

Computational Construction as a Means to Coordinate Representations of Infinity

Ana Isabel Sacristán · Richard Noss

Published online: 23 February 2008
© Springer Science+Business Media B.V. 2008

Abstract In this paper, we describe a design experiment aimed at helping students to explore and develop concepts of infinite processes and objects. Our approach is based on the design and development of a computational microworld, which afforded students the means to construct a range of representational models (symbolic, visual and numeric) of infinity-related objects (infinite sequences, in particular). We present episodes based on four students' activities, seeking to illustrate how the available tools mediated students' understandings of the infinite in rich ways, allowing them to discriminate subtle process-oriented features of infinite processes. We claim that the microworld supported students in the coordination of hitherto unconnected or conflicting intuitions concerning infinity, based on a constructive articulation of different representational forms we name as 'representational moderation'.

Keywords Computational microworld · Infinity · Infinite sequences · Logo · Representational moderation · Situated abstractions

“...let us remember that we are dealing with infinities and indivisibles, both of which transcend our finite understanding, the former on account of their magnitude, the latter because of their smallness. In spite of this men cannot refrain from discussing them, even though it must be done in a roundabout way ...

“difficulties [...] arise when we attempt, with our finite minds, to discuss the infinite, assigning to it those properties which we give to the finite and limited” (p. 26)

Galileo, *Dialogues Concerning Two New Sciences* (1638)

A. I. Sacristán (✉)
Department of Mathematics Education, Centre for Research and Advanced Studies (Cinvestav),
Av. Instituto Politécnico Nacional 2508, Col. San Pedro Zacatenco, Mexico, DF 07360, Mexico
e-mail: asacrist@cinvestav.mx

R. Noss
London Knowledge Lab, Institute of Education, University of London, 23-29 Emerald Street,
London WC1N 3QS, England, UK
e-mail: r.noss@ioe.ac.uk

1 Introduction

Infinity is intuitively contradictory (Fischbein et al. 1979). In this paper we present selected results from a computer-based experiment—a microworld—that aimed to provoke high-school students to reflect on and analyse infinity-related situations and apparent paradoxes (particularly related to infinite sequences and limit processes), by constructing and coordinating diverse representational models.

The concept of infinity is conventionally conceived as a mental construct of a highly abstract nature that depends—perhaps rather more so than many other mathematical ideas—on context and point of view: as Tall (1980, p. 281) points out, “our interpretation of infinity is relative to our schema of interpretation, rather than an absolute form of truth.” At the heart of the idea of the infinite, there is a tension. The infinitely large—in terms of process over time—is required for reaching the infinitely small. In fact, where the infinite is concerned, tensions abound. The elements of these tensions—big versus small, discrete versus continuous, potential versus actual—coexist in phenomena that involve the infinite. We will elaborate later how our approach aimed to exploit and build upon these tensions, rather than regard them as obstacles to be circumvented.

Historically, these tensions and the epistemological difficulties that ensue have led to infinity being a source of conflict and paradox. The great breakthrough of 19th century mathematics was to develop an adequate theoretical framework and operational field for this concept, by disaggregating some of these elements, so that in defining how they operate, their articulation could be explained. Infinity has many facets. There is the infinitely large, which we call an *outward* notion of infinity, and the infinite arising from repeated subdivisions, which we call the *inward* notion of infinity (the *infinitely small* or infinitesimals). In both cases, the idea of *iteration* plays a central role. While infinitesimals seem natural, they lead to paradoxical situations. For this reason, 19th century mathematicians banished completely the use of infinitesimals, instead adopting the epsilon-delta definition of limit. However, from a pedagogical perspective, several researchers (Tall 1980, 2001; Imaz 1991; Keisler 1986) have proposed the use of infinitesimals, such as is employed in non-standard analysis,¹ as perhaps more intuitive than the limit approach of modern calculus.

Other facets include the dual nature of infinity as *potential* or *actual*: these correspond to two ways of looking at infinity—as a *process* and as an *entity*—where, as Moreno and Waldegg (1991) explain, in the first infinity appears as something that *qualifies* the process, whereas in the latter it is an *attribute* or property of a set. Further, when conceiving actual infinite sets, there is the difficulty that there are different *sizes* or *powers* of infinity in that some infinite sets are *denumerable* (natural numbers) and some are not (the real numbers).

Finally, and most generally, common paradoxes of the infinite have been shown to arise when applying the mathematics of the finite to the infinite, or when mixing infinities of different types (e.g. when mixing the discrete and the continuous). In fact, this problem of the relationship between discrete and continuous is at the core of Zeno’s famous paradoxes, which highlight the difficulty of saying that the line is formed by points (Struik 1967): As Aristotle explained, when Zeno suggested that the infinite division of space requires an

¹ The best-known theory is Robinson’s semantic model, which extends the set of real numbers to include infinitely large and infinitesimal numbers in the set of hyperreal numbers (${}^*\mathbb{R}$), where an infinitesimal is defined as a number smaller than every positive real number and bigger than every negative real number (Robinson 1974/1996). Other approaches include Nelson (1977)’s axiomatic Internal Set Theory.

infinite amount of time to be completed, he was mixing two types of infinity by applying a (discrete) number to a (continuous) magnitude (Jones 1987).

2 A Brief Overview of Research on Learning About Infinity

Monaghan (2001) states, in a list of potential pitfalls for researchers, that “the real world is apparently finite and there are no real referents for discourse on the infinite” (p. 240). This is not just a pitfall for researchers; it is the crux of the problem for students as well. Lakoff and Núñez (2000) argue that, from a cognitive science point of view, the way in which the mind comprehends infinity is by analogies to the finite through a common inferential structure they call the ‘Basic Metaphor for Infinity’ or BMI: that is, the idea of actual infinity emerges from the knowledge that a finite succession of repeated actions will necessarily be completed. In other words, the notion of the infinite arises from knowledge in the finite; yet it is that knowledge of the finite which can also create difficulties. Contradictory situations arise, as Galileo pointed out, because “finitist” interpretations tend to prevail (Waldegg 1987), such as the use of the inclusion idea (Tsamir 1999): that the parts of an object (e.g. strict subsets of a set) should necessarily be smaller than, or properly contained in, the whole object (or set).

While mathematically (particularly in the nineteenth century) the notion of infinity and its related concepts (e.g. limits of infinite processes) have been refined and solidified, no similarly tidy outcome is generally enjoyed by students. Some authors (e.g. Sierpiska and Viwegier 1989; Cornu 1991; Waldegg 1993) have considered that the intuitions that people hold of infinite objects and processes can become “obstacles” for the adequate construction of formalised versions of these concepts.

Much research has been done in relation to limiting processes (as well as the comparison of infinite sets), and touching on several, if not all, of the facets described above.

A variety of early studies (Schwarzenberger and Tall 1978; Tall and Vinner 1981; Mamona-Downs 1987; Sacristán 1991; Cornu 1991) reported that students’ spontaneous conception of the limit notion is that of a *process* of getting close, with the limit perceived as unreachable. This is the problem of “the passage to the limit” (Cornu 1986) where the limit, or “that which happens at infinity”, seems to be isolated from the dynamic limiting process, and therefore acts as an obstacle to the possibility that what happens in the finite can allow us to predict what happens at infinity.

