them relax, soft drinks and cookies were provided. The students were assured that their identity would not be revealed, they became acquainted with the purpose of the study, and were asked for permission to audio tape. This was done orally in class, before the interview began and in writing at the top of the sheets with the tasks.

The Tasks

The students received two sheets with questions on limits, continuity and asymptotes, and plenty of empty sheets to write on. These topics were chosen as the students had recently had a midterm test on them. The topics were, furthermore, central to calculus. The sheets had seven tasks, but task 2 consisted of six questions named $f - k$. There were no illustrations provided. To allow for a deeper analysis within a single paper, only the following four tasks (original numbering kept) were discussed, including two questions (g, j) from task 2:

1. Define continuity for a function f at a given point.

2. Explain if, and where, the following functions are continuous ($x \in R$)

$$g(x) = \begin{cases} 0 & x < 0 \\ x & x \geq 0 \end{cases} \qquad j(x) = \begin{cases} x & \text{if x is rational} \\ 1-x & \text{if x is irrational} \end{cases}$$

4. Define horizontal and vertical asymptotes of a function f.
5. Can the graph of a function cross its vertical and horizontal asymptotes? Explain why/why not. If yes, how many times?

The tasks were similar to tasks in most calculus textbooks such as the one used in this course. The students did not have access to the textbook or notes during the interview. The tasks were created to expose the students' concept images and understanding of the definitions and to investigate if the students had two of the common misunderstandings in relation to continuity: confusion with differentiability and only perceiving continuity as something that relates to a graph not having any holes. Task 1, therefore, asked for the definition of continuity. Task 2g tested if they confused continuity with differentiability and task 2j detected if the students knew that continuity is not only related to functions whose graphs have no holes. Task 2j was more difficult as it required that they knew that between any two rational numbers is an infinite number of irrational numbers. Furthermore, task 2 also tested if the students knew that a continuous function is a function that is continuous on all points on its domain, hence, the specific formulation with 'and where' inserted. Task 4 asked for the definitions of asymptotes and task 5 aimed at exposing their individual concept images. The first part of task 5 could be answered by just finding an example of a horizontal asymptote being crossed many times, but the question about the vertical asymptote required more reasoning. The second part of task 5 asked for an explanation so the students had to show more than just an example of a function. Task 5 was almost identical to a task on the midterm test that had only asked about the horizontal asymptote. Taken together, tasks 1 and 4 asked directly for the definitions while tasks 2*gj* and 5 were 'indirect', as Vinner (1991) argued that one needs indirect tasks to expose a student's concept image.

Results

The Analysis

The analysis uses Kvale's (1996, pp. 214 – 217) contexts of interpretations. The first context is 'self-understanding', where the researcher in a condensed form summarises and rephrases what the students do and say. A second context is 'theoretical understanding', where a theoretical framework is applied to interpret the students' statements and actions. These two contexts are similar to Mason's (2002) distinction between 'accounting of' and 'accounting for', where the former is a brief description without explanation and interpretation, and the latter is when some theorising is added to explain and interpret the incident. The analysis, therefore, consists of three steps:

1. Identify situations where definitions of continuity and asymptotes are given (tasks 1 and 4) or where explanations are given (tasks 2 and 5).
2. Understand the students' self-understanding, e.g. an 'account of', summary, in the author's words of what the students are observed doing or saying.
3. Apply the notions of concept definition, concept image and partially correct understanding to get a theoretical understanding; e.g. when the students' actions are explained by reference to which of Vinner's (1991) three ways of applying the definition the students have used, or if they have partial understanding or not.

Students A and B

Two male students, A and B, write the following when answering task 1:

In the first context of interpretation—the self-understanding—one sees that the students believe that the definition refers to all x values and not just to one point $x = a$ (criteria 1, Fig. 1). The students also confuse the concepts 'defined' and 'exist' (criteria 2, Fig. 1). The fact that some mathematical entity can be defined does not mean that it exists. However, this difference is technical and one might argue that this error is not too serious for students at this level. Hence, in the theoretical understanding of the students' utterances, one can argue that the definition is partially correct since it has elements in common with the formal definition, but it also has some wrong elements.

