COURTNEY NAGLE, DEBORAH MOORE-RUSSO, JANINE VIGLIETTI and KRISTI MARTIN

CALCULUS STUDENTS' AND INSTRUCTORS' CONCEPTUALIZATIONS OF SLOPE: A COMPARISON ACROSS ACADEMIC LEVELS

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ABSTRACT. This study considers tertiary calculus students' and instructors' conceptualizations of slope. Qualitative techniques were employed to classify responses to 5 items using conceptualizations of slope identified across various research settings. Students' responses suggest that they rely on procedurally based conceptualizations of slope, showing little evidence of covariational reasoning. In contrast, instructors' responses demonstrated a multi-dimensional understanding of slope as a functional property, which applies to real-world situations and plays an integral role in the development of key calculus concepts. While relatively diverse, the instructors' responses seldom reported determining increasing or decreasing trends of a line from its slope. This conceptualization was used frequently by students and could help them better understand how slope ties to positive and negative derivatives. The most frequently used conceptualizations for students in this study align with past research findings on the emphasis of the secondary mathematics curriculum, supporting the possibility of cultural influences (academic and geographic) on individuals' conceptualizations of slope. Thus, this study provides valuable insight into conceptualizations of slope and provides direction for future research on slope and the broader topic of cultural influences on mathematical meaning.

KEY WORDS: instructors, school-to-university transition, slope, sociomathematical norms, students

Slope is an important mathematical topic emphasized primarily in the secondary mathematics curriculum. Slope is applied in many fields, including the sciences, but it is also important for the development of advanced mathematical topics. Calculus marks a significant transition in students' mathematical understanding of slope, progressing from linear functions to non-linear functions and from average rates of change to instantaneous rates of change. Moreover, students often first enroll in calculus courses during another important transition, as they move from secondary to tertiary studies. This transition marks a cultural change from secondary to tertiary mathematics and therefore presents the possibility of emergent cultural conflicts (Barton, Clark & Sheryn, 2010; Bishop, 1994; Clark & Lovric, 2009; Prediger, 2004).

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The diversity of slope conceptualizations presents a challenge for instructors, who should strive to build on and extend students' existing understandings of the concept. This study provides an initial investigation of the conceptualizations of slope that are commonly held by students and instructors, with particular attention to differences that may be linked to the academic cultures and mathematical emphases found in secondary versus tertiary mathematics. The research questions that drove this study are as follows:

1. How do students in tertiary calculus classes conceptualize slope?
2. How do tertiary mathematics instructors conceptualize slope?
3. Does culture (either academic or geographic) play a role in the conceptualizations of slope held by students and instructors?

Review of Literature Related to Slope

Slope is a key mathematical concept with implications far beyond its algebraic use as an indicator of the steepness of a line. Slope plays a critical role in the mathematics curriculum since it is: (1) an important prerequisite concept for advanced mathematical thinking (Carlson, Oehrtman & Engelke, 2010; Confrey & Smith, 1995; Noble, Nemirovsky, Wright & Tierney, 2001) and (2) represented and conceptualized in many different contexts and settings (Moore-Russo, Conner, & Rugg, 2011; Mudaly & Moore-Russo, 2011; Stanton & Moore-Russo, 2012; Stump, 1999, 2001a, b), requiring students and instructors to connect the various conceptualizations to form a complete, connected concept image.

Slope as a Prerequisite Concept for Advanced Mathematics

Slope is a fundamental topic because it extends to the concepts of rate of change in precalculus and derivative in calculus. In particular, understanding slope requires covariational reasoning to describe the relationship between two variables. Covariational reasoning, including the concepts of rate of change and growth rates of functions, has been identified as a key prerequisite for precalculus instruction (Carlson et al., 2010; Confrey & Smith, 1995). Additionally, the concept of derivative requires an extension of knowledge about average rate of change to situations involving instantaneous rates. Thus, slope is a fundamental mathematical notion introduced in algebra but with important implications extending into calculus.

Multiple Conceptualizations

Besides serving as a building block for more advanced mathematical ideas, slope also appears in many contexts and representations throughout the curriculum. Author and colleagues (Moore-Russo et al., 2011; Mudaly & Moore-Russo, 2011; Stanton & Moore-Russo, 2012) have suggested 11 conceptualizations of slope, outlined in Table 1, based on their own research and the earlier work of Sheryl Stump (1999, 2001a, b).

In addition to the 11 conceptualizations of slope, Zaslavsky, Sela & Leron (2002) described analytical and visual interpretations of slope. The analytical interpretations of slope described by Zaslavsky and colleagues include

TABLE 1

Conceptualizations of slope (adapted from Table 1 in Moore-Russo et al., 2011)

