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THE READING COMPREHENSION OF GEOMETRIC PROOFS: THE CONTRIBUTION OF KNOWLEDGE AND REASONING

Received: 22 October 2006; Accepted: 30 May 2007

ABSTRACT. A model of reading comprehension of geometric proofs (RCGP) has recently been proposed by the authors. This article further investigates students' development of such comprehension based on this model, looking at the relationship between students' reading comprehension and their prior knowledge and logical reasoning. The results show that (1) students' development of RCGP may follow two different learning trajectories; (2) the effect of logical reasoning on RCGP in ninth grade is larger and more complex than in tenth grade; (3) knowledge about geometric figures is not the main factor contributing to RCGP, and geometric knowledge that includes knowledge of figures and of verbal description and translation between the two distinguishes only the level of surface comprehension from the other levels of RCGP; and (4) regression analysis yields a two-variable model that includes logical reasoning and relevant geometric knowledge, and that accounts for 54% and 22% of the variance on RCGP data from the ninth and tenth graders, respectively.

KEY WORDS: geometric proof, knowledge, reading comprehension, logical reasoning

Introduction

The Special Role of Reading Mathematical Proofs

The 2003 Programme for International Student Assessment (PISA) survey focused primarily on students' mathematical literacy and only secondarily on reading and scientific literacy. One of the study's results showed a high correlation between mathematical and reading literacy (Organisation for Economic Co-operation and Development, 2005). However, mathematical proof is not included in the PISA study. Mathematical proof, especially geometric proof, is a special text genre in written discourse (see Pimm & Wagner, 2003), and the ability to read a mathematical proof is required for understanding proofs written by teachers or contained in textbooks.

The complexity of geometric figures and proofs has been closely analyzed by Duval (1998, 2006). The rigor and abstraction of mathematical proofs are markedly different from other subjects. There is a large gap between valid deductive reasoning in mathematics and general argumentation. A mathematical proof is validated on the basis of axioms, definitions, theorems, and deductive rules, while general argumentation can convince others on the basis of various forms of reasoning. For example, both undergraduate and high school students have difficulty discriminating between the validity of conditional statements and the truth of assertions (Morris, 2002; Wu Yu, Chin & Lin, 2004). In contrast, the comprehension of story-based or expository texts requires diverse inferences on the part of readers, e.g., elaborative, cohesive, knowledge-based, or evaluative ones (Bowyer-Crane & Snowling, 2005). Thus, we should have our own concerns about students' reading comprehension of mathematical proofs.

A Model of Reading Comprehension of Geometric Proofs

From a transactional perspective, reading is a process that goes beyond decoding; reading comprehension requires that readers work out the meaning of words, phrases and sentences, integrate several paragraphs into the major theme of the text, and thus infer implicit information in order to maintain text coherence (Kintsch, 1998; van den Broek, 1994). In order to map out a picture of reading comprehension of mathematical proofs, Yang & Lin (2005, 2007) have explored a model produced from mathematicians' and mathematics teachers' views and experiences, in addition to the analysis of proof texts themselves.

The four hypothesized levels of reading comprehension of geometric proofs (RCGP) are: comprehension of the surface, comprehension of recognized elements, comprehension of chaining elements, and comprehension of encapsulation. Comprehension of the surface level is characterized as epistemic understanding *without* analyzing the elements of an argument in a proof. The elements of an argument could be premises, conclusions, or applied properties. Comprehension of recognized elements level is characterized as recognition of premises, conclusions, or properties that may be implicitly applied in a proof. Comprehension of chaining elements level is characterized in terms of understanding the logical chaining of premises, properties, and conclusions in a proof and of viewing figures as referential objects. The comprehension of encapsulation level is characterized as interiorizing a proposition and its proof as a whole, which implies that one can apply it, as well as distinguish different premises related to other similar propositions. This model not only focused particularly on the transition from informal to more formal deduction, with a literacy perspective on understanding proof, but also took the features of visualization and analysis into account. This model aims to provide a structure for the reading comprehension of geometric proof while van Hiele's model aims at characterizing developmental stages of geometric thinking (van Hiele, 1986).

The structure of the five facets of RCGP is not only analogous to Bloom's taxonomy, but is also supported by students' performance (via multidimensional scaling methods; see Yang & Lin, 2007). The relationship between facets and levels are sketched in Figure 1. The similarity between students' performance on the facet of basic knowledge and those of logical status and summarization is greater than that between students' performance on the facet of basic knowledge and those of generality and application. The feature of developmental stages in van Hiele' s model is not present in this model. Therefore, how students actually develop among the levels of RCGP is still an unanswered question.

Aims of this Study

Yore, Craig & Maguire (1998) emphasized the importance of meta-level understanding in an interactive-constructive science reader. Readers access their pre-existing knowledge to make sense of texts and infer interactive interpretations by means of constructive cognition. Accordingly, readers' pre-existing cognition should predict their reading comprehension well. Much language research has focused on this issue. Three, central, interrelated factors that affect reading have been proposed: the reader, the text, and the context (Lipson & Wixson, 1991).