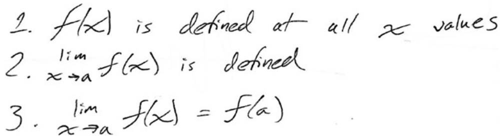

Fig. 1 The definition of continuity by students A and B

As seen below, criteria 1 creates problems in task 2*j*. Regarding task 2*g*, the students refer to the definition in Fig. 1 when they determine if *g* is continuous. After some discussion, they write the following:

From Fig. 2, one sees that the students use their definition of continuity to explain if *g* is continuous. They make a good attempt at using a precise mathematical language even though the use of 'defined' is not correct. Thus, it appears that the students follow the second option of Vinner (1991). When they reach task 2*j*, they get stuck. In the theoretical interpretation, one might argue that an incoherency in each of the student's concept images becomes visible, particularly after the interviewer (I) gives a hint after a period of observing the students are stuck [minutes and seconds into the interview are in brackets]:

I: If I say that it is continuous in only one point? [13:20]

7 s of silence.

A: Hmm, continuous, what do you say?

B: In, at, at, at *one* place? [sounds surprised]

I: Hmm-hmm, in one point.

A: THIS is continuous in only one point? [sounds surprised]

I: Hmm-hmm.

7 s of silence.

B: As in one set interval?

I: No, just one point, one *x* value.

5 s of silence.

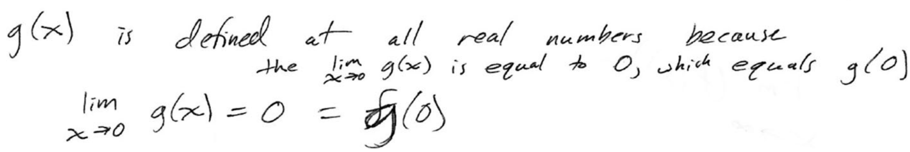

Fig. 2 Answer to task 2*g* by students A and B

I: Can a function be continuous just in one x value?

A and B: [A and B talk at the same time, interrupt each other.] No. It can't. I don't think so. It does not make sense [B shows an interval with his hands].

A: If it has to be continuous, it has to be continuous at every x value.

The definition seen in Fig. 1 stating that the first condition is true for *all x* values means that the students reject the idea that a function can be continuous at only one point. In the theoretical understanding, task 2*j*, therefore, exposes an instance where the partially correct definition cannot be used to reach the correct answer. They also both appear to rely on a concept image of continuity as something that refers to an interval, which might be related to the fact that in order for a function to be called a 'continuous function', it has to be continuous on its whole domain. Hence, the students consult each of their concept images, both of which state that it is not possible for a function to be continuous at only one point; but consulting the concept image creates problems here. Thus, here, they appear to follow the first of Vinner's (1991) three ways for applying the concept definition when solving tasks. Despite the fact that each of them only has a partially correct understanding of the definition of continuity, they are examples of students who *do* use the definition when solving tasks.

Student B leaves for another appointment at task 4, but student A gives correct definitions of the horizontal and the vertical asymptotes (see Fig. 3). He could have added '$-\infty$' to the definition of horizontal asymptote, but it is also correct as stated. The same can be said about the vertical asymptote for $x \to a-$ and $x \to a+$. The definition of the vertical asymptote obviously causes problems seen from the several attempts. In terms of task 5, student A gives the right answers to both questions and each time, he uses both the definitions and examples of functions such as $1/x$ and $\sin(1/x)$. Seen from the perspective of the theoretical understanding, Student A has a correct understanding of the definition and refers to this as well as his overall concept image while solving the

