ILHAN KOYUNCU, DIDEM AKYUZ and ERDINC CAKIROGLU

INVESTIGATING PLANE GEOMETRY PROBLEM-SOLVING STRATEGIES OF PROSPECTIVE MATHEMATICS TEACHERS IN TECHNOLOGY AND PAPER-AND-PENCIL ENVIRONMENTS

Received: 5 June 2013; Accepted: 6 January 2014

ABSTRACT. This study aims to investigate plane geometry problem-solving strategies of prospective mathematics teachers using dynamic geometry software (DGS) and paper-and-pencil (PPB) environments after receiving an instruction with GeoGebra (GGB). Four plane geometry problems were used in a multiple case study design to understand the solution strategies developed by 2 prospective teachers. The results revealed that although the participants mostly used algebraic solutions in the PPB environment, they preferred geometric solutions in the GGB environment even though algebraic solutions were still possible (the software did not preclude them). Furthermore, different proofing strategies were developed in each environment. This suggests that changing the environment may prompt students to seek for additional solutions, which, in turn, results in a deeper understanding of the problem. As such, using both environments simultaneously in solving the same problems appears to bring about important benefits.

KEY WORDS: dynamic geometry software, GeoGebra, mathematical problem solving

Introduction

In the last century, integrating technology into mathematics education has brought about many innovations in mathematics classrooms in terms of development and accessibility. Technology tools provide a powerful range of visual representations, which help teachers to focus students' attention to mathematical concepts and techniques (Zbiek, Heid, Blume & Dick, 2007). Computers are one of the most important tools of technology-supported learning environments. There are many studies that highlight the significance of using computers in teaching and learning of mathematics (Borwein & Bailey, 2003; Cuban, Kirkpatrick & Peck, 2001; Kokol-Voljc, 2007; Lee & Hollebrands, 2008; Zbiek et al., 2007). For example, Zbiek (2003) suggested that computers are used in mathematics education for various purposes such as gaining insight and intuition, discovering new patterns and relationships, graphing to expose mathematical principles, testing and especially falsifying conjectures, exploring a possible result to see whether it merits formal proof, suggesting approaches for formal proof, replacing lengthy hand derivations with tool computations, and confirming analytically derived results. Therefore, all schools need to have necessary equipment for active use of technology. Moreover, research suggests that for the successful integration of technology into classrooms, merely providing technology is not enough (Cuban et al., 2001). It needs to be supported with pedagogy and content knowledge for an effective technology-based instruction (Koehler & Mishra, 2005; Lee & Hollebrands, 2008).

Training prospective teachers about how to use technology during their teaching is an essential aspect of mathematics teacher education programs (Kokol-Voljc, 2007). In this sense, prospective teachers' content knowledge needs to be supported by using technology tools in teacher training programs. Mathematics teacher candidates should have sufficient conceptual understanding of mathematics topics to support it by using technology and understand how students learn mathematics and how technology influences their learning. In addition, they need to know the effective use of technology in teaching and learning mathematics and experience the use of a variety of technology tools to increase students' and their own mathematical learning.

Technology comes in many variants such as data handling and graphing software, computer algebra systems, programming languages, programmable calculators, and dynamic geometry software (DGS). Among these, the use of DGS has gained popularity in recent years in parallel to the development of various products such as Cabri, Geometer's Sketchpad, and most recently GeoGebra. Kokol-Voljc (2007) stated that in teaching and learning geometry, particularly Euclidean geometry, and solving problems related to geometry concepts, DGS are the most appropriate tools. Laborde (2002) pointed out that the use of DGS evolved over time from being a visual amplifier to a fundamental component that enhances conceptual understanding. Duval (1998) argued that DGS are superior to paper-and-pencil based (PPB) methods as they dissociate the "figure" from the "process of drawing." This allows students to understand the properties of the figure before it is being sketched on the screen.

However, since the use of DGS decreases the need for traditional methods, it was advised that DGS should not replace but improve and complement them (Kokol-Voljc, 2007). Although there are many advantages of constructions made with DGS, the construction activities with paper and pencil should not be lost because both DGS and paper-and-pencil environments make important contributions to students' conceptual development (Coşkun, 2011; Kokol-

Voljc, 2007). This was also supported by Gomes & Vergnaud (2010) who concluded that each set of artifacts used by students allow different geometric concepts to emerge.

In this study, we used GeoGebra as a dynamic geometry software that combines both algebra and geometry tools. Due to being open-source, multilingual, and free of charge, it is commonly used by a large number of institutions and researchers. In several recent studies, researchers have preferred to use GeoGebra in their studies instead of other DGS such as Cabri and GSP (Coşkun, 2011; Hohenwarter & Fuchs, 2004; Iranzo-Domenech, 2009; Preiner, 2008).

As we focused on the problem-solving strategies of prospective teachers in two different environments, the importance of problem solving in mathematics education should be emphasized. *Problem* is a situation that consists of exact open questions which will "challenge somebody intellectually who is not in immediate possession of direct methods/procedures/algorithms, etc. sufficient to answer the question" (Blum & Niss, 1991, p. 37). Problem solving is a process of engaging in a task or situation for which there is no obvious or immediate solution (Booker & Bond, 2008). It is a powerful and effective way of learning. Therefore, it plays an important role in teaching and learning mathematic, and so, it should not be kept apart from mathematics curricula (NCTM, 2000).

Technology can be a valuable tool in teaching problem solving, and the educational community has a general acceptance of the significant role of technology in mathematical problem solving (Yerushalmy, 2006). During the problem-solving process in a technology environment, teachers can realize students' difficulties in understanding mathematics and learn about their problem-solving tendencies (Zbiek, 2003). However, the literature still needs contributions from the studies that focus on the effect of technology on students' problem-solving preferences. In a technology environment, students are able to develop alternative strategies and explore different strategies that could not be easily explored in a PPB environment (Coşkun, 2011). Moreover, Iranzo-Domenech (2009) stressed that when students solve problems using technology, they tend to develop different competencies based on their mathematical knowledge.

The primary purpose of the present study was to test this assumption about whether prospective mathematics teachers indeed develop alternative strategies when solving plane geometry problems in two different environments. In addition, we also sought to understand how they present evidence for the correctness of their solutions, i.e. their proving strategies.

Methodology

In the present study, a qualitative research design was used to analyze the current situation in depth. Multiple case study was employed since solution strategies of multiple cases were analyzed at the same time (Creswell, 2007).

SAMPLING AND DATA COLLECTION. The participants were selected from 33 sophomore students who took the "Computer Supported Mathematics Education" course in 2011–2012 spring semester. During this course, all students were given a basic GeoGebra instruction. The reason for selecting sophomore students was because they took a geometry course in their second semester, and they were assumed to have sufficient capability for developing different problem-solving strategies for Euclidean geometry problems. In addition, they took a technology course; hence, they were capable of using computers at least at an average level according to their achievement in that course.