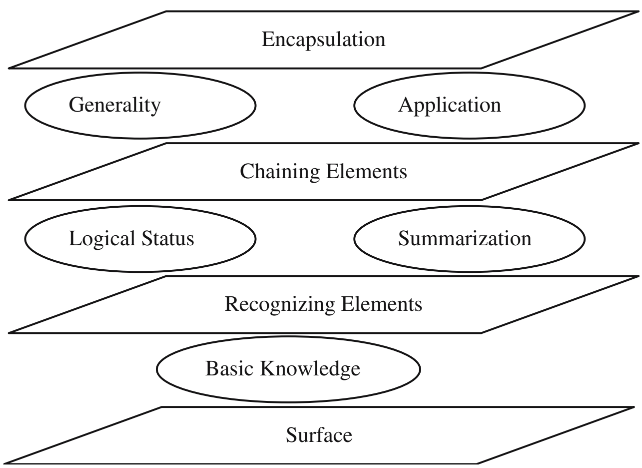


Figure 1. The integration of a theoretical model and five facets of reading comprehension of geometric proofs (RCGP)

In this study, there are three objectives focused mainly on the reader dimension of cognition: (1) to analyze students' development of levels of RCGP; (2) to contrast the difference between knowledge and reasoning among the various levels of RCGP; and (3) to predict RCGP based on knowledge and reasoning. Therefore, we present this study after briefly reviewing both theoretical and empirical evidence pointing to the importance of reasoning and knowledge in reading comprehension.

Predictors of Reading Comprehension

Prior knowledge denotes whatever one already knows about, such as facts, ideas, objects, and mediators; relevant prior knowledge denotes prior knowledge related to reading. Reading studies have found significant effects of relevant prior knowledge on prose comprehension (e.g., Fincher-Kiefer, 1992). In addition to prior knowledge, readers have particular difficulty in making inferences of different types (Yuill & Oakhill, 1991), and this obstacle could contribute to comprehension deficiencies (Nation & Snowling, 1998). Poor comprehenders may be less likely to initiate a contextually appropriate example, less able to use lexical cues, or less likely to recognize what can be inferred from text. Therefore, they ultimately fail to generate proper inferences (Oakhill, 1982, 1983; Yuill & Oakhill, 1988).

On the other hand, knowledge influences deduction (e.g., Oakhill, Johnson-Laird & Garnham, 1989). People's own knowledge interferes with logical deductions and reading performance, especially if this knowledge is inconsistent with text information (Markovits & Bouffard-Bouchard, 1992; Schmidt & Paris, 1983). For example, relevant background knowledge for a passage is a better predictor of fourth graders' ability to generate inferences from and elaborate upon this text than their comprehension skill (Marr & Gormley, 1982). No matter how prior knowledge, reasoning, and reading comprehension are intricately correlated with each other, readers' relevant prior knowledge and inference-making can provide two effective predictors of reading comprehension.

Similarly, understanding proofs requires readers to discriminate necessary conclusions reasoned for by deductive rules from probable conclusions reasoned for by induction or abduction. For example, suppose a reader encounters the following statement in a proof: *Because this triangle has a right angle, the square of the length of the hypotenuse is equal to the sum of the squares of the lengths of the other two sides.* The reader may recognize that the conclusion about the three sides of this triangle derives from a general property of *all* right triangles—one that is not only probable (or even one that happens to be true in this specific case) but one that is also logically necessary. Moreover, the reader may understand that a conclusion derived from measuring figures or folding paper is not logically necessary, just probable.

However, comprehending a proof runs into a tension between knowledge-based inference and deductive inference. The meaning of an argument or a statement is determined both by its logical value and by the connection between its logical and epistemic values (Duval, 2002). Selden & Selden (1995) used the term *validation* to describe the process an individual carries out to determine whether or not a proof is correct and actually proves the particular theorem it claims to prove. Validating a proof may benefit from knowing some properties; nevertheless, knowing properties may interfere with understanding a proof that proves just one property. Moreover, comprehending a proof not only requires readers to chain deductively each statement in this proof but also involves distinguishing what is applied from what is validated. While different kinds of inferences in literacy are made for text coherence (see, e.g., Graesser, Singer & Trabasso, 1994), deductive inference dominates reading comprehension of a mathematical proof.

Method

A test of reading comprehension of geometry proof was adopted from Yang & Lin (2007), a logical reasoning test was formulated based on a test of propositional logic with mathematical content (Jansson, 1986), and a relevant geometric knowledge test was designed by the authors. Based on the notion of reading to learn (Richardson & Morgan, 1997) and intertwining learning strands with prospective and retrospective connection (Freudenthal, 1991), students who are going to learn formal geometry proofs in school, as well as students who have just learnt such proofs, are potential subjects for this study.

These instruments were administered to 223 Grade 9 and 378 Grade 10 Taiwanese students. They were given a relevant geometric knowledge test and a logical reasoning test during the same class, and a RCGP test during a subsequent class. Almost all of the students had enough time to respond to all the items. The ninth graders had learnt some properties of triangles (including congruent triangles) by means of geometric calculation or manipulative reasoning and were going to learn about geometric proof in relation to some simple geometrical figures. The tenth graders had learnt about logical reasoning at the beginning of the term, as well as knowing what the ninth graders had learnt. The implemented curriculum in most Taiwanese secondary mathematics classrooms can be characterized as algorithms, well-chosen examples, and exercises for drill and practice (Lin & Tsao, 1999). Briefly, the content learnt by Grade 9 and 10 students who took part in this study belonged to informal and formal deduction, respectively.