Many other studies concerning limiting processes (Espinoza and Azcárate 1995; Sierpiska 1987) deal with the difficulties related to the properties of the set of the real numbers (including the structure and properties of decimal numbers) and of the continuum. In particular, many authors (e.g. Sierpiska 1987; Schwarzenberger and Tall 1978; Cornu 1991; Sacristán 1991) have considered whether $0.999\dots$ is perceived by students as equal to 1. It was found that many students use infinitesimal concepts: e.g. they consider that $0.999\dots$ can perhaps get infinitely close to 1, but not equal, with the limit being viewed as a boundary, rather than as the value at infinity (Cornu 1991), and that decimals are perceived as having “infinitesimal neighbourhoods” (Mamona-Downs 1987). In studies aimed at overcoming the difficulties inherent in challenging existing intuitive structures, it was found that formal mathematics teaching did not appear to modify students’ ideas of infinity (e.g. Fischbein et al. 1979; Waldegg 1993), but Fischbein (1987) proposed that by making students aware of their intuitive constraints and mental contradictions, it could help them deal with counter intuitive situations. This is the approach taken by Tsamir (2001) who designed tasks of comparisons of infinite sets, based on research-results that show students’ incompatible solutions when using

various representations, in order to raise students' awareness of such inconsistencies. (Even when the same problem is isomorphically constructed in different contexts, the context and forms of representation have been shown consistently to be crucially influential on students' responses and understandings—e.g. Fischbein et al. 1979; Waldegg 1987; Sacristán 1991; Núñez 1993; Tirosh and Tsamir 1996; Tsamir 2001.)

In an earlier research, Waldegg (1987) successfully used a context combining numerical and geometrical contexts through the use of algebraic language, to help students “overcome” the “obstacles” she had observed in previous cases (e.g. if a geometric set is bounded, this is a potential obstacle for its infinite quantification). More recently, Garbin and Azcárate (2002) focused on students' inconsistencies in relation to the concept of actual infinity, and stressed the importance of *connection tasks* between four different representational registers: verbal, geometrical, graphical, algebraic and analytical to develop more coherent thinking in students in relation to actual infinity. Below, we develop the idea that by building connections between different types of representations, some of the difficulties that arise when working in a single context might be avoided.

For the most part, however, the literature tends to support the negative finding that those areas of mathematics where infinity occurs naturally are precisely those that have traditionally been presented to students mainly from an algebraic/symbolic perspective, an approach that has tended to make it difficult to link newly-acquired formal knowledge with intuitive concepts. There is, however, an interesting challenge implicit in this corpus of findings. If algebraic infrastructure creates an extra layer of difficulty that deters the learner from linking different representations, might it be possible to seek alternative means of expression that facilitate, rather than obscure, the connections between them? A candidate for such a role is the formalism of a computer programming language, and it is this approach that is reported below. The computer presence creates, of course, a new epistemological domain, so that the objects constructed in that context will be given meanings according to the affordances of the computational environment (Balacheff 1994). Tall (1986) provided students with rich experiences through non-formal approaches for building “cognitive roots” (rich concept images) upon which the formal theories of Calculus could later be built. In particular, he found it particularly valuable to program the sum of a series procedurally, so that the sum function takes a value of n as an input, adds up n terms and returns the sum as output. This highlighted the fact that an infinite series is the limit of an associated sequence of partial sums, a fact that is not always evident to students.

Monaghan et al. (1994) found that a computer-based system they studied, highlighted and suppressed different facets of the concept of limit, emphasising the conception of limit as a process, and suggested that a system that produced symbolic limits as proper numerical expressions would lead to a more balanced view. More recently, many authors have used CAS environments, spreadsheets, and also Dynamic Geometry environments for the explorations of infinite sequences, iterative processes and chaos (e.g. Dugdale 1998; Abramovich et al. 1999; Kissane 2003).

In the study we report below, our task was to create situations for the investigation of infinity-related ideas through the use of constructive pieces of cross-representational tools within a computer-based microworld based on the Logo language.

3 A Computational Microworld to Study Infinity

In this section, we first outline the principles and objectives of the microworld design, before considering the microworld and its associated activities.

3.1 Design Objectives

We begin by sketching out two design preferences that have guided this and other studies in a similar vein (see Noss and Hoyles 1996). First, we maintain a predilection for the constructionist paradigm (see Harel and Papert 1991), which advocates—quite simply—that learning is facilitated when the learner has the opportunity to construct something “out there”, and reflect upon it as a public entity. Second, we consider it a methodological challenge to construct domains of abstraction, in which the abstractness of objects and relationships are recognised as in the eye of the beholder (or constructor) rather than inherent in the objects themselves (see, for example, Wilensky 1991 on abstract ‘versus’ concrete; Noss and Hoyles 1996, on the notion of ‘situated abstraction’).

An important corollary of this approach is that it is generally insufficient for the individual merely to be presented with diverse representations of a concept. A multiplicity of representations is not a guarantee of learning: others have pointed to the difficulties that students have in linking different types of representations and particularly in analysing visual information (Dreyfus and Eisenberg 1990). While many researchers (e.g. Cuoco and Goldenberg 1992) advocate incorporating more representations and types of thinking, particularly visual ones, into school mathematics, it is clear that the mere presence of multiple representations does not inevitably lead the learner to construct cognitive links between them.

In our view, it is by working with and re-constructing external representations and the relationships between them that the individual constructs his/her own mental representations of the objects, as well as the connections between them, thus giving them meaning in the wider conceptual web of ideas and concepts.

Translating these very general design preferences into practice, we aimed to construct a computationally-based approach that:

- a) focused on students expressing their ideas about the infinite by exploring, creating and modifying computer programs in Logo;
- b) afforded a means to build, work with and explore different forms of representations of infinite processes interconnected through the programming code.

The central idea, therefore, was that a range of infinite processes could be explored through the construction of different representational modalities, involving three types of representations, namely:

1. the symbolic code: very simple recursive Logo programs or procedures that describe the process and generate the visual and numeric representations;
2. different types of graphically-represented models;
3. numeric representations to complement and validate the visual observations.

In relation to the first modality, the symbolic code was proposed as a means to describe and reflect upon the process under study. Infinite processes can be described elegantly as recursive programs. This is a more than incidental observation: while iteration (repeating over and over) is the building block of infinite processes, recursive procedures can capture something else of the essence of infinity—not only are they implicitly endless, they can be designed to reflect the self-similar characteristic and structure of actual infinite objects. Thus our designed activities centred on self-referential Logo programs, i.e. that refer to themselves, whether tail-recursive (essentially iterative) or genuinely recursive (for a definitive description and rationale for recursive Logo programs, see Harvey 1997).

The second representational form, a range of geometrical or graphical models of infinite sequences, was aimed at approaching the possibility of visualising the infinite process as a whole. A picture can, after all, demonstrate a holistic image of an object, even if it is sometimes misleading. Computational graphics, unlike pencil-and-paper representations, are able to capture the change of a graphical object over time. Processes can be represented as they unfold; instead of viewing only the end-result of the process, the process itself and its behaviour can be visualised. This is particularly useful in the study of infinite processes since the result of the process, which could be said to be the behaviour at infinity, can only be deduced by analysing the behaviour of the process in the finite—the BMI metaphor (Lakoff and Núñez 2000)—this is true even in the formal definitions of a limit.

The third, numeric, representation gave students the possibility of adducing quantitative evidence of the processes under study, and was intended as a means to study at the micro-level the behaviour of the processes to confirm or dismiss conjectures derived from investigating the other representations. As we will elaborate later, one way in which this was achieved was by adding very simple commands to the programs that would output the numeric values of the terms of the sequences, as these were being generated and represented graphically.