At the end of the 2011–2012 spring semester, two students were selected for a pilot study, and five others were selected for a 3-week instructional and 1-week data collection period. The researchers preferred communicative students who were interested in Euclidean geometry to facilitate the data extraction process. The main purpose of the pilot study was to examine whether the activities that would be covered during the instruction and the questions in the instrument were appropriate for the present study. Based on the results of the pilot study, the activities were revised, and the instrument was found to be appropriate for the level of the selected students.

At the beginning of the 2012–2013 fall semester, the five prospective teachers that were previously selected were further trained on GeoGebra using the revised activities for 3 weeks (12 h). After this instruction, the instrument involving four questions were given to the participants (see Table 1 for the timeline of the study).

The students were allowed to work for 30 min on each question. It was observed that all the students attempted to solve the questions first using paper and pencil and then on GeoGebra although no such ordering was required. In fact, students were given the choice to choose in which environment they began to solve the problems. The reason for the

TABLE 1

Time schedule of the present study

Week	*Time*	*Activity*	*Duration*
1	9–28 April and 7–11 May 2012	The pilot study	2 h
2	1–5 October 2012	Instruction (basic GeoGebra tools)	4 h
3	8–12 October 2012	Instruction (GeoGebra activities)	4 h
4	15–19 October 2012	Instruction (GeoGebra activities)	4 h
5	22–27 October 2012	Data collection	2 h

students' tendency to choose the PPB environment could be caused by their relatively higher familiarity with the traditional method than the technological one. Among the data collected from the five prospective teachers, Merve and Kübra's (pseudonyms) solutions were analyzed in greater detail and presented in this paper. Merve and Kübra were selected because they were communicative participants and usually asked critical questions related to constructions. Focusing on communicative participants facilitated data analysis. For example, if the participant obtained solutions without explanations, it would not be possible to assess their solution processes and make interpretations. However, these participants explained every step of their constructions during the instructional period; as such, their data were found to be the most appropriate to be presented here.

All students were interviewed through the "clinical interview" technique described by Ginsburg (1981). The aim of the interviews was to explore prospective teachers' mathematical thinking by discovering and identifying their cognitive processes and evaluating their competencies.

The data collection period was recorded with a video camera. One of the researchers gave worksheets to the students for solving problems on them. For data triangulation, it is important to have data from video records, GeoGebra files, and worksheets. Hence, the students showed all of their works on these documents. Moreover, the researcher mostly preferred to ask questions such as "What do you mean? Why do you think so? How can you make sure that your solution is correct?".

THE INSTRUMENT. In the present study, data were collected through four plane geometry problems used and validated in earlier studies namely the root problem (Duval, 1998), the scaled triangles problem, the median problem, and the quadrilateral problem. These problems were also analyzed in depth in Iranzo-Domenech (2009)'s doctoral dissertation. We refer to these questions as validated because the difficulties that they pose as well as the thinking strategies they may prompt have previously been documented in literature, and these problems are known to have sufficient depth.

The problems were given in the order of increasing complexity to see how the participants developed different strategies in more complex situations. Although the participants solved all four problems in the actual study, here, the solutions for only the first two are reported for brevity. Furthermore, the solutions for the first two problems appeared to be the most informative as the participants used similar techniques that they developed for them in the subsequent questions.

The Root Problem. Let E be any point on the diagonal of a rectangle ABCD such that AB=8 units and AD=6 units. The parallel line to the line (AB) through the point E intersects the segment [AD] at the point M and the segment [BC] at the point O.

The parallel line to the line (AD) through the point E intersects the segment [AB] at the point N and the segment [DC] at the point P. What relation is there between the areas of the rectangles NEOB and MEPD in the figure? (Fig. 1).

The Scaled Triangles Problem. Let P be any point on the median [AM] of a triangle ABC. Let m and n be parallel lines through P to the sides (AB) and (AC) of the triangle. What relation is there between the segments [EM] and [MF]? (Fig. 2).

THREE-WEEK INSTRUCTIONAL PERIOD. The researchers designed the content of a 3-week instructional period for the actual study. Although participants had experience on GeoGebra from the Computer-Supported Mathematics Education Course, they were given an additional 4-h

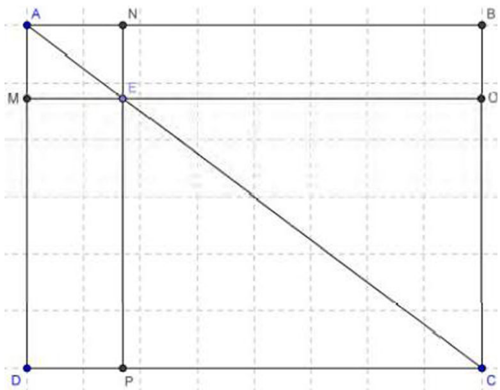

Figure 1. The root problem

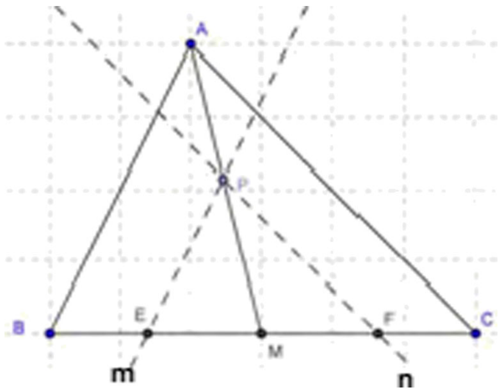

Figure 2. The scaled triangles problem

training program on the use of GeoGebra at the first week of the instruction period. The content of the instructional period was prepared by analyzing GeoGebra manuals, online tutoring videos, and the content of plane geometry taught at elementary level in the Turkish mathematics curriculum. The researcher prepared 10 activities by using objectives related to plane geometry problems in the curriculum (see Table 2). All of the GeoGebra menus that could be used in plane geometry tasks were introduced to the students.

After GeoGebra training, the instructor introduced the activities (that were previously piloted) by working with each participant simultaneous-

TABLE 2

Objectives for 10 GeoGebra activities covered in the instruction period

Objective	*Grade level*
Construct polygons	6
Draw the triangle whose measures of sufficient components are given	8
Construct medians, perpendicular bisectors of the sides, angle bisectors, and height of a triangle	8
Solve and pose problems related to area of planar regions	6
Explain conditions for equality of triangles	8
Explain conditions for similarity of triangles	8
Apply conditions for similarity of triangles to problems	8
Explain the relationship between area and length of sides	7
Determine and construct reflection of a polygon according to coordinate axes, translation along any line, rotation around the origin	7
Construct the graph of linear equations	7

ly. At the end of each activity, the participants were expected to accomplish a task related to student assessment. After the students finished all the tasks, the data collection phase was started.

In summary, the main aim of this period was to introduce the basic GeoGebra tools and train the students on the use of this software in carrying out plane geometry tasks. The students gained the required experience for using the software during the data collection period.

DATA ANALYSIS. The researchers analyzed GeoGebra and paper-and-pencil worksheets, GeoGebra files, and video records. The researchers viewed all video recordings and transcribed them into dialogues. In order to compare solutions, the participants' solution strategies were grouped based on the dominant characteristics inherent in them. Krutetskii's (1976) framework was used in order to categorize the solution strategies. In this framework, students are categorized as *algebraic*, *geometric*, and *harmonic* thinkers according to their relative predominance of using verbal-logical and visual-pictorial components of mathematical skills during the problem-solving process.