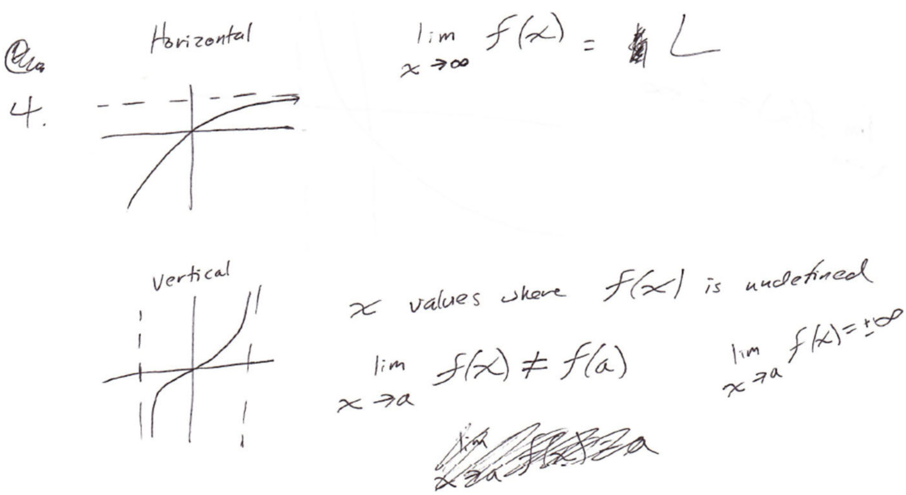

Fig. 3 Answer to task 4 by student A

tasks; hence, he applies the first of Vinner's (1991) ways of applying the concept definition when solving tasks.

Students C and D

Two female students, C and D define continuity as $\lim_{x \to a} f(x) = f(a)$, which is correct. In task 2*g*, they discuss the point $x = 0$ and draw some graphs while discussing if there are points of jump discontinuities. Student C states: "[4:13] It is not differentiable but it is continuous". They decide that *g* is continuous on all real numbers. Then, the interviewer challenges them to explain how they reached this conclusion:

> I: What made you decide [that it was continuous]—when you drew the graph, or did you look at how continuity is defined, or did you do something else? [4:54]
>
> D: Eh, well, I drew the graph because that's like, easier to visualise pictures, but I guess we could also have done the limits from both sides as *x* approaches 0 (C: Yeah) and then we would have found that it is 0.
>
> I: OK.
>
> C: Yeah, for me I think that if it is a less obvious one, then I *will* go back to the definition of a limit (I: Hmm-hmm) and solve for the limit from the left hand and the right hand and solve for *f* (a) and make sure they are all equal, but for this one, it was more clear (D: To this graph), yeah.

In a theoretical understanding, referring to their concept images by looking for jumps in the graph is the easiest method; hence, they use the third way of Vinner (1991). They *do* know the definition and can, in detail, describe how they could have used it. It appears that in their view, looking at the graph or using the definition are equally valid methods for determining if a function is continuous. The choice is a matter of convenience. In task 2*j*, they are stuck, but after hints from the author to search for one point, D states: "[13:58] That's where they connect … so I'd say it is continuous where *x* equals ½". Here, they refer to the piece wise function *g*, and it is, therefore, a reference to a concept image, but after some encouragement from the author to use the definition, they discuss the following:

> D: As it goes to ½. … The limit as it goes to what. I don't know how to start. [14:59]
>
> C: I think that when these two are equal, they are ½, right, so that's a critical point.

Hence, D does not know how to begin while C knows how to use the definition and can investigate the limits of $x \to ½$, but none of them chooses this approach on their own.

The students give a correct definition of the horizontal asymptote in task 4 (see Fig. 4). In task 5, about the horizontal asymptote, they explain the following: '[32:36] C: You can cross it like, if you, you can go, uhhh, as long as at its end, it is approaching that line'. Here, from the perspective of the theoretical understanding, the students just refer to their concept images (which here are similar for C and D) and rely on a graph seen in class.

