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PROOFS THROUGH EXPLORATION IN DYNAMIC GEOMETRY ENVIRONMENTS

ABSTRACT. The recent development of powerful new technologies such as dynamic geometry software (DGS) with drag capability has made possible the continuous variation of geometric configurations and allows one to quickly and easily investigate whether particular conjectures are true or not. Because of the inductive nature of the DGS, the experimental-theoretical gap that exists in the acquisition and justification of geometrical knowledge becomes an important pedagogical concern. In this article we discuss the implications of the development of this new software for the teaching of proof and making proof meaningful to students. We describe how three prospective primary school teachers explored problems in geometry and how their constructions and conjectures led them "see" proofs in DGS.

KEY WORDS: dynamic geometry, exploration, functions of proof, phases of proof, proof

The increasing use of computers in mathematics education is mostly reflected in the use of DGS. The fact that DGS has revitalized the teaching of geometry in many countries implies a radical change in the teaching of proof (De Villiers, 1996; Hanna, 1998). One of the most important facilities of dynamic geometry is its potential to encourage students' "research" in geometry (Luthuli, 1996). In such a research-type approach, students are inducted into theorem acquisition and deductive proof. In particular, students can experiment through dragging on geometrical objects they construct, and consequently infer properties, generalities, or theorems. However, these facilities have led some educators to believe that deductive proof in geometry should be abandoned in favor of an experimental approach to mathematical justification (Mason, 1993), indicating the theoretical gap that exists between the acquisition (inductive nature of DGS) and justification (proof) of geometrical knowledge. In this paper, we assert that DGS is complementary to the deductive nature of geometry, and discuss the pedagogical aspects of introducing DGS into the teaching of geometrical proofs. The main purpose of the present paper is to provide some indications of how DGS can be used to offer insight and understanding of proofs through investigation and experimentation.

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In the theoretical framework that follows, we deal with some of the functions that proof performs in mathematical practice, in order to provide the background to the case studies, which have been conducted mainly to bridge the gap between deduction and experimentation. Through these case studies, we provide examples of how the DGS can contribute to the need for analytical proof and help students to distinguish between proof and exploration. We also highlight the importance of the interplay between actions and dependent properties (Laborde, 2000) provided by the DGS environment.

THEORETICAL FRAMEWORK AND PURPOSE OF THE STUDY

The Gap between Proof and Exploration

The exploration of a problem is by its nature empirical, and, at first glance, it seems that it does not fit into the deductive character of geometrical proofs. When the empirical and inductive dimension is to be added to the pedagogical structure that is traditionally rooted in deductive logic, one has to combine these two seemingly opposite perspectives. The problem of combining inductive exploration with the deductive structure of geometrical proofs has been the subject of a number of research studies (Hanna, 1998, 2000; Jones, 2000; Mariotti, 2000).

The traditional teaching emphasizing that a mathematical statement is true if it can be proved, leads students to distinguish proof from exploratory activities. However, De Villiers (1996) and Hanna (2000) indicated that in actual mathematical research, mathematicians have first to convince themselves that a mathematical statement is true and then move to a formal proof. It is the conviction that something is true that drives us to seek a proof. In DGS, students can easily be convinced of the general validity of a conjecture by seeing its truth displayed on the screen while geometrical objects undergo continuous transformations (De Villiers, 1993, 2003).

A number of researchers showed that the passage from "exploratory" geometry to the deductive geometry is neither simple nor spontaneous. Hoyles and Healy (1999) indicated that exploration of geometrical concepts in a DGS environment could motivate students to explain their empirical conjectures using formal proof. They found that DGS helped students to define and identify geometrical properties and the dependencies between them, but when students worked on proofs, they abandoned the computer constructions. The latter point leads to the argument that DGS may be useful only in helping students understand problems in geometry but it does not contribute to the development of their abilities in proofs, reinforcing the idea that there exists a gap between dynamic geometry and proof. In a more recent research, Pandiscio (2002) also showed that preservice teachers believe that after using dynamic software high school students may not see the need for proofs. This kind of belief might also be the reason that some educators and researchers expressed their concerns and worries that DGS could lead to the "further dilution of the role of proof in the high school geometry" (Chazan, 1993, p. 359). However, the main discussion of recent research, and the main purpose of the present study were to find ways of effectively utilizing DGS to introduce proof as a meaningful activity to students. This can be achieved by reconceptualizing the functions of proofs.