In the present study, algebraic solutions included calculating or proving the results by solving equations that are derived from geometric relationships. Since the participants' solutions were comprised of either verbal-logical justifications or logical verifications without verbal explanations, the researchers divided this category in two subcategories, namely, *verbal-logical* and *logical* ones. In logical solutions, the participants mostly preferred to use paper and pencil without using verbal messages.

Geometric solutions consisted of mostly visual-pictorial components when compared to verbal-logical ones. Here, the participants tended to prefer one or both of the following approaches. They either verbally explained their solutions using a geometric property that they discovered or showed that their solution works by dynamically dragging the figure. Therefore, the researchers preferred to divide geometric solutions as verbal-pictorial and dynamic ones. The idea of using dynamic solutions emerged from Presmeg's (1986) imagery framework. Among five imagery types, the students who preferred dynamic imagery used moving images (Presmeg, 1986). That is, they moved or dragged a figure and deduced the result from particular cases. Since GeoGebra is a dynamic geometry software, using such a classification was found to be more appropriate.

In harmonic solutions, there is a relative equilibrium between verbal-logical and visual-pictorial components of mathematical skills. Krutetskii

(1976) divides this category into two subcategories: abstract-harmonic and pictorial-harmonic. The students who used abstract-harmonic solutions prefer less pictorial components for their mental operations than pictorial-harmonic ones.

In summary, data analysis in the present study was comprised of Krutetskii's (1976) and Presmeg's (1986) frameworks. A visual representation of the solution categories is given in Fig. 3.

Results

The results were categorized first according to the participant, then the problem, and finally the solution medium, namely whether it is a PPB solution or a GeoGebra (GGB) one.

The Case of Merve

The Root Problem; PPB SOLUTION. For the root problem, Merve summarized the problem and determined what was expected in the problem. First, Merve attempted to solve the problem by using the Pythagoras theorem. She labeled the lengths and tried to find hypotenuses (Fig. 4). She tried to find a relationship between the sides of the rectangles NEOB and MEPD. When she calculated the hypotenuses by using the theorem, she realized that it was hard to find the relationship in this way. This solution path was mostly an algebraic solution because Merve directly applied the Pythagoras theorem to calculate algebraic equations and find a relationship between two unknown variables.

This algebraic way of thinking made her insist on developing a strategy based on the relationships between the sides of the rectangle. Then, Merve figured out that there should be another way to find a relationship

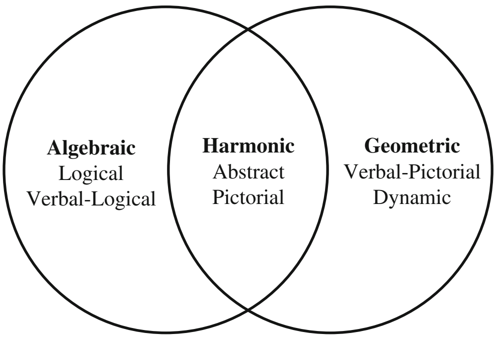

Figure 3. Classification of solution strategies

Figure 4. Merve's use of the Pythagoras theorem

between the sides of rectangles NEOB and MEPD. She explored the similarity between the triangles AME and ADC. Next, she found that if AM=3*k*, then ME=4*k* as explained in the following dialogue.

Researcher: How will you solve the problem?
Merve: If I find a relationship between the sides of the rectangles NEOB and MEPD, then I can find the relationship between the areas of these rectangles. For this purpose, I can use the similarity between triangles inside the rectangle ABCD. The triangles AME and ADC are similar. The ratio between the sides of the right triangle ADC is (6, 8). Therefore, the ratio between the sides of the right triangle AME is (6*k*, 8*k*), i.e. (3*k*, 4*k*), where *k* is a constant variable.
Researcher: Well, why are these triangles similar?
Merve: Because, the angle α is common and the other angles are equal due to the fact that the side ME is parallel to the side DC.

This paragraph showed that she understood the logical structure of the problem. As shown in Fig. 5, Merve calculated that the rectangles DPNA and MOBA have the same area. Since the rectangle ANME is common in two rectangles, she subtracted this rectangle from other rectangles and found the area equality of the rectangles NEOB and MEPD. This strategy is based on finding the equality of the areas of rectangles and subtracting the common rectangle.

This solution strategy was classified as a harmonic solution because she attempted to solve the problem by using both geometric and algebraic

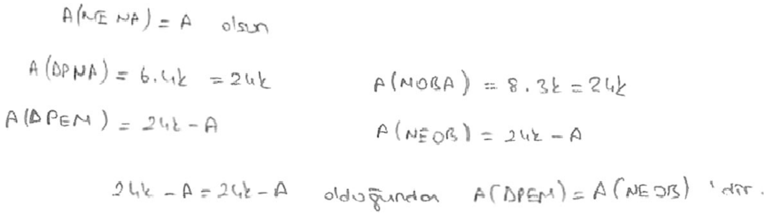

Figure 5. The solution based on finding the equality of the areas of rectangles and subtracting the common rectangle

approaches in different steps of the solution. In the first step, Merve found the relationship between the sides of rectangles based on the geometric approaches. She did not use any algebraic equation. However, in the next step, while calculating the area of the rectangles, she set equations and found them to be equal. Therefore, a relative equilibrium in the use of algebraic and geometric approaches was observed. Moreover, Merve preferred to use fewer visual-pictorial components than algebraic ones during her solution. Therefore, this solution can also be categorized as an abstract-harmonic solution.

GGB SOLUTION. Merve was confused about drawing either on the grid or using a blank graphic window. The main reason for this confusion was because she could not decide whether the lengths given in the problem (AB = 8 and AD = 6) were important or not. Initially, Merve thought that she must be careful about using accurate lengths in her construction. For this reason, she used the grid and placed the points on the grid intersections to set AB = 8 and AD = 6. The following dialogue shows how she thought about using the given lengths.

Researcher: How will you set the length of the sides?
Merve: I will use the grid view or measure the lengths on the graphic window. I prefer to use the grid view.
Researcher: Well, why do you measure the lengths? Will you use this information in your solution?
Merve: Because, they are given in the problem statement. I used them in my paper-and-pencil solution. I will also use them in the GGB solution.
Researcher: OK.

