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Interacting Parallel Constructions of Knowledge in a CAS Context

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Abstract We consider the influence of a CAS context on a learner's process of constructing a justification for the bifurcations in a logistic dynamical process. We describe how instrumentation led to cognitive constructions and how the roles of the learner and the CAS intertwine, especially close to the branching and combining of constructing actions. The CAS has a major influence on parallel constructions after branching and it facilitates combining. Hence, the CAS has the upper hand near branching points but the learner has the upper hand near combining points.

Keywords Abstraction · Bifurcation · CAS (Computer Algebra System) · Context · Epistemic actions · Knowledge construction

1 Introduction

The contextual factors that may influence a process of abstraction include elements known in advance such as students' prior learning history, the physical setting, the tasks presented to the learners. However, they also include dynamic factors such as interactions with peers, teachers and technology. The role of context is crucial in mathematics learning processes and the complexity of learning processes is, at least in part due to the contextual influences on the learner's construction of knowledge. Hence, a better understanding of the role of context is likely to lead to a better understanding of learning processes. Artigue (2002, p. 268) deals with the potential of Computer Algebra System (CAS) for learning and

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teaching mathematics, and reflects on the possible epistemic value of instrumented techniques. She claims that the epistemic value of techniques is not something that can be defined in an absolute way; it depends on contexts, both cognitive and institutional. Similarly, Noss and Hoyles (1996) situate abstraction in relation to the resources students have at their disposal: Students learn to attune practices from previous contexts to new ones. There is a process of webbing, in which students connect to previous similar activities and capitalize on the tools they have at their disposal to construct new mathematical knowledge. Hitt and Kieran (2009) describe a project aimed at understanding the complexity of the construction of knowledge in a CAS environment. Their work is based on the French instrumental approach, in particular the Task-Technique-Theory (T-T-T) theoretical frame as adapted from Chevallard's Anthropological Theory of Didactics (ATD). Whilst there is important literature that links instrumentation and ATD, they are separate theories. Hitt and Kieran analyze how the use of symbolic calculators along with a careful task design promote rich and meaningful learning:

> While many of the epistemic actions that we identified can be viewed as quite general mathematical reasoning processes (and thus definable in a certain sense), there is no doubt that the context, that is, the nature of the tasks that we designed, in combination with the technologies that were involved therein, was instrumental in provoking the epistemic actions that we observed. (p. 150)

The aim of the present paper is to contribute to the identification and analysis of the crucial role played by a CAS in a complex process of constructing knowledge. We consider a solitary learner, to be called L, who deals with bifurcations in a logistic dynamical system and attempts to justify the occurrence and position of the second bifurcation point (the transition point from 2-periodic to 4-periodic behavior of the dynamical system). The mathematical background, as well as the process by which L is constructing the justification have been described elsewhere (Dreyfus and Kidron 2006; Kidron and Dreyfus 2010) and will be briefly reviewed below. The nature of the mathematical problem and the complexity of the intended justification led L to make heavy use of a CAS, specifically Mathematica. The CAS thus became an integral component of the context of the processes of knowledge construction and justification, and played a crucial role in them. The analysis of this role will be a main aim of this paper. The next two sections are devoted to the background needed for appreciating this analysis: the nested epistemic actions model for Abstraction in Context is reviewed and the mathematical background of the study is described. Our methodology is discussed and a top-level description of the learning experience is offered as well as main results of relevant previous studies. Based on this background, Section 4 is devoted to the analysis of the relations between parallel constructions of knowledge and the CAS context. We summarize and complete the picture in the concluding section of the paper.

2 Background

Since the main purpose of this paper is to analyze, at a micro-analytic level, the role of a contextual factor in a process of abstraction, we have adopted the theoretical framework of Abstraction in Context (AiC) proposed by Hershkowitz et al. (2001). In this section, we present a brief introduction to the model as well as a description of previous relevant research.

2.1 The Nested Epistemic Actions (RBC) Model for Abstraction in Context

The view of abstraction underlying the nested epistemic actions model is based on Davydov's (1990) ideas, according to which the process of abstraction starts from an undifferentiated and possibly vague initial notion, which need not be internally and externally consistent. The development of abstraction proceeds by establishing an internal structure by means of links and results in a differentiated, structured, consistent entity. It does not lead from concrete to abstract but from an undeveloped to a developed form of abstract in which new features of the concrete are emphasized. In this sense, a process of abstraction is a process during which new knowledge constructs emerge and are differentiated.

AiC uses a limited number of epistemic actions for the purpose of describing and analyzing such processes, while taking into account the context in which the learning process occurs. Epistemic actions are mental actions by means of which knowledge is used or constructed (Pontecorvo and Girardet 1993; Schwarz and Hershkowitz 1995). Recognizing a familiar mathematical notion, inferring a consequence from data and appealing to a strategy for solving a problem are examples of epistemic actions. In certain social contexts, such as small group problem solving, there is quite a high likelihood that participants' actions or verbalizations attest to epistemic actions, thus making them observable and allowing the researcher to analyze the process of knowledge construction.

Hershkowitz et al. (2001) proposed a model based on three epistemic actions for describing processes of abstraction; these epistemic actions are Recognizing, Building-with and Constructing, in short RBC. The model is therefore often referred to as the RBC model. Constructing is the central epistemic action of abstraction. It consists of assembling or integrating elements of one's knowledge to produce a new construct. Recognizing a familiar mathematical notion, process or idea occurs when a learner realizes that the latter is inherent in a given mathematical situation. Building-with consists of combining existing knowledge elements in order to meet a goal such as solving a problem or justifying a statement.

In the RBC model, constructing incorporates the other two epistemic actions in such a way that building-with actions are nested in constructing actions and recognizing actions are nested in building-with actions and in constructing actions. Moreover, constructing actions may be nested in higher level constructing actions (see, e.g., Dreyfus et al. 2001). Furthermore, several constructing actions can evolve in parallel and interact with each other in complex manners such as branching, combining, interruption and resumption (Dreyfus and Kidron 2006; see the next subsection for more detail).

The RBC model is apt to describe processes of abstraction in their specific context. The contextual factors that may influence a process of abstraction include the physical setting, the tasks on which learners work and tools such as paper and pencil or computers and software that are available to them; they also include students' personal histories, ideas, conceptions, language and procedures that are outcomes of previous learning. Furthermore, any process of abstraction takes place in a particular social setting and thus the context also includes social relationships among students and between students and teachers. As a consequence, context becomes an inseparable component of the process because students act in a manner that seems relevant to them in the given context.

The role of several contextual factors in processes of abstraction has been investigated previously by researchers using AiC; in particular, the role of social interaction has been treated in detail by Dreyfus et al. (2001), as well as by Hershkowitz et al. (2007). On the other hand, we know of no AiC study in which computer tools, or more generally, tools for learning have been similarly investigated. In addition to its other aims, the present study is a first attempt to complete this missing aspect of AiC based research.

For a more detailed discussion of AiC, the theoretical background from which it arose, and its epistemic actions, as well as a detailed example, we refer the reader to Schwarz et al. (2009).