An important design criterion was that these different types of representations (symbolic, graphic and numeric) were explicitly linked with one another through the first, which is represented in the Logo code. In fact, the code was designed to act as an isomorphism between the different models, and serve as a link between the representations and the student; furthermore, the code could make the transformation between representations more transparent, or in Duval's (1999) words, more *congruent*. The key intention of this descriptive link between the representations was that, by expressing relevant relationships and constructing them in different representational modes, students would come to construct rich meanings arising out of their coordination. In this way, students were provided with a means to carry out the three requirements that Duval (1999) promotes as crucial for mathematical learning:

to compare similar representations within the same register in order to discriminate relevant values within a mathematical understanding, to convert a representation from a register to another one; and to discriminate the specific way of working in order to understand the mathematical processing that is performed in this register. (ibid, p. 24)

The central theme of the activities was the exploration of the convergence or divergence of infinite sequences and series through the use of recursive geometric models. Self-similar figures and fractals—which are the limits of infinite graphical sequences—were explored as a setting for the idea of the limit of a sequence, and for interrogating some of the paradoxes and tensions referred to above. By observing the movements of the turtle, a sequence of geometrical objects can be seen as a developing process; that is, through the geometric representation, the different steps of the process can be seen. Additionally, the possibility of producing successive levels of a figure as approximations to the “real fractal” can be thought of as a (potentially) infinite process, yet it is an exploration of a geometric object, which has already been described through the symbolic code. Reflecting on the different levels of the figure can thus be interpreted as approaching an object, which is “already there”.

3.2 The Design of the Activities

The activities were divided into two categories:

- a) Explorations of classical infinite sequences and their corresponding series, through geometric models (spirals, bar graphs, and staircases);
- b) Fractal explorations.

We consider these in turn:

- a) Explorations of infinite sequences, such as $\{1/2^n\}$, $\{1/3^n\}$, $\{(2/3)^n\}$, $\{2^n\}$, and $\{1/n\}$, $\{1/n^{1.1}\}$, ... , $\{1/n^2\}$, and their corresponding series

These explorations involved a range of geometric models in the form of spirals, bar graphs, staircases, and straight lines, the corresponding Logo procedures, with a complementary analysis of the numerical values. These models were chosen since they constitute a straightforward way of translating arithmetic series into geometric form. For example, in the spiral representation, each term of the sequence is translated into a length, visually separated by a turn, so that the total length of the spiral corresponds to that of the sum of the terms, i.e. the corresponding series. Thus, for instance, for the sequence $\{1/2^n\}$, the cumulative lengths of the spiral as it appeared, would represent the series: $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \dots$, a notation which is descriptive of the process involved—the ellipsis points indicating an indefinite continuity of the process (a potentially infinite process). On the other hand, in the symbolic computer code, the same series can also be represented by a notation corresponding to $\sum_{n=1}^{\infty} \frac{1}{2^n}$, which is an object in itself (an actual infinite object²). Our intention, therefore, was to illustrate how the same mathematical object can be represented both as an object and a process.

Students were introduced to these explorations by being given the Logo procedure below and asked to predict its behaviour.

```
TO DRAWING :L
  PU
  FD :L
  RT 90
  WAIT 10
  DRAWING :L / 2
  END
```

This procedure makes the turtle walk through a spiral with each arm having half the length of the previous one (see Fig. 1a, below). It is a first approach to the infinite sequence $\{1/2^n\}$, which was chosen because of its simplicity, and the ease with which it can be described in plain language (e.g. just keep halving). It should also be noted that this is a tail-recursive procedure without a stop condition, so the procedure could potentially continue indefinitely.

This procedure—which initially does *not* draw (PenUp) but *does* show the turtle moving—was designed to encourage students to reflect on the behaviour of the turtle as a process.

The above procedure was the generic activity for subsequent activities. This initial procedure was subsequently modified so that the turtle actually left a trace (by putting the

² This is independent of the convergence or divergence of the series, although when there is convergence it is easier to think of the series as a “complete” object, e.g. when $\sum_{n=1}^{\infty} \frac{1}{2^n} = 1$.

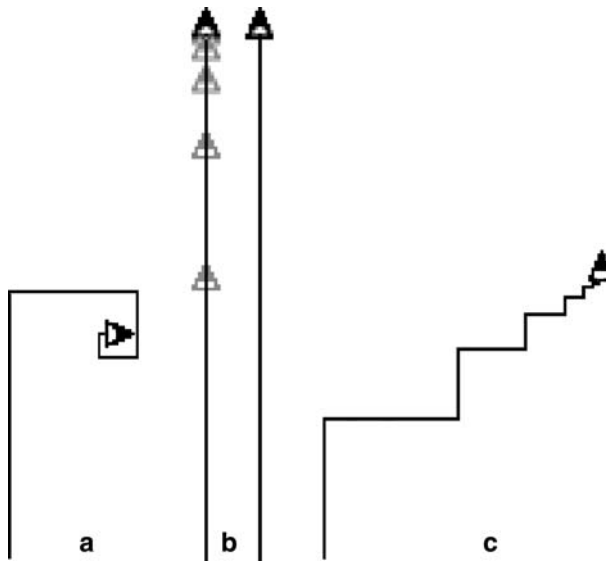


Fig. 1 The different graphical models of the sequence $\{1/2^n\}$ (or $\{L/2^n\}$). In (a), the spiral model, the turtle walks a length (L), turns, then walks half the previous length, and so on. In (b), the Line model, by eliminating the turns in the procedure, the Spiral model is “stretched out”, so that the sum of terms (the series) can be visualised; here, the left hand side of the figure depicts the movements of the turtle, while the right hand side shows the end result. The Staircase model (c) is another transformation, by adding turns, of the Spiral model procedure; here, each step represents a term of a sequence, while the total length of the staircase represents the corresponding series

turtle’s pen down), and a stop condition was added (this led to rich explorations of the relationship between the values used in the stop condition and the number of segments drawn in the geometric model).

The key innovation, however, was a more fundamental modification, which transformed the initial spiral representation into other isomorphic models of the sequence. The specific models represented were:

- A straight Line (see Fig. 1a): the spiral can be stretched out in order to observe its total length, which represents the value of the corresponding series; in the case of sequences with corresponding convergent series, this model can visually demonstrate the sequence’s convergence.
- The Staircase model (see Fig. 1c): this is a way of combining the Spiral model, where the different terms of a sequence can be discerned, with the Line model which allows the visualisation of the behaviour of the corresponding series.
- Bar Graph model (see Fig. 5 further below) in which each term of the sequence is separated so that consecutive terms of the sequence can be seen as adjacent bars.

Each model was instantiated as a very simple Logo program (see Table 1) that could be dropped into the initial program as a self-standing module. Any of these modules could then be used within the DRAWING procedure and easily changed depending on the model under investigation. For example, to investigate the spiral model, one would use the SPIRAL subprocedure within DRAWING:

Table 1 Subprocedures (modules) for the different visual models: Within the procedure DRAWING, various subprocedures can be used to produce the different visual representations of the sequence being modelled

Subprocedure for creating a spiral	Subprocedure for creating a bar graph	Subprocedure for creating a staircase	Subprocedure for creating a straight line
TO SPIRAL :L	TO BARGRAPH :L	TO STEPS	TO LINE :L
FD :L	JUMP	:L	FD :L
RT 90	FD :L	FD :L/2	WAIT 10
WAIT 10	END	LT 90	END
END	TO JUMP	FD :L/2	
	PU	RT 90	
	SETY -100	END	
	RT 90 FD 5 LT 90		
	PD		
	END		

```

TO DRAWING :L
SPIRAL :L
DRAWING :L / 2
END
    
```

This simple device pointed to an important piece of mathematical knowledge with which students could engage: that the recursive structure of the different programs was identical except for the ‘module’, highlighting the isomorphism between them.