The researcher did not give clues about the role of dimensions in the problem in order to avoid interfering with Merve's solution. After deciding that the actual lengths were important, Merve constructed the figure. Next, she decided that measuring the areas might help her to understand the relationship. Since she had solved the problem in the PPB environment, she expected that the areas should be equal. She verified this equality by using the measuring tool (Fig. 6). In order to justify her

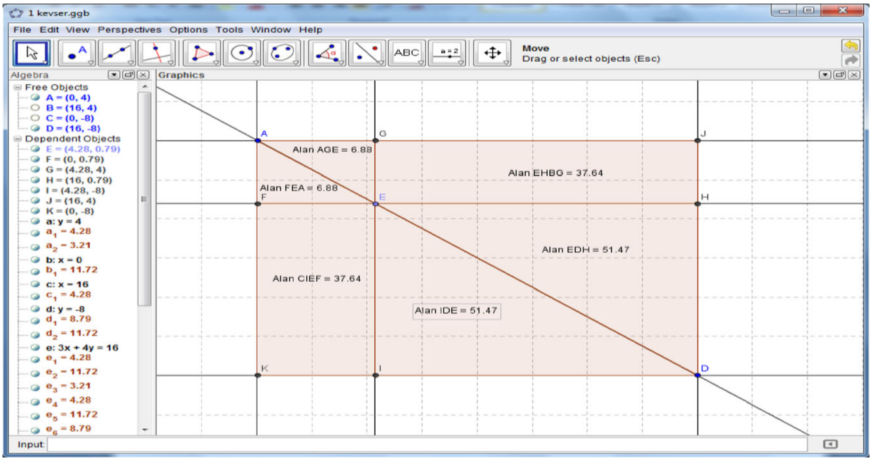

Figure 6. Merve’s GGB solution of the root problem

solution, she dragged the point E and showed that the equality of areas is satisfied along the diagonal of the rectangle.

Her explanations can be observed in the following dialogue:

Researcher: You measured the areas and found the equality of areas. So, why are they equal?
Merve: Because, when we drag the point E, the equality remains the same along the diagonal of the rectangle AJDK (She showed this situation on GeoGebra file).
Researcher: What is the main reason for this situation?
Merve: The point E is on the diagonal.
Researcher: So, what is the function of the diagonal?
Merve: It divides the rectangle in two right triangles with equal areas. When we apply this rule on this figure, we can see the equality.
Researcher: Well, another important point is that you did not use the lengths of the sides. Initially, you thought them as important.
Merve: If we measure the sides, we can see that the software used the ratio of the sides.
Researcher: So, the ratio (6, 8) is necessary or not? Do any other ratios satisfy this equality?
Merve: I think that the ratio will remain the same because the point A is on the diagonal and the lines EI and EH are parallel to the sides. The equality is true for all rectangles that satisfy this condition. We can show this by dragging feature as well.

According to the dialogue, she explored the equality from the diagonal property of the rectangles. She thought that the diagonal divides all rectangles into equal parts. Another important exploration for her was that the lengths of sides were not important in this case as the diagonal evenly divides any rectangle. Since she drew the figure and used geometric verifications, this solution can be classified as a geometric solution.

Moreover, she verbally and visually justified her solution; therefore it is classified as a verbal-pictorial geometric one.

The Scaled Triangles Problem; PPB SOLUTION. In this problem, Merve first labeled the sides using variables (Fig. 7) and then argued that it can be solved using the Thales theorem (i.e. by using similar triangles). She stated that the triangle EPM is similar to BAM. Accordingly, the triangles MPF and MAC are also similar.

However, she did not know how to use this information to solve the problem. Here, it was observed that Merve preferred to write all equations first without having a full understanding of which equation would lead her to the solution. However, after observing that the second set of equations share a common denominator, she used that information to conclude that EM and MF are equal (Fig. 8).

The following dialogue shows how she came to this realization:

Researcher: You expressed the sides in terms of unknowns. How do you use these unknowns?
Merve: I will write the ratio of similarities in terms of unknowns.
Researcher: Well, what do you expect to obtain by using these ratios with unknowns?
Merve: Actually, I exactly do not know, but I consider having a relationship between the equalities.
Researcher: You expect to have a relationship?
Merve: Yes. (After writing the equalities) I found the equality of the sides. In fact, I could show this equality on the figure, but this way is much easier.

As shown in this dialogue, Merve solved the problem after noticing that the second set of equations share a common term, and simplifying them leads to the equality of the parts in question. Therefore, this solution strategy could be classified as an algebraic solution. In addition, she

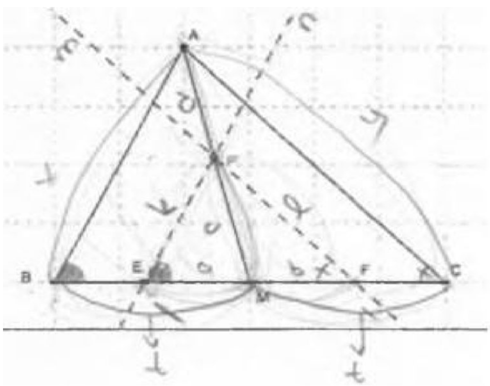

Figure 7. Expression of the sides in terms of unknowns

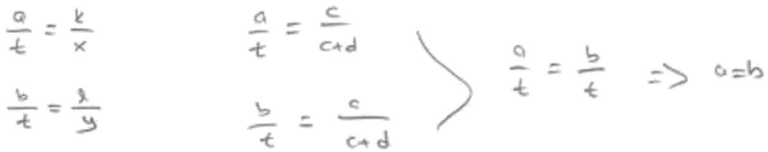

Figure 8. Merve's solution of the scaled triangles problem

justified her drawing and algebraic expressions (Fig. 8) by verbal explanations in above dialogues. Therefore, the strategy can also be classified as a verbal-logical algebraic solution.

GGB SOLUTION. In this problem, Merve again thought of using the measuring tool. However, she did not consider her solution in the PPB environment. She just focused on measuring the lengths of the line segments. The use of the similarity theory and algebraic equations was disregarded in this environment. The solution process could be observed in following dialogue:

Researcher: How did you solve the problem?
Merve: I measured the side FD and the side DG (see Fig. 9). They are equal.
Researcher: How can you verify this equality?
Merve: While dragging the point E, the equality remains the same. Moreover, moving the sides and vertices of the triangle did not affect this equality.
Researcher: What are the main reasons for this equality?
Merve: AD is the median of the triangle, FE and GE are parallel to sides AB and AC, respectively, and E is at the median and the intersection of parallel lines FE and GE.
Researcher: In other words, you summarized the information given in the problem.
Merve: Yes.

According to the dialogue, Merve measured and found the equality of the sides FD and DG (Fig. 9). She preferred to use the dynamic feature of

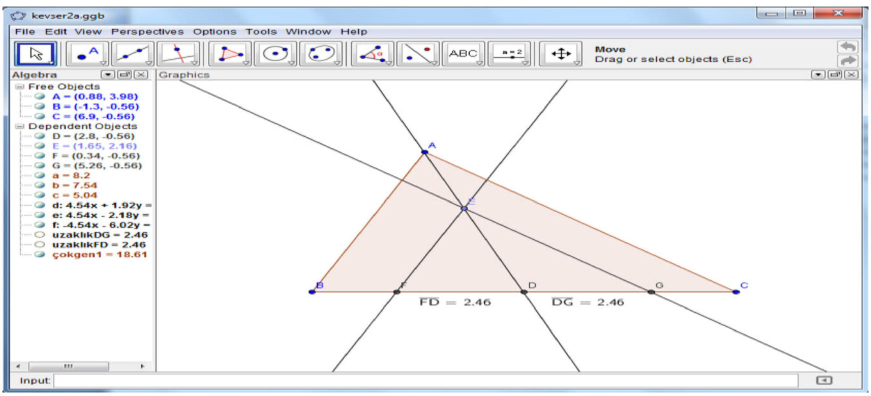

Figure 9. Merve's GGB solution of the scaled triangles problem

the software. She justified her solution by dragging the vertices and the sides of the triangle. However, that was just a verification of the equality. At this point, after thinking for a while, she claimed that the lengths of the line segments FD and DG are equal because point E is on the median, and the lines FE and EG are parallel to the sides of the triangle. This explanation was simply a repetition of the problem statement. At this point, it can be argued whether such an approach can be counted as a proper solution. Essentially, Merve only verified that FD and DG remain equal for different triangles without understanding why this relationship holds. However, due to the dynamic verification that was used, this solution was categorized as dynamic geometric.

