Int J of Sci and Math Educ (2017) 15:759–776
DOI 10.1007/s10763-016-9712-3

Students' Mathematical Reasoning and Beliefs in Non-routine Task Solving

Jonas Jäder[1,2] **· Johan Sidenvall**[2,3] **· Lovisa Sumpter**[4]

Received: 19 February 2015 / Accepted: 21 January 2016 / Published online: 29 February 2016

Abstract Beliefs and problem solving are connected and have been studied in different contexts. One of the common results of previous research is that students tend to prefer algorithmic approaches to mathematical tasks. This study explores Swedish upper secondary school students' beliefs and reasoning when solving non-routine tasks. The results regarding the beliefs indicated by the students were found deductively and include expectations, motivational beliefs and security. When it comes to reasoning, a variety of approaches were found. Even though the tasks were designed to demand more than imitation of algorithms, students used this method and failed to solve the task.

Keywords Beliefs · Mathematical reasoning · Non-routine tasks · Problem solving · Upper secondary school

Introduction

Research has shown that students adjust their solutions to perceived picture of solution strategies (to specific problems) and that lack of confidence combined with previous lack of

✉ Johan Sidenvall
johan.sidenvall@liu.se; johan.sidenvall@outlook.com

Jonas Jäder
jjr@du.se

Lovisa Sumpter
lovisa.sumpter@mnd.su.se

[1] School of Education, Health and Social Studies, Dalarna University, SE-791 88 Falun, Sweden

[2] Department of Social and Welfare Studies, Linköping University, Norrköping SE-601 74, Sweden

[3] School Administration, Municipality of Hudiksvall, SE-824 80 Hudiksvall, Sweden

[4] Department of Mathematics and Science Education, Stockholm University, SE-106 91 Stockholm, Sweden

success could make them stop working (Lerch, 2004). Hence, in mathematical task solving there are other factors involved than just cognitive ones. There is a fairly good picture of the relation between affect and problem solving and the conditions for being a successful problem solver. A student who has confidence and control is more likely to continue and ultimately succeed when solving problems (Hannula, 2006). However, previous research has identified beliefs that could influence students in their problem solving. For example, beliefs can be restraining when solving tasks as some students hold beliefs that tasks should be solved in an algorithmic manner and do not expect to understand, but only to memorise, and also that a task should be solved in 5 min or less (Schoenfeld, 1992). But beliefs can also assist a student to be persistent instead of giving up when working on a task (Carlson, 1999). In this way, beliefs play a part in problem solving.

According to the Swedish Schools Inspectorate (2010), Swedish students to a large extent are engaged in an education that emphasises rote learning and procedural knowledge. To receive an education in line with the curriculum and develop, e.g. problem solving and reasoning competencies, students must be given more comprehensive, better developed and more systematic opportunities to engage in activities than they receive from working on textbook tasks that heavily rely on procedures (Swedish Schools Inspectorate, 2010). Aspects such as curriculum and the character of the task (e.g. routine or non-routine) may influence the beliefs (Liu, 2010). This implies that beliefs may be highly contextualised and results depend on the tasks you give to the students. In a study of Swedish upper secondary school students' beliefs and reasoning when solving routine task, the results involved three themes of beliefs: safety, expectations and motivation (Sumpter, 2013). These beliefs seemed to interplay and connect to imitative mathematical reasoning. Other research has also pointed out the connection between reasoning and affect; it has been shown that an imitative approach is less stressful and secure (Sfard & Linchevski, 1994), and that combinations of beliefs shape the way a student develops solutions to tasks (Furinghetti & Morselli, 2009). During the past three decades, extensive research has been performed on students' beliefs. Yet more research is needed at a pre-college level, since most of the research performed is in a college context (Francisco, 2013). In this present study, we explore upper secondary school students' beliefs and reasoning when solving non-routine tasks. The research question posed is: What beliefs are indicated in students' reasoning when solving non-routine tasks?

Theoretical Background

This paper has two main concepts: beliefs and mathematical reasoning. We will here give a short background to these concepts in the context of students' task solving, and more specifically the solving of non-routine tasks. These two concepts will later be discussed in relation to each other and the results.

Beliefs

Depending on the research question, different aspects of the concept beliefs need to be emphasised. Beliefs also have components in both the cognitive area and the affective area and appear to work in symbioses with other affective concepts such as attitudes and emotions (Cobb, Yackel & Wood, 1989). This means that it could be hard to separate different concepts from each other. In this paper, we build upon the results from Sumpter (2013) and therefore the same definitions will be used. Beliefs are defined as "an individual's understandings that shape the ways that the individual conceptualises and engages in mathematical behaviour generating and appearing as thoughts in mind" (Sumpter, 2013, p. 1118). All beliefs are attributed (Speer, 2005), and we are aware of the impact methods and researchers play when we report data and draw our conclusions. Beliefs are stable since they are developed over a long period of time and less likely to change (Hannula, 2006). A motivational belief is a belief that includes an active goal, e.g. a correct answer (independent of strategy) would result in point on an exam. An emotional belief is a belief that involves an emotional component, e.g. my own reasoning is not a safe strategy, whereas an emotional reaction is a feeling, e.g. insecure. A longer discussion is provided in Sumpter (2013).

Another issue to recognise is that it is not possible to establish causality between specific beliefs and actions (Callejo & Vila, 2009). Still the notion of beliefs works as a model that can produce attributions and thereby help with predicting and explaining behaviour. Following this, we use the notion of *Beliefs Indications* (*BI*) introduced by Sumpter (2013). BI is defined as a "theoretical concept and part of a model aimed to describe a specific phenomenon, i.e. the type of arguments given by students when solving school tasks in a lab setting" (Sumpter, 2013, p. 1116). This means that we talk about beliefs as being indicated and attributed. BI is therefore a tool for studying arguments, and they could be local (e.g. about a specific choice when solving a task) or global (e.g. a belief about mathematical problem solving).

In relation to a greater classroom context, an individual's beliefs can be thought of as an understanding of the classroom norm, which is collectively negotiated among the classroom participants (Yackel & Rasmussen, 2002). Beliefs are then contextually bound (Francisco, 2013), and the context sets the rules operating together with other affective factors.