The Reading Comprehension Test

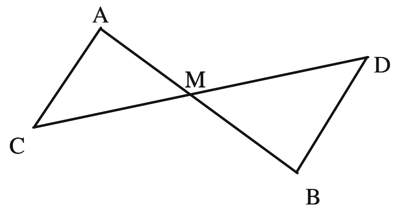
The geometrical content of the RCGP test is shown in Figure 2, while the structure of designed items regarding the operational definition of each facet is given in Table I. This proposition and its proof were adopted from the Taiwanese Grade 9 mathematics textbook. In addition, three mathematics education researchers and two experienced mathematics teachers were invited to allocate the items into corresponding facets. After discussing any discrepancies, some items or descriptions of definitions were modified. Sixteen revised questions were included in the test (Appendix). The Cronbach alpha reliability coefficient for this instrument was 0.84 for Grade 9 and 0.69 for Grade 10.

The Logical Reasoning Test

Jansson (1986) developed an instrument to measure students' logical reasoning in four geometric situations with properties of two line segments being parallel or of equal length. (Jansson's students were, on average, 13.6 years old.) He found that 16 different modes of propositions could be separated into seven levels. Six of his modes of logical reasoning (occurring in his second to seventh levels) were selected in this study.

Taking the diversified determination of inference into account, we classified responses to these logical reasoning items into three types of affirmation: *direct* (e.g., given that X is p or q, and X is not p, we can affirm that X is q), *equivalent* (e.g., given that X is not q if X is p, we can affirm that X is not p if X is q), and *non-contraventive* (e.g., given that X is not q if X is p, we can affirm that X is p or not p if X is not q). The framework of this logical reasoning instrument and the corresponding item numbers are shown in Table II. The Cronbach alpha reliability coefficient for each affirmation in this instrument was above 0.9 for both Grades 9 and 10 in the study.

Problem: As figure shown, $\overline{AB}$ and $\overline{CD}$ intersect at the point M, $\overline{AM}=\overline{BM}$ and $\overline{CM}=\overline{DM}$; then, must $\overline{AC}$ and $\overline{DB}$ be parallel with each other?



As to this problem, Tom gives the following proof:

In $\triangle AMC$ and $\triangle BMD$, because $\overline{AM}=\overline{BM}$ and $\overline{CM}=\overline{DM}$, Line 1

and because $\angle AMC = \angle BMD$, Line 2

$\triangle AMC \cong \triangle BMD$ (SAS) Line 3

So $\angle MAC = \angle MBD$ (corresponding angle) Line 4

Because $\angle MAC = \angle MBD$, $\overline{AC} /\!/ \overline{DB}$

(The alternate interior angles are congruent.) Line 5

Proof Content	Sign	Analysis of Proof Content
$\overline{AM}=\overline{BM}$	P1	P1,P2,P'3 →C1 "To derive a conclusion C1 from the three premises P1, P2, P'3 which is transformed from the intersection of two lines."
$\overline{CM}=\overline{DM}$	P2	
$\angle AMC = \angle BMD$	P'3	
$\triangle AMC \cong \triangle BMD$	C1	
$\angle MAC = \angle MBD$	C2	(C1)→C2 " To derive a conclusion C2 as to the premise C1 which is not clearly shown as a premise."
$\overline{AC} /\!/ \overline{DB}$	C3	C2 C3 " To derive a conclusion C3 as to the premise C2."

Figure 2. Analysis of the geometric proof given in the RCGP test

TABLE I

Structure of reading comprehension of geometry proof test

Facet	Object of comprehension	Operational definition	Item number	Score[a]
Basic knowledge	Content of premise or conclusion	Recognizing the meaning of a symbol in figure	(1)	0,1,2
		Explaining the meaning of a property	(2)	0,1,2
		Recognizing the meaning of a property	(3)	0,1
Total score of basic knowledge				5
Logical status	Status of premise	Recognizing a condition applied directly	(4)	0,1
	Logical relation between premise and conclusion	Judging the logical order of statements	(5)	0,1,2
			(6)	0,1,2
	Property applied to derive conclusion from premise	Recognizing which properties are applied	(7)	0,1,2
Summary	Multiple arguments and critical ideas	Identifying critical procedures, premises, or conclusions	(8-1)	0,1,2
			(8-2)	0,1,2
		Identifying critical ideas of a proof	(9-1)	0,1
			(9-2)	0,1
Total score of logical status and summary				13
Generality	Proposition or proof	Judging the correctness	(11)(12)	0,1; 0,1
	All arguments and attached figure	Identifying what is validated by this proof	(10)(13)	0,1; 0,1,2,3
Application	Knowing to apply in other situations	Application in the same premises	(14)(15)	0,1; 0,1,2,3
		Identifying the different premises	(16)	0,1
Total score of generality and application				10

[a]Some items require multiple right responses or choices, so partial scores of 1 or 2 are given.

The Relevant Geometric Knowledge Test

The space of mathematical knowledge is composed of results, concepts, and examples (Michener, 1978). For example, the statement *The sum of any two even numbers is even* is a result about the concept even, while the number two or a couple is an example of evenness. While reading geometric proofs, it is also important to translate verbal descriptions into aspects of figures. Therefore, relevant geometric knowledge in this study refers to both figures and verbal descriptions of geometrical concepts and properties, which are related to the proposition and its proof in the RCGP test, and the translations of verbal descriptions in several application items in the same test. The framework of this instrument and sample items are shown in Table III. In this study, the Cronbach alpha reliability coefficient of each component (figure, description, translation) of this instrument was good to excellent (0.74–0.81) for Grade 9 and acceptable (0.60–0.69) for Grade 10.