Category	*Slope as …*
Geometric ratio (G)	Rise over run of a graph of a line; ratio of vertical displacement to horizontal displacement of a line's graph
Algebraic ratio (A)	Change in y over change in x; ratio with algebraic expressions (often seen as either $\Delta y/\Delta x$ or $(y_2 - y_1)/(x_2 - x_1)$)
Physical property (P)	Property of line often described using expressions like grade, incline, pitch, steepness, slant, tilt, and "how high a line goes up"
Functional property (F)	(Constant) rate of change between variables; sometimes seen in responses involving related rates
Parametric coefficient (PC)	The variable m (or its numeric value) found in $y = mx + \mathrm{b}$ and $(y_2 - y_1) = m(x_2 - x_1)$
Trigonometric conception (T)	Property related to the angle a line makes with a horizontal line; tangent of a line's angle of inclination/decline; direction component of a vector
Calculus conception (C)	Limit; derivative; a measure of instantaneous rate of change for any (even nonlinear) functions; tangent line to a curve at a point
Real-world situation (R)	Static, physical or dynamic, functional situation (e.g. wheelchair ramp, distance versus time)
Determining property (D)	Property that determines if lines are parallel or perpendicular; property can determine a line if a point on the line is also given
Behavior indicator (B)	Property that indicates increasing/decreasing/horizontal trends of line or amount of increase or decrease; if nonzero, indicates intersection with x-axis
Linear constant (L)	Constant property independent of representation; unaffected by translation of a line; reference to what makes a line "straight" or the "straightness" of a line

notions of derivatives, difference quotients, and the coefficient *m*, aligning with the *calculus conception*, *algebraic ratio*, and *parametric coefficient* conceptualizations of slope, respectively. The visual interpretations of slope include the tangent of the angle formed between the line and a horizontal and the ratio of the vertical change to the horizontal change, aligning with the *trigonometric conception* and *geometric ratio* conceptualizations of slope, respectively. The multitude of ways to conceptualize slope and the related concept of rate of change is further illustrated by Noble et al. (2001) who described rate of change, which is related to the *functional property*, as representing both a directly perceived quantity (as in velocity) and a relationship between two varying quantities. In light of the important roles of slope and the various conceptualizations of slope that occur throughout the mathematics curriculum, it is important to study which conceptualizations are used by students and instructors and to investigate whether academic or geographic culture may influence the dominant conceptualizations.

Students' Understanding of Slope

Most research related to slope has focused on primary and secondary students. Findings indicate that the concept of slope presents challenges to students (Barr, 1981; Carlson, Jacobs, Coe, Larsen & Hsu, 2002; Lobato & Thanheiser, 2002; Orton, 1984; Stump, 2001b; Teuscher & Reys, 2010; Thompson, 1994). In particular, research shows that students' knowledge of slope does not transfer between problem types (e.g. mathematical versus real-world settings) and that students do not relate slope and rate of change (Hattikudur, Prather, Asquith, Knuth, Nathan & Alibali, 2011; Planinic, Milin-Sipus, Kati, Susac & Ivanjek, 2012; Stump, 2001b; Lobato & Siebert, 2002; Lobato & Thanheiser, 2002; Teuscher & Reys, 2010). Perhaps because of its important link to covariational reasoning, researchers have focused particularly on students' ability to reason about rate of change. Studies show that students of various ages struggle to make sense of variable rate of change (Ellis & Grinstead, 2008; Hauger, 1998; Orton, 1984), including high-performing, second semester calculus students (Carlson et al., 2002). Despite noted student difficulties, other researchers have reported success in building students' ability to reason about variable rate of change (Ebersbach, Van Dooren, Goudriaan & Verschaffel, 2010; Johnson, 2011; Stroup, 2002; Thompson, 1994). These findings support recommendations from the US National Council of Teachers of Mathematics (2000) to include analyzing and describing varying rates of change as goals for instruction of upper primary students (ages 9 – 11). The inconsistencies between students' potential for developing advanced notions of slope and the results of various studies showing this potential is not achieved warrant further investigation of students' conceptualizations of slope.

Secondary Teachers' Understanding of Slope

Few researchers have investigated the conceptualizations of slope that secondary teachers hold. Stump (1999) found that secondary teachers most frequently conceptualized slope as a *geometric ratio* focusing on rise over run. Results from a program intended to build pre-service teachers' understanding of slope, and their knowledge of students' conceptualizations of slope indicated that they developed lessons incorporating applications but their actual instruction focused more on graphs and equations (Stump, 2001a). Additionally, these pre-service teachers demonstrated hesitancy when teaching slope as a measure of steepness and rate of change. Coe (2007) indicated that practicing secondary teachers showed difficulty working with average rates of change and could not explain the use of division in the familiar slope formula.

Tertiary Instructors' Understanding of Slope

There is an absence of research on tertiary instructors' conceptualizations of slope, which is noteworthy in light of the important role of slope in calculus (Noble et al., 2001). In order to understand pedagogical issues related to slope, it is important to consider both instructors' and students' conceptualizations of this key concept. Researchers have described students' difficult transition from secondary to tertiary mathematics, acknowledging that students often face new conceptualizations of previously well-established mathematical ideas (Barton et al., 2010; Clark & Lovric, 2009; Selden, 2005). This is particularly evident of slope, which is familiar to precalculus students and is needed for derivatives, a main topic of introductory calculus. As a result, tertiary instructors face the challenge of building on students' pre-existing notions while promoting more advanced conceptualizations of slope.

Theoretical Underpinnings

This study draws on intercultural perspectives on mathematics and concept image theory as it relates to the slope concept to better understand how university instructors and calculus students conceptualize slope. By comparing their dominant conceptualizations of slope (see Table 1), we provide an initial description of differences that might be attributable to academic cultures.