The three themes of beliefs shown by Sumpter (2013), expectations, motivation and security, can be further divided into sub-categories. An expectation has been seen as being either personal, e.g. I can only solve tasks by memorising an algorithm (Schoenfeld, 1992), or subject oriented, e.g. doing mathematics is to memorise (facts, theorems) and reproduce (Furinghetti & Morselli, 2009). There could also be combination of these two types of expectations. For instance, it has been reported that a majority of students believe it more valuable to memorise than to think in mathematics classrooms (Boaler et al., 2000), which could be both a personal expectation and an expectation on mathematics education. The motivational beliefs indicated in Sumpter (2013) were either sprung from intrinsic or extrinsic motivation (Ryan & Deci, 2000). As to security, this may be seen as an emotional belief saying something about a student's view of his or her own degree of security in relation to a specific task-solving situation. Mercer (2010) argues that emotions may both establish and strengthen beliefs.

The different beliefs interplay to create an "internal dynamic of the belief systems that characterises how students' beliefs influence learning and problem solving" (Op' T Eynde, De Corte & Verschaffel, 2002, p. 33). Beliefs do not just interplay with each other but also interact with other affective notions such as emotions and attitudes (Hannula, 2006).

Reasoning

In the literature 'reasoning' is often defined as a skill of high deductive-logical quality (Lithner, 2003). At the same time, Ball and Bass (2003) state that "mathematical reasoning is no more than a basic skill" (p.28). The latter implies that reasoning can be found in all levels of mathematical understanding. The overall assumption in this study is that mathematical reasoning can be used at all levels of difficulty in solving non-routine tasks.

A broad definition of reasoning is applied: “reasoning is the line of thought adopted to produce assertions and reach conclusions in task solving. It is not necessarily based on formal logic, thus not restricted to proof, and may even be incorrect as long as there are some kind of sensible (to the reasoner) reasons backing it” (Lithner, 2008, p. 257). This definition provides flexibility when studying different types of reasoning since it does not have to be based on formal logic, and it even allows reasoning to be incorrect. We use the four-step reasoning sequence proposed by Lithner (2003): (1) a (sub-)task is met, which is denoted task situation; (2) a strategy choice is made where ‘choice’ is seen in a wide sense (choose, recall, construct, discover, guess etc.); (3) the strategy is implemented; and (4) a conclusion is obtained.

The characterisation of reasoning types is based on analyses of arguments for strategy choice and implementation. Empirical studies have shown it possible to extract these arguments from students when solving tasks both in pairs (Schoenfeld, 1985; Sidenvall, Lithner, & Jäder, 2015) and individually (Boesen, Lithner, & Palm, 2010). It has also been shown successful to use interviews to further strengthen the analysis (Bergqvist, Lithner & Sumpter, 2008; Boesen et al., 2010). There are two main categories of reasoning: imitative reasoning (IR) and creative mathematical founded reasoning (CMR) (Lithner, 2008). In IR, the task solver applies a recalled or externally provided solution method. In CMR, the solver constructs a solution method. There are three central aspects distinguishing CMR from IR (Lithner, 2008): (1) a new reasoning sequence is created, or a forgotten one is re-created; (2) there are arguments supporting the strategy choice and/or strategy implementation motivating why the conclusions are true or plausible; and, (3) the arguments are anchored in intrinsic mathematical properties of the components involved in the reasoning. IR contains no CMR, but CMR may contain parts of IR. For example, to solve a task, a student might need to use the formula for calculating the area of a circle, which is an example of IR, while the rest of the task solving requires the student to create a, to him novel solution, using CMR. In the framework (Lithner, 2008) both IR and CMR contain subgroups to further specify the reasoning used. These sub-categories become secondary in light of the aim of the study and will therefore not be operationalized.

Non-routine Tasks

The reasoning used by students is likely to be dependent on the type of task they meet. Also, the beliefs indicated by students are likely to alter depending on the type of task (Hoyles, 1992). In relation to the two types of reasoning mentioned above, we would like to distinguish two types of tasks. Routine tasks are tasks that a student has met before, maybe on several occasions, and have a ready algorithm to solve, while non-routine tasks to the student means that he or she has to create to him or her a new solution method to solve the task. Therefore, characterising tasks as being routine or non-routine also means considering the students and their previous experiences. There is evidence to the statement that students tend to use imitative reasoning when faced with routine tasks (Boesen et al., 2010). It also seems that a wider range of reasoning is used when students work on non-routine tasks (Boesen et al., 2010). Therefore, it is possible that a student could interpret a non-routine task as a routine task, a task that is not routine for them, and coherently to their belief they will use a routine-task approach.

Methods

Data was collected by video recording task-solving sessions and stimulated recall interviews, both of which were fully transcribed. The students' written solutions were also part of the data. The students participating in this study worked in pairs in a lab situation (c.f. Bergqvist et al., 2008; Schoenfeld, 1985). The students were encouraged to talk to each other while solving the tasks and this enabled us to extract their arguments from the communication. Apart from the encouragement to talk out loud and the possibility to use the textbook and the calculator, no further instructions or time constraints were given to the students. They were placed in an adjacent room during an ordinary class session with a video camera and microphone set up. Eight students from year 1 of the upper secondary school, equivalent to year 10 of schooling were selected from two programmes with different intensities of mathematics, the Building and Construction Programme (four boys) and the Social Science Programme (two girls and two boys). Two teachers were asked to select two pairs of students each that usually work together in their task solving and are likely to communicate verbally with each other. More than 50 % of the students taking this courses either fail or receive the lowest passing grade. Therefore, the teachers were also asked to select students from the group of students that were expected to barely pass the course. In order to pass the course, you need to be able to perform and follow mathematical reasoning.

Post-interviews were used to clarify issues concerning the task-solving sessions. Semi-structured, stimulated recall interviews were conducted individually since both the reasoning used and the beliefs indicted were analysed separately rather than in pairs. This is made possible since the method is based on the students' individual arguments and not their collective mathematical reasoning. In total, the data consisted of (I) four task-solving sessions with a total length of 1 h and 40 min (varying between 17 and 32 min/session), (II) eight interviews with a total length of 3 h and 20 min (varying between 12 and 34 min/interview) and (III) written solutions to all tasks from all eight students.