Scoring and Analysis

For the RCGP test, each student was assigned a vector triple. The value assigned for each dimension was either 0 or 1. If a student scored above 70% in the basic knowledge items, a 1 was placed in the first coordinate. In the same way, logical status and summarization items, generality and application items, determine respectively the second and third dimensions of the students' RCGP test result. Table I indicates that the total possible scores for the three dimensions were 5, 13, and 10, respectively, and the scores above 70% success rate in the three dimensions were 4, 9, and 7, respectively. Accordingly, students' RCGP was meaningfully classified into five categories: Beyond Chaining Elements (1, 1, 1),

TABLE II

Structure of logical reasoning test

	Affirmation		
Modes of logic; item	Direct	Equivalent	Non-contraventive
Disjunction p.q $v\ \overline{p}.q\ v\ p.\overline{q}$	13, 01, 08	13	13
Equivalence p.q $v\ \overline{p}.\overline{q}$	09, 07		
Incompatibility $\overline{p}.q\ v\ p.\overline{q}\ v\ \overline{p}.\overline{q}$	14, 10	14, 10	14, 10
Implication p.q $v\ \overline{p}.q\ v\ \overline{p}.\overline{q}$	15, 02, 11	15, 11	15, 11
Reciprocal implication p.q $v\ p.\overline{q}\ v\ \overline{p}.\overline{q}$	12, 16, 3	12, 16	12, 16
Exclusion $\overline{p}.q\ v\ p.\overline{q}$	04, 05, 06		

TABLE III

Structure and exemplary item of relevant prior knowledge test

Figure		Description		Translation	
Example	Non-example	Reading	Explanation	Symbol	From description to figure
Draw an example of $\overline{AB} = \overline{BC}$	Draw an example of $\overline{AB} \neq \overline{BC}$	Write how you read $\overline{AB} \neq \overline{BC}$	Explain why you think this figure is no example	$\Delta ABC \cong \Delta DEF$, $\angle$BAC=? (The matching angle)	Draw a quadrangle PURV that has two diagonals $\overline{PR}$ and $\overline{UV}$, Q is the midpoint of both $\overline{PR}$ and $\overline{UV}$

Beyond Recognized Elements (1, 1, 0), Beyond Surface (1, 0, 0), Surface (0, 0, 0), and Skipping Recognizing-Chaining (1, 0, 1). The other three possible response type—(0, 1, 1), (0, 0, 1), and (0, 1, 0)—we treated as errors, labeled *unsupported responses*. The logical reasoning test was composed of three types: direct, equivalent, and non-contraventive affirmations, with possible total scores of 29, 7, and 7, respectively. The relevant geometric knowledge test also had three components: figures, descriptions, and translation, with possible total scores of 6, 9, and 6, respectively.

To characterize broadly the difficulties of students who failed to comprehend geometrical proof in terms of their logical reasoning and geometric knowledge, such students were grouped on the basis of their scores on logical reasoning, on direct, equivalent and non-contraventive affirmations, and on the basis of their scores on geometric knowledge (figures, descriptions, translation). For each factor of logical reasoning or geometric knowledge, students who scored above 70% were assigned as 1; students who scored under 30% were assigned as 2; those who scored in between were assigned as 0. The Statistical Package for the Social Sciences (SPSS 7.5) was used for analyses. In addition to descriptive statistics, the main statistical methods used were analysis of variance and multiple regression.

TABLE IV

Result of students' performance in RCGP (%)

Comprehension category	Grade 9 (n=223)	Grade 10 (n=378)
Beyond Chaining Elements (1, 1, 1)	18.8	36.5
Beyond Recognizing Elements (1, 1, 0)	11.7	18.8
Skipping Recognizing-Chaining (1, 0, 1)	9.4	10.8
Beyond Surface (1, 0, 0)	26.0	18.5
Surface (0, 0, 0)	26.0	5.8
Unsupported Response (0, 1, 0)/(0, 0, 1)/(0, 1, 1)	8.1	9.5
Total	100.0	99.9

Results and Discussion

Reading Comprehension Performance

Table IV shows the results of students' performance on the RCGP test. It indicates that 18.8% of ninth graders scored in the top level (Beyond Chaining Elements) and a high percentage, 26.0%, who scored at the lowest level (Surface). This evidence shows the wide spread of student achievement in learning geometric proofs. It was no surprise that the tenth graders performed better than ninth graders (Mann–Whitney *U*-test, $p<0.001$), because most ninth graders have not yet learnt formal geometric proof (some of them had learnt about it in cram schools, which are popular in Taiwan), while tenth graders had learnt in school. However, this does not imply that the progression arises from the effect of students' learning experience, because the samples were not randomly selected from the population of the same grade, and the tenth graders were mostly from senior high schools, which have more higher-achieving students than vocational schools.

The category of Skipping Recognizing-Chaining (1, 0, 1) denotes that students understand the mathematical terms or concepts in these proofs and what these proofs validated, and can apply these statements properly in other similar situations. On the other hand, students in this category do not fully understand the logical relations among the arguments in this proof and some critical proof ideas. Before analyzing students' performance, we assumed that the number of students in this category would be small and statistically insignificant. However, the data showed 9.4% of ninth graders and 10.8% of tenth graders in this category.