Intercultural Perspectives on Mathematics

Whereas mathematics was once viewed as a "universal" subject that transcended cultural divides, research now recognizes the sociopsychological nature of mathematics as a socially constructed body of knowledge (Bishop, 1994; Gerdes, 1988; Voight, 1994; Yackel & Cobb, 1996). The role of social interactions in constructing mathematical meaning suggests that cultural conflicts may arise (Bishop, 1994; Prediger, 2004). When mathematics is viewed through a cultural lens, the same underlying mathematical concept may develop different meaning to an individual depending on the norms of the underlying culture. Yackel & Cobb (1996) explain the importance of sociomathematical norms (i.e. social norms pertaining to mathematical content) in shaping an individual's mathematical knowledge. For instance, while a social norm might be that a person will share the solution strategy when presenting findings, a sociomathematical norm would include an understanding of what constitutes a mathematical explanation (Yackel & Cobb, 1996). As a result of the potential cultural differences, researchers have described three distinct bodies of mathematics: school mathematics, real-world mathematics, and mathematicians' mathematics (Bishop, 1994; Civil, 2002; Prediger, 2004).

Other researchers have applied the social nature of mathematical knowledge to explain the difficult transition from secondary to tertiary mathematics. Clark & Lovric (2009) liken the secondary to tertiary transition in mathematics to a rite of passage, acknowledging that it is necessarily a painful process. Barton et al. (2010) and Clark & Lovric (2009) describe a void between secondary and tertiary instruction and discuss the negative consequences that can result from a lack of serious discussion regarding the mathematics taught at each level. Thus, it may be necessary to subdivide the body of knowledge known as school mathematics into secondary and tertiary mathematics, based on different content emphases and different sociomathematical norms that can promote alternate conceptualizations of the same underlying concept. We investigate this hypothesis by comparing the conceptualizations of slope held by introductory calculus students (representing secondary school culture) and college instructors (representing tertiary culture). Note that although the introductory calculus students in this study were no longer secondary school students, they had successfully navigated their way through the cultural expectations of secondary mathematics classes and had been relatively unexposed to tertiary mathematics culture since data were collected at the beginning of the semester.

Concept Image

Tall & Vinner (1981) defined an individual's *concept image* as the total cognitive structure related to a particular concept. Advancing a person's concept image requires reformulation of old ideas to fit with new information (Tall, 2008). As discussed above, past research suggests that students' concept images of slope are fragmented and that secondary mathematics instructors' concept images of slope might impede their ability to "mediate students' ideas, make choices about representations of content, modify curriculum materials, and the like" (Ball & Bass, 2000, p. 97). No known research has studied tertiary instructors' concept images of slope despite the critical role they play in advancing students' concept images.

Methodology

The data reported were gathered from a single, open-ended instrument that was administered to two groups as described below.

Participants and Data Collection

Data were collected from 65 students enrolled in two sections of an introductory calculus course at a university in the Northeastern United States in the fall 2010 semester. The majority of students at this university typically come from a 250-km radius. A 13-item survey was administered to the students on the first day of class to determine their understanding of slope and function. The instructor referred to the survey as a "quiz" and informed students that it was required and would be graded, not on correctness, but on completion of serious, relevant responses. The instructor used the quiz to guide her instructional planning then shared the de-identified data with the research team after the course's completion. In an interview, the instructor said the survey took the students "about 20 min" to complete. This study focuses on five items on the survey that were related to slope. Since the purpose of the instrument was to elicit evidence regarding how individuals in different academic cultures conceptualize slope, the open-ended items were phrased in common vernacular, intentionally vague, and allowed for a vast array of possible, likely repetitious, responses.

Data were also collected from 26 professors attending a regional postsecondary mathematics instructor conference in 2010. The conference was held in the same region as the university used to collect student data. The instructors attending the conference taught at 14 colleges within a 200-km radius of the conference location. The same items used to collect student data were also administered as a paper-and-pencil survey to the instructors at the conference, who were given 15 min to complete the five items. Participation in the survey was optional. Participants received no compensation; however, they might have been motivated to complete the survey to the best of their ability since they knew that these results would be disseminated at the conference in a plenary talk on instructors' conceptualizations of slope.

Data Analysis

For both groups, a participant's response to an item was considered to be the unit of analysis. The 65 students' responses to the five items provided 325 units of analysis. Working independently two members of the research team coded each student response using the 11 conceptualizations of slope. A single response could be coded as addressing multiple conceptualizations. For the purposes of calculating inter-annotator agreement, 19 blank or illegible responses were eliminated from the pool of 325. The overall percentage of agreement for coding the remaining 306 students' responses was 0.863. Cohen's kappa values, reported in Table 2, represented "good" agreement (Landis & Koch, 1977; Altman, 1991). Two additional members of the research team reviewed the data to focus on those instances where disagreements in coding occurred. While the 11 conceptualizations remained the same, a few additional details were written to help with coding the second data set of instructors' responses. The two members of the research team who reviewed the disagreements in coding for the student data independently coded the instructors' responses. The 26 instructors' responses to the five items provided 130 units of analysis. Working independently, the two members of the research team coded each response using the 11 conceptualizations of slope. The overall percentage of agreement for coding all 130 instructors' responses was 0.993. Cohen's kappa values are reported in Table 2. The members of the research team discussed each response for which there was a disagreement in coding until consensus was reached.