To answer the research question posed, we needed to identify and select tasks of a non-routine character that were in line with the course curriculum for the designated students. We choose four tasks from national tests specific to the courses the students were taking. Using the method of Boesen et al. (2010), we compared the tasks of the national test with the textbooks used by the students to conclude that the tasks were of non-routine character. According to the method used by Boesen et al. (2010), a task was considered to be of routine character if, in the students textbooks, there were tasks, templates or solved examples that needed the same algorithm to be solved as the solution to the task in the national test needed. If there was none, in one or two equivalent algorithms in the students' textbook, it was considered that the student had not had enough chances to remember the algorithm and the task in the national test was then categorised as non-routine. A further aim was to provide a progression of difficulty as to meet each of the students at an appropriate level. Therefore, tasks at different levels of difficulty were chosen. Dìaz-Obando, Plasencia-Cruz, & Solano-Alvarado (2003) show that the approach that students take to problem solving depends on his or her capacity, which could be linked to the level of difficulty on specific tasks. A motivational factor that has been proven important is the level of difficulty, where too easy or too hard tasks may bore or frustrate the students (Kloosterman, 2002). Here, we apply the same method as in Bergqvist et al. (2008) and use tasks considered to be of three different levels of difficulty. In relation to student's capacity, these may be called easiest (task 1), intermediate (tasks 2 and 3) and more difficult (task 4). All four tasks are presented in Appendix.

We analysed the reasoning using Lithner (2008) framework. In applying the framework to analyse student's reasoning and to conduct a more fine-grained analysis, we identified subtasks in the students' task solving and subdivided the subtasks using the four-step reasoning sequence. Following this, any observed predictive and verifying arguments were identified in each subtask situation. These arguments were used to identify the student's strategy choice and strategy implementation, and also on what basis these choices and implementations were made. The reasoning sequence was then classified according to the reasoning types in focus of this study, IR or CMR.

When studying affective issues in mathematics education, the issue of subjectivity is present; in relation to their study, Furinghetti and Morselli (2009) state that they are "aware of the fact that our methodological choice may introduce elements of questionability and subjectivity in our work" (p. 78). In a likewise manner, we acknowledge our method to be partly subjective. Therefore, it is necessary to show how the analysis was carried out by referring to detailed examples of data for the reader to follow. Following the categorisation of the reasoning used by the eight students we, after a first analysis, identified three students that all showed different behaviours in regard to the reasoning used. These three students also provided us with rich data in terms of verbal communication in the task-solving sessions and the interviews.

To analyse students' beliefs, we used a thematic analysis of the video recordings, transcripts and students' written solutions was conducted with equal attention to these data items (Braun & Clarke, 2006), where we focused on BI (Sumpter, 2013). Beliefs indications could be explicit meta-cognitive statements in the transcripts of the task-solving session, the transcripts of the interview, or in the students' task-solving notes. They could also have an emotional element such as a sigh or a gesture connected to emotions or an explicit mention of a feeling (e.g. 'I do not like this'). Passages when the BI was not clear were left out. The BIs were then interpreted in a wider context to be able to work deductively considering the three themes attributed by Sumpter (2013). The three themes of belief indications, *security*, *motivation* and *expectation* were used as a basis for consideration. Two of the authors analysed the data by discussing it in relation to the framework for categorising reasoning and to what beliefs were indicated in the students' statements. The discussions were in all cases ended with an agreement on both the reasoning used and what beliefs were indicated. There was also a possibility to discuss issues with the third author.

In this way, there were two separate analysis procedures: one for the reasoning used and one for the beliefs indicated. The next step in the analysis was to recognise what beliefs were indicated concurrently as a specific type of reasoning was used. In some cases, this connection was between an indication of beliefs and a change of reasoning or a pattern of reasoning sequences. The generated data might have been different if the data had been collected in another setting, e.g. with single students and using a think aloud protocol. We are also aware that the lab setting might have had an impact on the generated data and the results and conclusion only concern this particular setting.

Results

The results indicate that students expressed beliefs of security, motivation and expectation. We have been able to further distinguish between different kinds of expectations belonging to the subject-oriented category. The motivational beliefs have been further divided into positive and negative. This applies to security as well, which we have been able to see as both security and insecurity. In our presentation of the results, we will show how beliefs connect to the reasoning used. We will do this by first exemplifying the work of three students (Leila, Karl , & Eric) on one of the four tasks, task 2. The reason for exemplifying data using task 2, in line with Kloosterman (2002), is that this task appeared to be a task where all these three students met their challenge. By this we mean that the task was neither too easy nor too hard. The task-solving session and the interview rendered a lot of data concerning the chosen task. The task that we use to exemplify the students' work is the following: 'Which of the following expressions correspond to the perimeter of the figure (see Fig. 1)?

$$a+b, \quad 2a+2b \quad 3a+2b \quad 3a+3b, \quad 4a+2b.$$

Motivate your answer'. (Swedish National Agency for Education, 2010, p. 3, authors' translation)

Leila, Reasoning

Leila worked in pair with Anna. Leila's work on the task resulted in three reasoning sequences.

Part 1

Leila	[Reads the question]. Seriously, I don't know how to do this stuff. […] What? Is this whole line *a*? [Pointing at the most left vertical line.]
Anna	Yes.
Leila	And all of this, is *b*? [Pointing along the bottom of the figure.]
Anna	Yes.
Leila	Isn't it 2, or, eh, *2a* plus *2b*? Then it is …
	[…]
Leila	If you, [gesturing in the figure] just move these [indentations], kind of, then it will be, then you just take …
Anna	Yes…
Leila	No, I don't get it. [pause] But it, if you, even if you squeeze it together, it should be *2a* plus *2b*? […]
Anna	Yeah. […] It should be that one [pointing at the expression *2a+2b*].
Leila	Mmh.
	[…]
Leila	But then you could write, eeh: it is *2a+2b* since if you squeeze it together the sides becomes equal length.
Anna	Yeah.