Through interaction with the graphical (and numeric) behaviour of the models, students were able to explore the convergence or divergence of a sequence and that of its corresponding series, and predict the behaviour at infinity. The different geometric models for the same sequence provided different perspectives of the same process. A key element of the activities was that students carried out the transformations of the models themselves by modifying the computer code, so that they constructed for themselves an explicit link between representations through the programs.

b) Explorations of fractal figures

The second category of activity centred on the study of the recursive structures of the Koch curve (Fig. 2), the snowflake (formed by putting together three Koch ‘segments’; see Fig. 3) and the Sierpinski triangle (Fig. 4). Fractal figures, such as these, are *limit* objects generated through infinite visual sequences. These fractals exist as limits of infinite processes; yet, once produced they can also be conceived in terms of sets consisting of infinitely many parts. These figures therefore provide a rich ground for the exploration of infinite processes and of infinite objects: they afford an opportunity to study a different kind of limit: that of sequences that are constructed *visually, rather than algebraically*, although they can also be described, explored and understood symbolically. Once again, the programming code reflects in its recursive structure each of the steps of the sequence, and involves sequences such as $\{1/3^n\}$ that were studied in the first exploratory sessions. Through these activities we intended to confront students with the idea of “what happens in the infinite” by encouraging them to visualize the infinite processes under study through

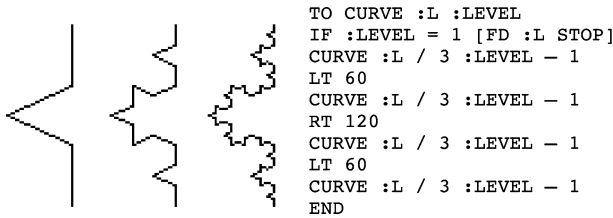


Fig. 2 The Koch curve, and the procedure to generate it. The Koch curve is constructed by replacing in each step, each (sub) line-segment with a figure similar to the generating one (figure at furthestmost left)

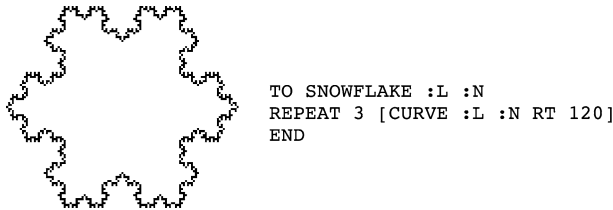
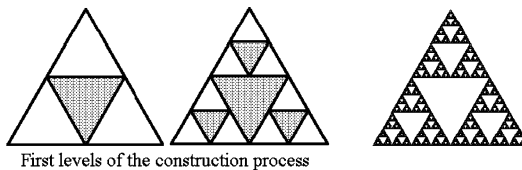


Fig. 3 The Koch snowflake. It is constructed by putting together three Koch curves. The procedure SNOWFLAKE therefore uses CURVE as a subprocedure



```

TO STRI :SIDE :LEVEL
IF :LEVEL = 0 [STOP]
REPEAT 3 [STRI :SIDE / 2 :LEVEL - 1 FD :SIDE RT 120]
END

```

Fig. 4 The Sierpinski triangle. It is constructed by a process that “takes away” the central triangle (one fourth of the area) of the original triangle, and repeating the process for each remaining triangle

the computational approximations they constructed on the screen, and to have them deal with apparent paradoxes at infinity such as the infinite perimeter of the Koch curve being formed by infinitesimal “zero-sized” segments, or the snowflake’s finite area that is bounded by an infinite perimeter.

Since programming the construction of these figures is quite complex, students were assisted in building the procedures. Nevertheless, in each case that we present, the final programs were indeed created by the students themselves. The relevant programs are given with the corresponding figures (see Figs. 2, 3 and 4).

4 The Microworld in Practice

The data³ we present here is derived from elaborated case studies (Sacristán 1997) of several Mexican students of varying ages and backgrounds. Here we focus on four

³ The study took place in Mexico, so all data was translated into English from Spanish.

Mexican middle- and high-school students: two 14 year-old girls (Consuelo and Verónica) from the same school and in the same grade, and two boys from different schools (Manuel, 17, who was about to enter his last year of high-school; and Jesús, 18, who had just finished high-school). These students were interested in learning Logo and volunteered⁴ for the study, which was carried out at a research institution in Mexico during the summer school holidays. The two girls were average mathematics students with no particular mathematical inclinations; in contrast, both Jesús and Manuel liked mathematics. However, none of the students had any prior experience or formal knowledge related to the concepts of mathematical infinity, or limits of infinite processes, though all had an adequate understanding of the concept of function, as well as that of sequence.

Prior to the study, the students participated in an intensive two-week Logo course (along with other students not involved in the study), so they were all able to write their own procedures, use recursion and were confident in making changes to procedures and exploring them for themselves. During the study, the participants worked in pairs (one team at a time), in a laboratory setting, using a single computer in order to facilitate the sharing and discussion of ideas (simultaneously providing the researcher with insights into their thinking processes—as proposed by Noss and Hoyles 1996) and to give them independence from the instructor. From a research methodology perspective, by working with only one pair of students at a time, we could carry out informal interviews and analyse students' experiences more easily. The researcher (first author) worked alone with each pair of students for 15 hours of hands-on activities (five 3-hour sessions) over the course of 3 weeks.

The role of the researcher was that of a participant observer, suggesting the field of work (the initial procedures and activities), as well as new ideas for exploration when needed. Students were encouraged to work as much as possible on their own, allowing them to be in control of the explorations, and giving them freedom to explore and express their ideas.

Students were informally interviewed during the sessions (formal interviews were also conducted at the beginning—in order to gain an insight into these students' prior intuitive conceptions of infinity—and at the end of the study, but discussion of that data is beyond the scope of this paper).

We now present one episode for each pair of students, reporting significant events from the elaborated case studies. We make no claim to the representativeness of our episodes; on the contrary, we have chosen these episodes carefully, to illustrate how the microworld shaped the ways the students expressed their ideas about infinity, and reciprocally, how they shaped the tools at their disposal to do so. In so doing, we hope to throw further light on this reciprocity between tool and learner, and particularly, the ways in which abstractions can be articulated within the expressive system defined by the tools-in-use (Noss and Hoyles 1996).

4.1 Episode 1: The “more is bigger” Intuition—Using the Decimal Structure of the Real Numbers for Justifying the Infinite Nature of a Bounded Process

This episode shows how the students involved coped, using the decimal structure of the numeric values, with the infinite nature of the processes, particularly when these processes

⁴ A Logo programming course was advertised in several schools across Mexico City. The course was free if students agreed to participate in the study that followed the course. Thus, the researcher never met beforehand any of the students, and did not chose them: these participants signed up for the Logo course and volunteered to participate in the study.

were bounded (or convergent). Throughout the study, a common confusion arose, particularly among the less mathematically-oriented students: the confusion that if a process is infinite then it will diverge. This intuition has been found by other researchers such as Núñez (1993), who explains that the problem arises when there are several competing components (processes) present; that is, when two types of iterations of perhaps different nature (cardinality vs. measure) are confused—the process itself and the divergent process of adding terms to the sequence. Thus, in the case where infinite sums are involved, this idea could appear as: “if an infinite number of terms or elements (*cardinality*) is added then the *measure* of the sum must be infinite, it must pass any preset value”.