As for the definition of a vertical asymptote, they say, '[27:54] D: As the x values approach that point, the y gets either infinitely, it approaches infinity in either direction'. Their orally stated definition is correct but they are unable to formally state it in writing, which in the theoretical interpretation makes it just partially correct. They also deny that a graph of a function can cross its vertical asymptote:

> D: If it were to cross the vertical asymptote, it wouldn't be a one-to-one function, because the asymptote itself means that you have values over here that are approaching somewhere, but you also have values over there for the x and so, and these ones continue on and would interfere. It wouldn't pass the vertical line test. [32:54]

The students show that they can use their orally stated definition, which is Vinner's (1991) second way, and that they have some understanding of what a function is. But D's understanding of a function is partially correct, as being 1 – 1 is not a criterion of a function.

Student E

Student E was interviewed alone. She correctly defined continuity as $\lim_{x \to a} f(x) = f(a)$. When solving task 2$g$, she nevertheless claimed that g is not continuous at $x = 0$:

> E: Technically, because of the shape of that, that wouldn't be continuous at that sequence. Well, no, it *is* continuous. The limit doesn't exist, though [3:12]
>
> I: What is the limit if x is zero?

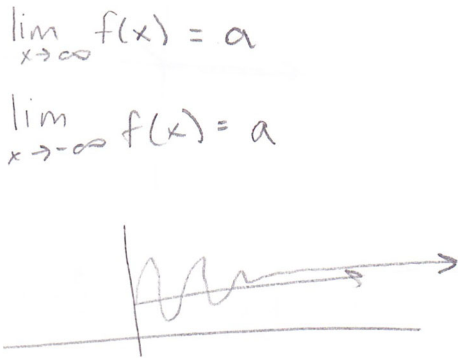

Fig. 4 Answer to tasks 4 and 5 by students C and D

E: It equals zero and that also equals zero but graphically …

In student E's self-understanding, continuity relates to the shape of a graph. Applying the theoretical framework, she has an incoherent concept image of continuity as she talks about 'sequence' and not points. She then draws some figures and compares the graph of g with the graph of the absolute value function $|x|$ and appears stuck. Then, the interviewer prompts: "[5:21] Maybe you are thinking about differentiability?" After a few seconds of silence she says, '[5:25] THAT'S what I am thinking about [laugh]'. She is the only one who confuses the concepts of continuity and differentiability. Hence, she writes the correct definition but she does not fully understand it, and she does not refer to it when determining if g is continuous. She relies on her (wrong) concept image of continuity and also gives a wrong answer to task 2*g*. In terms of task 2*j*, the student does not know the difference between rational and irrational numbers, so this task is abandoned.

In task 4, she gives a correct definition of a horizontal asymptote and a partially correct definition of a vertical asymptote (see Fig. 5). The definition of a vertical asymptote is partially correct, as she does not write that $x = a$ is the vertical asymptote and the notation 'DNE' (shorthand for 'does not exist') is not formally correct. In terms of the horizontal asymptote in task 5, she answers correctly that a graph of a function can cross its horizontal asymptote many times but does not give an explanation and says, '[22:25] NN [the lecturer] told us in class. Didn't really explain it. I just remember him drawing that'. Hence, in the theoretical understanding, she does not refer to the definition but to her concept image containing memorisations from class. Regarding the vertical asymptote in task 5, she draws a graph (see Fig. 5). The interviewer adds something wrong to this graph (showing a graph *crossing* a vertical line many times) in order to make her explain her answer. She then says:

E: I can define in terms of what kind of function generates a vertical asymptote. If you have $y = 1 / (3 - x)$, the vertical asymptote is $x = 3$ and it couldn't cross it because if you apply 3 as the x value into the denominator, you can't divide by 0 so that's undefined for that function. [23:33]

Hence, in the theoretical understanding she refers to a partially correct definition in combination with her concept image when correctly answering the question about the vertical asymptote, which fits the third option of Vinner (1991).

