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Mapping Variation in Children's Mathematical Reasoning: The Case of 'What Else Belongs?'

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Abstract Explaining appears to dominate primary teachers' understanding of mathematical reasoning when it is not confused with problem solving. Drawing on previous literature of mathematical reasoning, we generate a view of the critical aspects of reasoning that may assist primary teachers when designing and enacting tasks to elicit and develop mathematical reasoning. The task used in this study of children's reasoning is a number commonality problem. We analysed written and verbal samples of reasoning gathered from children in grades 3 and 4 from three primary schools in Australia and one elementary school in Canada to map the variation in their reasoning. We found that comparing and contrasting was a critical aspect of forming conjectures when generalising in this context, an action not specified in frameworks for generalising in early algebra. The variance in children's reasoning elicited through this task also illuminated the difference between explaining and justifying.

Keywords Comparing and contrasting · Generalising · Justifying · Reasoning · Variation theory

Introduction

Reasoning is prominent in mathematics curricula around the world and is regarded as entwined in the learning of mathematics for all years of schooling (Brodie, 2010; Carpenter, Franke & Levi, 2003; Hunter, 2006; Kilpatrick, Swafford & Findell, 2001; Visnovska & Cobb, 2009). Researchers and curriculum writers have articulated various actions of mathematical reasoning to explain the ways reasoning supports children's logical thinking or argument about mathematical concepts and the relations between

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them (Carpenter & Franke, 2001; Carpenter et al., 2003; Dreyfus, 1999; English, 1999; Nunes, Bryant, Barros & Sylva, 2012; Russell, 1999; Schifter, 1999). In particular, generalising is fundamental to mathematics (Kaput, 1999; Russell, 1999), and Carpenter et al. (2003) argued that carefully choosing the mathematical concepts and properties for children to explore, form conjectures about, and test for validity or generality, contributes to children's mathematical learning:

> We have found that students who learn to articulate and justify their own mathematical ideas, reason through their own and others' mathematical explanations, and provide a rationale for their answers develop a deep understanding that is critical to their future success in mathematics and related fields. (Carpenter et al., 2003, p. 6)

However, Lithner (2008) found that reasoning in mathematics classrooms is often imitated when algorithmic reasoning is memorised and contained within specific problem or exercise settings, in contrast to creative reasoning that enables understanding and construction of new knowledge by the learner. Studies of primary teachers' understanding of reasoning reveal that many teachers struggle to define it, confuse reasoning with problem solving or point to examples of children explaining as evidence of the reasoning occurring in their classroom (Clarke, Clarke & Sullivan, 2012; Herbert, Vale, Bragg, Loong & Widjaja, 2015; Loong, Vale, Bragg & Herbert, 2013). However, as Stebbing (1952) argued, an explanation of the process of forming a statement does not satisfy the listener in everyday situations who is actually seeking evidence:

> When we are told something that is startling or unpleasant we may be moved to ask our informant: 'How do you know that?' Usually such as question is a demand for reasons: we want to know the grounds for the statement rather than to inquire what were the processes of thought through which our informant was led to make the statement in question; we are asking for some assurance; we are not willing to accept the statement without evidence. The sort of answer that would satisfy such a questioner would take the form: 'Because it (ie. what was originally stated) follows from so-and-so'. (Stebbing, 1952, p. 1)

Research of students' mathematical reasoning typically has reported reasoning by students in the middle or secondary years of schooling in algebra. Notable exceptions involving reasoning by younger primary children have focused on generalising and justifying in both early algebra and number operations or relations (for example, Carpenter & Franke, 2001; Carpenter et al., 2003; Copper & Warren, 2008; English 1999). However, it is not clear how the classifications or characteristics of generalisation and justification arising from the studies that have focused on generalising number patterns in the middle years of schooling apply to other mathematics topics or tasks for primary children's reasoning. Nor do these frameworks make clear other reasoning processes that enable generalisation and justification to occur and be made visible to a teacher. If mathematical reasoning underpins mathematical learning (Carpenter et al., 2003), then we should expect it to be elicited and developed using a diverse range of tasks, not just tasks designed to generalise pattern or test conjectures. Swan (2011) calls for making reasoning "the object of attention" in the mathematics lesson and "visible and audible" through written and oral explanations.

Our aim in conducting this study is to explore these issues by examining 7–10-year-old children's reasoning actions when solving a number commonality problem: "What else belongs?" (Small, 2011). We use the notion of dimensions of variation (Lo & Marton, 2012; Marton & Booth, 1997; Marton & Pang, 2006; Runesson, 2005; Watson & Mason, 2006) to explore variation in primary children's reasoning and discuss affordances and constraints of the "What else belongs?" problem in eliciting children's reasoning. By attending to the variation in children's reasoning, this study hopes to identify the critical aspects of reasoning actions to inform teachers' awareness so that they can support their students' mathematical reasoning.

Background

In this section, we explain variation theory prior to reviewing studies of reasoning concerning generalising or justifying by students in the middle years of schooling.

Variation Theory

A central tenet in variation theory is that patterns of variance and invariance are necessary for learning. To guide and discern a particular feature of a phenomenon, a learner must experience potential alternatives of this feature by attending to variation (difference) and invariance (sameness) in this feature and in other features of the *object of learning* (Marton & Pang, 2006; Marton, Runesson, & Tsui, 2004; Runesson, 2005). Variation theorists describe the process of learning as a shift in awareness arising from instantiation, contrast, generalisation and fusion of the critical aspects of a phenomenon (Marton & Booth, 1997; Marton, 2014). "Meaning is derived from difference; not from sameness" (Lo, 2012, p. 93) so foregrounding variation in a feature against relative invariance of other features is more likely to lead learners to discern this aspect of a phenomenon (Lo & Marton, 2012; Watson & Mason, 2006). The action of comparing and contrasting to notice "what is the same?" and "what is different?" is crucial as it enables generalisation defined as "differentiating the critical aspects from those that are not" (Lo & Marton, 2012, p.11). Watson & Mason (2006) contend that when using an *object of learning* approach "tasks that carefully display constrained variation are generally likely to result in progress in ways that unstructured sets of tasks do not, as long as the learners are working within mathematically supportive learning environments" (p. 92).