The tendency to show that a relationship holds by dragging in the software but without understanding the mathematical reason behind it can be a possible fallacy that students might develop when using dynamic geometry software. Teachers should be careful about such uses of DGS.

The Case of Kübra

The Root Problem; PPB SOLUTION. Kübra developed a different strategy from other students in the experiment. She preferred to use a trigonometric approach. She thought that in order to compare the areas of the rectangles, she had to calculate the areas using the same unknowns. In the following dialogue, this thinking process could be observed.

Researcher: What is your plan to solve the problem?
Kübra: I will calculate the areas of the rectangle NEOB and MEPD. However, I need to express the areas in terms of the same unknowns in order to find a relationship between them.
Researcher: So, what will you do?
Kübra: First of all, I need to find a relationship between the sides of the rectangles. Actually, I can use trigonometry to find a relationship.
Researcher: How will you use trigonometry?
Kübra: The angle α is common in the triangle AME and ADC. We know the tangent value of the triangle ADC because the sides of rectangle ABCD were given. Hence, if I expressed the sides of the triangle AME in terms of unknowns x and y, I can find a relationship between x and y (Fig. 10).

First of all, she expressed the sides of the rectangle NEOB and MEPD in terms of the unknowns x and y (Fig. 10). Before calculating the areas of these rectangles, she needed to find the relationship between x and y.

Therefore, she looked for the triangles that she could use in her trigonometric approach. Then, she realized that the angle α is common in the triangle AME and ADC. Therefore, she could find the tangent value of this angle for the triangle AME. By using the tangent value 6/8 in the triangle ADC, she found the relationship between the unknowns x and y (Figs. 10 and 11). However, as shown in Fig. 10, she also expressed the hypotenuses of these triangles in terms of the unknown a. However, she did not use this information. She explained the reason for this expression and stated that "I thought that I might use Pythagoras theorem but I realized that I could easily find the relationship by using the tangent value of the triangle AME based on some mental operations that I quickly did. Then, I abandoned this strategy and used the trigonometric approach." This explanation showed that she was able to develop different approaches quickly in order to solve the problem.

Her solution process could be observed in Fig. 11. After finding the relationship between x and y, she calculated the areas separately. Then, she computed the areas in terms of x and found the equality of the rectangles NEOB and MEPD.

Kübra entirely used algebraic expressions to find the relationship between the sides and, eventually, the rectangles. Therefore, her solution was an algebraic one according to the framework of the present study. During the solution process, she used only a few verbal explanations. She supported her mental operations with logical explanations and justifications on the paper. Therefore, the solution strategy was a logical algebraic solution according to our subcategories in the framework.

GGB SOLUTION. Kübra solved this problem by calculating and comparing the areas of the rectangles NEOB and MEPD. However,

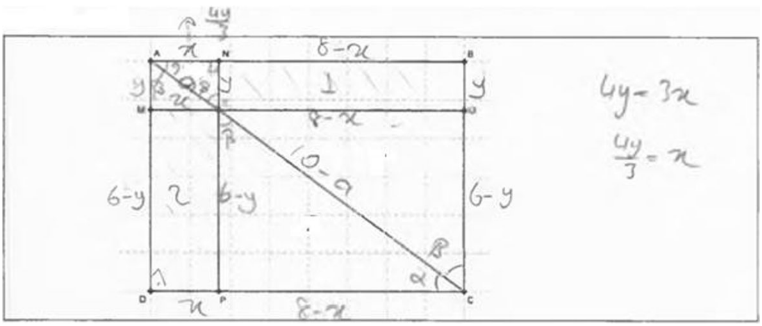

Figure 10. Expression of the sides in terms of unknowns

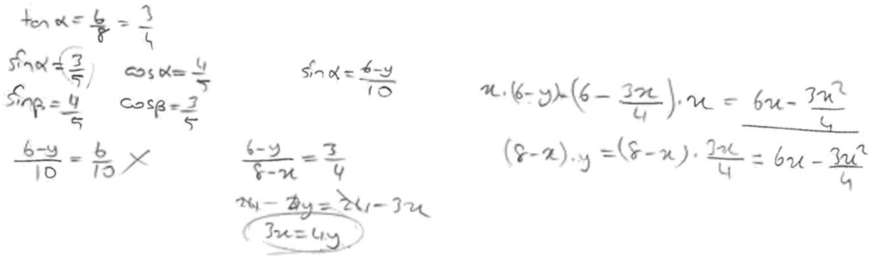

Figure 11. Kübra's solution of the root problem

when she drew the figure in GeoGebra, she realized her previous knowledge about rectangles. In the dialogue below, this exploration phase could be observed:

Researcher: How can you solve the problem?
Kübra: When I solve this problem in paper-and-pencil environment, I did not think of using the diagonal property of the rectangle. The diagonal divides the rectangle into two equal parts. I realized it during the constructions in GGB environment.
Researcher: So, how do you use this information?
Kübra: I will measure the areas. They are equal because the diagonal divides the rectangle EHDF and the rectangle AGEI into two triangles with equal areas. Therefore, the areas of rectangle GCHE and the rectangle IEFB will also be equal because this diagonal divides the rectangle ABCD into two triangles with equal areas at the same time.
Researcher: In your construction, you did not use the grid or measure the lengths of the sides. Why?
Kübra: Yes; because I used the diagonal property, the lengths of the sides are not important if the figure yields the conditions given in the problem statement.

Kübra explained her solution by verbal justifications. She did not know how to show the equality based on the diagonal property with the software. She just measured the areas of the triangle and colored the triangles and rectangles with the same color (Fig. 12). She explained her solution with the idea that the diagonal divides the rectangles into two equal triangles, and hence, their areas are equal. In addition, she realized that lengths of the sides are not important if construction was drawn according to the given conditions.

We categorized this solution as a verbal-pictorial geometric solution because she used geometric approaches supported by verbal-pictorial explanations as shown in the above dialogue.

The Scaled Triangles Problem; PPB SOLUTION. Kübra expressed the sides of the given triangle in terms of unknowns (Fig. 13). She thought of solving the problem by using the Thales theorem.