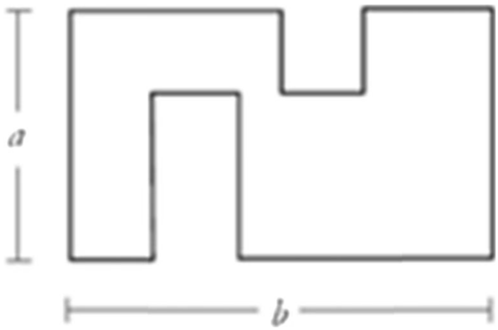


Fig. 1 Figure to task 2 (Swedish National Agency for Education, 2010, p. 3) The data have been translated from Swedish. stands for omitted passages not relevant to the solving process

Task situation 1: Which of the following expressions correspond to the perimeter of the figure and why?
Strategy choice 1: Leila argues that the indentations do not add any length to the perimeter of the figure. Therefore, it can be considered to be a rectangle and the algorithm of a rectangle perimeter can therefore be used.
Strategy implementation 1: Leila adds the side lengths of the rectangle $a \times b$, (without considering that the indentations add to the perimeter).
Conclusion 1: Leila's answers is $2a + 2b$, but hesitates in her motivation of the chosen algebraic expression.
Leila's reasoning: Leila does not consider necessary properties of the figure, the indentations, in her strategy choice. Instead she uses the algorithm for computing the perimeter of a rectangle. This algorithm was considered to be familiar to Leila. The reasoning used in the first sequence was therefore categorised as IR, according to the framework by Lithner (2008). Leila's conclusion from the first reasoning sequence results in a new reasoning sequence, since Leila hesitates when she is to motivate her choice of expression.

Part 2

Leila	Or? [pause] It can't *4a* plus *2b*, can it? [pause] Or, yeah, if you add these, [pointing at the vertical lines in the indentations] maybe. Look, this one and then you add it.
Anna	Uhm.
Leila	… and then you add it to this one.
Anna	But what?
Leila	Look. [pointing at the figure]
Anna	Yes.
Leila	This, [pointing above the left indentation] thing …, or, there is something missing here, …
Anna	Yes.
Leila	… then you can, kind of take this [pointing at the larger indentation] and this one [pointing at the smaller indentation] or, not. [Pause] I don't get it.

Task situation 2: Which of the following expressions correspond to the perimeter of the figure and why?

Strategy choice 2: Leila considers the indentations as important for the computation of the perimeter. She also finds a way to integrate this necessary property into her solution. The argument supporting her following implementation is that the smaller vertical line of an indentation is the missing part of a vertical line of the larger indentation. The sum of these two vertical lines would be the same as the length of a.
Strategy implementation 2: Leila adds one of the vertical sides of the larger indentation to one of the sides of the smaller one making up another a. She repeats this for the other side of the indentations which leads to Leila seeing four a's in the figure. Leila then simplifies the expression $a + a + a + a + 2b$ to $4a + 2b$.
Conclusion 2: Leila's answer is $4a+2b$. Before Leila writes a motivation for the choice of algebraic expression she states 'I don't get it'.
Leila's reasoning: Leila creates a to her a novel solution. She gives an argument for her strategy choice (adding the vertical sides of the indentations) based on necessary intrinsic mathematical properties. Her reasoning sequence is therefore categorised as CMR. Nevertheless, Leila abandons the correct solution and the answer, which results in a third and final reasoning sequence.

Part 3

Leila	Should we write $2a + 2b$?
Anna	Mmh.
Leila	Because you, if you squeeze together the, block …, or, yeah, the figure, then, it is …
Anna	It will be …
Leila	Mmh.
	[Writes down the solution]

Task situation 3: Which of the following expressions correspond to the perimeter of the figure and why?

Strategy choice 3: Leila reconsiders her interpretation of the relevance of the indentations. She, once again omits the indentations by 'squeezing' the figure. 'Because you, if you squeeze the blocks …, or yeah, the figure […] then the figure becomes a rectangle'.

Strategy implementation 3: Adds the vertical sides ($a + a$) and the horizontal sides, interpreted as being of length equivalent to b ($b + b$).
Conclusion 3: $2a + 2b$ with the motivation that squeezing the figure, a rectangle is formed and the sides of the rectangle are not equally long.

Leila's reasoning: Leila returns to her incorrect answer $2a + 2b$ and omits an important intrinsic property of the figure; the indentations in her strategy choice. This reasoning sequence is therefore categorised as imitative reasoning.

Leila, Beliefs Indicated

Using the notion of BI (Sumpter, 2013), there are several beliefs indicated in the data of Leila working on this task. The subsequent interview also clarified several of these belief indications. There are three main beliefs indicated in the data on Leila. Firstly, Leila says that she does not know how to do this, when meeting the task. Leila's statement reoccurs several times throughout the three reasoning sequences. In the interview, Leila also says that:

'I am unsure when I have to think differently, therefore I choose the simplest way'. Combined, this indicates that there is an expectation on the task to be solvable in a way where the amount of thinking is reasonable according to Leila. The simplest way here, according to Leila is to consider the figure as a rectangle (without the indentations).

Leila, Reasoning and Beliefs Indicated

The above presented example is signifying for Leila as it shows her use of IR. Leila uses IR solving all four tasks. In the interview, Leila expresses that the first task, to her, is of routine character. She also delivers a correct solution to this task. Of the other three tasks, one more was solved with a correct result, using IR via peer guidance, and the two others were incorrect.

Leila indicates similar intrinsic motivational beliefs when working on all four tasks, expressing that she does not understand. On two occasions she also indicates an insecurity regarding her own ability. In the interview she states: 'Because, I would have probably done something wrong. You know, not knowing how to really think and just skipped something […] it had become more complicated'. As shown in the example above, she also indicates an expectation on herself not to be able to solve the task. In the interview, Leila states that she 'can only solve tasks with normal shapes, and not with indentations […] it becomes too complicated. […] You have to think differently, […] but then I used what seemed easiest'. These three beliefs seem to interplay in a way that supports each other. The three beliefs support each other so that, for example, the personal expectation of only being able to solve tasks with familiar geometrical figures may strengthen her intrinsic motivation of not understanding the task and also her insecurity of the task being of an unfamiliar character and also of her own solution.