Marton et al. (2004) suggested that "[t]he critical features have, at least in part, to be found empirically - for instance through interviews with learners" (p 24). Variation theorists investigate *dimensions of variation* that is the "themes of expanding awareness" (Akerlind 2005, p. 122) of a phenomenon portrayed by learners. More sophisticated conceptions of a phenomenon are differentiated from less sophisticated conceptions when learners attend to variation within and between features or to additional features of a phenomenon. By analysing what it is that students' notice when working on tasks, and in their interactions with other students and the teacher, variation theorists are able to discern the dimensions of variation and hence the critical features of a phenomenon for learning.

The reasoning actions of contrasting, generalising and fusing (or abstracting) are prominent in studies of the application of variation theory for mathematics teaching and some attention is given to explaining the critical aspects usually by the teacher and sometimes by students (for example, Lo, 2012; Marton & Pang, 2006; Runesson, 2005). However, it is not clear where or how justification or logical argument fits into this sequence of learning experiences.

Generalising and Justifying

Kaput (1999) defined generalisation as "deliberately extending the range of reasoning or communication beyond the case or cases considered explicitly identifying and exploring commonality across cases" (p. 136). Adopting an actor-orientated theoretical perspective, Ellis (2007a, b) who studied grade 7 students' reasoning when generalising linear growth in realistic problems, developed a taxonomy to distinguish between generalising actions and statements of generality, which she called "reflection generalisations" (Ellis, 2007b, p. 244). She argued that making this distinction enabled a focus on the processes students use to generalise, rather than a focus on the structure of statements of generalisation typical of other studies (for example, Cooper & Warren, 2008; Fuji & Stephens, 2001; Lannin, 2005; Radford, 2008). The taxonomy also identified the following three levels of sophistication of generalising actions amongst she Grade 7 students she studied: (1) relating, (2) searching and (3) extending. Whilst the action of relating in Ellis' framework shows that students perceive features, this action involves finding a similarity, rather than noticing variance and invariance. Ellis found that it was only when students searched that identifying commonality occurred.

Lobato, Hohensee and Rhodehamel (2013), who also used an actor-orientated approach when researching middle years' students' generalisation found that what students notice "has significant ramifications for how they reason" (p. 810). By considering students' verbal interactions, gestures and written records including drawings, annotations, labelling and symbolic representations, they inferred the focus of students' noticing. They were influenced by Mason (1998) who recognised that students may notice perceptual or conceptual features when comparing and contrasting objects. However, other researchers of children's mathematical reasoning have not highlighted the action of comparing and contrasting quite so decisively.

"Reflection generalisations" were distinguished from generalising actions by Ellis (2007b) and defined as "the final statements of generalisation" (p. 244). Drawing on Harel's (2006) student proof scheme, Ellis proposed three levels of reflection generalisation: (1) statements, (2) definitions and (3) influences of generality. Statements of generality include statements of commonality, similarity or general phenomena, such as a general formula or fact or a meaning of an object or idea. A definition of generality proposes a "class of objects … satisfying a given relationship, pattern or phenomenon" (Ellis, 2007b, p. 245). Later, Lannin, Ellis and Elliot (2011) reorganised these categories and combined conjecturing and generalising to nominate four essential understandings of generalising: (1) developing statements, (2) identifying commonality and extending beyond original cases, (3) recognising a domain for which the generalisation holds and (4) "clarifying the meaning of terms, symbols and representations" (p. 12). This final category of generalising gives prominence to the capacity to communicate reasoning. However, this framework of essential understandings of generalising no longer identifies actions that lead students to form conjectures and generalisations.

Justifying enables students to make sense of the mathematics for themselves and to convince others that a procedure or strategy they have used is valid or that a conjecture or generalisation is justified (Carpenter et al., 2003; Dreyfus, 1999; Lannin et al., 2011; Lannin, 2005; Pedemonte, 2007; Schifter, 1999). Teachers tend to believe that children need to be able to communicate their reasoning orally, that is, explain their thinking verbally (Bragg & Vale, 2014), whereas children may use other representations, such as symbolic representations or drawings, to convince their peers (Brodie, 2010; Carpenter et al., 2003; Dickenson & Doerr, 2014; Schifter, 1999; Widjaja, 2014). Explanations during which students describe strategies or procedures might be considered informal reasoning, since Brousseau and Gibel (2005) argued that explanations need to be complete and logically connected to be considered formal reasoning. On the other hand, Lannin (2005) argued that some explanations demonstrate mathematical insight when they show how a generalisation works across cases, that is, verifies ageneralisation. Another view is that justification is more than explanation and involves argument used to validate and to convince (Carpenter et al., 2003; Stebbing, 1952).

Carpenter et al. (2003) identified three classes of justification to describe the ways in which primary school-aged students justify and argue for the truth of mathematical statements: "appeal to authority; justification by example; and generalizable arguments" (p. 87). When appealing to authority primary, students may ask the teacher or another student to confirm that their solution or strategy is correct. Justifying by example means that they give one or more than one example, but examples are not sufficient to validate a conjecture or for proof, unless it is to refute or disprove (Carpenter et al., 2003; Widjaja, 2014). Carpenter et al. (2003) identified the most sophisticated form of justification as "generalizable argument" (p. 87), the capacity to build on prior knowledge or previously justified conjectures to generalise, justify and test conjectures. Other studies of justification have defined when justification is evident, and formalised using logical argument or not (for example, Brousseau & Gibel, 2005; Dickerson & Doerr, 2014; Lannin et al., 2011; Lithner, 2008), rather than categorising justification from least to most convincing forms of justification.

In this study, variation theory is used to frame our analysis of children's reasoning when solving the commonality problem: "What else belongs?" (Small, 2011). Using the previous literature and frameworks, we report on the variation in children's reasoning to generate a view of the critical aspects that primary teachers need to be aware of when mathematical reasoning is a learning objective. One element missing from existing generalising frameworks is comparing and contrasting, yet this element is seen to underpin variation theory. We also seek to clarify the distinction between explaining and justifying in this problem context. Mapping variation in children's reasoning by building on existing frameworks, leads to identification of the critical aspects of children's reasoning. Reflecting on the variation of children's reasoning enables refinement of the task and teacher actions.