2.2 Previous Relevant Research

In Dreyfus and Kidron (2006), we have used the RBC-model of abstraction in context in order to describe how L, a strongly motivated, mathematically mature, solitary learner constructed complex knowledge about bifurcations of dynamic processes while using a rich array of internal and external resources (but no human help from either teacher or peers). The resources at her disposal were books, the World Wide Web and a mathematical software package. These resources provided a rather unstructured collection of results, from which L constructed a mathematically coherent justification for the bifurcations. The research led to a refinement of the three epistemic actions of recognizing, building-with and constructing that have been identified in previous research based on the same model. While there is, as in previous research, an overarching constructing action C, the component constructing actions nested in it are more complex, more parallel, and interacting in more complex ways than in any of the previous research studies that use the model. Interactions include branching of a new constructing action from an ongoing one, combining or recombining of constructing actions, and interruption and resumption of constructing actions. These interactions of parallel constructions are explained in Dreyfus and Kidron (2006) in terms of the refined epistemic actions.

Only after the description of the process of abstraction was complete, did we realize the particular meaning L associated with the notion of justification (Kidron and Dreyfus 2010). We analyzed the relationship of this meaning of justification and the different patterns of interactions of epistemic actions that were observed in our previous paper. In particular, we analyzed the close relationships between the interacting pattern of combining constructing actions and justification as enlightenment. We discovered that, in this study, each additional degree of enlightenment occurs with a combination of two constructions, and each combination of two constructions indicated an additional degree of enlightenment.

2.3 Aims of the Present Research

We are now in a position to give a more precise formulation of the aim of the present paper and to precise what it adds to our previous papers. The occurrence of interacting parallel constructions that were observed in the previous papers might be due in part to the highly structured, advanced mathematics being learned but it might also be due to the interaction of the learner with the computer. While we were acutely aware of the crucial influence of the CAS on the learning process, we did not offer in the previous papers an in depth analysis of the influence of this particular component of the context, nor did we analyze the relationship of the CAS to the different patterns of interactions of epistemic actions that we discovered. This analysis constitutes the main aim of the present paper. Moreover, in analyzing the role of the CAS context, we refer to the subject-tool dialectic (Trouche 2005). In the interaction between the learner and the CAS we observe two directions: on the one hand, the CAS shapes the actions of the learner and on the other hand the initiatives of the learner shape the way the CAS is used. We investigate this interaction between the learner and the computer and analyze how it is related to the different patterns of interaction of epistemic actions that we observed in Dreyfus and Kidron (2006). It is important to note that we analyze the role of the CAS in a very tight relation with a theory of construction of knowledge. We are especially interested in the analysis of the role of the CAS in relation to the specific patterns of interaction of epistemic actions that constitute the process of construction of knowledge of the learner. The main aim of this analysis is to answer the question, whether certain patterns of epistemic actions (like branching and combining) are characteristic for certain contextual factors.

3 Mathematical and Cognitive Bifurcations

In this section, we briefly review the mathematical background as well as the previous research on which the present one is based (see Dreyfus and Kidron 2006 for details, or Kidron and Dreyfus 2004 for a short version).

3.1 Mathematical Background

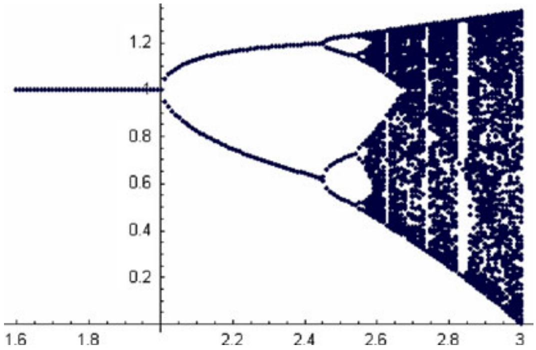
The previous as well as the present research deal with the learning experience of an experienced mathematician L (for 'the Learner') during a two-week period when she was homebound and had strong intrinsic motivation to learn about bifurcations of the dynamical system based on the logistic equation $dx/dt = r \cdot x\ (1 - x)$. During that time, she was solitary in the sense that she had no curriculum, no explicit goals set by outside sources such as a teacher or textbook, nor any guidance by peers or experts. On the other hand, she was able to take advantage of external resources such as books, the web, and the Mathematica software.

L approached the dynamical system from a differential equations point of view, but for the purpose of numerical calculations, she also used the iterative processes approach, which leads, for $\Delta t = 1$, to the recursive equation $x_{n+1} = f(x_n) = x_n + r\, x_n\ (1 - x_n)$. The phenomena of interest arise as r increases from $r = 1$ to $r = 3$. Using the CAS, L plotted x_n as a function of n, for different values of r (see Fig. 4 for two examples of such plots); these plots suggested that, independently of the choice of x_0, $0 < x_0 < 1$, for $r = 1.8$ the process tends to the final state $x = 1$, for $r = 2.3$ the final state is a periodic oscillation between two values, a 2-period, for $r = 2.5$ the process approaches a 4-period and for $r = 3$ it does not appear to become periodic at all.

L also found in the literature, and reproduced on her screen, the so-called bifurcation diagram (Fig. 1), which shows the final state (vertical axis) for different values of the parameter r (horizontal axis) and thus provides an overview of the different types of behavior engendered by the logistic process.

L saw, and read that $r = 2$ is the value of r at which the transition from a single point final state to the 2-periodic regime occurred. She did not know why this transition occurred at $r = 2$, nor did she know where the transition from the 2-periodic to the 4-periodic regime occurred and why. Nevertheless, it was intuitively clear to her that the rich and surprising images on the computer screen describe a mathematical reality, and that this reality must have mathematical reasons. She had also read about the fact that the stability of the final state is connected to the value 1 of the derivative, without being fully aware what function was differentiated. She thus accessed a large number of interesting and important results that appeared disjointed and without proper justification. She was challenged to find this justification.

Fig. 1 The bifurcation diagram produced with Mathematica



3.2 Methodology

Gathering data about learning processes is methodologically non-trivial. Gathering data about the learning process of a solitary learner presents even greater challenges because there is usually no need for the learner to report about her learning. In the present study, L's epistemic actions were inferred from the detailed notes she took, her Mathematica files, and computer printouts. Like many mathematicians, L wrote, graphed, drew and sketched a lot, some by hand and some by computer. As is her habit, she carefully dated and kept these notes as well as all computer files and printouts. These documents later served as a window into her thinking for the researchers.

L's collection of her notes, files and printouts had nothing to do with a plan for the present or any other research. In fact, the idea of using them as raw data for a research study was only conceived of several months later. We then constructed a report of the learning process, following an elaborate procedure of several cycles of description by the first author and challenges by the second author. The accuracy of the report may be verified by observing its close correspondence with the raw data, some of which have been published (Dreyfus and Kidron 2006). Excerpts from the report will be used throughout the paper. The complete version of the report is available from the authors.