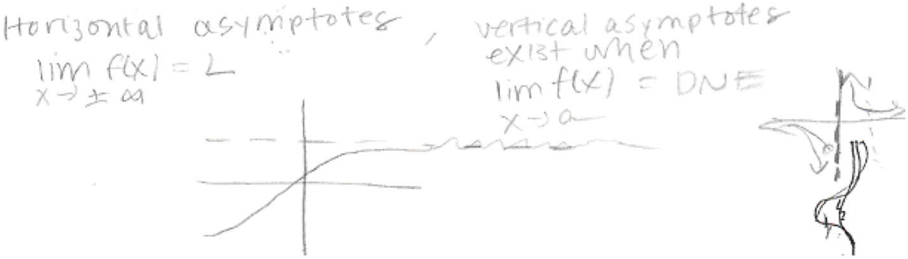

Fig. 5 Answer to tasks 4 and 5 by student E. Wrong additions to the graph were made by the interviewer. *DNE* does not exist

Student F

Student F was interviewed alone. In task 1 she answered, 'Continuity for a function is whereby the points do not break, they continue through the *x*-axis', which is not correct. Figure 6 shows more thick marking when $x < 0$, so she appears to believe that as the function was identically 0 when $x < 0$, and therefore 'on' the *x*-axis, this makes it not continuous here. She gives a wrong answer to task 2*g* while referring to her erroneous definition; this is Vinner's (1991) second way. From a theoretical understanding, she is able to refer to *a* definition but this (wrong) definition gives her the wrong answer. Furthermore, her understanding of the definition is not partially correct but incorrect. Due to the weak nature of her understanding of continuity, task 2*j* is skipped.

Student F gives a correct definition of a horizontal asymptote, but the definition of vertical asymptotes is partially correct and lacks formal mathematical language (see Fig. 7). In her answer to task 5, she states that a graph can cross its horizontal asymptote and gives sin(1/*x*) as an example. She then draws a graph to illustrate that a graph can cross its vertical asymptote. When the interviewer challenges her if such a graph is a graph of a function, she refers to the vertical line test to explain why it indeed is not.

In a theoretical understanding, *F* does not refer to her definitions of asymptotes but to her concept images in her answers to task 5. Regarding the horizontal asymptote, she refers to a concept image of sin(1/*x*), but her knowledge of the vertical asymptote is very weak.

Discussion and Conclusions

How Did the Students Apply the Definitions or Concept Images when Solving the Tasks?

In terms of task 1, three students provided a correct definition (C, D, E), two a partially correct definition (A, B) and one a wrong definition (F) of continuity. In task 2, the students were asked to determine if two given functions (*g* and *j*) were continuous. Four students correctly stated that *g* is continuous on its whole domain (A, B, C, D) while two did not provide a correct answer (E, F). The reasoning behind the right and wrong