The Study

The study reported here is part of a larger research project the Mathematical Reasoning Professional Learning Research Program (MRPLRP), which was designed to support primary teachers' developing understanding and teaching of mathematical reasoning. Three primary schools in Australia and one elementary school in Canada participated in this project. Two of the Australian schools were small rural schools whereas the others were located in large metropolitan cities, in lower socio-economic communities. This study involved 80 7–10-year-old students who were members of a grade 3 or composite grade 3–4 class at the four schools where the demonstration lesson using the open-ended task "What else belongs?" (Small, 2011) took place. We chose this grade level as previous studies typically focused on students in the middle schooling grades (5 to 8) or early grades. At each school a teacher participating in the project volunteered their class for the demonstration lesson. The demonstration lessons were taught by two of the authors and one other educator experienced in demonstration lessons.

This task was selected because it invites mathematical reasoning associated with the idea of same (invariant) and different (variant), and forming and testing a conjecture about a common property for a set of numbers. We chose {30, 12, 18} for the group of numbers for this task. As an open-ended task, it was appropriate to use as a "jump in" lesson for children with diverse prior knowledge of number properties and facts. Hence, we could use it in these diverse schools as the task did not need to fit into a specific sequence of lessons. Furthermore, it offered the opportunity to explore children's reasoning using a task not commonly used for generalising.

It was expected that the lesson would provide opportunities to elicit children's capacity to compare and contrast, form conjectures, generalise, explain and justify. The learning objectives were as follows:

- Students will notice and describe properties of number, such as size, order, composition, place value, multiples, factors, even or odd.
- Students will notice and explain common properties and justify.

The lesson was structured to support children's collaboration when reasoning and to communicate their reasoning during pair-work and then whole class discussion. At the beginning of the lesson, the set of numbers {30, 12, 18} was displayed to the whole class and the problem posed. When launching the problem the teacher did not take verbal responses but rather used stimulating questioning to encourage children to compare and contrast what they know about these numbers to find what is the same about these numbers:

> Think about what you know about each of these numbers. [Pause] I wonder could these numbers belong together. [Pause] I'm wondering what reasons there might be for these numbers belonging together in this group. [Pause] Why do you think these numbers belong together? [Pause] What is your reason? [Pause] I wonder if there is more than one reason …

These questions were intended to encourage children to form conjectures about possible common properties of these numbers. The children then worked with a partner to solve the problem and respond to three questions:

1. These numbers belong together because…
2. Other numbers that belong with this group are…

3. How do you know that all these numbers fit with your reason? Use words, numbers or drawings to explain.

The first of these questions calls on students to generalise, that is, to explore and identify a common property for the numbers in this set; the second invites them to extend the generality by identifying other cases. We chose this set of numbers because it provided the opportunity for students to identify one, or more than one common property. This set of numbers also provided children with the opportunity to notice relations between these common properties and use this to find another common property. The third question required students to justify their solutions to the previous two questions, that is, to justify their generalisation, and called on them to explain the common property.

During the paired working time, the teacher of the demonstration lesson observed the children's reasoning, reminding children that they need to be able to convince others that their reason works, and noted the different statements of generalisation and varying levels of complexity in the children's reasoning to share these later with the class. After 10 min the whole class was brought together to share and discuss their conjectures and justifications. The teacher invited the selected pairs of children using a statement such as:

> You selected [insert number, e.g. 6]. Tell us why. Convince us that this reason works for all the numbers [30, 12, 18, 6].

The lesson plan indicated that the teacher would use scaffolding questions to draw out the children's understanding of their nominated reason, that is, their generalisation, for example, how did they know the numbers were even or a multiple of 3. During this discussion, the teacher highlighted the action of contrasting by drawing the children's attention to counter examples, "What numbers would not belong if this is the reason? Why not?" The range of common properties identified by students, along with other numbers that belonged and examples of numbers that did not belong, were documented on the white board using the children's language during the whole class orchestrated discussion of solutions.

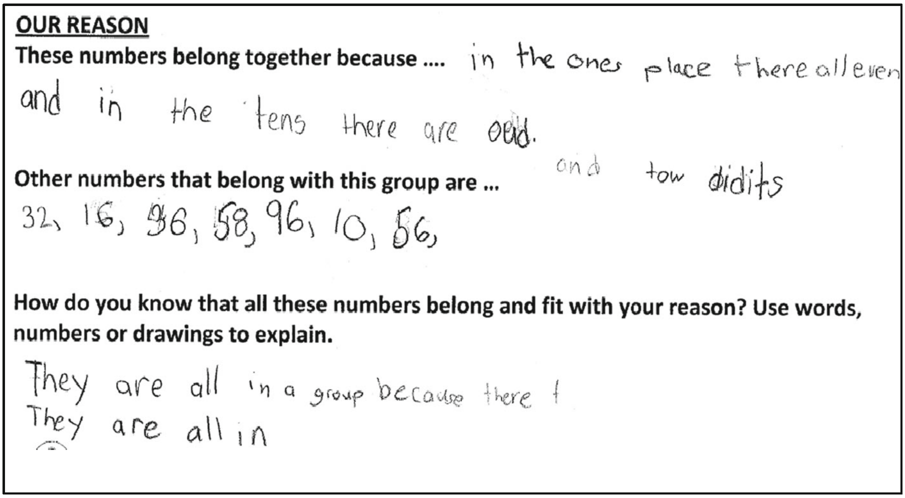
Methods of Data Collection and Analysis

Each demonstration lesson was video-recorded. Two video-cameras, operated by professionals, were used in the three Australian schools; one recorded whole class discussions, whilst the other roamed the room recording pairs of children working on the task. The written responses of the children (pairs) to the reasoning task were collected. One author observed the lesson, took photographs of the children's written work as they worked on the task and kept field notes at each Australian school. One video-camera was used in the Canadian classroom where one author taught the lesson.