Karl, Reasoning

Karl's work, in pair with Ian, resulted in one reasoning sequence.

Karl	I thought that it should be […] If we were to just do like this [showing in the figure]. Then we will get…
Ian	Yeah ….
Karl	… then we will have *2b* and *2a*. And then we pull these [indentations] together, put them opposite each other, then we will get, 1, 2 …
Ian	4, 5.
Karl	We [have], 1, 2, 3, 4 at least got four a's. And two b's, It should be *4a* and *2b*.
Ian	What? I don't get what you are getting at!
Karl	If we were to move this, [points at the smaller indentation], and put it here instead. Instead of here. Then it would be, 1, 2, 3, 4, four lengths.
Karl	I'll just… ehh
Ian	You can't write a solution to this, can you?
Karl	No, we've been talking so much, anyway

> Task situation: Which of the following expressions correspond to the perimeter of the figure and why?

Strategy choice: Karl immediately sees that the vertical sides in the indentations can be expressed by variable 'a'. 'We pull these [the vertical sides in the indentations] together, put them opposite each other. Then we get 1, 2, […] 4 a's'.
Strategy implementation: Karl simplifies the expression $a+a+a+a+2b$ to $4a+2b$.
Conclusion: $4a+2b$. Did not give a written motivation to the choice of algebraic expression
Karl's reasoning: Karl constructs a novel solution method to solve the task. The assumption that the method is novel is strengthened by Karl's statement in the interview where he says that he has not seen this type of task before. To meet a new, or non-routine task does not necessarily mean to create a novel solution method as we have seen. The assumption made by the authors is strengthened by Karl's work with the task. The arguments that the length 'a' can be found also as parts of the indentations are made plausible by his pointing in the figure. Karl anchors the properties of the indentations and the different forms of representation of the variables a and b intrinsically. Therefore the reasoning sequence is categorised as CMR.

Karl, Beliefs Indicated

Karl indicates motivational beliefs in two different ways solving the second task. First, he expresses that he does not understand but later in the interview formulating that he nevertheless thought that the solution was ok even though he was not certain. Altogether, he shows insecurity about whether the results were correct or not.

Karl, Reasoning and Beliefs Indicated

Karl uses CMR to solve all tasks except the last and most difficult one. The last task is also the only task where Karl' solution and answer is incorrect. Summarising Karl's BI on all tasks, he shows greater confidence than Leila but also both security and insecurity. On the first task he indicates, in a similar way to the task in the example above, a negative intrinsic motivational belief saying that he does not understand. In the interview, he stresses a different motivational belief, saying that '[the solution] just came to me'. Karl here has a clear picture of the complete solution and uses CMR to solve the task. Karl also indicates an extrinsic motivational belief regarding the written computation of a solution on paper. He finds it necessary to present a solution to us, in a similar manner as to his class teacher, but also finds it hard to written these written computations. In one task solution (task 3), Karl indicates a belief of expectation that the task should be solvable by using an algorithm when he in a response to Ian, who says 'But I don't remember how to do this', replies 'Neither do I'.. This is also the only task where Karl actually uses IR in his attempt to solve a task. The other subject oriented expectation that is indicated on two occasions in Karl's task solving, is that he expects tasks to be of a certain level of difficulty. He states in the interview 'I understood that it was too simple', and by this he signals that he had an expectation on how complicated the solution should be to meet the level of the tasks. On the other occasion, concerning the fourth and final task, Karl is puzzled when he says 'It's probably a trick question that is actually really simple to answer'. Here he expresses an expectation that it should be a difficult question but his simple IR solution does not fit his expectation of complexity of the task. The relation between the beliefs held and the reasoning used in the case of Karl is a relation between a mix of negative and positive intrinsic motivation and the relative security he shows regarding his ability to create novel solutions and to use intrinsic mathematical properties and his use of CMR.

Eric, Reasoning

Eric's work, in pair with Axel, with task 2 resulted in one reasoning sequence.

Eric	We can make our own ruler, [using] a piece of paper. […] Two [squares on the paper] is 1 cm.
	[Eric and Axel measures in the figure using the piece of squared paper.] […]
Eric	Let's guess that this [pointing at the larger of the indentations and referring back to a measurement of the figures height as 3.5 cm] is maybe two to three centimetres.
	[Eric and Axel continue to measure in the figure.] […]
Eric	Yes, listen, listen, listen. This side is as long as one *b*. This [pointing at the lower part of the figure] is *b*.
Axel	Mmh.
Eric	Mh. Plus one *a*, if you compare with that one [the figures upper part]. This is just one *b*, and this is one *b* and one *a*. [lower part of the figure].
Axel	But, look, this is what they show, this side [the figures left side] is a. Then, it just that side [the figures right hand side] …
Eric	Mmh.
Axel	… and that side [the figures right hand side] that is *a*.
Eric	[inaudible] 3.5 centimetres longer here [bottom part of the figure]. That is why I use *a* instead. [pointing at the bottom right horizontal line] Do you understand what I mean?
Axel	No
Eric	But that side [bottom] is 12.5, that [upper] 9.5
Axel	Mmh.
Eric	[counting on his fingers] It differs three, almost an a.
Axel	But, that …
Eric	I know, but I know what that side [left] is, what I mean is that the they [bottom] are just as long as that [upper], but one more of those [left side, *a*].
	[…]
Eric	Yeah. Or you could write three a's, not three of those. […] and one b. But that doesn't work [sees that there is no such algebraic expression. It is *2b*, exactly, *3a* and *2b*.
Axel	*3a*, *2b*?
Eric	Mmh. Because it's still *b* [bottom], *b* [upper], *b* plus *a*, 1, 2, 3.
	[…]
Eric	Can I copy your notes, I haven't written anything, because I've just been trying to work it out.
Axel	But I didn't have the time to write anything.
Eric	Mh. But they have it recorded. […]
Eric	That is about a half less. I doesn't matter if we round off a bit.
	[…]
Eric	Damn, I don't know how to explain on paper, it sounds better when you just talk.
	[…]
Axel	Tell me what you're writing.
Eric	I wrote that '*1b* is 1*a* longer than the other *b*'. Because that is the way it is!