TABLE 2

Coding agreement coefficients for five-item survey

Item		*Cohen's kappa*	
		Students	*Instructors*
1	What is slope?	0.739	0.968
2	List all the ways that slope can be represented.	0.689	0.862
3	How is slope used?	0.696	0.855
4	When is it appropriate to use slope?	0.730	0.882
5	Give three examples of slope.	0.729	0.968

Higher values of kappa were calculated for the instructors' responses than for the students', which might be in part to the additional details; however, the research team noted that the instructors' responses were easier to code than the students', which tended to be less typical and at times impossible to code. It was agreed that even if the instructors' responses had been coded first, the coefficient values would have been higher for the agreement between coders for the instructors' responses than for the students'. Table 3 provides sample responses showing how they were coded as well as which items were answered. As shown on the last row, the research team was not able to code all responses due to ambiguity in some responses.

Results

The results indicated that participants used a variety of responses to the same item and that the same participants used different conceptualizations of slope on different items. For instance, Table 4 provides sample responses to item 5: *How is slope used?* These responses highlight the diversity of conceptualizations used in response to a single item. Table 5 illustrates that different items elicited various conceptualizations for an individual participant, suggesting that the person's concept image contained all such conceptualizations.

Conceptualizations by Survey Item

To determine whether particular items elicited certain conceptualizations, the results are summarized individually for each survey item. The percentages of students and instructors using each conceptualization are compared.

TABLE 3

Sample responses with corresponding conceptualization codes

Code	*Sample responses*	*In response to item*
Geometric ratio	Slope is the ratio of vertical displacement to horizontal displacement between two points	1
	Graphically, showing the vertical change and the horizontal change	2
Algebraic ratio	It is the difference in the y coordinates over the difference in the x coordinates	1
	$(y_2 - y_1)/(x_2 - x_1)$	2
Physical property	A measure of how steep a line is	1
	When you are considering the "tilt" of a line	4
Functional property	How fast one thing changes as something else changes	1
	Whenever you need to discuss rate of change	4
Parametric coefficient	It is the m in $y = mx + b$	2
	In $y = 2x + 10$, $m = 2$ is the slope of a line	3
Trigonometric conception	It is a way to quantify the angle of a line (with the x-axis)	1
	The tangent of the angle	2
Calculus conception	As tangents to curves	2
	If $y = x^2$, $2x$ is the slope of the parabola at x	5
Real-world situation	Regulations for ladder construction/placement, wheelchair ramp specifications, the grade of road	3
Determining property	To check if lines are parallel or perpendicular	3
	When you are going to determine if lines are parallel (if they have the same slope)	4
Behavior indicator	A number that tells you if a line is increasing, decreasing, or staying flat	1
	When you want to see if a line is going up or down as you move left to right	4
Linear constant	When working with a line. You can use any pair of points for the slope since it stays the same	3
	When discussing any linear (steady) phenomenon, not appropriate when the phenomenon is not linear	4
No code	As any type of number—even a fraction or a decimal	2

Responses to Item 1: What Is Slope? Responses to item 1, reported in Table 6, indicate significant differences in students' and instructors' conceptualizations. Students' responses most often contained evidence of a *behavior indicator* conceptualization (i.e. considering slope as a gauge for determining increasing or decreasing trends of a line). Their responses were coded with 10 of the 11 conceptualizations, excluding only *real-world situation*. Instructors' responses, on the other hand, most often

TABLE 4

Sample responses to item 5: how is slope used?

Person	*Response*	*Code*
1	Line passing through (1, 2) and (3, 4) has slope $\frac{4-2}{3-1} = \frac{2}{2} = 1$	Algebraic ratio
	$y = -½\, x + 3$, slope = −½	Parametric coefficient
	Slope of a horizontal line is 0	Behavior indicator
2	What is your average speed (velocity) if you have driven 300 miles in 6 h?	Real-world situation
	Which line increases at a greater rate, $y = 2x + 1$ or $y = 5x-6$?	Behavior indicator
	Show tan θ is the slope of the line connecting (0, 0) to (cos θ, sin θ)	Trigonometric conception
3	Steepness	Physical property
	Increasing or decreasing slope	Behavior indicator
	Plane taking off or landing	Real-world situation

contained evidence of a *functional property* conceptualization (i.e. the constant rate of change between two variables). The instructors' responses only showed evidence of seven of the 11 conceptualizations, and no instructors used the students' most popular conceptualization (*behavior indicator*). These results highlight important characteristics of how students and instructors understand and commonly think about slope. First, students holding more diverse views of the definition of slope may have resulted from their less mathematically precise descriptions of slope. While instructors were more likely to provide "textbook" definitions of slope, students were more likely to reference various ideas they recalled about slope, even if the notions did not provide a concise definition for the term. This is illustrated by the sample student and instructor responses in Table 7. Notice the brevity and exactness of the instructor responses

TABLE 5

Coding for three participants' responses to the five items

	Role	*Item 1*	*Item 2*	*Item 3*	*Item 4*	*Item 5*
Person 1	Instructor	F	G, A	B	–[a]	PC
Person 2	Student	A	G, A, PC, T, C	F, PC, T, C	C	R
Person 3	Instructor	G, F	G, A, PC, T	G, F	F	P, R

[a]No code was assigned to this vague response: "There is no limit to where slope can be appropriate, as long as it is mathematically correct."