The Functions of Proof

Proof performs a wide range of functions in mathematical practice, which are reflected to some extent in the mathematics curricula. The NCTM Standards (2000) emphasized in a special section the importance of students developing reasoning and proving abilities, forming conjectures, the evaluation of arguments and the use of various methods of proofs. Within the NCTM's document it appears that proof is not only understood in the traditional rigid and absolute way, but it also embraces many other functions. Hanna (2000), based on recent research on proof, provided a list of the functions of proof and proving: verification, explanation, systematization, discovery, communication, construction, exploration, and incorporation. She also considered verification and explanation as the fundamental functions of proofs, because they comprise the product of the long historical development of mathematical thought. Verification refers to the truth of a statement while explanation provides insight into why this statement is true.

Traditionally, the function of proof has been seen almost exclusively in terms of the verification of the correctness of mathematical statements. Hanna (1995) and Hersh (1993) characterized this kind of proof as "proofs that convince", based on the idea that proof is used mainly to remove either personal doubt and/or those of others; an idea which has one-sidedly dominated teaching practice and most of the research in the teaching of proof. Hanna and Hersh both argued that "proofs that convince" are often not appropriate for the mathematics classroom and therefore they advocated using the "proofs that explain." Based on the same idea, De Villiers (2003) proposed other important functions such as explanation, discovery, intellectual challenge and systematization, which in some situations are of greater importance to mathematicians than that of mere verification. In addition to these functions of proof, researchers identified various other roles that proof plays in mathematics such as to communicate mathematical knowledge, and to systematize statement into an axiomatic system (see Knuth, 2002).

Edwards (1997) defined the term "conceptual territory before proof" by indicating that conjecturing, verification, exploration and explanation constitute the necessary elements that precede formal proofs. The conceptual territory provides the arena for the construction of intuitive ideas that may subsequently be tested and confirmed through formal methods, and it is the basis of a richer understanding of a proof. This approach reflects the "quasi-empirical" view of mathematics in which understanding proceeds from students' own conjectures and verifications to formal proofs (Chazan, 1993). Simpson (1995) differentiated between "proof through logic", which emphasized the deductive nature of proof, and "proof through reasoning", which involved most of the functions of proofs as were listed by Hanna (2000). Proof through reasoning is accessible to a greater proportion of students, because it is closer to students' learning style, it makes mathematics more useful and enjoyable, and it reflects the quasi-empirical view of mathematics and the process adopted by mathematicians when they invent mathematics (Simpson, 1995).

The Functions of Proofs and DGS

The availability in the classroom of DGS gave a new impetus to the teaching of geometry based on students' investigations and explorations. This does not mean that proof is replaced by exploration. On the contrary, exploration is not inconsistent with the view of mathematics as an analytic science or with the central role of proof. DGS has the potential to encourage both exploration and proof, because it makes easy to pose and test conjectures (Hanna, 2000). Polya (1957) emphasized the connection between deductive reasoning with exploration. He pointed out that solving a problem amounts to finding the connection between the data and the unknown, and to do it, one must use a kind of reasoning based on deduction.

In the DGS environment students acquire understanding through verifying their conjectures and in turn this understanding solicits further curiosity to explain "why" a particular result is true. However, students working in the DGS environment are able to produce numerous configurations easily and rapidly, and thereby they may have no need for further conviction /verification (Hölzl, 2001). Although students may exhibit no further need for conviction in such situations, it is important for teachers to challenge them by asking why they think a particular result is true (De Villiers, 1996, 2003). Students quickly admit that inductive verification merely confirms but the why questions urge them to view deductive arguments as an attempt for explanation, rather than verification (Hölzl, 2001). Thus, the challenge of educators is to convey clearly to the students the interplay of deduction and experimentation (Hanna, 2000).