Task situation: Which of the following expressions correspond to the perimeter of the figure and why?
Strategy choice: Eric considers the perimeter of the figure to be affected by the indentations. He chooses to measure all the sub-lengths of the figure. The argument used is that some sub-length measured in centimetres can be transformed into variable a.
Strategy implementation: Eric measures all sub-lengths. The lengths are rounded off to fit the, by Eric measured length of a. Eric adds all sides: left side $= 3.5$ cm $= a$, right side $= 3.5$ cm $= a$, upper side (including small indentation) $= b$, lower side (including indentation) $= b + a$. Eric simplifies the expression $a + a + b + (b + a)$ to $3a + 2b$. Eric acts as if variable b could have different lengths.
Conclusion: $3a + 2b$. Motivation: 'One b is an a longer than the other b. That was why it ended up being $3a$ and $2b$'.
Eric's reasoning: Eric in this reasoning sequence uses a type of reasoning that has the character of CMR. He does not at any point refer to an algorithm or other familiar solution method, but constructs a, to him, new solution method. He argues for his solution together with his peer, basing the arguments on intrinsic properties of the task such as the indentations and what these do to the perimeter of the figure. He also realises that the answer must consist of only 'a' and 'b' terms, meaning that the measured sub-lengths must be represented in terms of the two variables. What is lacking in his reasoning is a correct handling of variables. Nonetheless, this reasoning is categorised as being CMR.

Eric, Beliefs Indicated

Eric indicates several beliefs in working on the example task. He indicates a security in the regard to his solution to the task, at the same time as he shows insecurity in regard to how he came about this solution. His insecurity is apparent in the interview: 'I don't know how we thought, we got that answer anyway'. This may interplay with another expressed BI, an extrinsic motivational belief saying that there should be a written solution to the task. This latter BI is stated when Eric argues for not including a written computation because he finds it difficult. Eric also shows a general frustration about the hassle to present a written solution of the task.

Eric, Reasoning and Beliefs Indicated

The reasoning used by Eric on the four tasks have the character of CMR in that he approaches the tasks by creating novel reasoning sequences and hence also solutions. These solutions are vaguely based on mathematical arguments. He expresses a confidence in these arguments based on the relevant mathematical properties. It can be Eric's mathematical ability that hinders him from always grounding the arguments in the intrinsic mathematical properties, but nevertheless he reasons in a way that is reasonable and plausible to him, in this context. Therefore the reasoning used by Eric on all task but the first task is categorised as being CMR. The first task is to him, just like it was to Leila, of routine character as stated by Eric in the interview. He reaches a correct answer on solely one task, the third one.

During the work on all four tasks, Eric expresses a positive intrinsic motivational belief when he says in the interview that the task-solving process feels rather good and that he thought the solution to be correct (e.g. commenting one of the task solutions: 'It felt right at that moment'.). Working on the last, and most difficult task, he also shows another intrinsic motivational belief expressing that guessing is a valid method to approach a task. In the interview he states: 'well, it's a little difficult, but I try to guess, it feels rather ok'. Eric indicates differentiated beliefs of security. Statements such as 'I know that the answer is correct' and 'I don't know what our thought were, but we finished it anyway' and 'we were really unsure' all indicate different levels of security or insecurity. The use of CMR with an incorrect result is in Eric's task solving connected to mostly positive intrinsic motivational beliefs and security, but also to insecurity and different beliefs relating to the written computations and to the belief that also guessing is valid.

Summary

In the results, we have presented three belief systems connected to three different ways of approaching non-routine tasks, see Table 1.

From Table 1, we can see that the difference in the way the students approach the non-routine problems and also which BIs are connected to their approach.

Discussion

Even though beliefs are contextually bound (Francisco, 2013), studies indicate similarities between different countries (Dìaz-Obando et al., 2003; Furinghetti & Morselli, 2009). Students expect mathematical tasks in school to be solvable by memorised algorithms. Broadening the picture from just memorised to IR, using Lithner's (2008) framework, Swedish upper secondary school students indicated beliefs about mathematics as being of an imitative character, where CMR is not necessary to solve school tasks (Sumpter, 2013). It has also been reported that a majority of students believe it is more valuable to memorise than to think in mathematics classrooms (Boaler, Wiliam & Brown, 2000). What we have seen in this study is that results from previous research are also valid when students work on non-routine tasks. Leila in many ways exemplifies several of the above-mentioned beliefs. Yet, this present study also shows that there are other kinds of beliefs represented among the

Table 1 Beliefs indicated and reasoning used by Leila, Karl and Eric

Student	Beliefs indication	Reasoning
Leila	Insecurity—low personal expectations—negative intrinsic motivation subject expectation	IR, abandoning CMR (correct solution)
Karl	Negative and positive intrinsic motivation	CMR (correct solution)
	Security and insecurity	
	Extrinsic motivation	
Karl	Subject expectation	IR
Eric	Positive intrinsic motivation security and insecurity extrinsic motivation	CMR (incorrect solution)

upper secondary school students in Sweden compared with previous research (c.f. Sumpter, 2013). Two of students in our study indicate beliefs that CMR is indeed a valid method of approach, at least when a complete solution is within reach. These results seem to be an addition to the present picture of students' beliefs about problem solving and mathematical reasoning. However, these results also somewhat contrast previous research that has shown that students tend to focus on familiar algorithms when engaged in problem solving (Carlson, 1999). IR is very fruitful when you want to solve a lot of mathematical task of routine character quickly and (most likely) with a correct answer, but it is not so helpful when facing mathematical problems (Lithner, 2008). We can only speculate why the students in the present study use CMR to the extent shown in the results. The task design might trigger the use of CMR, or it may be that the selected students are weak procedurally. These students, all expecting to barely pass the course are likely to have limited procedural as well as conceptual knowledge. Having this limitation leaves you with two options, either try to fit a well-known algorithm to the task solution or try to create a new solution.