Early on in the activities, the two girls, Consuelo and Verónica, expected the line model (see Fig. 1b) representing the series $\sum \frac{1}{2^n}$ (the sum of the segments corresponding to the sequence $\{1/2^n\}$) to grow without bounds, since an infinite number of segments were being added. Verónica, had predicted that if they “stretched the spiral”, and did not use a stop condition, then the resulting line would go all the way past the top of the screen because it would become very long (perhaps infinitely?). She and her partner, Consuelo, were quite surprised to see that the line got “stuck” at a length about twice the initial value. Puzzled, and still convinced that the line would grow indefinitely, they increased the scale, but always ended up with a line that eventually “got stuck”. Consuelo then observed that they were using a stop condition in the procedure, and believed this was the justification for the turtle stopping at a certain length—somehow the program just ‘stopped’ at some point.

Consuelo: It only stops because it has an IF. It continues straight up. The spiral is stretched. [But it doesn’t go all the way] because we have an IF.

However, when she removed that STOP instruction, the behaviour of the line model was unchanged: the turtle appeared to stop, vibrating in the same place even though the program continued running. In order to get a sense of what was happening, the girls added a PRINT command that would print out the value of the last segment walked by the turtle; they also added a COUNT variable to count the total number of segments the turtle had walked—the number of iterations (see Table 2). In this way, the vibrations of the turtle were complemented by a numerical count of segments, and both gave evidence that the process continued even though the graphical model seemed to become invariant. Looking for a means to coordinate the apparent contradiction between the ongoing process and the invariant model, Consuelo began to realise that as the process progressed the added segments became very small (“it must be that it walks very, very little and it can no longer be seen”). She thus decided to look at the bar graph model of the sequence (Fig. 5) and, together with the numeric values, which they recorded in a table (Table 3), confirmed her suspicions.

However, although Consuelo seemed to have discovered and made sense of how a process can *continue* indefinitely yet not *grow* to be infinite, it was still a fragile realisation, bound to the particular situation in which they were working, and tightly connected with the specific tools involved. In fact, when these two students returned for their next session, their original intuition had resurfaced and they chose to repeat the entire process. At the beginning of the next session, both students repeatedly maintained that if the process was allowed to continue indefinitely, the line should continue growing past the edge of the screen. The students again associated the infinite nature of the process with an expectation that the sum of the terms (segments) represented through the line model would show indefinite growth, maintaining this position even when they perceived otherwise in the line models. But then unfolding events began to lead to a change of view.

Table 2 Modified DRAWING procedure: Verónica and Consuelo modified the DRAWING procedure that in this case generates the line model of the sequence $\{1/2^n\}$ and its corresponding series, by adding a COUNT variable used to print out the number of terms that have been traced by the turtle: each time the turtle walks (LINE :L), the total number of iterations is printed out (PRINT :COUNT)

```

TO DRAWING :L :COUNT
IF :L < 1 [ PRINT :COUNT STOP]
LINE :L
PRINT :L
DRAWING :L * 1/2 :COUNT + 1
END
    
```

Fig. 5 Bar graph model of the sequence $\{1/2^n\}$ with numeric output. Each vertical bar, and each corresponding value, represents a term of the sequence. The numeric output is produced simultaneously to its corresponding bar (in the Logo procedure, a “Print value” command follows each “Draw (FD) value” command)

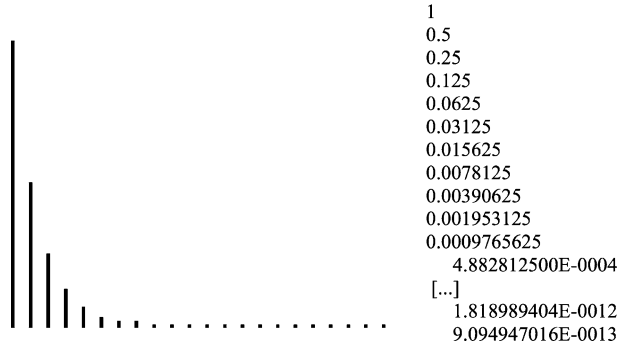


Table 3 Table constructed by Verónica and Consuelo recording the length of the segments corresponding to certain terms (as given by the variable COUNT) of the sequence $\{1/2^n\}$

COUNT	Length walked
95	0.00... -26 zeros-...02524354...
1857	0.0.... -557 zeros-...09711...

First, the visual behaviour (with the turtle vibrating on the same spot) reminded the students that the elements of the sequence became very small, as Consuelo confirmed by looking at the list of the values of the segments, although she still thought the line would eventually grow past the edge of the screen: “It is going to take a long time, it is going to take a very, very, very long time, because now it is doing very, very little.” Her idea was “many small steps implies a very long time”. Later, the students were surprised when they observed that the partial sums (see Table 4) eventually became a constant value. This led to an investigation of the values of the last segments and they observed how small those values were with tens of zeros after the point in the decimal expansion (Table 3), which served as a first justification for the convergence of the sum, even though the existence of a bound or limit was not yet fully realised. Further analysis of the values of the partial sums finally led Consuelo to conjecture that there was indeed a value that would not be reached nor surpassed, and she was able to explain the ongoing nature of the process through the numeric decimal representation where more digits can always be added. For Verónica this realisation would take longer, as she still believed that the line should keep extending, despite all the evidence to the contrary (which she dismissed—reasonably enough—by saying they were computer rounding errors).

Table 4 List of the first 100 partial sums, with a precision of 8 decimal digits, of the values of the segments corresponding to the sequence $\{1/2^n\}$, with a scale of 100, obtained using the procedure PARTIALSUMS (see Table 5)

50	75	87.5	93.75	96.875	98.4375	99.21875			
99.609375		99.8046875		99.90234375		99.95117188			99.97558594
99.98779297		99.99389648		99.99694824		99.99847412			99.99923706
99.99961853		99.99980927		99.99990463		99.99995232			99.99997616
99.99998808		99.99999404		99.99999702		99.99999851			99.99999926
99.99999963		99.99999981		99.99999991		99.99999995			99.99999998
99.99999999		99.99999999		100	100	100	100		100
100	100	100	100	100	100	100	100		100
100	100	100	100	100	100	100	100		100
100	100	100	100	100	100	100	100		100
100	100	100	100	100	100	100	100		100
100	100	100	100	100	100	100	100		100
100	100	100	100	100	100	100	100		100
100	100	100	100	100	100	100	100		100

Table 5 Procedures for generating sequences and their corresponding partial sums: In order to investigate the values of the series and partial sums of a sequence, the procedures PARTIALSUMS and SUML were used in combination with another procedure, SEQUENCE, which generated a list of n terms of a sequence

```

TO PARTIALSUMS :LIST                               TO SEQUENCE :N
IF :LIST = [ ] [ OP [ ] ]                          IF :N = 1 [ OP FN 1]
OP SE (PARTIALSUMS BL :LIST) (SUML :LIST)          OP SE (SEQUENCE :N - 1) (FN :N)
END                                                 END
TO SUML :LIST                                       TO FN :N
IF :LIST = [ ] [ OP 0]                             OP 1 / POWER 2 :N
OP (FIRST :LIST) + SUML BF :LIST                  END
END
    
```

The PARTIALSUMS procedure gives as output the *list* of partial sums of a sequence of n terms; while SUML is a procedure that adds all the terms in a list (sequence) (e.g. by typing SUML SEQUENCE <number of terms>). In this case, the list of terms of the sequence is generated using a function, the Logo procedure FN (in this example representing the function $f(n) = 1/2^n$) that outputs the n th term of that sequence

However, Verónica too began to change her view during her conversation with Consuelo: she started to realise that because the added segments became very small, the sum would not grow much.