TABLE 6

Conceptualizations evidenced in responses to item 1: what is slope?

	Percentage of respondents using each conceptualization	
Conceptualization	*Students (n = 65)*	*Instructors (n = 26)*
Behavior indicator	43	0
Geometric ratio	35	31
Algebraic ratio	26	23
Functional property	18	46
Physical property	15	38
Trigonometric conception	15	4
Parametric coefficient	14	8
Calculus conception	6	15
Determining property	3	0
Linear constant	2	0
Real-world situation	0	0

compared with student responses, which tended to include examples and descriptions of the usefulness of slope, often describing rather than defining the concept.

Students' use of *behavior indicator* more than any other conceptualization suggests that students conceptualize slope as a characteristic of a linear graph. Instructors, on the other hand, conceptualize slope as a relationship between two covarying quantities through the *functional property* conceptualization. Furthermore, the absence of any use of *behavior indicator* among instructors is striking in comparison to its use

TABLE 7

Student versus instructor responses to item 1: "what is slope?"

Participant		*Response to "What is slope?"*
Student	1	A slope tells whether the graph of the equation is going in a negative or positive way. The slope of a line can be put into an equation to find the y-intercept. Derivatives can also tell the slope of a graph
	2	Slope is the rise/run of a line, which gives the distance between points. When graphing a function (ex. $y = mx + b$, $y = 3x + 2$), the slope would be used (m) to plot points
Instructor	1	Description of the relationship between the rates of change of 2 variables linearly related
	2	Graphic version of rate of change of a linear function

TABLE 8

Conceptualizations evidenced in responses to item 2: list all the ways that slope can be represented

	Percentage of respondents using each conceptualization	
Conceptualization	*Students (n = 65)*	*Instructors (n = 26)*
Parametric coefficient	51	46
Algebraic ratio	38	62
Geometric ratio	11	77
Behavior indicator	8	8
Linear constant	8	8
Calculus conception	3	42
Trigonometric conception	2	15
Real-world situation	2	8
Physical property	0	4
Functional property	0	19
Determining property	0	0

by students. The use of *functional property* by less than one fifth of all students is significant since this conceptualization involves interpreting slope as a relationship between two covarying quantities. Since covariational reasoning has been described as a critical concept for precalculus instruction (Carlson et al., 2010; Confrey & Smith, 1995), its absence among calculus students' definitions of slope is concerning.

Responses to Item 2: List All the Ways That Slope Can Be Represented. Students' and instructors' responses to item 2 were similar. As seen in Table 8, *geometric ratio* (e.g. "rise over run"), *algebraic ratio* (e.g. "change in y divided by the change in x"), and *parametric coefficient* (e.g. "m the coefficient of x") were among the top three conceptualizations evidenced in both groups' responses, although the orders were reversed. Two important implications follow. First, despite providing different definitions for slope on item 1, students and instructors used similar representations on item 2. As an example, recall the very different "definitions" of slope provided by student 2 and instructor 2 (see Table 7). On item 2, both participants used m, $\frac{\text{rise}}{\text{run}}$, and $\frac{\Delta y}{\Delta x}$ as representations of slope, with instructor 2 also including "tangent of an angle" as a representation. These results suggest that students and instructors may use similar representations of slope but that the underlying meaning of the representations could be interpreted quite differently. Second, despite being used by 19 % of instructors, no students used the *functional property* conceptualization (i.e. the constant rate of change between two variables) as a representation of slope. This provides further evidence of the weakness of students' covariational reasoning with regards to slope.

Responses to Item 3: How Is Slope Used? On item 3, the differences between students' and instructors' responses persisted. As can be seen in Table 9, the most frequently noted conceptualization among students' responses was *determining property* (e.g. "when considering if lines are parallel or perpendicular"). Yet this conceptualization was the least frequent among instructors. *Functional property* topped the list of conceptualizations evidenced in instructors' responses, while only 11 % of students' responses were found to show evidence of the covariational reasoning in this conceptualization. Since this item focuses on uses of slope, it is striking that only 6 % of students and 23 % of instructors made reference to a *real-world situation*.

Again, the results highlight students' view of slope as relating to the graph of a line. Besides describing slope via the *behavior indicator* conceptualization (e.g. "slope is used to show how a line is increasing (or decreasing)"), students also emphasized graphical interpretations via the *determining property* conceptualization. Students described the role of slope in determining whether lines are parallel or perpendicular, as well as slope and a point determining the graph of a unique line. Once again, the lack of evidence in student responses for the *functional property* conceptualization points to students' weak covariational reasoning and

TABLE 9

Conceptualizations evidenced in responses to item 3: how is slope used?

	Percentage of respondents using each conceptualization	
Conceptualization	*Students (n = 65)*	*Instructors (n = 26)*
Determining property	20	4
Behavior indicator	0	12
Algebraic ratio	15	4
Parametric coefficient	14	8
Functional property	11	58
Trigonometric conception	8	8
Geometric ratio	6	12
Physical property	6	23
Real-world situation	6	23
Calculus conception	3	31
Linear constant	3	12

demonstrates how students' responses about slope indicated a procedural emphasis.