The research team analysed the data, that is, the primary school children's written responses to the three questions and the children's reasoning captured on video-recordings of each demonstration lesson, to identify the variation in children's reasoning. As explained above, variation theorists map the variation of students' work to discern the critical aspects of the object of learning and to refine tasks and teacher actions (Lo 2012; Marton & Booth, 1997; Mason 1998; Runesson, 2005; Watson & Mason, 2006). Mapping the variation in children's reasoning actions enabled us to discern the critical aspects of mathematical reasoning in the context of this number commonality problem and to evaluate the task and teacher actions.

Employing constant-comparative analysis (Glaser, 1965), data were analysed in three phases: analysis of written responses, analysis of interactions between children and teacher during pair-work and finally analysis of interaction between children and the teacher during whole class discussion. Throughout these three stages, our analysis focused on reasoning actions, rather than the children's content knowledge, though for some primary children, their content knowledge did enable them to notice particular common properties and use particular reasoning actions.

Analysis of Written Responses. Initially, each of the written responses to the three questions by pairs of children was compared and contrasted to determine the variation in the responses and categorised according to reasoning action and further coded according to possible levels of reasoning by two members of the research team. The three generalising actions and three levels of reflection generalisation documented by Ellis (2007a, b) and Carpenter et al. (2003) classes of justification were used as initial categories and codes in the process of coding the data. We identified the variation in how and what the children noticed about these numbers from their annotations, drawings and written statements, whether or not they formed one or more conjectures about possible common properties and whether or not and how they had justified their conjecture. The initial categories identified independently by two authors for the written responses were compared and discussed to reach agreement. Where data varied from the categories documented by Ellis (2007a) or Carpenter et al. (2003), we proposed a new code or category. This process of analysis of written responses is further explained using responses from two pairs of students Naomi and Dennis (pseudonyms, school B, Fig. 1) and Dane and Lloyd (school D, Fig. 2).


OUR REASON

These numbers belong together because .... in the ones place there all even and in the tens there are odd.

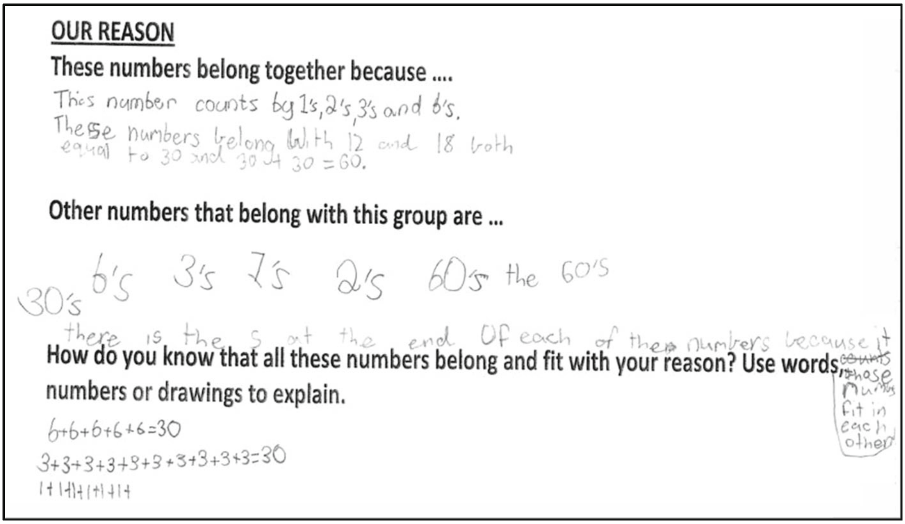
Other numbers that belong with this group are ... and tow didits

32, 16, 36, 58, 96, 10, 56,

How do you know that all these numbers belong and fit with your reason? Use words, numbers or drawings to explain.

They are all in a group because there t
They are all in

Fig. 1 Written responses by Naomi and Dennis (school B)


OUR REASON

These numbers belong together because ....

This number counts by 1's, 2's, 3's and 6's.
These numbers belong With 12 and 18 both equal to 30 and 30+30 = 60.

Other numbers that belong with this group are ...

30's 6's 3's 7's 2's 60's the 60's

there is the s at the end Of each of thes numbers because it ~~counts~~ those nums fit in each other

How do you know that all these numbers belong and fit with your reason? Use words, numbers or drawings to explain.

6+6+6+6+6=30
3+3+3+3+3+3+3+3+3+3=30
1+1+1+1+1+1+1

Fig. 2 Written responses by Dane and Lloyd (school D)

Naomi and Dennis (Fig. 1) attended to the digits and place value of the three numbers, to notice what was common and what was different: "in the ones place there (sic) all even and in the tens there are (sic) odd". They also noticed that the numbers are two-digit numbers. We can infer that these two children drew on their knowledge of place value, and the notion of odd and even to form this conjecture. They recorded three properties about these numbers which indicates that they continued to search for common properties after first noticing that the digits in the ones place were all even. Dane and Lloyd's response to the first question (Fig. 2) also shows that they noticed more than one common property for this group of numbers. Their conjecture was "counts by 1 s, 2 s, 3 s and 6 s". They identified the factors 1, 2, 3 and 6, though it is not clear whether their noticing was automatic recall or a consequence of searching for common factors. They also noticed an additive relation between the three numbers "12 and 18 both equal to 30…" These two pairs of children provided evidence of noticing relations, noticing commonalities and searching for commonalities. These two pairs of children also recorded statements of generality for the commonalities that they noticed or found. Variation in their statements arising from comparing and contrasting is also potentially related to variation in their prior knowledge of number facts and properties of number.

Responses to the second question revealed further variation in the children's reasoning. Naomi and Dennis (Fig. 1) extended their generalisation to include other cases by using their "rule" to find other numbers. Dane and Lloyd (Fig. 2) did not identify other individual numbers that belonged with the group, rather they repeated their conjecture. Naomi and Dennis (Fig. 1) did not complete their response to the final question but Dane and Lloyd attempted to justify their conjectures: "there is the s at the end of each of the numbers because [each of] those numbers fit in each other". Figure 2 shows that Dane and Lloyd documented repeated addition facts to validate their conjecture that

30 is in the 3-s and also in the 6-s counting pattern. However, they did not validate that their conjecture held for the other two numbers in the group. So their justification is used to explain or show "count by 3 s" and "count by 6 s" another way rather than to verify that these two properties hold for all three numbers. These cases provided examples of *no justification* and *using examples to explain* rather than to verify or justify.