Consuelo: It won't reach 100, by a few digits ...

Verónica: It does pass it, but because there are too many numbers, then it rounds it; the computer cannot write down so many numbers ...

Consuelo: I believe that: no, it is not going to pass 100. I think it is always going to stay where it is, because it doesn't pass 100, so the nines are going to keep increasing: nines, nines, nines, and so on ...

Verónica: Right, it is not going to pass 100.

Verónica, at this point, seemed to be focusing on the process as indefinite, and therefore felt that the line should keep growing (go off the screen); Consuelo, on the other hand, had come to realise that the process could continue without necessarily passing the observed bounds, and she found a numerical explanation for this in terms of “you can always add more nines to the decimal expansion 99.9999... and therefore never reach 100!”.

The crucial insight for Consuelo was the focus on the decimal structure of the real numbers—which were conceived by Cantor as an infinite sequence of digits that can be

seen separately from the geometry of the real line—realising that in the decimal expansion of the values under study, the number of digits would increase more and more as the sequence progressed. By disassociating from the geometrical appearances, she was able to shift her attention from “more is bigger” to uncovering that the “more” (infinite nature) could be found in the decimal structure of the numeric values—in the “infinitely small”—rather than in the bounded representation.

Thus, the construction and exploration of the numeric representations played an important role in shaping these students’ knowledge regarding the result of an infinite process (i.e. that the result of an infinite process cannot necessarily be characterised as being infinite itself): the different subprocesses involved (cardinality vs. measure), were progressively discriminated. Furthermore, justification for how the process could continue indefinitely, in that more digits can always be added in the decimal structure provided a way of making sense of the property of density of the real numbers.

It is worth contrasting this finding with that of Ferrari et al. (1995) who reported students’ difficulties with the idea of density; they related this to students’ problems in accepting that a bounded set can contain an infinite number of points. We argue that this is a problem that arises from a failure to coordinate geometry and number (Ferrari et al. acknowledge the presence of confusions between measure and cardinality). In our study, we think it was the act of construction that led the students to discover and make use of the property of the density of the real numbers for making sense of the bounded geometrical situation. By constructing the different representations (symbolical, graphical, and numeric) and having to express explicitly the relationships between these, new meanings emerged (e.g. they were able to see in the numeric values something they had not seen before) and were articulated with existing knowledge.

However, the dependency on the numeric structure also seemed to reinforce the dynamic approach to the limit, with the conception of the limit as unreachable. Consuelo and Verónica stated that the process of “taking halves” could continue indefinitely as more and more zeros could always be added to the decimal expansion; but because this process of adding zeros to the decimal expansion was seen as potentially infinite, they concluded that the value of $:L$ would never become zero:

Consuelo: It’s going to keep going, isn’t it? Afterwards it will be [in the decimal expansion of the length of the last segment] more zeros and more zeros, and more zeros and so it would never get to zero. And so we can use a condition that says that when we get to 5 decimal digits it should stop.

Verónica: Yes. So it is going to keep increasing each time the zeros to its decimal list. So it is never going to reach zero.

Consuelo: It is never going to reach zero.

The increase in the number of zeros in the decimal expansion justified the approach of the sequence to the limit zero, but the endlessness of the process (the process seen as potentially infinite) dominated, preventing the limit from ever being reached.

4.2 Episode 2: Koch Curve ‘paradoxes’: Solving an Indeterminate Case by Coordinating Two Simultaneous Infinite Processes

The two boys, Manuel, 17, and Jesús, 18, found the Koch curve explorations, and in particular, the idea of an infinite perimeter formed by an infinite number of “zero-length” segments initially confusing.

Jesús: I was analysing it, and the size of the segment becomes very small; but what always increases is the amount of segments, and that does go off until infinity so there are infinite segments ...

Manuel: Which measure zero ...

Jesús and Manuel: So they are points.

R: What about the perimeter?

Manuel: Well ... by watching its behaviour ... Why don't we use the formula as a guide?

Jesús: Well ..., by observing the numerical behaviour it should give us the idea that the perimeter will become very large.

R: How large?

Manuel: Well, if the number of terms is infinite, then it will be infinite.

Jesús: That's according to the numbers, to how it is growing ... But there is a problem: the number of segments increases, but they also become very small. And in fact we already saw that this function has a limit when it is $1/3^N$, it goes to zero.

Manuel: If there are infinite segments ...

Jesús: Then there is no perimeter.

Manuel: It would be 0.

Jesús: There wouldn't be a perimeter. It is like there wasn't any perimeter. It would be zero.

Manuel: What I say is that the segments would no longer be line segments, they would be points, and so it would no longer be a star-like shape, it would form a "curve", to call it something, but I don't know what shape it would have after that

Jesús was aware of the problem of having two types of processes involved in the change of the perimeter: the increase in the *number* of segments, and the decrease in the *size* of those segments. He realised that *the behaviour of the numerical values* pointed towards the perimeter becoming very large, infinite. But when they considered that the segments *at infinity* measured zero, this seemed to indicate to them that at infinity the perimeter would measure zero! In fact, by focusing on the latter process, Jesús challenged the idea of the divergence of the perimeter: "The segments are getting smaller ... The perimeter cannot be infinite ...". His partner had a different perspective: he focused more on how the zero-sized segments would affect the *shape* of the figure, initially concluding that it would become a "smooth" curve with no segments, "an infinite sequence of points":

Manuel: Then it will evidently be a curve. It wouldn't have segments. It would be a curve or a line ... It would be an infinite sequence of points.

Here the students were dealing with what is formally defined as an indeterminate case (an infinite number of segments of size zero: $\infty \times 0$). When Manuel decided to go back and look at the process from an algebraic perspective he soon discovered this:

Manuel: Better think of where we are going, an infinite number:

$100/3^{N-1}$, if N is infinite, then it is zero, right?

And 4^{N-1} , if N is infinite, it is infinite. And how much is zero by infinity?... Oh! How awful! What is infinity times zero?

Initially they were unable to resolve this situation, which proved a source of anxiety:

Manuel: I don't really know about the perimeter: one theory says it will be infinite, and the other that it is zero ... I can't even imagine it.

Jesús: The problem is we are multiplying an exaggeratedly small number by an exaggeratedly big number ...

After having carried out numeric and algebraic explorations during their explorations of the infinite sequences and series of the previous activities (e.g. $\{1/2^n\}$, etc.), Manuel and Jesús decided it was necessary to do the same in this case in order to solve the paradoxes. Thus, a breakthrough came in the next work session: Jesús brought with him a written list of conjectures for solving the paradoxical situation and was convinced the perimeter would tend to infinity. He wrote:

- 1) The perimeter will be infinite, because the length of the segments will never be equal to zero, and their number increases permanently.
- 2) Because the perimeter is obtained by multiplying the number of segments by their length, then a product is obtained where an extremely small number is multiplied by another that is extremely large, therefore it is always increasing.
- 3) The sequence $1/3^n$ tends to a limit but 4^n does not have one, therefore the perimeter must be infinite.
- 4) Perhaps we need to observe how much [multiplying by] $1/3^n$ reduces 4^n ; that is, how many decimal places move to the right after doing the product.