Once the report was agreed upon, we adopted the RBC methodology for identifying epistemic actions (Hershkowitz et al. 2001), while keeping in mind that the different context may lead to phenomena that had not been observed in earlier RBC model based studies. The report of the learning experience was divided into episodes, numbered 1-16. Each episode forms a cognitively coherent unit. For the purpose of analysis, each episode was further divided into subunits, called events, and denoted by Latin letters a, b, c, …. Events are the equivalent to utterances for the case of a solitary learner; they form the minimal units that can be categorized as epistemic actions according to the RBC model.

3.3 Top-level Description of the Learning Experience: The Story

In this subsection, we present the main trends of thought of L's learning experience about the transitions from fixpoint to 2-period and from 2-period to 4-period as it transpires from the previous research study (Dreyfus and Kidron 2006). The second of these transitions will form the object of our detailed analysis in the remaining sections of the paper.

3.3.1 The Transition from Fixpoint to 2-Period (Episode 1)

Using information from several web-based resources, L understood that the 4th order equation $f^2(x) - x = 0$ will yield the 2-periodic points, and the quadratic equation $p(x) = \frac{f^2(x)-x}{f(x)-x} = 0$ will yield those 2-periodic points that are not fixpoints. ($f^2(x)$ means $f(f(x))$). She solved $p(x) = 0$ for general r. She found that the discriminant $D = 0$ for $r = 2$, and checked that for $r < 2$ there are no real solutions, whereas for $r > 2$ there are two real solutions—the 2-period. These results made it clear to her that period doubling occurs where $D = 0$, and that this happens at $r = 2$.

3.3.2 Algebraic and Numerical Reasoning (Episodes 2-9)

Assuming an analogy, L set up the corresponding equation $p(x) = \frac{f^4(x)-x}{f^2(x)-x} = 0$ for the 4-periodic points. The CAS showed that this equation is of order 12 (episode 3) and cannot be solved for general r. The strategies that worked for the previous transition became inapplicable (episode 4). Web-resources led L to the notion of discriminant for a general polynomial (episode 5). In these Web-resources, she learned that the transition from period 2 to period 4 could be found by computing the discriminant of $p(x) = \frac{f^4(x)-x}{f^2(x)-x}$ and by comparing it to zero. She also learned that the discriminant of a polynomial is defined as the product of the squares of the differences of the polynomial roots. This definition was of no help to her since she had no chance to find the polynomial roots of this order 12 polynomial with parameter r. The discriminant could only be helpful if it offered a way to avoid finding the roots. In another web resource, she learned that the discriminant could be obtained using the coefficients of $p(x)$ and of its derivative $p'(x)$, without knowing the roots. L was therefore motivated to look for a 'built-in' function in Mathematica for this purpose. The 'built-in' function was a bit complex but, at that stage, L did not want to devote much time to understand it. She wanted first to find out whether it could help her make progress and find where the 4-period begins. She wanted to confirm that the theoretical approach is in accordance with the previous results obtained experimentally (numerically and graphically) with which she felt confident. The CAS (episode 6) helped her to factor the discriminant and to find the value $r = \sqrt{6}$ of the transition point to the 4-period. Encouraged by this numerical success, she began to search for the reasons behind the connection between multiple roots, $D = 0$, and transition points (episode 7). When using the CAS to compute the solutions of $p(x) = 0$ for $r = \sqrt{6}$, her attention was caught by the complex roots rather than by the more relevant real double roots. In an attempt to understand, she refreshed her knowledge about complex roots but soon realized that these attempts at algebraic justification turned out to be dead ends (episodes 8 and 9).

3.3.3 New Visual Intuition and the Bifurcation Diagram (Episode 10)

Her failed algebraic attempts at justification led L to a renewed study of the bifurcation diagram, and to an emerging understanding of the dynamic behavior of the x-values, as r varies. She realized that, at the bifurcation points, a branching of the x-values occurs.

3.3.4 Considerations with Derivatives (Episodes 11–14)

Now (episode 11) it was even clearer to L that answering the questions of episode 7 would provide the justification she looked for. A further search on the web helped her to connect

D to the coefficients of the polynomial $p(x)$ and its derivative $p'(x)$, as well as to connect multiple solutions of $p(x) = 0$ to solutions of $p'(x) = 0$ (episode 12). However, this relationship was not useful since it did not single out the real roots. On the other hand, exploiting the specific structure of $p(x)$ yielded a condition on the derivative that she knew to be characteristic for the stability of fixpoints. She tacitly made a connection between stability and bifurcation (episode 13).

3.3.5 Completion of the Dynamic View (Episodes 15–16)

Finally, L was able to combine her dynamic view of the transition with the link between the derivative and the stability of fixpoints (episode 15). This constituted for her the completion of the connections between the zeros of the discriminant of $p(x)$, the transition point from the 2-period to the 4-period and the existence of double real solutions of the equation $p(x) = 0$ (episode 16).

3.4 Main Results of the Previous Study

In the previous study (Dreyfus and Kidron 2006), we identified four constructs that emerged for L during the learning experience. These constructs arise from

C1 The process of constructing the solution of the polynomial equation $p(x) = 0$ in order to find the 4-periodic points. The solution process is considered algebraically and numerically. The focus is on the solutions for each value of the parameter r, and relationships between the solutions for different values of r

C2 The process of constructing algebraic connections between the existence of multiple real solutions of the equation $p(x) = 0$, the zeros of the discriminant D of $p(x)$ and the transition point between different periodic regimes, particularly the one from period 2 to period 4

C3 The process of constructing the links between the derivative of a polynomial and the zeros of its discriminant; this includes:
  a. the fact that the discriminant of a polynomial can be expressed by means of the coefficients of the polynomial and those of its derivative,
  b. the link between this derivative and the stability of fixpoints and periods

C4 The process of constructing of the dynamic view of the bifurcation in which the final state values of x are considered as functions of r.

The symbols C1–C4 are used in two meanings: the first meaning is the process of constructing, the second meaning is its product, the construct.

The episodes, in which each of the processes leading to these constructs is active, and the interaction of these processes are summarized in the construction diagram (Fig. 2) and will be discussed subsequently.

The diagram in Fig. 2 has been called the interacting parallel constructions diagram, based on the observation that constructing actions are not linearly ordered but often go on in parallel and interact as follows (see the analysis in Dreyfus and Kidron 2006):

- The constructions C1, C2, C3, and C4 go on largely in parallel; for example, in episode 13, both C2 and C3 are active.
- A construction may branch off from an ongoing construction. For example, at the beginning of episode 5, C3 branches off from C2.