This implies that students' RCGP may not develop from micro to local to global understanding. There are at least two different types of development. One is to develop from surface, to recognizing elements, to

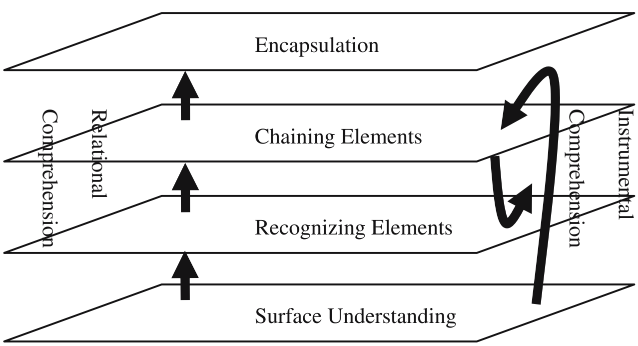


Figure 3. Two routes of development of RCGP

TABLE V

Distribution of students' logical reasoning score for direct affirmation (%)

Score		0–9	10–20	21–29
Direct	Grade 9	11.2	51.1	37.7
	Grade 10	3.2	30.7	66.1

chaining elements to encapsulation, which we see as a type of *relational* comprehension. The other is to jump from surface to encapsulation and then to come back to recognizing and chaining elements, which we see as a type of *instrumental* comprehension.

These two types of comprehension can be compared with relational and instrumental understanding, as sketched in Figure 3. Relational understanding denotes students who know how to apply a process or a rule, as well as understand why it works. Instrumental understanding denotes students who only know how to apply a process or a rule (Skemp, 1976). Relational and instrumental comprehension mean very different things. Relational comprehension of geometric proofs denotes students who come to know how to apply a proposition or a proof after knowing why a proof validates this proposition, while instrumental comprehension of geometric proof denotes students who know why a proof validates this proposition after knowing how to apply it. Relational understanding is supposed to be better than instrumental understanding. However, relational or instrumental RCGP both denote the process of reading to learn proofs; but neither is suggested as the better method.

TABLE VI

Distribution of students' logical reasoning score for equivalent and non-contraventative affirmation (%)

Score		0–2	3–4	5–7
Equivalent	Grade 9	81.2	3.6	15.2
	Grade 10	51.1	7.9	41.0
Non-contraventive	Grade 9	84.3	5.3	10.3
	Grade 10	75.4	9.0	15.6

TABLE VII

Mean Scores and Standard Deviations for Affirmations of Logic in Each Category of RCGP in Grades 9 and 10

	Directive		Equivalent		Non-contraventive	
Affirmations of logic	9M (SD)	10M (SD)	9M (SD)	10M (SD)	9M (SD)	10M (SD)
Surface	12.7 (4.6)	14.8 (6.3)	0.5 (1.2)	1.8 (2.3)	0.7 (1.3)	1.4 (1.9)
Beyond Surface	17.2 (5.7)	21.7 (4.9)	0.8 (1.8)	2.1 (2.9)	0.8 (1.6)	0.7 (1.8)
Skipping Recognizing-Chaining	19.6 (5.6)	22.0 (4.5)	2.0 (3.0)	3.1 (3.1)	2.0 (2.7)	1.2 (2.0)
Beyond Recognizing Elements	19.4 (7.2)	21.4 (4.6)	1.7 (2.8)	2.8 (2.90)	0.9 (2.0)	1.7 (2.3)
Beyond Chaining Elements	23.3	23.0	2.5	3.8	1.7	1.8
All (including Unsupported Response)	17.8 (6.6)	21.7 (5.1)	1.3 (2.4)	3.1 (3.1)	1.1 (2.0)	1.5 (2.4)

Relationship between Logical Reasoning and Reading Comprehension

Tables V and VI present the distribution of students' logical reasoning scores. Only 38.0% of ninth graders and 65.4% of tenth graders could pass 70% of the questions on direct affirmation (scores above 21), but 80–85% of ninth graders failed at least 30% of the questions in equivalent or non-contraventive affirmations (scores 0–2). Also, a high percentage of tenth graders failed to make correct inferences from non-contraventive affirmations.

To compare the characteristics of students in Grades 9 and 10 in each category of RCGP regarding logical reasoning, the mean scores and standard deviations for different affirmations in these categories were analyzed. These categories include Surface, Beyond Surface, Beyond Recognizing Elements, Beyond Chaining Elements, and Skipping Recognizing-Chaining, as shown in Table VII. Both ninth and tenth graders in the Surface category show the lowest mean scores of each affirmation, except non-contraventive affirmation of tenth graders. On the other hand, both ninth and tenth graders in the Beyond Chaining Elements category showed the largest mean scores of each affirmation, except non-contraventive affirmation of ninth graders.

Two sets of analysis of variance involving the five categories of students were conducted, one on Grade 9 data and one on Grade 10 data. The analysis of scores of each affirmation shows that the comprehension category factor is significant at both Grades 9 and 10 ($p < 0.05$). In Grade 9, a post hoc analysis with Dunnett T3 revealed that students in the Surface or Beyond Surface categories show significantly lower mean scores of direct and equivalent affirmations than do students in Beyond Chaining Elements. There is no significant difference in the mean score of non-contraventive affirmations between any two categories of students. In Grade 10, a post hoc analysis with Dunnett T3 revealed that students in either the Surface or Beyond Surface categories had significantly lower mean scores of direct, equivalent, and non-contraventive affirmations than students in Beyond Chaining Elements. These patterns indicate that students' logical reasoning improved from Beyond Surface, Recognizing Elements, to Chaining Elements. However, students' logical reasoning in Beyond Recognizing Elements and Skipping Recognizing-Chaining had no significant difference in either Grade 9 or 10.