The frequency of reasoning actions discerned from analysis of all written responses are recorded in the tables below. However, if we had limited our mapping of children's reasoning to the evidence of their written responses, we would have failed to reveal the extent and variation of children's reasoning that is elicited through their interactions with others.

Analysis of Interactions. The second stage of analysis focused on children's attention and actions, including gestures (Lobato et al., 2013) when generating these written responses as captured on video or observations and field notes. One author coded the video data using the categories previously generated, another reviewed the coding of data for level of reasoning and disagreements were resolved through discussion by the research team.

For example, during pair-work, the teacher at school B challenged Naomi and Dennis to justify their conjecture. In response, Naomi used the analogy of "partner" to explain the meaning of odd and Dennis followed up using gesture (see Fig. 3) to explain this analogy for even numbers.

Fig. 3 Student using gesture to show partners for even numbers (school B, Dennis)

Teacher: OK. How do you know these are odd numbers?

Naomi: Because two of them have a partner and there is one left out.

Teacher: Oh, so then how do you know this is an even number?

Naomi: Because on the back it has an even number.

Dennis: Like 2, 4, 6, 8.

Teacher: OK so can you just show me again how you are doing that?

Dennis: [pairing his fingers on each hand, see Fig. 3] Two partners, two partners, groups of two …

This interaction confirmed that the students could explain even and odd number but it did not provide a justification for their generalisation. To find out if the children could extend their generalisation, the teacher then asked them to identify another number that would join the group. Naomi nominated 32 and Dennis nominated 15. Attending to these children's analogy and use of gesture, the teacher conducted a gesture conversation with these children to clarify their use of analogy for explanation of even. In the excerpt below, Dennis tested a counter example and Naomi continued to use her analogy and gesture to explain why 15 cannot join the group:

Teacher: OK [Dennis] can you think of another number that fits with this group?

Dennis: Perhaps they're all even, I'll have an odd number, perhaps 15 [writes 15 down] because they're near these.

Teacher: So does this fit your reason (reads it from their sheet)?

Naomi: Five is not even.

Teacher: How do you know it's not even?

Naomi: Because it has one person left out.

Teacher: So can you say a bit more.

Naomi: There's like 5 people and you have to group up and there is one person left out (holds pointer finger up).

This excerpt shows the significant role of teacher questioning and revoicing in prompting children's reasoning, particularly explaining and testing their conjecture about the common property. Other children in this class also used the analogy of partner and the teacher used this analogy and gesture to clarify children's understanding of odd and even numbers during the whole class discussion. The reasoning of these two children illustrated *using an example* and *counter example* as a means of explanation rather than justification.

This stage of analysis illuminated the variation in *comparing and contrasting* actions used by the children more clearly, such that three levels of reasoning for this category were proposed that were related to, but distinct from, Ellis' (2007b) first two generalising actions. This stage of analysis also contributed findings related to different levels of justification that could not be discerned from children's written responses.

Analysis of Whole Class Discussions. In the final stage of analysis, verbal responses and gestures of pairs of children selected to contribute to whole class discussion of solutions captured on the video-recording of whole class discussion were coded using the same process followed for the other video data. For example, during whole class discussion, Dane and Lloyd (Fig. 2) explained their analogy of "fits in" and justified their conjectures. Following a choral count by 6 s, suggested by one child during the whole class discussion to verify that each number was in the 6 s counting pattern, Dane (school D) explained the meaning of "fits in": "6 goes into that [pointing at 30] 5 times, 6 goes into that [pointing at 12] 2 times, 6 goes into that [pointing at 18] 3 times". By providing these facts for each number, Dane *verified* their conjecture for each number in the group. So even though these two students did not verify their conjecture for each number in their written response, they did so when asked to convince others.

Analysis of whole class discussion enabled further distinction to be made between explaining and verifying to illuminate different levels of justification. As illustrated here, the discussion also enabled children to hear different explanations and justifications for the same conjecture. The whole class discussion also provided an opportunity to deepen the justification using a generalisable argument regarding multiples of 1, 2, 3 and 6. However, these students were not called upon to use a generalisation to justify that if 6 is the factor of a number, then 2 and 3 will also be factors of the number.

The analysis presented above illustrates the process followed to identify the variation in reasoning actions and degree of completeness or level of reasoning. The findings arising from analysis of all the data are presented in the following section.

Findings

In this section, we map the variation in children's reasoning actions elicited when forming a generalisation about the properties of a set of numbers and justifying this generalisation. We confirmed three main actions: *comparing and contrasting, generalising* and *justifying*. Mapping the variation in children's responses according to thoroughness or completeness revealed levels of reasoning within each of these categories. The levels of reasoning discerned for each reasoning action are defined and illustrated for the 40 pairs of children who completed this task (see Tables 1, 2 and 3). Frequency data for written responses are included in these tables, and further evidence from written responses and interactions is presented to validate the framework and justify classification of children's reasoning.

Table 1 Comparing and contrasting: levels and frequency by pairs of children recorded on worksheets

Level of comparing and contrasting	Descriptors	Frequency (N = 40)
1. Noticing (seeing) similarities or relations	Children use knowledge of number to compare and contrast numbers by order, size or digits used or to form a number sentence to relate the numbers	19
2. Noticing commonalities and differences	Children automatically recall knowledge of numbers or number facts to identify or recall properties that are common.	31
3. Searching for commonalities	Children actively work through various processes such as recalling and documenting facts or counting patterns randomly or systematically to identify common properties. Their search is to consider possibilities that are not immediately apparent, so that comparing and contrasting continues.	4

Comparing and Contrasting

In order for the children to form and document a reason (conjecture) for grouping the numbers, they needed to notice *commonalities* or *common properties*. Recalling knowledge to notice a common property is an act of comparing and contrasting. The two pairs of children discussed above recalled knowledge to notice common properties and to notice relations; they also continued to recall or search for other common properties. These three actions were confirmed through analysis of all the written responses and used to define three levels of thoroughness when *comparing and contrasting* as shown in Table 1. The frequency of levels of *comparing and contrasting* identified from the children's written responses are also recorded in Table 1. The total frequency is more than the number of pairs of children, indicating that some pairs of children provided evidence of more than one level of reasoning in this category.