THE STUDY

This article presents an account of the thinking exhibited by three prospective primary school teachers while attempting to answer proof problems. All three student teachers had completed high school geometry and algebra courses and one semester of university-level calculus.

It is conjectured that DGS provides an appropriate context where the significance of proof may be un-forcefully recognized. To this end, the development of "appropriate" tasks was necessary. By "appropriate" we mean tasks where proof may be providing insight-illumination into why a result, which can be seen on the screen, is true. Open-ended problems seemed more "appropriate" for two main reasons: (a) statements are short and do not suggest any particular solution methods, and (b) questions are different from traditional closed expressions such as "prove that ...", which present students with an already established result (Jones, 2000). Open-ended problems give students the opportunity to engage in a process, which utilizes a whole range of proof functions: exploring a situation, making conjectures, validating conjectures and proving them. The implicit assumption is that during this process students will not have to prove something that they are presented with and do not understand, but something that they have discovered, validated and which is meaningful to them.

Two main hypotheses were examined in this study. First, whether the use of DGS could help the students identify conjectures based on their constructions, and second whether the use of DGS could help the students search for mathematical arguments to support their conjectures and thus providing reasonable explanations. To this end, the participants were asked to work on the following open-ended problem suggested by De Villiers (1996):

Problem: Construct a kite and connect the midpoints of the adjacent sides to form an inscribed quadrilateral. What do you observe in regard to this inscribed quadrilateral? Write down your conjecture. Can you explain why your conjecture is true? Change your kite into a concave kite. Does your conjecture still hold?

After the exploration of this problem, the students were engaged in proving similar geometrical theorems. The aim of these additional problems was for them to utilize the proving process in systematizing and generalizing their results.

Students' Proofs

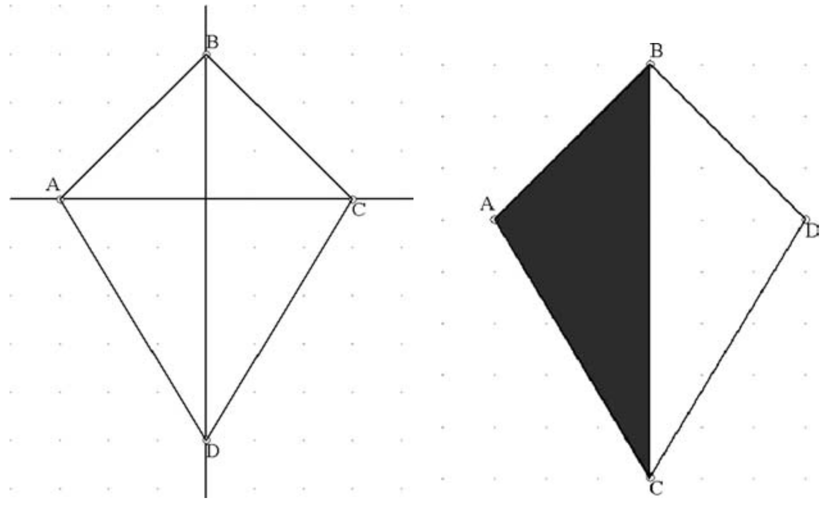
Three prospective primary school teachers with prior experience in dynamic geometry participated in this study. These students teachers had attended a course on the integration of computers into elementary school mathematics, and thus they had a basic understanding of Sketchpad's drawings, menus, and construction features. All three participated on a voluntary basis and were interviewed while working on the problem. The interviews were conducted in the mathematics laboratory equipped with computers loaded with the Greek version of the Geometer's Sketchpad. The setting was informal with the students being able to analyze and build geometric constructions that they thought would help them solve the problems without any time constrains being set.

In the following, we analyze the students' strategies and try to underline the different aspects and functions of proof. The discussion of the students' solutions to the problem is organized around three phases: (a) the phase before proof, (b) the proof phase, and (c) the phase of intellectual challenge of extending proof to similar problems.