As for ninth graders' scores of RCGP, three-way analysis of variance (direct by equivalent by non-contraventive affirmations with the level of significance set at $p < 0.05$) revealed one significant main effect (direct affirmation), two significant two-way interaction effects (direct by

TABLE VIII

Distribution of students' geometrical knowledge score for figure and translation (%)

Score		0–2	3–4	5–6
Figure	Grade 9	13.4	16.1	70.4
	Grade 10	1.9	5.6	92.6
Translation	Grade 9	22.4	32.8	44.9
	Grade 10	4.8	21.2	74.1

equivalent affirmations, equivalent by non-contraventive affirmations), as well as a significant three-way interaction effect (direct by equivalent by non-contraventive affirmations). As for tenth graders' scores on the RCGP test, one significant main effect (direct affirmation) was found. These results imply that the effect of logical reasoning on RCGP of Grade 9 is larger and more complex than on RCGP of Grade 10.

Relationship between Relevant Knowledge and Reading Comprehension

Tables VIII and IX present the distribution of students' relevant geometric knowledge scores. About 70% of ninth graders could pass about 70% of questions involving figure or description knowledge, but 22.4% of ninth graders did not pass at least 30% of the questions in translation knowledge. A high percentage of tenth graders succeeded in passing most of the questions involving figure, description, or translation knowledge.

The mean scores and standard deviations for figure, description, and translation knowledge in these categories were analyzed, as shown in Table X, in order to compare the characteristics of students in Grades 9 and 10 in the previously mentioned categories of RCGP regarding relevant geometric knowledge. Both ninth and tenth graders in the Surface category showed the lowest mean scores of each kind of

TABLE IX

Distribution of students' geometrical knowledge score for description (%)

Score		0–3	4–6	7–9
Description	Grade 9	13.0	25.1	61.9
	Grade 10	3.2	14.6	82.3

TABLE X

Mean scores and standard deviations for knowledge in each category of RCGP in grades 9 and 10

	Figure		Description		Translation	
Knowledge	9 M (SD)	10 M (SD)	9 M (SD)	10 M (SD)	9 M (SD)	10 M (SD)
Surface	3.4 (2.5)	5.1 (1.9)	4.4 (3.0)	7.0 (2.7)	2.1 (2.5)	3.3 (2.4)
Beyond Surface	5.4 (1.2)	5.6 (1.1)	6.8 (2.0)	7.4 (1.7)	3.6 (2.1)	4.6 (1.6)
Skipping Recognizing-Chaining	5.1 (1.4)	5.8 (0.6)	7.6 (1.9)	7.6 (1.8)	4.4 (1.7)	5.3 (1.3)
Beyond Recognizing Elements	5.5 (1.2)	5.7 (0.9)	7.9 (1.4)	7.5 (1.7)	4.9 (1.4)	5.0 (1.5)
Beyond Chaining Elements	5.7 (1.0)	5.9 (0.5)	7.7 (1.7)	8.1 (1.3)	5.2 (1.4)	5.2 (1.2)
All (Including Unsupported Response)	4.9 (1.8)	5.8 (0.6)	6.6 (2.5)	7.7 (1.7)	3.8 (2.3)	5.0 (1.5)

knowledge. On the other hand, both ninth and tenth graders in the Beyond Chaining Elements category showed the largest mean scores of each kind of knowledge, except translation of tenth graders.

Two sets of analysis of variance involving the five categories of students were carried out, one on the Grade 9 data and one on the Grade 10 data. The analysis of scores of each kind of knowledge showed that the comprehension category factor is significant at both grades ($p < 0.05$). At Grade 9, a post hoc analysis with Dunnett T3 revealed that students in the Surface category had a significantly lower mean score on each kind of knowledge than students in each of the other categories. At Grade 10, a post hoc analysis with Dunnett T3 revealed that students in either Surface category had a significantly lower mean score on description or translation knowledge than students in each of the other categories, while there was no significant difference in the mean score of figure knowledge between any two categories of students. These patterns indicate that figure knowledge is not the main factor contributing to RCGP, and geometric knowledge distinguishes only the category of Surface from the other categories of RCGP.

For ninth graders' RCGP scores, three-way analysis of variance (figure by description by translation with the level of significance set at $p < 0.05$) revealed two significant main effects (description, translation). For tenth graders' scores of RCGP, three significant main effects (figure, description, translation) and one significant two-way interaction effect (figure by translation) were found. These results imply that geometry knowledge of description and translation plays a crucial role for both ninth and tenth graders.