The first argument shows how Jesús conceived the process as only *potentially* infinite, and thus the limit zero of the size of the segments would never actually be reached. His reasoning was that if the number of segments of non-zero (no matter how small) measure increased, then the total must always increase without bounds; an argument which—however wrongly expressed—*intuitively* helped support his idea that the total length tended to infinity. A similar type of reasoning is shown in his second argument. In his third statement he seems to argue that because $1/3^n$ has a limit and 4^n does not, it is like multiplying something that tends to infinity by a finite value that would not change this tendency. But the most interesting, and apparently a conceptual breakthrough, is his last observation: he was interested in how each of the factors ($1/3^n$ and 4^n —the rate of decrease in the size of each segment vs. the rate of increase of the perimeter in the number of segments) behaved *in relationship to each other*. He recognised this is a key issue since he saw that it is the different rate at which each of the two sequences changes that determines the final outcome. Jesús was aware of this, as shown in his oral explanations below (these explanations also clarify his thoughts behind his second argument above). He used numerical explorations (see Table 6) to explore the behaviour of the perimeter, verify his hypothesis, and become convinced of the divergence of the perimeter (by observing that the perimeter's increase was faster than the segments' convergence to zero).

The transcript below illustrates how Jesús gave meaning to the two processes under study (the one that defines the length of the segments that form the Koch curve vs. the one that defines the number of segments) leading to the conclusion, expressed in the context of the exploration that they have different rates of convergence.

Jesús: Yes, I now have the total conviction that the perimeter of the curve is infinite. I was analysing what happened with both elements in the product:

On the one hand, the length of the segment: even if N is very, very large, that function, $L/3^{N-1}$, never gets to be equal to zero. It would always be an extremely small number.

Table 6 Table constructed by Manuel and Jesús in their investigation of the Koch curve. With the use of this table, these students were able to deduce the algebraic formula for the measure of the perimeter of the Koch curve at a level N

For L = 100				
Level	Size of each segment	Number of segments	Total distance (Perimeter)	=Total
1	100	1	100	100
2	100/3	4	(100/3) * 4	133.3 $\bar{3}$
3	100/9 (3 ²)	4 × 4 = 16	(100/9) * 16	177.7 $\bar{7}$
4	100/27 (3 ³)	16 × 4 = 4 ³ = 64	(100/27) * 64	237.0 $\bar{37}$
5	100/81	256	(100/81) * 256	316.0493827
N	$\frac{100}{3^{N-1}}$	4N-1	$\left(\frac{100}{3^{N-1}}\right) \cdot (4^{N-1})$	
	$\frac{L}{3^{N-1}}$		$\left(\frac{L}{3^{N-1}}\right) \cdot (4^{N-1})$	
100			233848680765595.64783	

And the other element, which is 4^{N-1} , that is going to be a very, but very, very large ..., too large, way too large, number ... So the number of segments is increasing, and it will be multiplied by a very small number, which will reduce it a bit, but the increase is more than the decrease ... so even though the segments are extremely small, the perimeter will always increase.

So that is why I say that the perimeter will be infinite.

He continued by explaining how he had used a calculator to verify his hypothesis:

I even did some computations using a scientific calculator, and I was able to get as far as 320 [for N], and that number is already very, very big. So with that result and the reasoning I did, I can say that the perimeter will be infinite.

Finally, the problem of dividing by an infinite value (an indeterminate case) is expressed explicitly.

So the segments will be very small, but they are never going to be equal to zero. The limit is supposedly zero, but that is when we divide by infinity or an infinite value, and that cannot be done.

It is interesting to observe how these two students conceived the formula for the length of the perimeter—as the size of each segment (determined by $1/3^N$) multiplied by the number of segments (4^N)—in that they did not abstract their reasoning from the resulting formula. That is, they did not consider that from a purely algebraic perspective it can be deduced that $\frac{1}{3^N} \cdot 4^N = \frac{4^N}{3^N} = \left(\frac{4}{3}\right)^N$ which solves the indeterminate case; and, since $\frac{4}{3} > 1$, the length of the perimeter clearly diverges as N tends to infinity.

Whereas the indeterminate forms of limits are solved traditionally through algebraic manipulation, in this case Jesús overcame the indeterminacy through analysis of the *behaviour* of each of the elements involved, observing specifically the difference in the *rate* of divergence or convergence of each of the elements and coordinating the two processes involved. This leads to the question of whether this type of analysis could perhaps help solve the intellectual misgivings that a mere algebraic proof does not resolve.

On the other hand, his partner, Manuel, still had a conflict between what his intuitions told him, and his attempt to apply (finite) mathematics and “logical” principles (“a number multiplied by zero is zero” vs. “a number multiplied by infinity is infinite”), and his confusions would resurface during the explorations of the Koch Snowflake perimeter: “*if the number is infinite the perimeter is zero, and what will happen? That all of this will become a point!*” He is considering that at infinity, the segments forming the curve would measure zero implying a sort of “collapse” of the curve into a point. His difficulty is related to the epistemological obstacle described by Cornu (1986) of “the passage to the limit” where “that which happens at infinity” seems to be isolated from the dynamic limiting process.

The situation faced by these students is analogous to Zeno’s paradoxes. There are, correspondingly, two components present here: the *number* of segments, and the *measure* of the segments. As in Zeno’s paradoxes, the construction of the Koch curve involves infinite subdivisions of the continuum, and the problem thus touches on many mathematical areas related to the infinite: limits of infinite processes, infinite sets, the nature of the continuum. Manuel wanted to conceive the infinite process as completed, considering that at that point the segments forming the curve would measure zero, implying a sort of “collapse” of the curve into a point. For Manuel, it would take a long process of (particularly numeric and algebraic) explorations and reflections to become convinced of the divergence of the perimeter and some of his doubts may not have been clearly resolved. Interestingly, Manuel had no difficulty with the bounded area of the snowflake which he attributed to the *shape* of the figure being such that the perimeter simply folds up as it increases, not letting the area grow any further.

Jesús on the other hand, did not accept that the segments could ever *equal* zero; for him the segments would become very, very (perhaps infinitesimally) small, but never zero. Jesús had a potential view of the process. But what is important in his approach is that he took into account both of the processes present and considered the idea that “relative to the perimeter” the segments would never be zero: he considered that the perimeter’s increase was faster than the segments’ convergence to zero.

Overall, this episode illustrates how, by constructing for themselves, not only the object under study itself (the Koch curve), but also the elements to analyse and study that object (e.g. the table of values and the algebraic generalizations contained within it), Jesús and Manuel were able to articulate and understand these representations, and coordinate the meanings they derived from their use. By coordinating their visual observations, their numeric explorations and their particular conjectures about what happens to each element involved (e.g. the number of segments vs. the size of the segments) they were able to solve the paradoxical situation they faced.

5 Discussion and Concluding Remarks

We now draw together some of the key issues that emerge from the two illustrative episodes, as well as from the wider corpus of data that is reported fully in Sacristán (1997).

There were three main representational elements involved in the microworld: symbolic (the programming code), visual/graphical (geometric figures), and numeric (tables of numerical values). What characterised the microworld, and what perhaps sets it apart from some other attempts to deal with different representations, is how these representations were articulated through the computer code, as illustrated in Fig. 6.