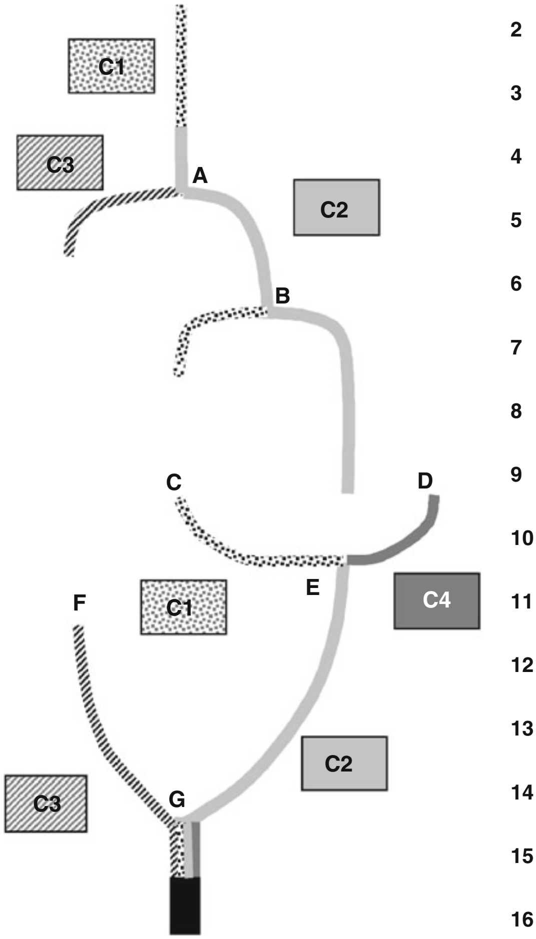


Fig. 2 The interacting parallel constructions diagram

- Constructions may combine. For example at the end of episode 10, C1 and C4 combine.
- Constructions may interrupt and resume. For example, C3 is interrupted in episode 5 and resumed in episode 12.

Altogether these patterns of interaction between constructions give rise, in the learning experience under discussion, to seven critical points, which will play a role below:

- Point A—C3 branches off from C2 at the beginning of episode 5,
- Point B—C1 branches off from C2 at the beginning of episode 7,
- Point C—C1 is resumed at the beginning of episode 10,
- Point D—C4 starts at the beginning of episode 10,
- Point E—C1 and C4 combine at the end of episode 10,
- Point F—C3 is resumed at the beginning of episode 12,
- Point G—All four constructions combine at the beginning of episode 16.

These interactions of parallel constructions are explained by Dreyfus and Kidron (2006), mostly in terms of refined epistemic actions. Indeed, we found that building-with actions appear in two types: The ordinary P (problem solving) type and a new V-type. This new type of building-with consists of the formulation or reformulation of questions, tasks or intentions in an effort to organize the problem space so as to enable further investigation. P- and V-type building-with actions each appear in a regular kind of limited local scope, denoted P_r and V_r, and a deep kind with potentially far-reaching implications, denoted P_d and V_d. Examples of P_d and V_d actions are offered in Dreyfus and Kidron (2006). Building-with actions of the deep investigative type (V_d) may lead to branching of constructing processes, and may be precursors of the new constructing actions (or the resumption of interrupted ones). Building-with actions of the deep problem-solving type (P_d), on the other hand, are indicators of imminent combining of constructing actions. In addition, Dreyfus and Kidron (2006) noted a similarity between the two combinations, the combination of C1 and C4 in episode 10 and the merging of constructions C1–C4 in episodes 11–15: In the first case, vague knowledge (about the bifurcation map) intervened, and in the second case elusive knowledge (about the derivative). In both cases, the activation of the elusive knowledge was a facilitating factor in the combining of the parallel constructions. Moreover, the connections between the fragile knowledge of the weak branch and the more established knowledge of the stronger branch reinforced the weak branch and contributed to a combination of the two branches.

4 Parallel Constructions and the CAS Context

In this section, we analyze the influence on the learning process, of a particular component of the context, the CAS, and the way it mediates the parallel constructions in the process of justification. The aim of this analysis is a contribution towards the effort to answer the question, whether certain patterns of epistemic actions (like branching and combining) are characteristic for certain contextual factors. The section is structured as follows: We first consider aspects of using the CAS that are related to the branching of constructing actions, then aspects that are related to the combining of constructing actions. In subsection 5.1, we then draw conclusions about the relationship between patterns of interaction of epistemic actions and specific contextual components. In terms of branching, we use two examples to consider the phase leading up to the branching as well as the development of the constructing actions after branching. In terms of combining, we use again two examples to consider the phase leading up to the combining as well as the integration of different modes of thinking following the combining.

4.1 The Relation between the CAS Context and the Pattern of Branching

Branching is a transition from a single construction to two parallel constructions. In the construction diagram, we observe two cases of branching: C3 branching off from C2 in episode 5 (point A in Fig. 2) and C1 branching off from C2 in episode 7 (point B in Fig. 2). We first describe the constructions near these branching points and then the parallel development of the two constructions after branching.

4.1.1 The First Branching Point

In episode 4, L learned from a web resource that the equation $D = 0$ will yield the transition point from period 2 to period 4, the discriminant D being defined as the product of the squares of the differences of the roots of $p(x)$. These roots being unknown, she was not able to continue C2, the process of constructing algebraic connections between the transition point, the roots and $D = 0$. She further learned that D could be obtained using the derivative of $p(x)$, without knowing the roots. This information belonged to a different mode of thinking - analysis. It was the starting point of a new construction, C3, which led in a different direction. There was, as a consequence, a potential for branching. Interestingly, the information at the beginning of C3 led to the continuation of C2: L used the CAS to compute the discriminant, without understanding how it was computed. The CAS enabled her to ignore the C3 information about the derivative and return to the algebraic mode of C2. The CAS thus led to the realization of the potential for branching.

4.1.2 The Second Branching Point

At the beginning of episode 7, L knew the value $r = \sqrt{6}$ of the transition point, but could not explain it and did not understand its connection to $D = 0$. For this purpose, she needed to either find out how D was computed (which would have meant to resume C3) or go back to the definition in terms of polynomial roots and observe the structure of the set of solutions, and this was possible only with a return to C1. In contrast to the first branching point, the second one was thus more than a possibility, it was a necessity. The fact that L knew that the CAS could numerically compute the solutions of the degree 12 equation $p(x) = 0$ helped her decide to return to C1 rather than resume C3.

In summary, the CAS did not initiate either of the two branching points but was instrumental for both: At both, L was aware of the potential of the CAS to make the required computation.

4.1.3 The Influence of the CAS on the Parallel Development of the Two Constructions after Branching

In both cases of branching, there was an ongoing construction, and another construction branching off from it. In both cases, the branching was essential for the ongoing construction to continue. In both cases, each of the two branches held promise to contribute to the justification. And in both cases, the constructions were closely interrelated and should therefore ideally have developed in parallel. On the other hand, they were too substantial for that. It was too high a demand for L to think simultaneously in two different modes. Therefore, after the branching point, there was a cognitive need for L to decide which of the two branches to continue. As a consequence, one of the constructions was interrupted at the end of the episodes that started with branching—C3 at the end of episode 5 and C1 at the end of episode 7.

The decision which construction to continue with was, in both cases, strongly influenced by the potential offered by the CAS. We will illustrate this using event 7f from the less obvious second case where L had used the CAS to solve $p(x) = 0$:

7f In the resulting output of the Mathematica file I obtained two pairs of double solutions, each listed twice.