Responses to Item 4: When Is It Appropriate to Use Slope? The responses to item 4 were very similar to the responses to item 3. The rankings for students and instructors changed minimally between items 3 and 4, although the frequency of the most popularly evidenced conceptualizations decreased on item 4. The results are reported in Table 10.

Responses to Item 5: Give Three Examples of Slope. When asked to give three examples of slope, students' responses provided evidence of *parametric coefficient* (e.g. "$y = mx + b$ where m is slope") and *behavior indicator* (e.g. "constant [$y = 6$], increasing [$y = 2x + 5$], decreasing [$y = -x + 1$]") conceptualizations most frequently, followed by *algebraic ratio* (e.g. "(1,0), (0,1), $m = \frac{1-0}{0-1} = -1$). All other conceptualizations were infrequently evidenced in students' responses to item 5. For this item, instructors' responses provided evidence that they relied heavily on *real-world situation*, *functional property*, *parametric coefficient*, and *calculus conception* (e.g. "slope of tangent lines"), as shown in Table 11. The *real-world situation* conceptualization dominated instructors' responses, doubling the frequency of the next three conceptualizations.

Students' conceptualizations once again provided little to no evidence to support their conceptual understanding of slope as the relationship between two covarying quantities. Instead it seems they

TABLE 10

Conceptualizations evidenced in responses to item 4: when is it appropriate to use slope?

	Percentage of respondents using each conceptualization	
Conceptualization	*Students (n = 65)*	*Instructors (n = 26)*
Determining property	12	0
Behavior indicator	12	4
Algebraic ratio	11	4
Parametric coefficient	9	4
Functional property	8	38
Trigonometric conception	8	8
Geometric ratio	5	8
Physical property	3	12
Real-world situation	3	19
Calculus conception	3	31
Linear constant	2	4

TABLE 11

Conceptualizations evidenced in responses to item 5: give three examples of slope.

	Percentage of respondents using each conceptualization	
Conceptualization	*Students (n = 65)*	*Instructors (n = 26)*
Parametric coefficient	38	31
Behavior indicator	29	27
Algebraic ratio	12	8
Geometric ratio	6	12
Real-world situation	6	62
Functional property	3	31
Calculus conception	2	31
Determining property	2	4
Linear constant	2	0
Physical property	0	23
Trigonometric conception	0	8

were thinking of slope as the coefficient m or as an indicator for the behavior of a line. For instance, responses often provided an equation in slope-intercept form with some indication of which value represented slope (e.g. "$y = \boxed{\frac{3}{4}}x + b$"). Although students did reference an *algebraic ratio* conceptualization, this was usually restricted to some variation of the "$y_2 - y_1$ *over* $x_2 - x_1$" mnemonic or an example that used the algebraic ratio to find the slope between two points (e.g. "slope between (3,4) and (6,5) is $\frac{5-4}{6-3} = \frac{1}{3}$). There was little evidence that students actually considered a relationship between the variables x and y. Students' infrequent use of *real-world situations* as examples of how slope is used is also of some concern, particularly in light of its heavy use by instructors.

Conceptualizations Used on at Least One Item

The percentage of students and instructors who used each conceptualization at least once on the survey was calculated since a conceptualization may be part of an individual's concept image even if it is only evoked in particular situations (Tall & Vinner, 1981). The results are summarized in Table 12. Students used *behavior indicator* most often, followed closely by *parametric coefficient*. Four conceptualizations were used by nearly three fourths of all instructors, namely *geometric ratio*, *functional property*, *real-world situation*, and *calculus conception* (e.g. "$f'(c) = \lim_{h\to 0} \frac{f(c+h)-f(c)}{h}$").

TABLE 12

Relative frequency of participants with a conceptualization on at least one item

Students (n = 65)		*Instructors (n = 26)*	
Conceptualization	*Percentage*	*Conceptualization*	*Percentage*
Behavior indicator	71	Geometric ratio	85
Parametric coefficient	68	Functional property	77
Algebraic ratio	54	Real-world situation	77
Geometric ratio	45	Calculus conception	73
Determining property	28	Algebraic ratio	65
Functional property	26	Parametric coefficient	65
Physical property	22	Physical property	54
Trigonometric conception	20	Trigonometric conception	35
Linear constant	18	Behavior indicator	35
Real-world situation	17	Linear constant	15
Calculus conception	12	Determining property	8

The results indicate that a larger percentage of students used *behavior indicator* and *determining property* conceptualizations than instructors, mostly to denote that slope helped (1) identify the increasing/decreasing behavior of lines and (2) determine if two lines were parallel or perpendicular. In fact, students were over three times more likely to show evidence of *determining property* conceptualizations and over twice as likely to show evidence of *behavior indicator* conceptualizations on at least one item as instructors. Students were also more likely to show evidence of the *parametric coefficient* and *linear constant* (i.e. the unchanging constant property of a line) conceptualizations, although these gaps were fairly narrow. All other conceptualizations were evidenced by a greater percentage of instructors than students.