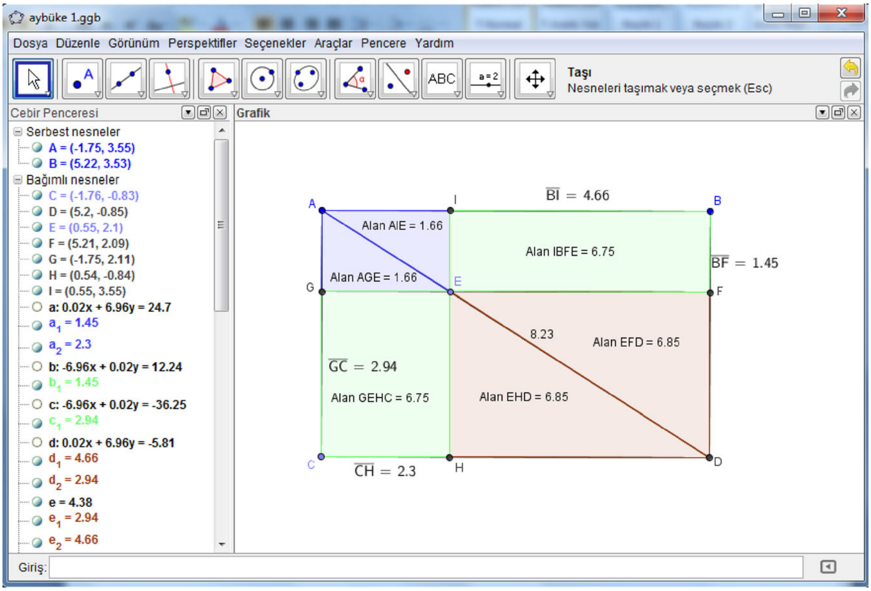

Figure 12. Kübra’s GGB solution of the root problem

However, there were different triangles that are similar, and she could not decide how to use the unknowns at the first glance.

In the following dialogue, Kübra’s process of developing her strategy could be observed based on her verbal explanations:

Researcher: You expressed the sides in terms of unknowns. How will you use this expression?

Kübra: There are similar triangles in this figure. For example, EFP and the triangle that exists between the lines EP, EC, and AC are similar. When I applied similarity theory to these triangles, I obtained the equality $y/(y + a) = (2x - a - b)/(2x - b)$ (Fig. 14). However, this equality will not help to find a solution.

Researcher: Why?

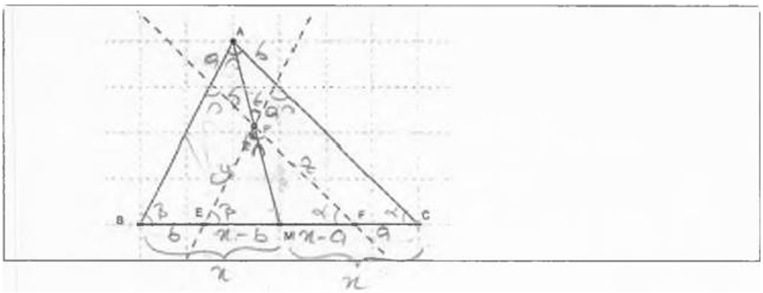

Figure 13. Expression of the sides in terms of unknowns

Kübra: Because, if I calculate this equality, there will be an equation with three unknowns. Maybe, I will get a solution in this way, but there other similar triangles. I will look for more simple similarities, then I will decide about which will be helpful for my solution.
Researcher: So which triangles are similar?
Kübra: The triangle MFP and the triangle MCA are similar. In addition, the triangle MEP and the triangle MBA are also similar. The median AM of the triangle ABC is a common side for all these triangles. By using this knowledge, I get ME/MB = MP/MA and MF/MC = MP/MA. Since MP/MA is common in two equalities, I obtain the equality ME/MB = MF/MC. We know that MB = MC due to the median, I have ME = MF.

While Kübra was explaining her strategy verbally, she was writing the equalities based on the similar triangles at the same time (Fig. 14). Although she began with the similarity shown in Fig. 14, she continued with using the similarities on the triangles MFP~MCA and the triangles MEP~MBA. According to the dialogue above, she stated that "Since ME/MB = MP/MA and MF/MC = MP/MA, then ME/MB = MF/MC. And because MB = MC (due to the median) ME = MF."

Kübra realized that the side AM is common for the triangle AMB and AMC, and if she wrote the equalities, there would be a common ratio, which is MP/MA, in the equations. However, Kübra observed this situation after writing the equations. This showed that Kübra was more preoccupied with writing equations than observing the geometrical properties of the figure. She was able to solve the problem by using algebraic equations and supported them with verbal explanations. Therefore, this can be considered as a verbal-logical algebraic solution strategy.

GGB SOLUTION. Kübra thought of measuring the lengths directly. However, her justification for the equality was different when compared

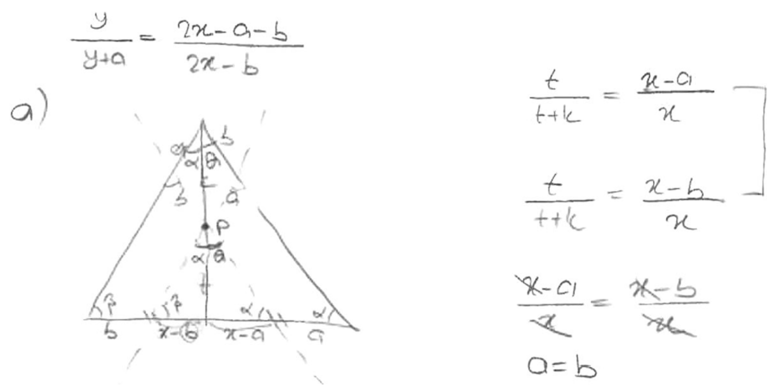

Figure 14. Kübra's solution of the scaled triangles problem

to Merve's solution. She used the areas in order to verify her solution. In the following dialogue, Kübra summarized her solution strategy:

Researcher: How do you solve this problem?
Kübra: By using the measuring tool, I can find the lengths. For this reason, I will measure the areas of the triangle EFD and the triangle EDG and try to find a relationship between the areas. For these two triangles, the heights that belong to the bases are equal. Therefore, the ratio of the areas is equal to the ratio of the bases.
Researcher: (After Kübra measured the areas) The areas are equal. And what about the bases?
Kübra: The bases are also equal.
Researcher: So, how do you show your solution is always true?
Kübra: If I drag the point E, the sides and vertices, the equality is always satisfied as was in the first question.