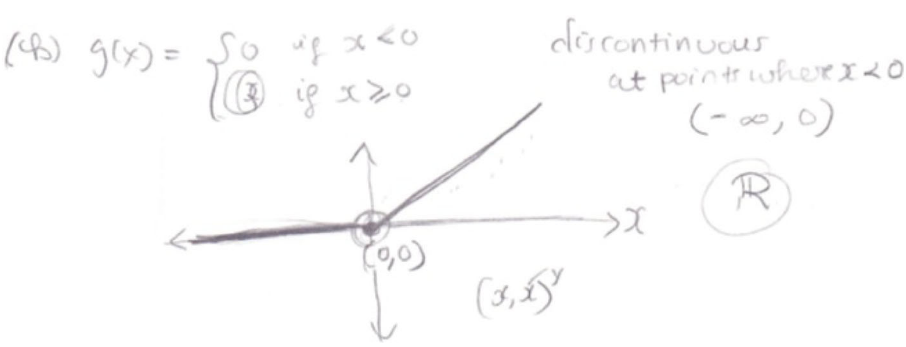

Fig. 6 Answer to task 2*g* by student F

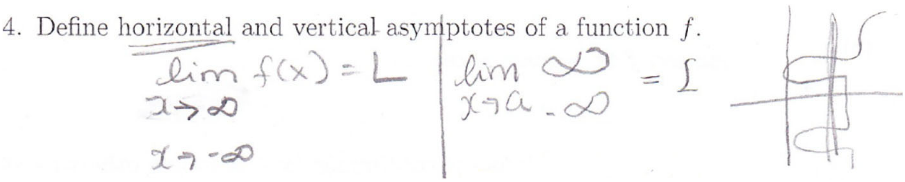

Fig. 7 Answer to tasks 4 and 5 by student F

answers varied. Three referred to the definition by themselves (A, B, F), two needed a hint to use the definition (C, D) and one (E) only used the concept image. In relation to function *j*, the study also found a mix of answers as two students referred to the definition (C, D, after a hint) or referred to a definition together with the concept image (A, B). Overall, in relation to the continuity tasks, the study found all combinations of providing a correct definition (or not), right/wrong answers and used/used not the definition. All six students at some point referred to the definition they stated. It also appeared (not surprisingly) that when the definition was correct and applied, the answer to the task was correct. The use of a wrong definition led to a wrong answer (also not surprisingly), while the use of a partially correct definition sometimes led to the right answer, sometimes the wrong answer. The latter is discussed in the next section about the quality of the partial correct definitions.

Regarding the answers to the tasks about asymptotes, the answers showed much less variety. All students gave the correct definition of the horizontal asymptote, but only one (A) gave a correct definition of the vertical asymptote. In relation to the first part of task 5, all students answered the question about the horizontal asymptote correctly, but four students (C, D, E, F) only relied on their concept image while one (A) used both the definition and his concept image. Hence, none of them relied solely on the definition. The answers to the task on the vertical asymptote in the first part of task 5 varied. Four (A, C, D, E) answered it correctly using either the definition in combination with the concept image by referring to a graph (A, correct definition; E, partially correct definition), or used a property of a function (C, D, partially correct definition). The student (F) who gave the wrong answer used the concept image of asymptote. It therefore appeared that four students did not use the definition in solving the task about the horizontal asymptote but successfully relied on their concept images, while five used the definition together with the concept images and property of a function when solving the task about the vertical asymptote. See Table 1 for an overview.

Summing up, the six students were different. In line with studies referred to above, two students who only referred to their concept images in some tasks failed to get the right answer (E in 2*g*, F in 5*v*), but what is different from these studies is that four students *were* successful in answering some tasks only using their concept images (C, D, E, F; all in 5 h). It also appears that two students (A, B) successfully just referred to the definition when solving task 2*g*. All students referred to their concept image at some point when solving the tasks, which often helped them to get the correct answer. The students furthermore used all Vinner's (1991) three ways of using the definition, when they applied a definition.

Results that are Different from Other Studies. This study gave examples of non-STEM students who were *successful in solving tasks when just referring to their*

Table 1 Overview of answers

	Task 1	Task 2*g*	Task 2*j*	Task 4*h*	Task 4*v*	Task 5*h*	Task 5*v*
A B	Partial correct	Correct. Use cd	Wrong. Use cd and ci	Correct	Correct	Correct. Use cd and ci	Correct. Use cd and ci
C D	Correct	Correct. Use ci, use cd when prompt	Correct with help. Use ci, use cd when prompt	Correct	Partial correct	Correct. Use ci	Correct. Use cd and property of a function
E	Correct	Wrong. Use ci	NA	Correct	Partial correct	Correct. Use ci	Correct. Use ci, use cd when prompt
F	Wrong	Wrong. Use cd	NA	Correct	Partial correct	Correct. Use ci	Wrong. Use ci