Overall, students evidenced fewer conceptualizations than instructors. Table 13 provides the relative frequencies for the number of conceptualizations evidenced in students' and instructors' responses on at least one item. Although the mode was five conceptualizations for both groups, the mean was 3.6 for students and 5.9 for instructors. The relationship between the mode and mean for students indicates that the data are left-skewed and that more students made use of a smaller number of conceptualizations. In contrast, the relationship between mode and mean for instructors indicates that the data are right-skewed and that more instructors made use of a larger number of conceptualizations. The range for the number of conceptualizations used by students went from zero to eight, and the range for instructors went from three to nine. Taken together, these statistics reveal that instructors held more conceptualizations than students.

Discussion

The responses provided a great deal of information about how each group conceptualized slope. The discussion is comprised of four sections. In the first and second sections, the students' and instructors' conceptualizations of slope are discussed individually. The third section compares the conceptualizations of slope revealed by the students' and instructors' responses. The fourth section discusses the findings in terms of potential cultural differences.

Students' Conceptualizations of Slope

Students rely heavily on *behavior indicator* and *parametric coefficient* conceptualizations of slope, with *algebraic ratio* and *geometric ratio* in close proximity. Students used all other conceptualizations less often. As shown in Table 12, only four conceptualizations were used by over one third of the students on at least one item while it is shown in Table 13 that 19 % of students used two or fewer conceptualizations of slope. These results suggest that students have limited diversity in their use of conceptualizations of slope.

Students appear to interpret slope as a coefficient or ratio that describes a line's behavior. Students' responses to the individual items suggest that their concept images of slope focus on the procedures involved to (1) generate the numeric value associated with the slope of a line (e.g. "$m = y_2 - y_1/x_2 - x_1$") and (2) create or identify the line's corresponding graph (e.g. "slope is used as a piece of information to form a line on a graph or an equation"). Students' use of these conceptualizations implies a procedural focus, with little indication of meaning for the covarying quantities involved or the physical and real-world applications. It is exactly the covariational aspects of slope as well as the applications of slope to describe physical phenomena that students should develop before entering calculus (Carlson et al., 2010). In order to grasp the concept of the derivative, students need a conceptual understanding of slope beyond what was evidenced in the majority of their responses. The findings

TABLE 13

Number of conceptualizations evidenced by students and instructors on at least one item

Number of conceptualizations of slope	*Percentage of students (n = 65)*	*Percentage of instructors (n = 26)*
11	0	0
10	0	0
9	0	8
8	2	8
7	0	19
6	6	19
5	31	27
4	20	12
3	23	8
2	11	0
1	6	0
0	2	0

indicate that students may not build critical covariational reasoning in the particular case of the concept of slope.

Instructors' Conceptualizations of Slope

The results provide evidence that the instructors have relatively diverse conceptualizations of slope using *geometric ratio*, *functional property*, *real-world situation*, and *calculus conception* interpretations of slope most often. Nine of the conceptualizations were evidenced by over one third of the instructors on at least one item, and 92 % of the instructors were found to have used at least four conceptualizations of slope in their responses.

Based on responses to all items, instructors appear to hold a more robust view of slope as a ratio of covarying quantities (e.g. "rate at which a function changes wrt its argument"), while recognizing the utility of slope in applied contexts. Instructors' responses to the individual items provide evidence that their concept images of slope do not focus solely on its procedural calculation; rather, the instructors seem to have a multi-dimensional understanding of the concept. Despite this diversity, *behavior indicator* was infrequently evidenced in instructors' responses. The first derivative test in calculus uses the sign of the derivative (the slope of the tangent line) to determine the increasing or decreasing behavior of a function. In light of frequently demonstrating a *calculus conception*, it is puzzling that the closely related *behavior indicator* conceptualization was not evidenced more frequently in instructors' responses, particularly considering students' heavy use of this conceptualization.

Comparing Students' and Instructors' Conceptualizations of Slope

The most striking finding of the comparison between instructors' and students' conceptualizations of slope is the rarity of *behavior indicator* and *determining property* among the conceptualizations used by instructors compared to students. Recall that 71 % of students used the *behavior indicator* conceptualization compared with only 35 % of instructors. Even more telling, *behavior indicator* was the most common conceptualization evidenced at least once for students, but ranked in the bottom four for the instructors. Likewise, 28 % of students used the *determining property* conceptualization compared with 8 % of instructors. *Determining property* was fifth among students' conceptualizations but last among instructors.

While it is expected that instructors show evidence of more conceptualizations than students, it is also important that instructors build on the conceptualizations frequently used by students. This is particularly important for tertiary instructors responsible for building advanced notions of slope, an already familiar mathematical concept for university students (Selden, 2005). In order to build advanced ideas from students' existing conceptualizations, instructors must work comfortably and frequently with the conceptualizations commonly held by students. In fairness, this study did not investigate the conceptualizations emphasized during instruction, nor whether a hierarchy of slope conceptualizations exists. It is possible that an instructor who cited the derivative as indicating instantaneous rate of change might emphasize the *behavior indicator* conceptualization in instruction by interpreting instantaneous rate of change as indicating the increasing, decreasing, or constant behavior of the tangent line. Future research should investigate if a hierarchical relationship exists between conceptualizations and how teachers' personal concept images relate to their instructional focus.