The Phase before Proof

In this phase the students explored the problem through constructing the kite and rearranging the constructed figure by dragging it in different directions. This exploration led them to form their own conjectures about the solution of the problem by visualizing the transformations that resulted by the dragging facilities of the software. Figure 1 shows the way in which students constructed the kite and consequently the inscribed quadrilateral. Two of the students (Student 1 and Student 2) constructed the kite using the property of perpendicularity of its diagonals (see Figure 1(a)), while the third one (Student 3) used the property of equal adjacent sides by firstly constructing a triangle and then reflecting it on one of its sides (see Figure 1(b)). All students managed to find the midpoints of the adjacent sides and connected them with line segments using the appropriate tools provided by the software.

The students conjectured that the inscribed quadrilateral (see Figure 2) might be a rectangle and confirmed their conjecture by dragging the vertices of the kite to new positions. The students also realized that their conjectures also hold in the case of the concave kites. All the students evaluated their mathematical conjectures not only visually but also numerically by measuring the sides and angles of the inscribed quadrilateral, confirming that it was a rectangle, and thus verified their conjecture. It is also important to note that these students used the measuring tools for slope to show that the opposite sides of the inscribed shape were parallel.



(a) (b)

Figure 1. Students' constructions of kite.

Furthermore, they noticed that the diagonals of the kite were also parallel to the sides of the inscribed shape.

The Proof Phase

The exploration of the problem as it was done in the "phase before proof" led students to the conviction about the validity of their conjecture. This conviction was achieved solely by the use of the dynamic geometry environment. During the "proof phase" the role of proof is not to convince or remove individual or social doubt about a proposition but primarily to find ways to explain why a result that can be seen on the screen is true (Jones, 2000). One of the students (Student 3) in this study showed no further need for conviction that the inscribed quadrilateral was a rectangle, while the other two students (Students 1 and 2) felt the need to explain why they thought this particular result was true. These two students admitted that the inductive verification they provided for the mathematical statement was not satisfactory in the sense that the inductive process was not a consequence of other familiar results. Furthermore, they proceeded to view a deductive argument as an attempt for explanation, rather than for verification.

In this phase, the DGS enabled students to pass from "exploratory" geometry to deductive geometry, bridging in this way the gap between dynamic geometry and proof. Specifically, the two students, who successfully solved the problem, based on the measurements they made earlier

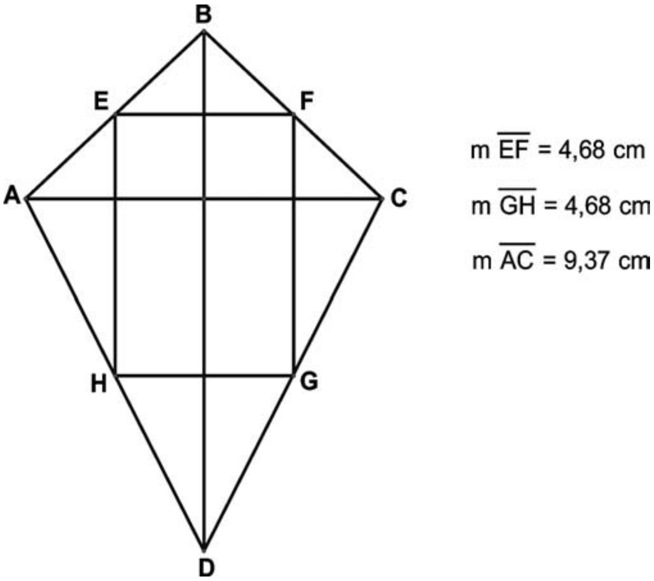


Figure 2. The proof that the inscribed quadrilateral is a rectangle.

on in the exploration phase (the pre-proof phase), defined and identified the geometrical properties and the dependencies between them, and provided a deductive proof of the problem. In fact, they realized from their measurements that EF, and HG are equal to 1/2 AC (see Figure 2). This directed them to what they needed to look for in their geometry books, where they found the respective theorem. Based on this property they showed that EF is equal and parallel to HG as well as EH is equal and parallel to FG, and therefore EFGH is a parallelogram. The next step was to prove that the parallelogram was a rectangle, i.e., at least one of the angles of the parallelogram was a right angle. Based on the property of the perpendicularity of the diagonals of the kite, students observed that since BD $\perp$ AC, then EF $\perp$ EH (since BD is // to EH and AC is // to EF), which implies that EFGH is a rectangle (the dragging facility of the software enabled students to conceive that their explanations hold even in the case of concave kite).