Note. For Tasks 4 and 5, *h* and *v* stands for horizontal and vertical, respectively

NA the task was skipped, *cd* concept definition, *ci* concept image

concept image and students who were *able to use the definition* when solving tasks. Both are different from the main conclusions in the studies of STEM students referred to earlier, where students often do not use the definitions when solving tasks but instead mainly use their concept images, which cause problems for them. This study, therefore, both adds different examples to the large body of literature of how students use, or do not use, the definition or concept image when solving a task, but it also adds to the smaller body of literature on non-STEM university students' learning of mathematics. One might ask why the results about the use of definitions and concept images are different from the other studies. One reason is that these other studies did not state that it *never* happened that students used the definitions or successfully used the concept images; they reported on main trends. Another reason might be that the students in this study are indirectly led to use the definition as they are asked about the definition before the tasks. This does not change that some of them show that they *are* able to use the definition. A reason why some students are successful only applying their concept images might be that they have coherent concept images. Earlier, Vinner (1991) is referred to for arguing that students *should* always refer to the definition, but coherent concept images might be enough when tasks are routine. Further study is needed here.

The Students' Correct, Wrong and Partial Understandings and their Qualities

Continuity. Three partial or incorrect understandings appeared: (1) none of the six were aware that continuity had to do with property at a *point* and not property of a *graph*; (2) one student confused differentiability with continuity. Both types were documented in previous literature (e.g. Nuñéz, Edwards & Matos, 1999; Selden, Selden, Hauk & Mason, 1999; Tall & Vinner, 1981). A third (3) type appeared when F stated that points that were 'on' the *x*-axis, which meant that the graph was not continuous at that place. These partial or incorrect understandings were not equally wrong. The first was a partial understanding that was still usable in determining if some functions were continuous. This was not possible with types two and three, which were incorrect. The third type was probably also related to a general lack of understanding of graphs and coordinate systems. Referring to the discussion above about learning being a process with incoherencies (Vinner & Dreyfus, 1989, p. 365), one could argue that although the concept images of continuity of A and B were not completely coherent, they were more useful in solving tasks than the concept image of E, who correctly recalled the definition but did not fully understand it.

Asymptotes. It appeared that the definition of the horizontal asymptote was easier than the vertical asymptote. The students had difficulties with the formal language of the vertical asymptote. The study found an opposite pattern among the six students: the students all knew the definition of the horizontal asymptote, but did not use it much in correctly answering task 5, while most students only had a partial understanding of the definition of the vertical asymptote, but they nevertheless often used this definition alongside their concept image of a property of a function. A reason for not always relying on the definition was offered by C and D who said that had it been a 'less obvious' (i.e. more difficult or non-routine) function, they would have relied on the definition. C and D obviously thought that using the graph was an equally valid method as referring to the formal definition. Nevertheless, as argued earlier, the learning of asymptotes has not been studied extensively, but this study showed differences in the difficulty of understanding the horizontal and the vertical asymptote, and to some students the latter appear related to the notion of a function.

Implications for Teaching

Continuity. Nuñéz et al. (1999) argue that the informal/intuitive definition of continuity characterises it as a process without gaps or sudden changes. As described earlier, the intuitive definition is used by the textbook and by the lecturers. A continuous function is a function that is continuous at every point of its domain, but a rational function such as $f(x) = 1/(x + 3)$, for example, is continuous on its whole domain but its graph exhibits a hole at the restricted value ($x = -3$), hence one needs to 'lift your pen' to draw the graph. Ergo, the intended help might have contributed to the confusion or at least to form some partially correct understanding and incoherent concept images. The formal definition deals with static and discrete elements, and Nuñéz et al. (1999) argue that the two definitions are different and it is thus a problem that students are often not told this. In this study, C and D obviously think that using the graph (i.e. the informal intuitive definition, concept image) is equally valid as referring to the formal definition. Leron and Hazzan (2006) argue that students need to be made aware of the different modes of thinking and how they operate. For teaching, one can, therefore, argue that teachers need to make the students aware of this, which might help the students understand the definition and develop their concept image and have a better understanding of their mutual relationship. Students also need tasks that are too difficult or non-routine to solve solely relying on the concept image, such as tasks like task 2*j*. In line with this, Vinner (1991) argues for non-routine tasks. As suggested by Edwards and Ward (2004), it is essential to spent time on definitions to aid the students in using them.