Another result of the present study is that students have indicated a belief of expectation regarding the level of difficulty of the tasks, a subject expectation. This expectation could be connected to an understanding of what type or reasoning should be used, e.g. Leila expressed that one of her solutions to task 2 was too complicated while on the fourth and last task several students indicated a belief that their solution was not complex enough. Kloosterman (2002) also argues that students seem to want to have a clear picture of tasks' level of difficulty and also how the students act in relation to this expectations. A possible explanation to why the students in this present study turned to IR when facing more difficult tasks could be the level of difficulty following Kloosterman (2002). Another possible explanation could be that if students do not see CMR as an option (Sumpter, 2013), then it is difficult to evaluate and control your own reasoning (Schoenfeld, 1992) and thereby judge the 'correct' level of difficulty or simply see if the solution is correct. Boaler (1998) describes the same phenomena: students have expectations on the expected difficulty level of tasks.

All three students indicate beliefs of insecurity and just as in Mercer (2010) the emotions supports beliefs (in either direction). Here, it is illustrated by Leila who shows a negative intrinsic motivation and low personal expectations interplaying with insecurity. Leila's interplay of beliefs may strengthen the connection to her use of IR, a behaviour that is in line with previous research (c.f. Lerch, 2004). Related to this Leila, with a low personal expectation, is the one using IR on all tasks. Callejo and Vila (2009, p. 116) describes a similar situation as a student being "focused on recurrence, without getting into the overall analysis of the situation". Karl shows a mix of positive and negative intrinsic motivation while Eric indicates more positive intrinsic motivation. The motivational belief is what distinguishes the three students from each other, rather than the insecurity. Motivational beliefs can be viewed as the engine, the driving force, of the mathematical work (Hannula, 2006). Without it, it can be hard to sustain the reasoning.

The successful students in Carlson's (1999) study showed high levels of patience, trusting their own thinking even though it did not go smoothly. In this present study, both Karl and Eric use CMR without abandoning it. However, they did not indicate a belief of expectation that a task should be solved using a known algorithm, i.e. using IR. The BI:s connected to CMR are to a large extent the same for Karl and Eric. Eric's indicate beliefs of positive intrinsic motivation and of security that seems to overrule insecurity. Eric may lack the conceptual knowledge that is necessary to be able anchor his solutions in intrinsic mathematical properties. Another possible explanation to his incorrect solutions may be his seemingly unreflective manner. This behaviour is similar to what Callejo and Vila (2009) categorises as being naive, impulsive or unthinking and is characterised for example by quick answer without justifications. Unlike Eric, Karl indicates both a negative and a positive intrinsic motivation. In this particular episode, Karl has a more reflective approach to the task solving than Eric. It appears that students' cognition and beliefs are intertwined following the results from previous research (Furinghetti & Morselli, 2009). The reasons for a student to use IR on non-routine tasks could be that the student holds expectations on the subject, or more specifically, does not know that CMR is an option. Another reason could be that the student argues CMR not being an option (c.f. Sumpter, 2013), which can be paralleled to Sfard and Linchevski (1994) findings that an imitative approach is less stressful and secure. Here it is illustrated by Leila when saying 'I am unsure when I have to think differently, therefore I choose the simplest way'.

To conclude, we see that students when working on non-routine tasks use a variety of approaches including both CMR and IR. Compared with previous studies on students' beliefs when solving routine tasks, this indicates that the task influences the way student's reason (c.f. Liu, 2010). Nevertheless, contrasting with previous research we see that similar themes of beliefs are indicated in this partly new setting. Furthermore, even though the tasks are designed to demand CMR, students still use IR (without success). Combined, this would imply that if you as a teacher would like to support your students to a broader view of mathematics and encourage them to use CMR, you need to do more than just giving non-routine tasks to students and leave it to them to makes sense of it.

Appendix

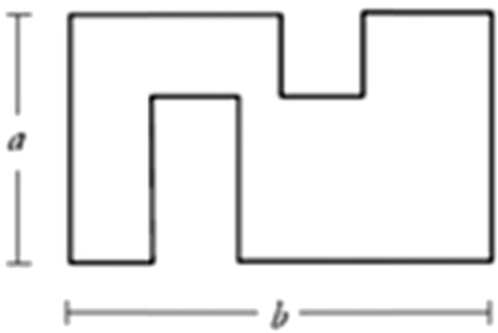
Authors' translation of tasks from Swedish.

1. 'When using 6 kg of apples Astrid gets 2.8 l of apple juice. How many litres of apple juice will she get using 15 kg of apples, of the same sort?'

2. 'Which of the following expressions correspond to the perimeter of the figure?

$$a+b \quad 2a+2b \quad 3a+2b \quad 3a+3b \quad 4a+2b$$

Motivate your answer'.



3. 'The average age of five employees at a sporting goods store was 24 years. A woman of age 36 years was hired as shop manager. What will be the new average age of the employees at the sporting goods store?'

4. 'A circular one person American pizza has a diameter of 21 cm. What should the diameter be for the pizza to be a two person pizza?'

Task 1: Swedish National Agency for Education (2005a). *Nationellt kursprov i matematik, kurs A, våren 2005 Del I* [National test in mathematics Course A Spring 2005 Part I]. Available at http://www.su.se/primgruppen/matematik/kurs-1/tidigare-prov. Swedish.

Task 2: Swedish National Agency for Education (2010). *Nationellt kursprov i matematik, kurs A, våren 2010, Del I kortsvar* [National test in mathematics Course A, Spring 2010, Part I short answers]. Available at http://www.su.se/primgruppen/matematik/kurs-1/tidigare-prov. Swedish.

Task 3: Swedish National Agency for Education (2005b). *Nationellt kursprov i matematik, kurs A, våren 2005 Del II* [National test in mathematics Course A Spring 2005 Part II]. Available at http://www.su.se/primgruppen/matematik/kurs-1/tidigare-prov. Swedish.

Task 4: Swedish National Agency for Education (1996). *Nationellt kursprov i matematik, kurs A, våren 1996* [National test in mathematics Course A, Spring 1996]. Available at http://www.edusci.umu.se/np/np-b-d/tidigare-prov/. Swedish.