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{{x → 0.0712841 - 0.0374608 i}, {x → 0.0712841 + 0.0374608 i},
 {x → 0.236884 - 0.116138 i}, {x → 0.236884 + 0.116138 i},
 {x → 0.619573}, {x → 0.619573}, {x → 0.712718 - 0.265841 i},
 {x → 0.712718 + 0.265841 i}, {x → 1.19692}, {x → 1.19692},
 {x → 1.38736 - 0.0111918 i}, {x → 1.38736 + 0.0111918 i}}
```

> I observed the other solutions. There were four pairs of complex numbers and their conjugates. I had no idea how to interpret the meaning of these complex solutions, not even the fact that they were not real.

L expected real solutions only, possibly multiple ones; the CAS provided an answer that included complex solutions and, to her, went far beyond answering the question. She did not know how to interpret the answer, but it influenced her further work. She was interested to know more about the complex roots. As a consequence, C1 was interrupted and L's attention diverted to C2. In the case of the first branching point, Mathematica's computation of D enabled L to (interrupt C3 and) continue C2.

4.1.4 Summary

We summarize the findings of this section by means of the diagram in Fig. 3, in which A and B denote the branching points. As mentioned, L's awareness of the potential offered by the CAS permitted her to take the initiative at the branching points, while the CAS itself thus acted as an agent that enabled the branching. After the branching, the decision which branch to continue with, was strongly influenced by the CAS. We conclude that the tool had the upper hand near the branching points.

Finally, we note that in both cases the constructions did develop in parallel during the episode following the branching point, allowing the establishment of connections between the two constructions. This mutual influence finds its full expression later, when these constructions combine.

4.2 The Relation between the CAS Context and the Pattern of Combining

Constructions combine in point E at the end of episode 10 and in point G at the beginning of episode 16 (see Fig. 2). In each case, one of the two combining constructions had been interrupted earlier, after branching, and resumed shortly before the combining point: C1 was interrupted at the end of episode 7 and resumed at the beginning of episode 10; C3 was interrupted at the end of episode 5 and resumed at the beginning of episode 12. In each case, the resumption was an effect of L's initiative to consider information which had been provided by the computer at an earlier time but had then been inaccessible to her. And in each case the resumption led to two constructions developing in parallel toward the combining point. We will now show that in each case, the CAS had a direct influence on the parallel development of the two constructions toward combining.

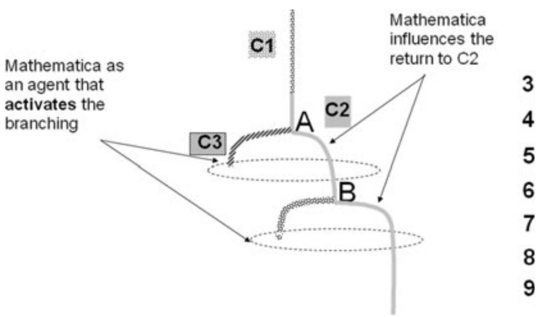

Fig. 3 The influence of the CAS before and after branching

4.2.1 The First Combining Point

The transition from episode 9 to episode 10 was crucial because at that moment L realized that her resources had been exhausted. Her algebraic attempts at justification failed and construction C2 was interrupted. The natural thing to do was to return to C1. Having returned, in episode 10, to construction C1, the search for solutions of $p(x) = 0$, and exhausted its numerical and algebraic potential, L observed the graphic representations of several time series plots (see Fig. 4) together with the bifurcation diagram (Fig. 1), focusing, in this order, on:

- The two repeating solutions for $r = 2.3$,
- The double real solution for $r = \sqrt{6}$,
- The four repeating solutions for $r = 2.5$

Thus she initiated C4, and this led her to move with her eyes along the r-axis of the bifurcation diagram (Fig. 1). Now, the transition from period-2 to period-4 became prominent:

10d Looking at the values of the double solutions in the bifurcation diagram, my attention was focused on the transition from the 2-period to the 4-period. This focus was different from the one I had previously when each time series plot gave a partial picture corresponding to a specific value of the parameter r.

10f I looked at the fork-like shape and associated its splitting with the fact that the discriminant vanishes. Suddenly, the bifurcation diagram seemed different, endowed with a new meaning. I looked at it and I could not understand how I failed to see it this way before.

L's static graphical mode of thinking about the time series plots thus turned into a new view, a dynamic graphical one. The parameter r turned from a discrete into a continuous one. The connection between C1 and C4 is expressed by the transition from finding the solutions (numerical aspects of C1) to a graphical mode of thinking about the solutions, first a static one and then a dynamic one, which allowed L to begin constructing a dynamic view of the bifurcation (C4). As pointed out previously, Dreyfus and Kidron (2006) discerned new varieties of epistemic actions-the deep B-actions. The events—10d and 10f—are deep epistemic actions of the P_d type: the new meaning L attributed to the bifurcation diagram in these events is a B-action of a crucial importance for the learning process.

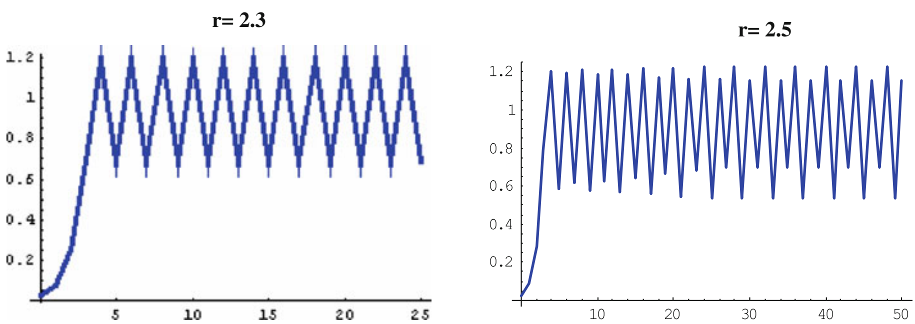

Fig. 4 Time series plots for $r = 2.3$ and for $r = 2.5$

The initiative for this process was L's. She took command and used the CAS as an agent that facilitated the combination, so that the numerical and the graphic–dynamic views could intertwine. The experimentation in a new setting with information obtained in a previous setting enabled the integration of the two modes of thinking associated with C1 and C4.

Nevertheless, L's dynamic view of the bifurcation was not yet complete; it was uni-directional from the left to the transition point and from there to the right (Fig. 1). We will return to this when discussing the second combining point.

4.2.2 The Second Combining Point

Here we will show that the second combining happened in two stages, and that Mathematica had a direct influence on the second stage.

After episode 10, L had a dynamic view of the transition; the relation between the second bifurcation point and the two pairs of double solutions was intuitively, visually clear to her. Moreover, she had associated the equation $D = 0$ to this connection. Construction C2 was thus enriched: It now carried the marks of C1 and C4. In the previous episodes, L did not succeed to obtain an algebraic expression giving the x-values as functions of r. A new view, a different type of thinking was needed. This need initiated C4, leading to a graphic equivalent of the requested algebraic expression and thus to the consideration of the numerical values in a graphical framework. A parametric algebraic expression may be considered as a dynamic process that attributes different x-values to different values of the parameter r. The graphic equivalent thus had to show the x-values as they depended on r. This representation was provided by the new graphical dynamic view L developed in episode 10.

In episode 12, L attempted to understand how the CAS used the coefficients of the polynomial p and of its derivative p' for computing D without solving the equation $p(x) = 0$, thus resuming C3. By means of an intricate sequence of building-with action, both of the V_d-type as well as of the P_d-type, she came to realize that multiple roots of $p(x)$ are multiple roots of the numerator $f^4(x) - x$, and that for such roots the derivative of the numerator vanishes and therefore $(f^4(x))' = 1$. For L, this step amounted to much more than only a computation:

13i At this moment, I connected the last equality with my previous vague knowledge that fixpoints change stability when the (absolute value of the) derivative move across the border 1 as the parameter r varies

13 k At last, I found some connection between the fact that there exists a multiple root (therefore, the discriminant equals 0) and the way fixpoints change stability. Now I understood why at the bifurcation points the discriminant should equal zero

Events 13i and 13 k are of the P_d type: deep learning takes place. The deep learning is characterized by the establishment of new connections. The algebraic connections between the zeros of the discriminant, the existence of multiple roots and the transition point are now almost established.

Here the basic issue of C2, namely why there the transition occurs where $p(x)$ has a multiple root, was explained by means of reflections based on derivatives (C3): multiple roots of p link to $p' = 0$, which links to $(f^4)' = 1$, which links to a change of stability, which links to a transition point! This constitutes a large step toward the combination of C3 with the enriched C2. However, as we will see shortly, the combination was incomplete.

L's knowledge about the role of derivatives in the description and explanation of bifurcations stemmed from reading rather than from some earlier activity or deep reflection. It was therefore elusive:

14a My background included some knowledge about analytical properties. For example, I knew that the k-periodic orbit is stable (attracting) if p is an attracting point for f^k or if $\left|(f^k)'(p)\right| < 1$

14b This knowledge was "theoretical" and in order "to feel" the analytical properties I decided to use Mathematica to check their validity in specific cases

Intuitively, L connected her elusive C3 knowledge about stability to transition between regimes, more precisely to her unidirectional C2 conception of transition. In this sense, the C3 construction was a much weaker branch than the C2 construction. We will show that L's work with the CAS reinforced this weak branch in the process toward combining, and that this enabled the branches to approach and combine fully. However, the CAS check held a surprise:

14f Considering $(f^4(x))' = -1$ instead of the equation $(f^4(x))' = +1$ I had written earlier, I realized that my mathematical construction collapsed!

15a But I did not give up; I decided to use Mathematica once more to check the value of the derivative $(f^4(x))'$ for $r = \sqrt{6}$

L remembered having read that a period doubling bifurcation leads to derivative -1 at this orbit, and the problem of the wrong sign constituted an obstacle between C2 and C3. However, not only did she not give up but she was able to use the checking with the CAS to lead to a full combination of the constructions into the sought justification: Using the CAS, L found that at the bifurcation point the value of the derivative of the previous cycle tends to -1 and the value of the derivative of the new cycle tends to $+1$, thus strengthening C3. This checking enabled her to understand that her mathematical construction did not collapse. Moreover, it also enabled her to change her previous view of the bifurcation which was unidirectional—from the left to the bifurcation point and from there to the right. Now, she was able to look from the bifurcation point in both directions—to the right, to the new cycle, and also to the left, to the previous cycle.

In episode 16, L drew the part of the bifurcation diagram containing the transition point, marking the x-values as well as the values of the derivatives on either side and thus obtained an image transcending the transition point from the 2-period into the 4-period.

L's notes corresponding to the crucial events 16a and 16b are presented in Fig. 5. The corresponding part of the report is

16a I drew with my pencil the bifurcation map and I observed the bifurcation points: Now, I understood why the solutions $x = 0.619573$ and $x = 1.19692$ of $f^2(x) - x = 0$, are also the double solutions of $f^4(x) - x = 0$. On my drawing of the bifurcation map I wrote the word ONSET with an arrow to the transition point at $r = \sqrt{6}$

The completed dynamic view of the transition (C2, enriched by C1 and C4) thus combined with the link between derivatives and stability (C3). And it was the central component of C3, namely $(f^k)' = 1$, which enabled L to complete the justification:

16b I also wrote on my right the results obtained for $f^4(x)$and on my left the results for $f^2(x)$. Connecting the picture with the concrete writing, I realized that at the

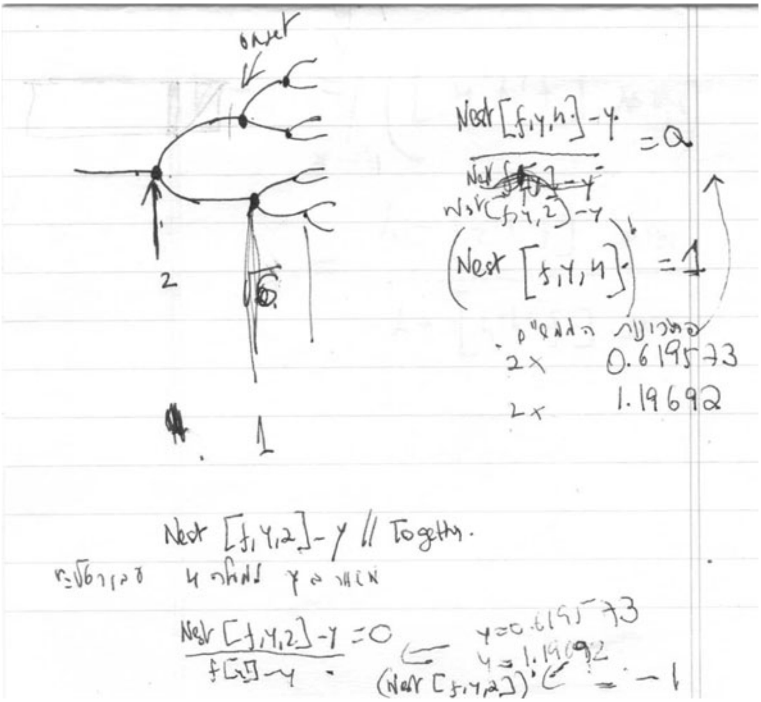

Fig. 5 L's notes on the onset of the 4-period

bifurcation point, −1 is the value of the derivative of the last cycle and not the value of the derivative of the new cycle that is just beginning (which is +1). Now, I was sure that my mathematical construction will not collapse any more.

The deep epistemic P_d action is characterized in event 16b by the way L changed her previous view of the bifurcation map which was unidirectional—from the left to the bifurcation point and from there to the right. Now, she was able to look from the bifurcation point in both directions - to the right, to the new cycle, and also to the left, to the previous cycle.

4.2.3 The CAS and the Integration of Different Modes of Thinking

A salient feature of combining constructions is the integration of different knowledge structures and of different modes of thinking. The CAS played a crucial role in supporting this integration; more specifically, it offered the potential

- to experiment in a new setting with information obtained in a previous setting: This was the case, for example, in episode 10 in which the CAS enabled the learner to experiment in both settings, the numerical and the graphical, first the static graphical mode and then the dynamic graphical mode. This potential offered by the CAS was especially crucial in episodes 15–16 in which we observed the constructive interference between the constructions, especially the fact that a computation in an algebraic mode belonging to C1 was utilized in an analytic mode belonging to C3 and then transferred into a dynamic graphical mode belonging to C4; while this was essential to the integration of the different modes of thinking, the CAS by itself is neither doing any kind of integration nor is it creating new views. In the process of combining L took command and used the CAS as an agent facilitating combination;

- to use multiple graphical representations: Specifically, this enabled L to carry out the transition from the static graphical view to the dynamic graphical view in episode 10;
- to establish previously fragile knowledge by means of checking: the CAS enabled L to check that the derivative tends to -1 from the left and to $+1$ from the right. It helped her establish her knowledge, and this was done on her initiative. It also gave her a lot of confidence (see 16b, above). Checking, taking actions to convince oneself that a result indeed does answer the question that was asked, and does answer it correctly, is an essential part of mathematical activity;
- to establish connections between the elusive knowledge in one construction and the more explicit knowledge in the other construction towards the combination between the two constructions: In episode 13, L used the CAS to firm up her fragile C3 knowledge toward the second combination point. Similarly, in episode 10 vague knowledge about the bifurcation diagram intervened. Mathematica's activation of L's vague knowledge was a facilitating factor towards the first combination point.