According to the dialogue, she measured the areas of the triangle EFD and the triangle EDG and found that the areas were equal. Since the respective heights of these triangles for the side FD and the side DG were common, these sides must be equal (Fig. 15). By solving the problem in this way, she did not need to explain the function of the median in this equality. However, the main reason for the equality of areas was the median ED of the triangle EFG. She also justified her solution by moving the point E and the sides of the

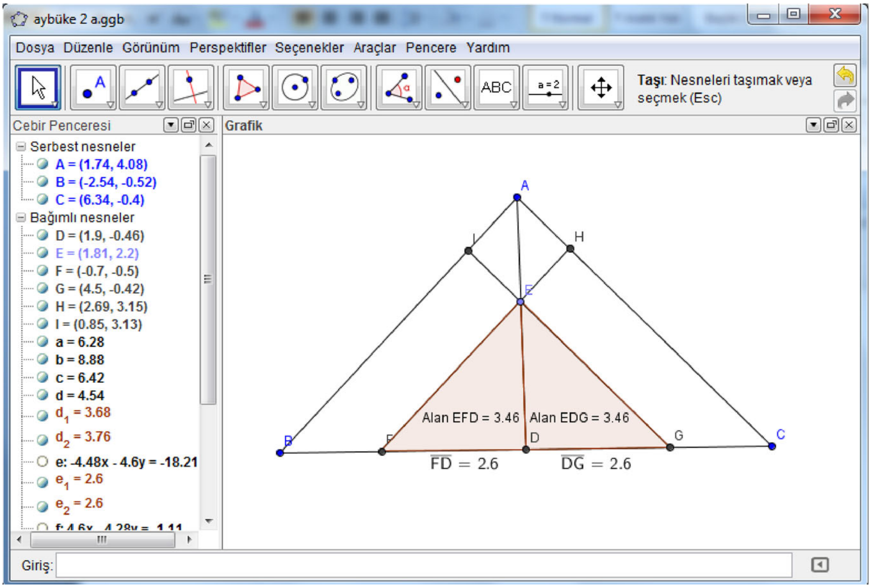

Figure 15. Kübra's GGB solution of the scaled triangles problem

TABLE 3

An overview of solutions developed by the students for the root problem

Root problem	*PPB solution method*	*PPB solution category*	*GGB solution method*	*GGB solution category*
Merve	Equality of area of rectangles and subtracting the common rectangle	Abstract-harmonic	Properties of diagonal of the rectangles	Verbal-pictorial geometric
Kübra	Calculating area of rectangles based on a trigonometric approach	Logical-algebraic	Properties of diagonal of the rectangles	Verbal-pictorial geometric

triangle ABC. This solution was considered as a geometric solution verified by verbal explanations in the dialogue.

She used dynamic features as a secondary verification method in response to the questions asked by the researcher. For example, when the researcher asked her to show that her result was always true, she responded by dragging the points and showing the changes in the results.

Participants' solutions in PPB and GGB environments are summarized in Tables 3 and 4.

Discussion

In this study, it was observed that the characteristics of the environments affect prospective mathematics teachers' solution strategies. Both partic-

TABLE 4

An overview of solutions developed by the students for the scaled triangles problem

Scaled triangles problem	*PPB solution method*	*PPB solution category*	*GGB solution method*	*GGB solution category*
Merve	Equality of ratios based on similarity theory	Verbal-logical-algebraic	Measuring lengths and justifying by dragging tool	Dynamic geometric
Kübra	Equality of ratios based on similarity theory	Verbal-logical-algebraic	Equality of areas by using measuring tool	Verbal-pictorial-geometric

ipants had a tendency toward using algebraic solutions in the PPB environment, whereas they used geometric solutions in the GGB environment. This is interesting because in the instructional period prior to the data collection, the algebraic capabilities of the GGB software were emphasized as much as its geometric capabilities. This might be attributed to the fact that the software opens up with a large canvas inviting the user to draw figures on it. The algebraic capabilities, although quite powerful, are not immediately visible to the user. Another reason for this could be the fact that the nature of the problems prompted them for geometric solutions in the GGB environment. In other words, all problems were related to plane geometry, and the students needed to construct the figures; therefore, they found the software more effective in solving these problems without using the algebra window.

The observation that students mostly used algebra in the PPB environment is also interesting in the sense that the students were comfortable in setting up and solving equations than exploring geometric relationships. For instance, in the root problem, neither student could observe the diagonal property of the rectangles in the PPB environment. Instead, they immediately preoccupied themselves with solving algebraic equations. This might stem from the fact that both students felt more comfortable in algebra than in geometry based on their past experiences. Indeed, in Turkish schools, the emphasis has traditionally been placed on algebra than in geometry (MoNE, 2009).

However, despite this background, students were able to develop geometric solutions in the GGB environment. This might be attributed to the deeper understanding of the problem and the geometric relationships that it contains when students construct figures in the GGB environment. A construction in the GGB environment takes more time than a construction in the PPB environment. This helps students to better observe and absorb the conditions of the problem. For instance, while drawing a diagonal could be second nature in a PPB environment, it requires a more thought process in the GGB environment. This thought process could result in a better understanding of the subconfigurations that are crucial for solving the problem. This finding supports earlier ideas in literature. For instance, Duval (1998) argued that for solving the root problem, it is important for the students to identify two important subconfigurations. The first subconfiguration involves the two triangles that are created by the diagonal of the main rectangle. The second subconfiguration includes two concave shapes when the areas in question are subtracted from these triangles. Duval (1998) argued that this second subconfiguration especially is difficult to distinguish as its parts are noncomplementary and nonconvex. In this study, we observed that while reconstructing these shapes in the GGB environment, students became aware of these subconfigurations more easily compared to the PPB environment.

Also Iranzo-Domenech (2009) stated that students are able to encounter with deep information about the logical structure of the problem in a dynamic environment. In other words, dynamic solutions help students to understand the logical structure of the problem and make the solution more meaningful (Christou, Mousoulides, Pittalis & Pitta-Pantazi, 2004; Iranzo-Domenech, 2009). Similarly, Gal & Linchevski (2010) found that most problems in geometry appear to be caused by students' initial visual perception of the figure. This perception, although requisite for the next steps, may hinder students to go further from the first glance of the figure and thus identify the important geometric relationships that can be used to solve the problem. In our study, we observed that not only the dynamic solutions but also the process of constructing the figure in a dynamic geometry environment gives a deeper insight into the logical structure of the problem. That is, by reconstructing the figure in the DGS, they could go beyond their initial perception of the figure as a whole.

The fact that construction helps in finding solutions is also supported by Duval's (1998) categorization of geometric reasoning. Duval suggests that geometric reasoning is comprised of three independent activities namely *visualization*, *construction*, and *reasoning*. Although these activities can be performed separately, they may also support each other. For instance, construction can help visualization which in turn can help in reasoning (although the process is more complex than a simple linear ordering). In this study, we clearly observed that construction indeed helped reasoning. Furthermore, construction using a tool (GeoGebra) appeared to be more effective than construction using paper and pencil. We hypothesize that this is because the former requires a higher intellectual effort than the latter.

In our study, we also observed a very important pitfall that students experienced when solving problems in the GGB environment. The students had a tendency to use the measuring tool and dynamic features of the software (e.g. dragging) to show that some relationship holds. However, such solutions should clearly not be accepted as valid because they do not explain why this relationship holds. Such approaches should be considered as auxiliary methods that help understand a relationship, but teachers should prompt students to logically analyze these observations. Otherwise, the use of the GGB environment could instead be counterproductive.

In this study, we observed that each environment had different contributions to the students' mathematical thinking and problem-solving skills. This result supports Coşkun's (2011) study about the effectiveness of technology in developing visual and nonvisual solution methods in different environments since she had found that the use of each environment has different influences on students' thinking styles. It is also in line with Mariotti's (2000) findings that dynamic geometry enhances theoretical thinking. In our study, most students were able to develop more theoretical solutions by observing the properties of the figure rather than procedural computations as was mostly done on paper. In addition, Iranzo-Domenech (2009) stressed that different environments help students develop different competencies as a result of her study about the synergy of environments; a finding corroborated by our study.