Another finding that raises concern is students' infrequent use of *real-world situations*. Since students struggle applying the concept of slope in real-world situations (Lobato & Siebert, 2002; Lobato & Thanheiser, 2002), it is possible they do not mentally associate the real-world contexts with the mathematical concept of slope. Another possible explanation for the disparity between instructors' and students' use of *real-world situations* might be because secondary teachers do not emphasize applications of slope. Such an explanation would correspond with previous findings, which indicate that pre-service teachers prepared lessons on slope with real-world references but omitted these examples in actual instruction (Stump, 2001a). The results of this study suggest that future research should investigate the role of secondary instructors and secondary instruction in the conceptualizations developed by secondary students.

Initial Cultural Comparisons

The findings presented above, together with past research, indicate the potential influence of culture on an individual's conceptualizations of slope. The comparisons highlighted key differences between the conceptualizations of slope held by tertiary instructors and calculus students, which may also be interpreted in terms of tertiary and secondary school culture.

The calculus students were surveyed on the first day of class in order to determine the dominant conceptualizations of slope that could be attributed to secondary school culture. Past research links the conceptualizations of slope commonly held by students in this study to the secondary school culture. A study of secondary mathematics standards documents for each of the 50 states in the USA found that the standards called for curricular focus on slope as a *geometric ratio*, *behavior indicator*, *determining property*, *functional property*, and *algebraic ratio* (Stanton & Moore-Russo, 2012). Students in this study relied heavily on *behavior indicator*, *geometric ratio*, and *algebraic ratio* conceptualizations of slope, while using *determining property* and *behavior indicator* conceptualizations much more frequently than their instructors. Findings that these conceptualizations are emphasized by standards documents for the USA's secondary curriculum supports the possibility that differences in academic cultures might be behind the different conceptualizations demonstrated by the two groups of participants.

Recall that in response to item 1 (*What is slope?*), the students demonstrated a greater diversity of conceptualizations than instructors (see Table 7). It was suggested that the diversity of student responses may have stemmed from heavy use of conceptualizations that, although true, did not completely define slope. Although only preliminary findings, students' liberal use of conceptualizations that described, but did not necessarily define slope, may be a result of secondary sociomathematical norms. In particular, students were content to provide a series of true statements like a checklist of characteristics of slope, while instructors focused on defining interpretations. This is in sharp contrast to item 2 (*List all the ways that slope can be represented*), in which the two groups responded similarly. Here, there is less room for cultural influence since the notation used to denote slope is rather universal.

Since the instructors and students in this study were from the same geographical region, the differences in their conceptualizations suggest that different academic cultures contributed to the different interpretations of slope. Although data are limited, a prior study of historically disadvantaged South African secondary teachers' conceptualizations of slope (Mudaly & Moore-Russo, 2011) showed that *trigonometric conception* (i.e. responses related to slope as the tangent of the angle of inclination) was among the top three conceptualizations used (along with *parametric coefficient* and *behavior indicator*). Since *trigonometric conception* was infrequently used by the American students and instructors in this study and was among the least common conceptualizations in the states' standards documents (Stanton & Moore-Russo, 2012), one might infer that geographic cultural differences may also influence an individual's concept image of slope.

Implications and Future Work

The results described above imply (1) that incoming tertiary students demonstrate procedural and graphical interpretations of slope with little evidence of covariational reasoning and (2) university instructors demonstrate diverse, conceptual interpretations of slope but infrequently use students' most prevalent conceptualizations. These findings suggest that incoming tertiary students may not possess a solid foundation of covariational reasoning on which to build more advanced conceptualizations of slope. Tertiary instructors' infrequent use of students' dominant conceptualizations of slope suggests that either (1) a hierarchical progression of slope conceptualizations exists or (2) instructors do not capitalize on students' dominant conceptualizations as a foundation on which to build more advanced ideas. The findings presented here warrant further investigation into students' concept images of slope and instructors' use of conceptualizations during instruction. Since secondary teachers were not participants in this study, future research should investigate the extent to which the learning outcomes for secondary slope instruction align with teachers' intended emphasis.

The findings also suggest a need for future investigation of the role of culture in developing interpretations of slope. While only an exploratory investigation, the findings suggest that secondary school versus tertiary mathematics culture and geographic cultural differences may influence the dominant conceptualizations of slope. For this reason, additional investigation of secondary and tertiary students and instructors in other cultures is needed. More specifically, if the same 11-category system for identifying conceptualizations of slope could be used in a variety of studies across cultures, the global mathematics education community would learn more about how this key concept develops, is taught, and is understood.

Conclusions

Instructors, especially those teaching courses offered in the first 2 years of university, should recognize the conceptualizations of slope their students do and do not hold as they plan instructional activities. Students' and instructors' abilities to conceptualize slope in a multifaceted way are of critical importance. However, little research has been conducted on either the concept of slope in general or the manner in which it is conceptualized in tertiary education. This study is of value as it adds to existing research concerning slope, which has primarily addressed development of this key concept at the K-12 level. The results also support the importance of future investigations of the mathematics culture in secondary and tertiary schools.

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Courtney Nagle

Penn State Erie, The Behrend College
4205 College Drive, Erie, PA, 16563, USA
E-mail: crt12@psu.edu

Deborah Moore-Russo and Kristi Martin

University at Buffalo, State University of New York
Buffalo, NY, USA
E-mail: dam29@buffalo.edu
E-mail: krm33@buffalo.edu

Janine Viglietti

Buffalo State College, State University of New York
Buffalo, NY, USA
E-mail: vigliejm@buffalostate.edu