The important point that we would like to make clearer is that the two constructions, the two branches, the "strong" and well established branch as well as the "weak" branch were both necessary in the justification process. The CAS has an important role in firming up the fragile knowledge in the weak branch: the way the learner uses the CAS in order to "reinforce the weak branch" towards combination of constructions of knowledge is consistent with the description of the genesis of abstraction as expressed by Davydov's method of ascent. According to Davydov (1990), abstraction starts from an initial, undeveloped first form, which need not be internally and externally consistent. Davydov's move from an undeveloped to an elaborate form of abstraction is compatible with our description of the learner's progressing towards justification by reinforcing her previous elusive knowledge.

We summarize this subsection by means of the diagram in Fig. 6, in which E and G denote the combining points. The P_d actions, which occur close to the combining points underline the learner's initiative and the fact that the learner took the command; contrary to the previous subsection in which we concluded that the tool had the upper hand near the branching points, we here conclude that the learner took command near the combining points. However, the CAS has a direct influence towards the process of combining: The CAS is an agent that facilitates the combining of constructions (this is indicated in the diagram by the dashed ellipses before the combining points).

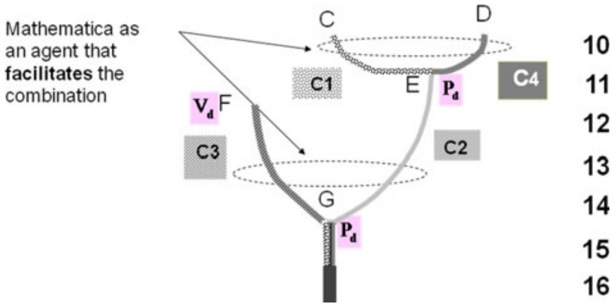

Fig. 6 The influence of the CAS in the process of combining

5 Discussion and Concluding Remarks

We now turn to the general question whether certain patterns of interaction of epistemic actions are characteristic for specific contextual components. Although contextual factors such as the mathematical topic under consideration, the learner's history of learning, the social context, books, and the CAS are closely interrelated and it might be difficult to make a clean separation between them, we focus on the influence of the computer and in particular of the CAS.

5.1 Patterns of Interacting Parallel Constructions When Working with a Computer

The interaction between the learner and the computer was crucial for enabling and promoting the entire learning sequence. When L was stuck, she often used the potential offered by the CAS to continue like for example in episode 6; but occasionally, there were unexpected consequences like the unexpected complex roots in event 7f. The computer provides information according to its own logic rather than according to the learner's. This may lead to situations where the learner is not ready for the information that is being presented, for example because the algorithms used by the computer are beyond the reach of the learner. The learner may register the information and make sense of it later. This happened for example, in episode 12 in which L attempted to understand how the CAS used the coefficients of the polynomial p and of its derivative p' for computing the discriminant D. This explains why some constructions were interrupted and resumed later, like for example, construction C3 in episode 12. Interruptions and resumptions are important for justification. In fact, in our study, resumption was usually preceded by V_d actions, in which L formulated the connections she had to establish towards justification like for example in the resumption of C3 in episode 12. Indeed, the required justification was possible only by means of establishing connections between different knowledge structures. Different settings were necessary, not optional. Here, the possibility offered by the computer to experiment in a new setting with information obtained in a previous setting was crucial. We encountered this phenomenon in episodes 15 and 16 in which the CAS permits the constructive interference between the constructions, especially the fact that a computation in an algebraic mode belonging to C1 was utilized in an analytic mode belonging to C3 and then transferred into a dynamic graphical mode belonging to C4.

The different capabilities of the CAS were instrumented by L in different ways. We have already noted how the multiple graphical representations offered by the CAS enabled L's transition from her static graphical view to a dynamic graphical view. L was looking for an algebraic expression giving the x-values as functions of r. The new dynamic graphical view developed in episode 10 which shows the x-values as they depend on r provided a "graphic equivalent" to the algebraic expression L was looking for. In addition, it is especially interesting to notice how the numerical approximations offered by the CAS contributed to both patterns, branching and combining, of L's parallel constructions of knowledge. For example, in the second branching point, L used the CAS to numerically compute the solutions of the degree 12 equation $p(x) = 0$. The numerical approximations of the real solutions gave her some confidence. She obtained the same values like in the graphic plots and this was especially important since exact symbolic results were not available. But at the same time, the numerical solutions that were offered included complex solutions and therefore contributed to exploration inviting L to know more about the complex roots which she did not expect. When examining the second combining point, the numerical approximations offered by the CAS were used by L to consolidate previously fragile knowledge by means of checking. Checking the value of the derivative by means of numerical approximation helped to reinforce her previously fragile knowledge and gave her a lot of confidence. As a consequence, it enabled her to change her previous view of the bifurcation which was unidirectional and enabled her to look from the bifurcation point in both directions—to the right, to the new cycle, and also to the left, to the previous cycle.

5.2 The Instrumentation Process and Cognitive Constructions

According to the instrumental approach (Trouche 2005), the instrumentation process describes how the tool becomes an effective instrument of mathematical thinking in the hands of the learner. The instrumentation process led to cognitive constructions by L. For example, obstacles which L encountered during the instrumentation process like information on complex roots that she had not asked for as in event 7f offered opportunities for construction of knowledge and what might have been considered as technical difficulties offered to L a wider conceptual aspect (Guin et al. 2005). More generally, as shown above, the pattern of branching was activated by the CAS permitting the flow of additional ideas, which then served as the starting points or seeds for the future combinations. The pattern of combining was invited by the way the CAS was used by L, namely to help establish connections between the fragile weak branch and the more established stronger branch, reinforcing the weak branch and achieving a combination of the two branches. Observing the cognitive constructions by L illuminates how the CAS artifact became an instrument—a mixed entity, in part artifact, and in part cognitive schemes. Quoting Vérillon and Rabardel (1995) "no instrument exists in itself… it becomes an instrument when the subject has been able to appropriate it for himself and has integrated it with his activity" (p. 85).We should note, however, that in the present case the learner, L, is quite expert in using the Mathematica instrument.