Noticing Similarities. Some children noticed similarities, rather than common properties, indicating a perceptual or beginning level of comparing and contrasting. For

Table 2 Generalising: levels and frequency by pairs of children recorded on worksheets

Level of generalising	Descriptors	Frequency (N = 40)
1. Forming conjectures about common properties	Recording a statement about the common property using words, symbols, number sentences or rules.	22
2. Extending a common property through further examples	Recording further examples of the common property to show that the property exists beyond the examples as a generalisation rather than a coincidence.	30
3. Generalising properties	Notices and records relations between common properties to identify a further common property or to define the domain (or boundaries) within which the commonality holds.	0

Table 3 Justifying: levels and frequency by pairs of children recorded on worksheets

Level of justifying	Descriptors	Frequency (N = 40)
0. No justification	Does not show that the common property holds true or believes that the common property is so obvious it does not need to be justified.	5
1. Appealing to authority or others	Checks with others that their generalisation holds, that is, their statement is correct.	5
2. Explaining a common property using an example; explaining a common property using a counter example	States a number fact or pattern or uses diagrams, analogy or gesture of one example to show what the property means; states a number fact or uses diagrams, analogy or gesture of a counter example to show what the property does not mean.	7
	Incorrect explanation	1
3. Verifying that the common property holds for each member of the group	Shows that the property is common to each example using number facts or counting patterns.	11
	Incomplete verification [counting all but did not start at 2, 3 or 6]	5
4. Extending generalisation using logical argument	Use an "if … then …" logical argument to use one generalisation to verify the common property.	0

example, Talia and Britany listed the digits used to make these numbers, and other students noticed the closeness of the numbers to other numbers (for example, Emily, Brad and Jack, Fig. 4). These children drew upon knowledge of symbols and order of numbers rather than properties.

Noticing Commonalities and Differences. Three-quarters of the children noticed common properties of these numbers, such as that the numbers were even or belonged to a counting pattern, as illustrated earlier. The action of comparing and contrasting was stated explicitly by Ryan when responding to the teacher during pair work in the following excerpt:

> *Teacher*: Who'd like to explain what you've been doing?
>
> *Ryan*: We've been looking at the numbers and working out what's different and we've so far worked out that they're all over 10, under 40 and they each have 2 digits.

Searching for Commonalities. A small proportion of the children (about 10 %) went beyond what they could see in the numbers to explore what they knew about each of these numbers in search of a common property. They explored counting patterns to

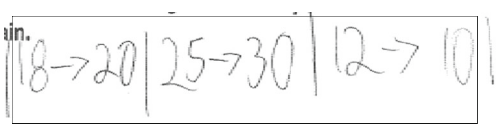


Fig. 4 Noticing closeness to multiples of 10 (school B, Emily, Jack and Brad)

determine if each of these numbers were included; they documented number facts or operated on the numbers looking for something in common. For example, Shanti (school D) explained to the teacher: "Well we're trying to figure out well like something about the numbers like see if we can like split them to see if there is something to do with them." Another two children from school C (Madison and Ron) searched for common properties of these numbers. They randomly recorded the following number facts on the worksheet: "12 × 1 = 12, 4 × 3 = 12, 12 × 5 = 60, 2 × 14 = 60 (sic), 3 × 6 = 18; 5 × 6 = 30, 3 × 10 = 30". However, whilst their documentation shows that they searched looking for a common property, they did not notice that 3 was a factor in at least one of the multiplication facts they recorded for each number in the group. A challenging prompt from their partner or the teacher about "what is the same?" and "what is different?" in their list of number facts may have helped them to notice the common factor. However, their response shows that the process of searching for a common property also involves testing to verify that the property holds.

The children's responses indicate that they were using knowledge of numeration, such as the order of numbers and place value, and knowledge of counting patterns, multiples or factors when comparing and contrasting these numbers to notice similarities and commonalities.

Generalising

Evidence collected indicated that children's conjectures not only depended on whether they had noticed commonalities or relations but also varied in level of response depending on whether they were able to extend the generality with further examples. Levels of generalising along with definitions and frequencies of written responses are recorded in Table 2.

Forming Conjectures About Common Properties. The children formed conjectures, with more than half of participants recording a statement about a common property: "all two digit"; "each number is [created] by 2 s they are evan (sic) numbers", "all in the threes [twos, sixes] counting pattern", "all in the count by 2 s pattern", "count by 6 s", "in the 6 column 6 times tables", "all in the groups of 6", "6 fits into each number" or "in the times patterns". Whilst students did not use the term multiple or factor, the fact that some students identified that the numbers belong to the 6 column in 6 times tables or 6 fits into each number indicates their understanding that 6 is a common factor. Many children found more than one reason for grouping these numbers and recorded more than one conjecture about the group of numbers.

Some children did not record a response and children who noticed the additive relationship between the numbers generally did not express a common property for these numbers.

Extending a Common Property Through Further Examples. The action of finding further examples of numbers with the common property was used by the children to consolidate their conjecture and begin to test and verify their generalisation. Three quarters of the pairs of students recorded further examples on the worksheet. This reasoning action functioned as the first test for their conjecture, that is, whether or not there were other numbers with the same property. Some children recalled or thought of other numbers fitting their conjecture, for example, “6, 24” for “count by sixes” (Noel and Ben, school C). Others used counting patterns to find and record other numbers belonging to the group; some starting at 12 and others at 2, 3 or 6 depending on the counting pattern.

Ryan and Jeff (school D) combined the common properties they identified “over 10, under 40 and they each have 2 digits” and “even” and found two other numbers that could be members of the group which satisfied each property not just one. This generalising action is similar to defining the domain for which the conjecture holds when forming a conjecture (Lannin et al., 2011). However, aside from this pair, defining a domain was not observed.