The phase of intellectual challenge of extending proof to similar problems

In this phase we discussed two categories of problems: (a) problems that have a similar context to the kite problem, and (b) problems that require the same type of reasoning. The purpose of the problems in the first category was to help students generalize the results they had reached in the kite problem to quadrilaterals of various types. To this end, the three students tried to systematize their experimentations by investigating first the more familiar quadrilaterals such as parallelograms, rectangles, rhombi, squares, rectangles and then they proved, using the same explanations as they did in the kite problem, that in any quadrilateral the shape resulting from the midpoints of its sides is always a rectangle.

The purpose of the problems assigned in the second category was to ensure that students could easily transfer the proofing process to problems with different structure. Thus, the following problem was assigned to students: "What figure is formed by the angle bisectors of the interior angles of a parallelogram?" This problem is a common geometrical problem found in most geometry textbooks. In this problem it was expected that students would follow the same procedures for verification and proof as they did in the previous problem.

The software facilitated the students construction of the parallelogram as well as the bisectors of the interior angles. Figure 3(a) shows the construction of the angle bisectors of the interior angles of a parallelogram as it was built by two students (Students 1 and 3). These students observed that the figure they formed seems to be a rectangle. Of course, this construction could be done without the computer, and the students could also prove this conjecture. However, without the use of the dynamic software they would not be able to add experimental evidence to this conjecture as they did by dragging any of the flexible points of the parallelogram and noticed, as previously conjectured, that the figure might be a rectangle. What is most important was the fact that by dragging one of the flexible points of the parallelogram until it becomes a rhombus, they observed that the figure formed is not still a rectangle but a point (see Figure 3(b)). This shows that their first conjecture does not always hold. This led them to consider a point as a degenerate rectangle! Again, it was evident that the dragging capabilities of DGS allow students to consider extreme cases of a geometric configuration, cases that textbook authors fail to consider.

A special case of the problem was to start with a rectangle instead of a parallelogram, as Student 2 did. Student 2 suggested that the figure formed by the angle bisectors of the interior angles of a rectangle is possibly a square (see Figure 4). The discussion that followed students' investigations led them to consider several conjectures that could be explained only by proving. At that point, we asked the students to prove their conjectures using paper and pencil. They were allowed 15 minutes without reaching a reasonable solution. Then, we allowed them to use the software. They all proceeded to measure the angles formed by the bisectors and the

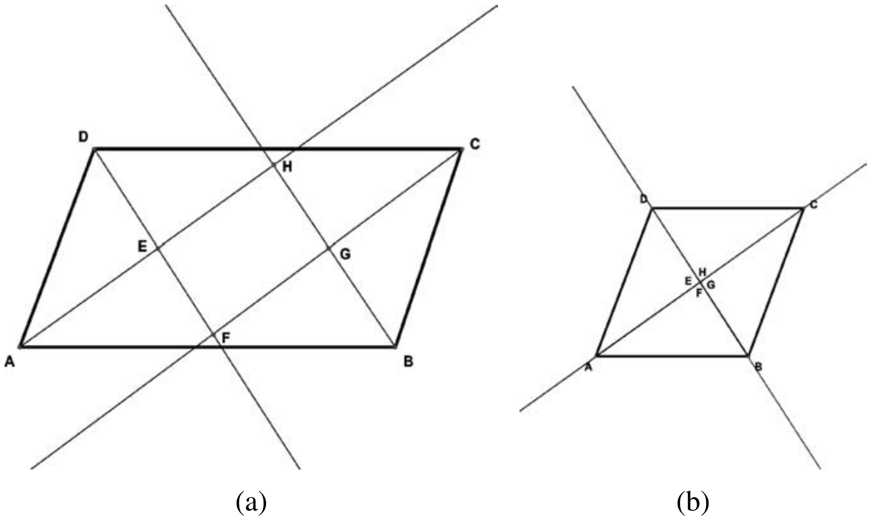


Figure 3. The figure formed by the bisectors of a parallelogram and a rhombus.