5.3 Is the Initiative with the Learner or Does the Tool have the Upper Hand?

The subject-tool dialectic in the instrumental approach is described in Trouche (2005). Since we have above characterized the CAS tool as an agent that has a major influence on the parallel constructions after branching and that facilitates combining, we might want to treat it as a second participant in the learning process. Nevertheless, there are important actions that are not in the realm of the CAS as a partner. For example, the CAS facilitates the integration of different modes of thinking but by itself is neither doing any kind of integration nor creating new views. It is only a part of the context in which this might happen. In this sense, as already mentioned in our analysis of the influence of the CAS context on the parallel constructions in the previous section, the tool had the upper hand near the branching points, and the learner took command near the combining points. This fact was demonstrated by the nature of the learner's deep epistemic actions (P_d) close to combinations or in episodes in which the active constructions approach each other.

Indeed at the critical points (A, B, … G) in Figs. 3 and 6 or very close to these points, V_d or P_d actions appear, showing that L took the initiative (like for example the deep epistemic actions in event 10d and 10f that were described earlier). In other words, the CAS had no direct influence on her decision. This does not mean, however, that the CAS was absent or unimportant, because in several cases, L is aware of the potential of the CAS and makes use of this potential. The CAS has a direct influence on her decision in which branch to continue after branching. It also facilitates the combining. And the way the CAS is used by L is a consequence of her desire for a justification. On the one hand, L's initiative depends on the potential promised by the CAS. On the other hand, there could not be any expression of this potential without her initiative.

5.4 Conclusion

The main contribution of the research presented here lies in the identification that certain patterns of epistemic actions (in our study, the specific patterns of branching and combining) have been facilitated by certain contextual factors (in our case the CAS context). Indeed, we found that the branching and combining patterns have been enabled by the work with the CAS. This is due to the fact that the computer provides a context which is very rich in resources, "une mosaïque non structurée de résultats" (Michèle Artigue, personal communication, June 19, 2005). The potential promised by the CAS activates the branching of constructions even if the learner is unable to make immediately sense of this "unstructured mosaic of results". Constructions are interrupted by lack of knowledge. Nevertheless, the seeds for the future combinations are already present. The fact that the computer can perform the computations even if the learner does not really understand its mechanism encourages her to make sense of the rich resources offered by the computer. Therefore, in this study, the branching, interruptions after branching and resumptions of the interrupted constructions were a necessary stage preceding the integration of the knowledge structures. The combining process which ends in the integration of knowledge structures was facilitated by the potential offered by the CAS and the learner's ability to make sense of the resources offered by the computer. The relations between the learner and the computer as a "dynamic partner" were different at the branching and at the combining phases: the impression is that the computer had the upper hand at branching and the learner took command at combining. In fact, the situation is more complex and we described in the paper how the roles of the learner and computer intertwined during the process of constructing the justification.

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References

Artigue, M. (2002). Learning mathematics in a CAS environment: The genesis of a reflection about instrumentation and the dialectics between technical and conceptual work. *International Journal of Computers for Mathematical Learning, 7*, 245–274.

Davydov, V. V. (1990). *Soviet studies in mathematics education: Vol. 2. Types of generalization in instruction: Logical and psychological problems in the structuring of school curricula* (J. Kilpatrick, Ed., & J. Teller, Trans.). Reston, VA: National Council of Teachers of Mathematics. (Original work published in 1972).

Dreyfus, T., Hershkowitz, R., & Schwarz, B. B. (2001). Abstraction in Context II: The case of peer interaction. *Cognitive Science Quarterly, 1*, 307–368.

Dreyfus, T., & Kidron, I. (2006). Interacting parallel constructions: A solitary learner and the bifurcation diagram. *Recherches en didactique des mathématiques, 26*, 295–336.

Guin, D., Ruthven, K., & Trouche, L. (2005). *The didactical challenge of symbolic calculators: Turning a computational device into a mathematical instrument. (Introduction, pp. 1–8)*. New York: Springer.

Hershkowitz, R., Hadas, N., Dreyfus, T., & Schwarz, B. B. (2007). Processes of abstraction, from the diversity of individuals' constructing of knowledge to a group's "shared knowledge". *Mathematics Education Research Journal, 19*, 41–68.

Hershkowitz, R., Schwarz, B. B., & Dreyfus, T. (2001). Abstraction in context: Epistemic actions. *Journal for Research in Mathematics Education, 32*, 195–222.
Hitt, F., & Kieran, C. (2009). Constructing knowledge via a peer interaction in a CAS environment with tasks designed from a task-technique-theory perspective. *International Journal of Computers for Mathematical Learning, 14*, 121–152.
Kidron, I., & Dreyfus, T. (2004). Constructing knowledge about the bifurcation diagram: Epistemic actions and parallel constructions. In M. J. Høines & A. B. Fuglestad (Eds.), *Proceedings of the 28th annual conference of the international group for psychology of mathematics education*, Vol. 3 (pp. 153–160). Bergen: Bergen University College.
Kidron, I., & Dreyfus, T. (2010). Justification enlightenment and combining constructions of knowledge. *Educational Studies in Mathematics, 74*(1), 75–93.
Noss, R., & Hoyles, C. (1996). *Windows on mathematical meanings: Learning cultures and computers*. Dordrecht: Kluwer.
Pontecorvo, C., & Girardet, H. (1993). Arguing and reasoning in understanding historical topics. *Cognition and Instruction, 11*, 365–395.
Schwarz, B. B., Dreyfus, T., & Hershkowitz, R. (2009). The nested epistemic actions model for abstraction in context. In B. B. Schwarz, T. Dreyfus, & R. Hershkowitz (Eds.), *Transformation of knowledge through classroom interaction* (pp. 11–41). London: Routledge.
Schwarz, B. B., & Hershkowitz, R. (1995). Argumentation and reasoning in a technology-based class. In J. F. Lehman & J. D. Moore (Eds.), *Proceedings of the 17th annual meeting of the cognitive science society* (pp. 731–735). Mahwah, NJ: Lawrence Erlbaum Associates.
Trouche, L. (2005). An instrumental approach to mathematics learning in symbolic calculators environments. In D. Guin, K. Ruthven, & L. Trouche (Eds.), *The didactical challenge of symbolic calculators: Turning a computational device into a mathematical instrument* (pp. 137–162). New York: Springer.
Vérillon, P., & Rabardel, P. (1995). Cognition and artifacts: A contribution to the study of thought in relation to instrumented activity. *European Journal of Psychology of Education, 10*, 77–101.