TABLE XI

Multiple regression summary

Dependent variable: Y (RCGP)Predictor variables:X1 (knowledge) X2 (logical reasoning)	R-square (Adjusted R-Square)	
Grade 9	Y=.66X1	.44
	Y=.59X2	.35
	Y=.49X1+.37X2 (VIF=1.266)	.54 (.54)
Grade 10	Y=.36X1	.13
	Y=.37X2	.14
	Y=.32X1+.30X2 (VIF=1.028)	.22 (.22)

Relationship between Logical Reasoning, Relevant Knowledge, and Reading Comprehension

Multiple regression analysis was adopted to examine the contributions of total logical reasoning scores and relevant prior knowledge scores to the prediction of RCGP scores for ninth and tenth graders. Table XI summarizes these results. Relevant geometric knowledge accounts for 44% of the variance in RCGP at Grade 9 and 13% at Grade 10. Logical reasoning accounts for 35% of the variance in RCGP at Grade 9 and 14% at Grade 10. In combination, the relevant geometric knowledge scores and the logical reasoning scores account for 54% of the variance in RCGP scores at Grade 9 and 22% at Grade 10. Prediction of the RCGP test score is strengthened by the addition of relevant prior knowledge scores and logical reasoning scores. The variance in RCGP accounted for by logical reasoning and geometric knowledge decreased from Grade 9 to 10. This pattern may result from the fact that only one significant main effect (direct affirmation) was found as for tenth graders' scores of RCGP, 66.1% achieved scores above 21 on direct affirmation (full score=29), and 25.4% achieved a full score on geometric knowledge. Moreover, this implies that some other factors related to RCGP, e.g., reading strategy or metacognition, should be taken into account in addition to logical reasoning and geometric knowledge.

Conclusion and Suggestions

Originally, we implicitly believed that one should understand a proof step by step. Geometric proof content in textbooks also follows a sequence of proposition, then proof, then application. However, the evidence from this study shows that there are at least two types of student development of RCGP. One develops from the surface, through recognizing elements, chaining elements to encapsulation, which is labeled relational comprehension. The other jumps from the surface towards encapsulation and then goes back to recognizing and chaining elements, which is labeled instrumental comprehension. The two types of comprehension seem analogous to pure and applied mathematicians; pure mathematicians intend to validate conjectures, while applied mathematicians intend to apply what has been validated.

Two different developments of RCGP reflect two different learning approaches, and different teaching designs should be proposed regarding different learning approaches. Relational comprehension and instrumental comprehension point out approaches of analyzing proof process by steps and understanding proof process with application orientation, respectively. Moreover, 18.8% of Grade 9 and 36.5% of Grade 10 students displayed their RCGP in the top level, Beyond Chaining Elements. This evidence, as opposed to students' poor performance in proof construction (e.g., Soucy McCrone & Martin, 2004), discloses approaching geometric proof from reading may be better than from writing regarding students' development of understanding proof.

On the basis of the results of analyzing the mean score and standard deviation for logical reasoning and relevant geometric knowledge in each category of RCGP, it supports that (1) students' logical reasoning improves from the category of Beyond Surface, Recognizing Elements, to Chaining Elements; however, students' logical reasoning in the categories of Beyond Recognizing Elements and Skipping Recognizing-Chaining makes no significant difference in either Grade 9 or 10; and (2) figure knowledge is not the main factor contributing to RCGP, and geometric knowledge distinguishes only the Surface category from the other categories of RCGP.

While further investigating the difficulties of students' RCGP, we found that the three kinds of logical reasoning play a more complicated role for ninth graders than tenth graders and, contrarily, the three kinds of geometric knowledge play a more complicated role for tenth than ninth graders. Regarding logical reasoning, we assumed that the logical reasoning of ninth graders to be spontaneously developing and, therefore, the diversity of students' logical reasoning may have resulted in these interaction effects among factors on ninth graders' RCGP, and the difficulties of students' RCGP could be mostly overcome by learning the logic of direct affirmation. Regarding geometric knowledge, we suppose that most of students who are familiar with geometric knowledge may not comprehend geometric proof so well. Hence, relatively good performance on the knowledge test and relatively poor performance on the RCGP test may result in these interaction effects among factors on tenth graders' RCGP scores, and the difficulties of students' RCGP could be partially overcome by greater attention to description and translation knowledge.

Relevant geometric knowledge is the entry point for understanding geometric proof. Reading a statement expressing properties or transforming attributes of geometrical objects is necessary for RCGP. If students cannot translate descriptions into geometric figures, they will find it hard to apply a learnt proposition to solve similar problems presented descriptively. As classroom instruction evolves to include reading to learn proof, educators will need to provide more precise factors that influence mathematical reading comprehension.

Relevant prior knowledge scores and logical reasoning scores accounted for 54% of the variance in RCGP at Grade 9 and 22% at Grade 10. At least two cognitive reasons can explain this result. First, students who are already familiar with relevant knowledge may tend to read each proof step from the epistemic point of view rather than logical value (Duval, 2002). Second, tenth graders who are capable of directive affirmation do not use this capability during the reading of proofs because they do not actively recognize what is given and what is claimed.

Although these two variables (prior knowledge and logical reasoning) in our study predict a significant and considerable amount of the variance in RCGP, other language factors should be further investigated. For example, reading span and conceptual span accounted for 19% of the variance in text comprehension (Haarmann, Davelaar & Usher, 2003); and inference-making ability, comprehension monitoring, and knowledge about story titles explained unique variance in reading comprehension, after the contribution made by working memory, verbal ability, and component skills had been taken into account (Cain, Oakhill & Bryant, 2004). Factors such as inference-making ability and comprehension monitoring might be worth investigation.