References

Ball, D. & Bass, H. (2003). Making mathematics reasonable in school. In J. Kilpatrick, W. G. Martin & D. Schifter (Eds.), *A research companion to principles and standards for school mathematics* (pp. 27–44). Reston, VA: National Council of Teachers of Mathematics.

Bergqvist, T., Lithner, J., & Sumpter, L. (2008). Upper secondary students' task reasoning. *International Journal of Mathematical Education in Science and Technology, 39*(1), 1–12.

Boaler, J. (1998). Open and closed mathematics: Student experiences and understandings. *Journal for Research in Mathematics Education, 29*(1), 41–62.

Boaler, J., Wiliam, D. & Brown, M. (2000). Students' experiences of ability grouping—disaffection, polarisation and the construction of failure. *British Educational Research Journal, 26*(5), 631–648.

Boesen, J., Lithner, J. & Palm, T. (2010). The relation between types of assessment tasks and the mathematical reasoning students use. *Educational Studies in Mathematics, 75*(1), 89–105.

Braun, V. & Clarke, V. (2006). Using thematic analysis in psychology. *Qualitative Research in Psychology, 3*, 77–101.

Callejo, M. & Vila, A. (2009). Approach to mathematical problem-solving and students' belief systems. *Educational Studies in Mathematics, 72*, 111–126.

Carlson, M. (1999). The mathematical behavior of six successful mathematics graduate students: Influences leading to mathematical success. *Educational Studies in Mathematics, 40*(3), 237–258.

Cobb, P., Yackel, E. & Wood, T. (1989). Young children's emotional acts while engaged in mathematical problem solving. In D. B. McLeod & V. M. Adams (Eds.), *Affect and mathematical problem solving. A new perspective* (pp. 117–148). New York, NY: Springer.

Dìaz-Obando, E., Plasencia-Cruz, I. & Solano-Alvarado, A. (2003). The impact of beliefs in student's learning: An investigation with students of two different contexts. *International Journal of Mathematical Education in Science and Technology, 34*(2), 161.

Francisco, J. (2013). The mathematical beliefs and behavior of high school students: Insights from a longitudinal study. *Journal of Mathematical Behavior, 32*(3), 481–493.

Furinghetti, F. & Morselli, F. (2009). Every unsuccessful problem solver is unsuccessful in his or her own way: Affective and cognitive factors in proving. *Educational Studies in Mathematics, 70*, 71–90.

Hannula, M. (2006). Affect in mathematical thinking and learning: Towards integration of emotion, motivation and cognition. In J. Maasz & W. Schloeglmann (Eds.), *New mathematics education research and practice* (pp. 209–232). Rotterdam, Netherlands: Sense Publishers.

Hoyles, C. (1992). Mathematics teaching and mathematics teachers: A meta-case study. *For the Learning of Mathematics, 12*(3), 32–44.

Kloosterman, P. (2002). Beliefs about mathematics and mathematics learning in the secondary school: Measurement and implications for motivation. In G. C. Leder, E. Pehkonen & G. Törner (Eds.), *Beliefs: A hidden variable in mathematics education?* (pp. 247–269). Dordrecht, Netherlands: Kluwer Academic Publishers.

Lerch, C. (2004). Control decisions and personal beliefs: Their effect on solving mathematical problems. *Journal of Mathematical Behavior, 23*, 361–372.

Lithner, J. (2008). Students' mathematical reasoning in university textbook exercises. *Educational Studies in Mathematics, 52*(1), 29–55.

Liu, P.-H. (2010). Are beliefs believable? An investigation of college students' epistemological beliefs and behavior in mathematics. *Journal of Mathematical Behavior, 29*(2), 86–98.

Mercer, J. (2010). Emotional beliefs. *International Organization, 64*(1), 1–31.

Op' T Eynde, P., De Corte, E. & Verschaffel, L. (2002). Framing students' mathematics-related beliefs. In G. C. Leder, E. Pehkonen & G. Törner (Eds.), *Beliefs: A hidden variable in mathematics education?* (pp. 13–37). Dordrecht, Netherlands: Kluwer Academic Publishers.

Ryan, R. & Deci, E. (2000). Intrinsic and extrinsic motivations: Classic definitions and new directions. *Contemporary Educational Psychology, 25*(1), 54–67.

Schoenfeld, A. (1985). *Mathematical problem solving*. Orlando, FL: Academic Press.

Schoenfeld, A. (1992). Learning to think mathematically: Problem solving, metacognition and sense-making in mathematics. In D. Grouws (Ed.), *Handbook of research in mathematics teaching and learning* (pp. 334–370). New York, NY: Macmillan Publishing Company.

Sfard, A. & Linchevski, L. (1994). The gains and the pitfalls of reification—the case of algebra. *Educational Studies in Mathematics, 26*(2–3), 191–228.

Sidenvall, J., Lithner, J., & Jäder, J. (2015). Students' reasoning in mathematics textbook task-solving. *International Journal of Mathematical Education in Science and Technology, 46*(4), 533–552.

Speer, N. (2005). Issues of methods and theory in the study of mathematics teachers' professed and attributed beliefs. *Educational Studies in Mathematics, 58*(3), 361–391.

Sumpter, L. (2013). Themes and interplay of beliefs in mathematical reasoning. *International Journal of Science and Mathematics Education, 11*(5), 1115–1135.

Swedish National Agency for Education (2010). *Nationellt kursprov i matematik, kurs A, våren 2010, Del I kortsvar* [National test in mathematics course a, spring 2010, part i short answers]. Available at http://www.su.se/primgruppen/matematik/kurs-1/tidigare-prov. Swedish.

Swedish Schools Inspectorate (2010). *Undervisningen i matematik i gymnasieskolan* [Mathematics education in upper secondary school] *(No. 2010:13)*. Stockholm, Sweden: Swedish Schools Inspectorate.

Yackel, E. & Rasmussen, C. (2002). Beliefs and norms in the mathematics classroom. In G. C. Leder, E. Pehkonen & G. Törner (Eds.), *Beliefs: A hidden variable in mathematics education?* (pp. 313–330). Dordrecht, Netherlands: Kluwer Academic Publishers.