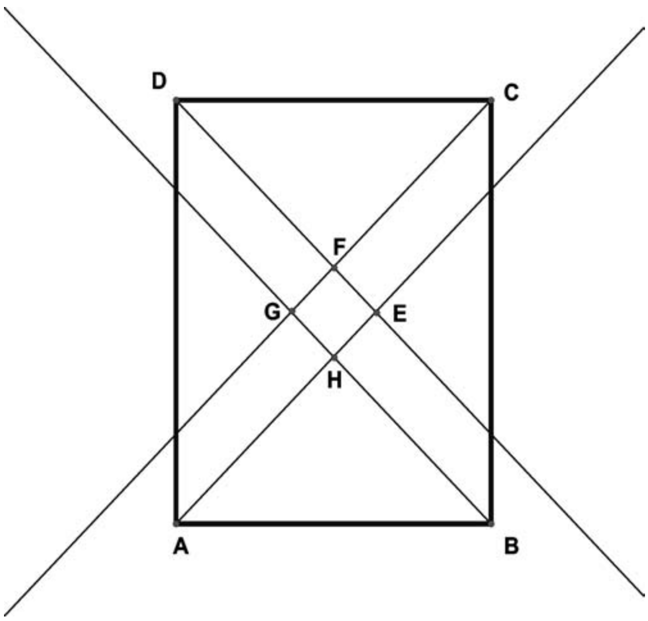


Figure 4. The figure formed by the bisectors of a rectangle.

parallel lines of the parallelogram (or the rectangle) as well as the slopes of the bisectors. In the following we describe their proof in the case of parallelograms, noting that the same procedures were used for the cases of rectangles, rhombi and squares.

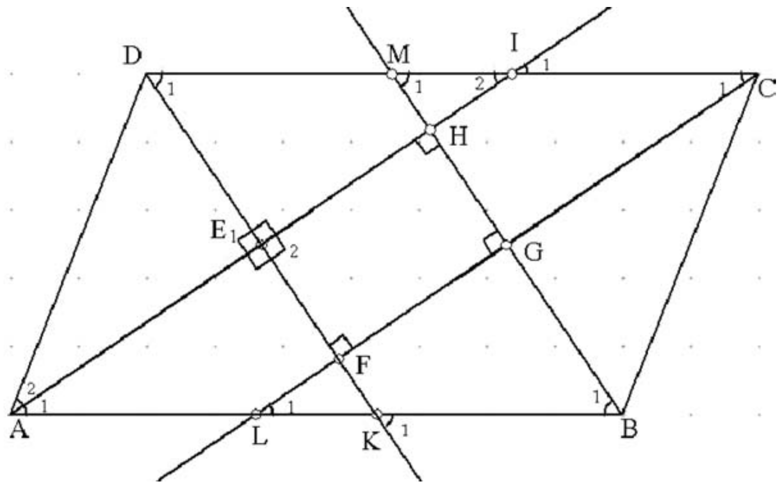


Figure 5. The figure formed by the bisectors of a parallelogram.

The measurements (see Figure 5) led Students 1 and 2 to realize that angles A1 = I1 (from parallelogram ABCD) and that C1 = I1 (C1 = A1 as angles of parallelograms), which implies that EH is parallel to FG. In the same way they explained that EF//HG (since B1 = D1, D1 = K1, then B1 = K1), and thus they proved that EHGF is a parallelogram. They also realized that they needed to show that at least one of the angles of the parallelogram should be a right angle in order to explain that EHGF is a rectangle. This created a lot of difficulties for Student 2, while the other one (Student 1) was motivated by the observation that angle E1, which is equal to E2 (as vertically opposite angles), is a right angle. In order to explain why E1 is a right angle, Student 2 moved back to the computer program just to confirm that the bisectors of two adjacent angles of a parallelogram always form a right angle triangle. Student 2 constructed a new parallelogram to validate this new conjecture.