Asymptotes. As seen earlier, all students knew the definition of a horizontal asymptote whereas the definition of a vertical asymptote caused problems. However, as noted by Kidron (2011) referred to earlier, the definition of the horizontal asymptote can also be difficult for students. In this study, the definition and concept image of the vertical asymptote was often related to how students defined a function; hence, teaching of a vertical asymptote should put students in situations that expose them to the relationship between these concepts as well as how an asymptote relates to the concept of limit.

What to Expect Mathematically of Non-STEM Students? Given that this calculus teaching is without epsilon-delta definitions, and hence, less precise, this in itself makes it more difficult for the students to get to the depth of understanding the concepts of limits, continuity and asymptotes. Ergo, when the students are introduced to less precise definitions, it is not surprising that the study then reports many partial understandings. However, even within the frames of the present course and textbook, it is possible to arrive at correct understandings if, for instance, statements such as ‘graphs with no holes’ are properly explained and coherent concept images are developed. It is worth noting that some students appeared stronger than others and had correct understanding, coherent concept images and good partial understandings, while others were weaker and did not even know, for instance, the difference between rational and irrational numbers. This study is qualitative, hence, one cannot make generalisations on behalf of all non-STEM university students based on these six students. Knowing how large a portion of all non-STEM students would resemble the six students is beyond the scope of this study. One may ask why some of the six students were able to get the relatively high level of understanding, particularly since, as also stated earlier, Randahl (2012) reports that first-year engineering students have major difficulties using definitions provided as formal definitions, and Morgan (1990) states that engineering students have difficulties performing deductive reasoning. Based on these studies, the perhaps obvious assumption is, as also stated earlier, that non-STEM students would have even greater difficulties as they are even farther away from mathematics than engineering students. But this might not always be the case. Hasse argues that in Italy, physics is not seen as a ‘hard’ discipline and many physics students have a humanistic background, “[T]he ‘classical’ students become especially apt physics students because they, through their knowledge of philosophical and classical subjects, learn to think in the abstract lines of thought of importance to both the natural sciences and the humanities” (2009, p. 123). This does not mean that all non-STEM students would understand epsilon-delta definitions but some could. One suggestion, therefore, is to stop suggesting a slower-pace calculus course for *all* non-STEM students as it is not *one* group. Some non-STEM students are able to take courses with epsilon-delta definitions and enjoy it even though they are not going to use it later.

Closing Comments

The study shows that the notions of concept image and concept definition are useful to interpret students’ behaviours as they put names to what students do. As also stated earlier, not much research has been done on the *non*-STEM students who are required to study mathematics as part of their degree programme in humanities or social science.

The six students in this study do not represent all non-STEM students, also given the fact that they attend a top US university. Nevertheless, going back to the concept of 'fittingness' (Schofield 1990) discussed earlier, the results might be used as a point of reference for other studies addressing similar issues not only in relation to non-STEM students but also in relation to the extent to which STEM students might be more successful in solving tasks even when they only apply the concept image, contrary to the main results in many studies referred to earlier. The study shows cases of benefits of using concept image over concept definition when such images are coherent. This study also shows cases of students who are able to refer to the definitions when solving tasks, which is different from the main conclusions of many previous studies. Further study is therefore needed to understand the learning process of the non-STEM students at other types of universities as well.

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