Generalising Properties. Dane and Lloyd (discussed above) appeared close to exploring or noticing the relation between factors, whilst Brett and Jarrod (school A) used a more general property to find other numbers to join the group. They identified “all in the 3 s counting pattern” as the common property and listed “3, 6, 9, 15, 21, 24, 27, 60, 72, 84, 96, 108, 120” as other numbers that could join the group. These two boys counted by 3 leaving out the numbers already in the group, doubled 30 to list 60 as another number that belonged, and then counted on by 12 to list more numbers that belonged. When this pair of students were invited to report and justify their reason for grouping the numbers during the whole class discussion, Brett offered 216 as another number that was in the counting pattern and explained: “Yeah, and I just got to 108 and doubled it.” His response revealed that rather than using pattern Brett used a conjecture, that is, the doubling of any number in the counting pattern to find another number that belonged. They had also used the conjecture that multiples of 12 are also multiples of 3 when they counted on by 12 s from 60 to find other numbers on their worksheet. Their responses on the worksheet and during whole class discussion show that these children demonstrated knowledge and use of relations between properties and a more general structure, but the teacher missed an opportunity to challenge children to justify their general conjecture.

The variations in generalising observed show that generalising is an outcome of comparing and contrasting that involves at least some testing when formulating the statement of the common property. Generalising is more likely to be achieved if the task does not provide examples that distract from identifying commonality. The data also revealed the potential for the task to enable generalising beyond the situation if supported or encouraged by the teacher.

Justifying

Justifying, the final reasoning category or action was prompted by the last question on the worksheet. The levels of justification discerned are recorded and defined in Table 3, along with frequency of these actions identified in written responses. One quarter of the children either did not record a justification or *appealed to authority* (Carpenter et al., 2003). Some children did not provide an answer to the last question or repeated their response to the first question. Other children sought confirmation from the teacher, another student or materials available in the classroom such as multiplication fact posters. In these instances, the children *appealed to authority.* Evidence of levels 2 to 4 of justification are presented and discussed below.

Explaining a Common Property Using One Example. Children used known properties or number facts to justify their conjecture for at least one number in the group. They used number sentences (Fig. 2), analogy and gesture (Fig. 3) or diagrams of concrete materials (Fig. 5) to explain their conjecture. Tanika and Trish (Fig. 5) argued that 18 is an even number because you can find even numbers by adding 2, though they have not established that 16 is an even number. Their diagram shows that even numbers can be split equally. "Even to each" is interpreted as meaning the two groups are the same. Other children also used "even" and "equal" to mean "the same". These findings show that children communicate their explanations using a range of representations.

Explaining a Common Property Using Counter Examples. Aside from negative cases revealed by children when searching for a common property or extending the generalisation (e.g. Naomi and Dennis, above), the teacher introduced the idea of counter examples during the whole class discussion of the first task by collecting and recording numbers that do not belong. For example, at school D during whole class discussion, the teacher asked: "How did we know that 33 was not even?" Hence, rather than using counter examples, or negative cases, as a means of refuting a conjecture (Lannin et al., 2011), counter examples proved useful in challenging the children's understanding to clarifying the meaning of even, multiple or factor. In this way, the children under the guidance of the teacher re-engaged students in contrasting actions to deepen their explanation of the generalisation.

"Explaining a common property using one example" and "explaining a common property using a counter example" were used by 20% of children when responding to the last question on the worksheet. We considered using examples and counter examples to be at the same level because these actions clarified the meaning of terms used in their statements of commonality (Lannin et al., 2011) but did not verify that this property held for each number in the set.

Verifying That the Common Property Holds for Each Member of the Group. Verifying was the most frequent form of justifying, used by 40 % of children. Almost all of the pairs of students categorised as *verifying* used a count all/record all strategy starting at 2, 3 or 6 in their recorded response to the second question. Some children more clearly verified that each number belonged by crossing out 30, 12 and 18 in these counting patterns. Sequentially, listing all members of the group is an effective strategy to verify if the group is finite. Indeed at each school, the teacher performed a

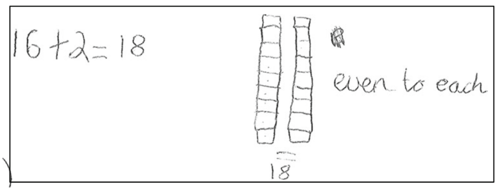


Fig. 5 Explaining conjecture of evenness using one example (school A, Tanika and Trish)

choral count during the whole class discussion to verify generalisations phrased as counting patterns. This strategy is convincing because it verifies that each number belongs. Five pairs of children recorded their count all strategy starting at 12 or 18 or 30 so were not as convincing in verifying that all three numbers belonged to the counting pattern nominated. So rather than verifying their conjecture, they were extending the generalisation. These are recorded as incomplete verification in Table 3.

A much smaller proportion of children verified that each number belonged using other known facts, for example, the whole class discussion of Dane and Lloyd's generalisation of "fits in" presented earlier. Jordan and Elton (school C) wrote that these numbers belonged together because they were "all in the 6 groups of" and justified this generalisation using multiplication facts: "$2 \times 6 = 12$, $3 \times 6 = 18$, $5 \times 6 = 30$". Bonnie and Eve (school C) (Fig. 6) used diagrams to show that each number was even because it could be split into two equal groups. Their drawing of 30 shows it split into three groups of 10 with each group of 10 represented as two groups of five.

Generalisable Argument. Whilst some students in this study used relations and logical argument to extend the generalisation beyond the cases, there were no examples of children using logical arguments to justify their generalisation. Pairs of students who identified multiple factors were more likely to have attempted such a justification, for example, Dane and Lloyd, but the teacher needed to provide a challenging prompt for this to occur.

Most children in this study used a count all strategy to extend the generalisation and in so doing verified that all the numbers belonged to the group, thereby justifying their conjecture. But it is not clear whether or not these children would verify that each number belonged if a count all strategy would not achieve this goal.