Studies in second language reading comprehension mainly focus on research questions about the variation between and within groups of variables and not about predicting performance on a dependent variable via independent variables (Cindy, 2004). These studies compared differences between poor and good readers in order to understand why poor/good readers are poor/good. We, on the other hand, tried to evaluate factors that significantly influence RCGP to understand why readers may fail. Although in this study the amounts of variance in RCGP accounted for by logical reasoning and relevant prior knowledge at Grade 10 are less than those at Grade 9, future researchers may find a stronger relationship in another form of proof, e.g., the indirect proof method, because these kinds of proofs are unfamiliar to tenth graders.

At present, the evidence of empirical research about learning proof has mainly focused on students' responses to questions requiring proof or their choices of different argument modes from the point of view of either teacher or student (e.g., Healy & Hoyles, 2000). In this study, we approached it from reading comprehension of geometric proofs and extended this perspective to exploring its predictive factors. Researchers may be further interested in investigating the differences between reading and writing geometric proofs. Intuitively, the gaps among prior knowledge, reasoning ability, and *writing* geometric proofs should be far from those among prior knowledge, reasoning ability and *reading* geometric proofs. While it is reasonable to suppose that more than 50% of tenth graders in Taiwan can prove that *As the figure shows, if* $\overline{AB}$ *and* $\overline{CD}$ *intersect at the point M*, $\overline{AM}=\overline{BM}$ *and* $\overline{CM}=\overline{DM}$; *then*, $\overline{AC}$ *and* $\overline{DB}$ *are parallel to each other*, only about 38% of tenth graders in this study reached the comprehension level of beyond chaining elements. Therefore, the conclusion that someone who can write geometric proofs can also read and comprehend such a proof is still under inquiry.

Overall, this study not only investigated students' RCGP and its predicting variables but also initiated a practical study of RCGP. Nevertheless, related factors affecting RCGP were not completely explored in this study. For example, comprehension strategies should be critical for students' RCGP. In research on problem solving (Pape, 2004), students' strategies for comprehending geometry proof can be classified, and the relationships among their strategies, RCGP, and even proof construction can be further explored with different propositions and within proof formats. Students' beliefs about and preferences for propositions and proofs are other factors influencing their RCGP. A series of research studies about reading comprehension of geometric proof will shed further light on formulating hypothetical learning trajectories and stimulating teachers' perception of understanding proof from reading comprehension perspectives.

ACKNOWLEDGEMENTS

The authors wish to thank anonymous reviewers and the editor, Larry Yore, for commenting on earlier drafts. We also thank Michael Tang and Shari Yore for proofreading our English manuscript. This paper is part of a research project funded by the National Science Council of Taiwan (NSC 94-2521-S-003-002). The views and opinions expressed in this paper are those of the authors and not necessarily those of the NSC.

APPENDIX

Respond to the following questions on the basis of this task and the proof process.

(1) Label ∠AMC of this figure as 1 and ∠MAC of this figure as 2.
(2) Do you agree ∠AMC=∠MAC? Explain why.

(3) While ΔAMC and ΔBMD are congruent, what is the corresponding angle of ∠MAC?
(4) Besides the known conditions ($\overline{AB}$ and $\overline{CD}$ intersect at the point M, $\overline{AM} = \overline{BM}$, $\overline{CM} = \overline{DM}$), which conditions can be directly applied without any explanation?
(5) If someone thinks that the proof process of line 1, 3, 2, 4, 5 is correct after line 2 and 3 are interchanged, do you agree with that opinion?
(6) If someone thinks that the proof process of line 1, 2, 4, 3, 5 is correct after line 3 and 4 are interchanged, do you agree with that opinion?
(7) Which properties are applied in this proof?
(8) On the basis of the question and the proof,

(8-1) Which conditions are necessarily used?
(8-2) What is derived from this proof?

(9) From this proof process, it firstly derives an important result from $\overline{AM} = \overline{BM}$, $\overline{CM} = \overline{DM}$ and other conditions, and then derives a condition used to confirm $\overline{AC}//\overline{DB}$.

(9-1) What is this important result?
(9-2) What is this condition used to confirm $\overline{AC}//\overline{DB}$?

(10) Which statements can be validated from this proof?
(11) Choose the correct statements.
(12) Do you agree that this proof process is correct?
(13) Statement A: If $\overline{AB}$ and $\overline{CD}$ intersect at the point M, $\overline{AM} = \overline{BM}$, $\overline{CM} = \overline{DM}$, then $\overline{AC}$ is parallel with $\overline{DB}$.

(13-1) Do you agree that this proof process can prove that Statement A is always correct?
(13-2) Do you agree that this proof process can prove that Statement A is sometimes correct and sometimes incorrect?

Answer the following questions on the basis of what you know.

(14) A quadrangle PURV has two diagonals $\overline{PR}$ and $\overline{UV}$, Q is the midpoint of both $\overline{PR}$ and $\overline{UV}$, and then is this quadrangle a parallelogram?
(15) $\overline{PR}$ and $\overline{UV}$ intersect at a point Q, Q is the midpoint of both $\overline{PR}$ and $\overline{UV}$, and which conclusions can be derived?

(16) If $\overline{XY} = \overline{YZ}$, $\overline{MY} = \overline{YN}$ and ∠XYM=∠ZYN, then are $\overline{XM}$ and $\overline{NZ}$ parallel to each other?

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