Student 3 provided a different solution. She noticed from the constructed shape that ADI (see Figure 5) is an isosceles triangle, since A1 = I1 (from parallel lines), A1 = A2 (AE is bisector of angle A), and therefore A2 = I2. She also found that BCL is an isosceles triangle using the same reasoning and that the bisectors DE and BG are simultaneously altitudes on bases AI and LC, respectively. The latter leads to the conclusion that angles E and G of quadrilateral EFGH are right angles. Then she explained that angle H is a right angle, since A1 = I1, M1 = B1 = D1, and D1 + A1 = 90. In this way she reached the conclusion that the quadrilateral EFGH is a rectangle (all its angles are right angles).

DISCUSSION

In this paper we tried to show some of the ways in which DGS can provide not only data to confirm or reject a conjecture, but also ideas that lead to a proof. To this end, the results of the study were presented in three phases: the phase preceding proof, the proof phase, and the phase of intellectual challenge of extending proof to similar problems.

In the phase preceding proof it is quite necessary for students to understand the problem based on their own intellectual efforts. In the kite problem the students encompassed their informal reasoning and argumentation that came into play when they worked from their own investigations (Edwards, 1997). To construct the kite, which was a challenge by itself, they first investigated its properties and then tried to apply them on the computer screen. The graphing and validating capabilities of DGS enabled the students to explore the problem and make mathematical conjectures. In turn, they checked specific cases of kites, using the dragging facility of the software, to see if their conjecture held, i.e., the shape formed by connecting the midpoints of adjacent sides of a kite is always a rectangle. In other words, the phase preceding proof helped the students to build up empirical evidence for the plausibility of their conjectures. Thus, in this phase, DGS offered the students the possibility of constructing the appropriate shape and exploring properties of the changes of distinct geometric configurations.

A number of research studies indicated that engaging students in the phase preceding proof did not necessarily lead them to an awareness of the need for proof (Chazan, 1993; Edwards, 1997). On the contrary, in the present study, we found that DGS and appropriate questions prompted and/or motivated the students to seek justifications for their conjectures. During the proof phase, two of the three students in this study justified their conjectures for the kite problem based on the screen outputs. For this purpose, the measurement facilities of the software provided the means of finding explanations and the means for gathering information for justifying their results. The relations between the measurements in conjunction with the invariant properties of the shapes functioned as students' hints towards explaining their conjectures. Measurements also provided the students with specific examples that formed the basis of further conjectures and generalizations. It is in this area that the computer contributed to students' attempts toward proof and bridged the gap between inductive explorations and deductive reasoning.

Specifically, measurements in the proof phase helped the students see the dependencies between properties (Laborde, 2000). For example, in the kite problem, measurements helped the students observe that the opposite sides of the interior shape were equal and thus they tried to explain this property by finding and studying the relevant theorem. Measurements led the students to move from empirical to formal justifications and illuminated how proof is not separated from action. Action is expressed by the construction of shapes and measurements, while proof is expressed by dependencies between properties. Action seemed to push the students to look for explanations, while explanations were based on or initiated by construction and measurements (Laborde, 2000; Mariotti, 2000). This became more apparent during the phase of intellectual challenge of extending proof to similar problems. During the last phase, the students felt a strong desire to explain their conjectures and understand how one conclusion is a consequence of other familiar ideas, results or theorems. The students found it quite satisfactory to view a deductive argument as an attempt for explanation rather than for verification (De Villiers, 2003).

CONCLUSIONS

The main purpose of this study was to identify the functions of DGS, which may enable students to move from the empirical exploration of a problem to proof. We presented a set of observations that illustrate mathematical relationships that emerged from students' interactions with the provided tasks. In the pre-proof phase, we showed how routine tasks that appear in traditional geometry could be approached with the use of technology as open problems. In this phase, the software enabled the students to construct the appropriate figures and then act on them using the dragging facilities of DGS in order to identify conjectures that are not easy to observe in advance (Laborde, 2000). These actions were important for the students during the proof phase and the phase of intellectual challenge, because they enabled them to search for mathematical arguments to support their conjectures. The interplay between action (constructions and measurements) and dependent properties provided students the motive and the context to explain their conjectures and reached proof through reasoning (Hanna, 2000).

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