Conclusions and Future Work

According to the findings of the present study, technology gives the opportunity to develop alternative strategies. After solving each problem, the students attempted to find alternative strategies, and they usually found alternative solutions. The result that the students were able to develop alternative strategies in technology environment is consistent with the related literature (Cai & Hwang, 2002; Christou et al., 2004: Coşkun, 2011). Nevertheless, Meydiyev (2009) found that the students with little conceptual understanding are not able to develop new strategies. In other words, the students with little knowledge of concepts face difficulties in developing additional solutions. This did not apply to Merve and Kübra in the present study because they had a good conceptual understanding of the relevant topics.

In the present study, the researchers analyzed plane geometry problem-solving strategies of two particular cases at a public university to understand how technology affects their solutions. Future studies may focus on different cases at different universities and with other types of geometry problems to explore different aspects of the issue. In addition, the number of problems and their complexity can be increased. In the current study, the participants were allowed to use the software whenever they want during the problem-solving process. However, they solved the problem in PPB environment at first and then they used the software. Therefore, their GGB solutions might be affected by their PPB solutions. To avoid this interaction, future studies may involve a larger number of students some of which only solve problems in PPB and others in GGB after ensuring that both groups of students have similar mathematical competencies.

References

Blum, W. & Niss, M. (1991). Applied mathematical problem solving, modeling, applications, and links to the other subjects—State, trends and issues in mathematics instruction. *Educational Studies in Mathematics, 22*, 37–68.

Booker, G. & Bond, D. (2008). *Problem-solving in mathematics*. Greenwood, Western Australia: R.I.C. Publications.

Borwein, J. M. & Bailey, D. H. (2003). *Mathematics by experiment: Plausible reasoning in the 21st century*. Natick, MA: AK Peters.

Cai, J. & Hwang, S. (2002). Generalized and generative thinking in U.S. and Chinese students' mathematical problem solving and problem posing. *Journal of Mathematical Behavior, 21*(4), 401–421.

Christou, C., Mousoulides, N., Pittalis, M., & Pitta-Pantazi, D. (2004). Proofs through exploration in dynamic geometry environments. 28th Conference of the International Group for the Psychology of Mathematics Education, Norway.

Coşkun, Ş. (2011). *A multiple case study investigating the effects of technology on students'visual and nonvisual thinking preferences: comparing paper-pencil and dynamic software based strategies of algebra word problems*. Unpublished doctoral dissertation, University of Central Florida, Florida, USA

Creswell, J. W. (2007). *Qualitative inquiry and research design: Choosing among five approaches*. Thousand Oaks, CA: Sage.

Cuban, L., Kirkpatrick, H. & Peck, C. (2001). High access and low use of technologies in high school classrooms: Explaining an apparent paradox. *American Educational Research Journal, 38*(4), 813–834.

Duval, R. (1998). Geometry from a cognitive point of view. In C. Mammana & V. Villani (Eds.), *Perspectives on the teaching of geometry for the 21st century* (pp. 37–52). Dordrecht: Kluwer.

Gal, H. & Linchevski, L. (2010). To see or not to see: Analyzing difficulties in geometry from the perspective of visual perception. *Educational Studies in Mathematics, 74*(2), 163–183.

Ginsburg, H. P. (1981). The clinical interview in psychological research on mathematical thinking: Aims, rationales, techniques. *For the Learning of Mathematics, 1*(3), 4–11.

Gomes, A. S., & Vergnaud, G. (2010). On the learning of geometric concepts using dynamic geometry software. RENOTE, 2(1)

Hohenwarter, M., & Fuchs, K. (2004). *Combination of dynamic geometry, algebra and calculus in the software system GeoGebra*. Computer Algebra Systems and Dynamic Geometry Systems in Mathematics Teaching Conference, Pecs, Hungary

Iranzo-Domenech, N. (2009). *Influence of dynamic geometry software on plane geometry problem solving strategies*. Unpublished doctoral dissertation, Universitat Autonoma de Barcelona, Spain

Koehler, M. J. & Mishra, P. (2005). What happens when teachers design educational technology? The development of technological pedagogical content knowledge. *Journal of Educational Computing Research, 32*(2), 131–152.

Kokol-Voljc, V. (2007). Use of mathematical software in pre-service teacher training: The case of *GeoGebra. Proceedings of the British Society for Research into Learning Mathematics, 27*(3), 55–60.

Krutetskii, V. A. (1976). *The psychology of mathematical abilities in schoolchildren*. Chicago: The University of Chicago Press. Eds.: J. Kilpatrick & I. Wirszup.

Laborde, C. (2002). Integration of technology in the design of geometry tasks with Cabri-Geometry. *International Journal of Computers for Mathematical Learning, 6*(3), 283–317.

Lee, H. & Hollebrands, K. (2008). Preparing to teach mathematics with technology: An integrated approach to developing technological pedagogical content knowledge. *Contemporary Issues in Technology and Teacher Education, 8*(4), 326–341.

Mariotti, M. A. (2000). Introduction to proof: The mediation of a dynamic software environment. *Educational Studies in Mathematics, 44*(1–2), 25–53.

Meydiyev, R. (2009). *Exploring students' learning experiences when using a dynamic geometry software tool in a geometry class at a secondary class in Azerbaijan*. Unpublished master's thesis, Universiteit van Amsterdam, The Netherlands

Ministry of National Education [MoNE] (2009). İlköğretim Matematik Dersi 6–8. Sınıflar Öğretim Programı. Ankara, Turkey: Talim ve Terbiye Kurulu

National Council of Teachers of Mathematics (2000). *Principles and standards for school mathematics*. Reston, VA: NCTM.

Preiner, J. (2008). *Introducing dynamic mathematics software to mathematics teachers: The case of GeoGebra*. Unpublished doctoral dissertation, University of Salzburg, Salzburg, Austria

Presmeg, N. C. (1986). Visualization in high school mathematics. *For the Learning of Mathematics, 6*, 42–46.

Yerushalmy, M. (2006). Slower algebra students meet faster tools: Solving algebra word problems with graphing software. *Journal for Research in Mathematics Education, 37*(5), 356–387.

Zbiek, R. M. (2003). Using technology to foster mathematical meaning through problem solving. In H. L. Schoen & R. I. Charles (Eds.), *Teaching mathematics through problem solving* (pp. 93–104). Reston: The National Council of Teachers of Mathematics, Inc.

Zbiek, R. M., Heid, M. K., Blume, G. W. & Dick, T. P. (2007). Research on technology in mathematics education. In F. K. Lester (Ed.), *Second handbook of research on mathematics teaching and learning* (pp. 1169–1207). Charlotte, NC: Information Age Publishing.

Ilhan Koyuncu, Didem Akyuz and Erdinc Cakiroglu
Middle East Technical University
Ankara, Turkey
E-mail: dakyuz@metu.edu.tr