Discussion

Mapping the variation in children's reasoning confirmed different reasoning actions and levels of reasoning when solving "What else belongs?" This study showed that open-ended tasks that involve forming conjectures about a common property provide an opportunity for the reasoning actions: comparing and contrasting, generalising and justifying and one in which children can appreciate the need to convince (Hunter, 2006). We noticed variation in the way reasoning was communicated including oral

Fig. 6 Verifying that each number is even (school C, Bonnie and Eve)

and written statements and arguments, symbolic recording of numbers and number sentences, as well as through gesture, analogy and diagrams of concrete materials consistent with other studies of children's reasoning (Carpenter et al., 2003, Dickerson & Doerr, 2014; English, 1999; Lobato et al., 2013; Radford, 2008). Analysis of reasoning arising from this common properties task revealed variation in degrees of thoroughness or completeness of reasoning actions. We used this variance in responses to identify levels of reasoning, or at least to tease out and clarify levels previously documented (Carpenter et al., 2003; Ellis, 2007a, b; Lannin et al., 2011) for this problem context (Table 4).

We have used *comparing and contrasting* (Lo, 2012; Lobato et al., 2013) rather than generalising actions (Ellis 2007b) to name the processes of noticing and searching for similarity, relations and commonality across cases. However, we agree with Ellis that going beyond properties that were easily noticed to search for common properties exemplified higher order reasoning when comparing and contrasting. This task did not elicit much evidence of contrasting. This was more likely to occur during the whole class discussion when prompted by the teacher to identify numbers that did not belong. It is likely that the task "What does not belong?" (Small, 2011) would provide more opportunities for contrasting, that is noticing variance or "what is different".

In line with Kaput (1999) and Lannin et al. (2011), we used *forming conjectures* to describe the first level of generalising action as the initial step when generalising common properties. In this study, some students recorded more than one common property which might be interpreted to be a domain for their conjecture, but this did not constitute a higher level of reasoning as presented by Carpenter et al. (2003) and

Table 4 Reasoning actions and levels: commonality problem "What else belongs?"

Reasoning Actions	Reasoning Levels	Source
Comparing and contrasting	1. Noticing (seeing) similarities or relations	Ellis (2007b); current study
Lo (2012); Lobato et al. (2013)	2. Noticing commonalities and differences	Current study
	3. Searching for commonalities	Ellis (2007b); current study
Generalising	1. Forming conjectures about common properties	Kaput (1999); Lannin et al. (2011); current study
	2. Extending a common property through further examples	Lannin et al. (2011); current study
	3. Generalising properties	Ellis (2007b)
Justifying	0. No justification	Carpenter et al. (2003); current study
	1. Appealing to authority or others	Carpenter et al. (2003); current study
	2. Explaining a common property using an example or counter property	Carpenter et al. (2003); current study
	3. Verifying that the common property holds for each member of the group	Current study
	4. Extending generalization using logical argument	Brousseau and Gibel (2005); Carpenter et al. (2003); Dickerson and Doerr (2014); Lannin et al. (2011); Lithner (2008); Stebbing (1952)

Lannin et al. (2011). A few students were on the cusp of extending their generalisation to relate the common properties as a general structure as previously observed (Carpenter et al., 2003; Cooper & Warren, 2008; Ellis, 2007a, b; Radford, 2008) but we found that eliciting this level of reasoning would require further challenge by the teacher, perhaps in a follow-up lesson if not in the moment.

The context of common number properties might also account for the use of examples as justification, when compared with other studies more focused on proof (Carpenter et al., 2003; Lannin et al., 2011; Widjaja, 2014). Other researchers have shown that giving one example is not sufficient to provide a complete justification (Brousseau & Gibel, 2005; Carpenter et al., 2003). We argue that one or more examples are the basis of an explanation concerning the meaning of the conjecture rather than an argument to verify or validate a conjecture. Variation theory gives prominence to the discernment of critical aspects by attending to variance and invariance, however unless the generalisation is explained and verified the critical aspects may only be perceived rather than rationalised.

Mapping the variation in children's reasoning re-iterated the importance of designing the task so that the action of comparing and contrasting guides student awareness of features that matter (Lo, 2012; Watson & Mason, 2006). The choice of the three numbers {30, 12, 18} distracted some children who noticed an additive relation between the three numbers which limited their capacity to generalise. In a subsequent demonstration of this lesson conducted at another school not participating in this study, the numbers 30, 12 and 24 were used and the children were able to focus attention on common properties of these numbers. The terms used and order of the questions in this task may also have limited children's opportunity to justify or test their conjecture about the common property. If we had asked "How do you know that all these numbers fit with your reason?" as the second question, instead of the third question, we may have gathered convincing arguments from children who used only examples when attempting to justify. Awareness of the potential of the task to provoke generalisation of properties using logical argument would enable teachers to be prepared to provide a challenging prompt to explain why these numbers could join the group.

Conclusion

Mapping the variation in children's reasoning for this number commonality problem has generated a view of the critical aspects of reasoning when solving a number commonality problem, a task not commonly researched for eliciting generalisation or justification. Our findings confirm the importance of comparing and contrasting to generalise and identifies verifying, a level of reasoning not included in reasoning frameworks (see Table 4). A distinction between explaining, used to define the property or conjecture, and verifying, to provide a convincing argument was presented. Carpenter et al. (2003) levels of justification do not include a level involving validating or verifying other than by generalisable argument, and logical argument is the only means of justifying according to Lannin et al. (2011).

Attending to the variation in children's reasoning enabled us to evaluate and improve the task where the learning objective is mathematical reasoning. Our findings imply that teachers need to be aware of the possible variation in students' reasoning, that is, the different categories and levels of reasoning, to notice and act in the moment to develop their students' mathematical reasoning. Questions such as "What is your reason for …?", "How do you know …?" and "Why …?" prompted students to notice and search for commonalities, form conjectures and provide explanations using examples. Prompts such as "Convince me/us …" and "Show that it works for all cases" were needed for students to verify and hence justify their conjecture. Other specific prompts, or an extended task, are needed for children to generalise properties or use logical argument to justify. Further research is needed to identify teacher actions, and other mathematics contexts and tasks, for children to compare and contrast, form conjectures and develop generalisations, and where explaining and verifying may foreground the use of a convincing argument.

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