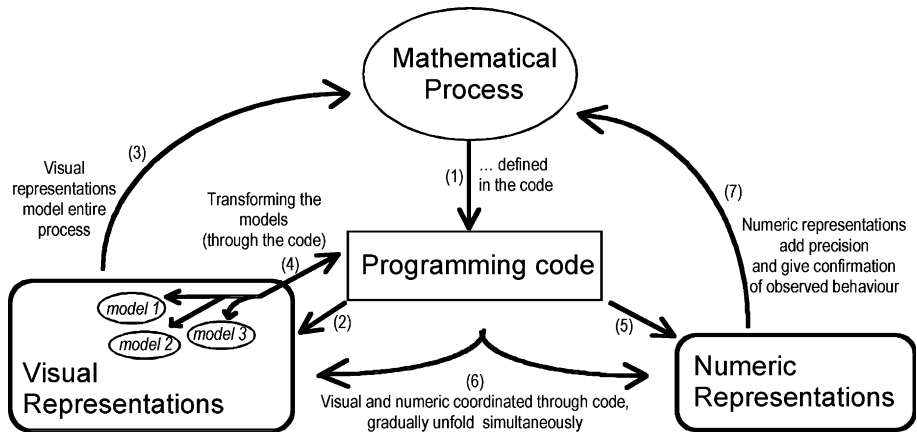


Fig. 6 The representational elements of the microworld and their interactions

Figure 6 is a schematic of a typical paradigm case of the use of the representational tools of the microworld. It is a theoretical structure underlying the design of the microworld, a design that was further refined as a product of the data analysis. In it, we see the following (refer to numbers in diagram): (1) the mathematical process is symbolically defined in the code; (2) by running the programming code, a visual representation is produced, which (3) models the entire process. Additionally, the visual representation is produced by the *movements* of the turtle, gradually unfolding: this allows the process to be observed sequentially and for its characteristic *behaviour* to be highlighted. (4) There are different visual models (e.g. spirals, bargraphs, steps, lines) that can be produced, each providing a different visual perspective on the process. The switch between models is effected through the code whose structure, and the process it defines, remain invariant: the models are *isomorphically* constructed. (5) Numeric values can also be produced through the same code; this links them to the visual representations and (6) both simultaneously gradually unfold. (Numeric values are also produced through complementary procedures—also representing the same process). (7) The numeric representations serve to add precision and give confirmation of the observed behaviour of the process.

This schematic description depicts the rather special role that the programming representation is called upon to play within the activities. The students built, modified and controlled the different representations through the code: beginning with the initial procedures such as those given in Sect. 3.2., they continuously changed the different procedures to test their conjectures, and to build new models and tools; this was illustrated briefly in the first episode when Verónica and Consuelo modified the DRAWING procedure with STOP, PRINT, or COUNT commands. We have argued elsewhere that a significant point in favour of programming (see, for example, Noss and Hoyles 1996, Chap. 3), is that it entails explicit and rigorous expression of relationships: the learner can describe mathematical objects and relationships in a language that can be communicated, extended, and reflected upon. This means that one can focus on the structures, relationships and allowable actions on objects, at the same time as engaging with the figurative or physical features of the computational objects (or rather, computational instantiations of mathematical objects) under consideration.

The general case for programming is nicely summed up by diSessa (2000) who argues that it “turns analysis into experience and allows a connection between analytic forms and

their experiential implications that algebra and even calculus can't touch" (ibid., p. 34). The notational systems of mathematics, developed over thousands of years using inert media, were designed for the establishment of mathematical truth: a purpose which suits mathematics well, but mathematics learners badly. The compactness and rigour of expression, together with the deliberate fragility of chains of mathematical reasoning, makes mathematical notation far removed from that which makes sense in everyday settings. It seems likely that this disjuncture between the norms of mathematical and everyday discourses, is responsible for at least some of the difficulties experienced by children in learning mathematics. Understanding the connection between analysis and experience takes us a step towards a more nuanced view of what may have occurred in the infinity microworld. diSessa draws attention to the functionality of programming for learning, not merely its ontological nature as a kind of alternative to algebra.

This functional view is supported by Edwards (1998) who makes a useful distinction between a structural definition of microworlds, which focuses on epistemological facets of microworld design, and a functional definition that points to the ways in which students may learn within them. We share her view that a fundamental facet of programming-based environments is the imperative of formalisation inherent within them, an imperative that simultaneously demands a degree of rigour that characterises the mathematical endeavour, and provides expressive power, in the sense that actions are expressed within a language (this language need not be text-based; see e.g. Noss and Hoyles 2006). We saw how this expressive power was articulated in the creation of numerical, graphical and programming representations. In fact, programmable worlds are only an instance of a more general class of *autoexpressive* environments, those in which the only way to manipulate and reconstruct objects is to express explicitly the relationships between them (see Noss and Hoyles 1996).

Our claim, then, is that the act of construction through articulation via the program is what provides the all-important coordination between representations, and this in turn, afforded students the possibility to coordinate their own, sometimes conflicting, knowledge of infinity. For example, the problem of applying the rules of the finite to the infinite (in which case the knowledge of the finite has a dominating role) or the influence of the context (e.g. the dominating influence of visual appearances, where the visual component is not coordinated properly with other factors present) was—at least to an extent—overcome in the relations expressed between the different representations, and the possibility of choosing the right representation to examine and reflect upon how the rules might work. Similarly, we discussed the intuition that links the idea of “more is bigger” with the idea that “things get infinitely big if you add long enough”, leading to paradoxes in the presence of bounded infinite processes. The evidence of the episodes above, considered alongside the wider corpus of data, indicates that as the students worked with the different representations they were able to form a connection: for instance, as in the first episode, giving meaning to the visual boundaries and the infinite nature of the process through the numeric decimal representations.

We thus argue that a powerful way in which the microworld assisted in the construction of helpful mental representations for infinity, was in its affordance for expressing and representing in different ways fragments of knowledge that were hitherto only weakly connected: it is evident that the students were able to express as situated abstractions their developing state of knowledge of the properties and relationships between the objects they created. Moreover, out of this expression emerged a coordination of these abstractions, a constructive articulation of diverse representational forms we name as *representational moderation*.

Representational moderation is sensitive to the direction of representations, or rather, to the trajectory between them. For example, whereas in the context of sequences and series, the approach was in the direction process to visual/numeric—which led to the belief that adding more would imply a larger figure/value—in the context of the fractal explorations, the approach began with the figure. In the latter cases, the dominating factor was the visual image (which was *visually* invariant), particularly with the younger (thus less experienced) students (possibly because these students were more likely to be influenced by the most salient feature of the representation). The achievement of a balance between these representations, the result of moderation, was facilitated by the structure of the system and, of course, by the activity structures presented to the students.

In cases where there was a procedural focus of the activities, where the processes (and procedures) were presented as potentially endless, this was accompanied by the prevalent view that the limits of convergent processes will never be reached—this is fairly straightforward to explain, since, as Tall and Vinner (1981), point out, the dynamic model of limit, with the limit as unreachable, tends to be the prevalent conception among students, particularly when the limit is expressed as the result of an infinite *process*. The students who participated in the study were no exception.

Nevertheless, it is also interesting that some of the students (Manuel and Jesús) acknowledged explicitly that infinity is different from “a very large number” (a common confusion in children—see Monaghan 2001) and that it is not an “amount”, even if they were not quite sure how to define it. They also seemed to be clear that the infinite cannot be quantified in the way that finite quantities can (e.g. “we cannot say ‘this is half of infinity’”) which indicates their (new?) awareness that finite operativity and logic cannot be applied to the infinite—the same realisation that Galileo had more than 300 years ago (quoted at the beginning of this paper).

A final point about representational moderation is that it elaborates a little further the precise role that programming can play. This involves a generalisation of the body syntonic view, suggested by Papert (1980)—which he derived from Freud—that being able to identify oneself as part of the action is a powerful way to make sense of experience. In fact, the articulation of representations through the programming code generalises this from the purely visual element, to the numerical and the programming representations themselves: precisely because the learner is constructing these representations themselves, they can identify with what is happening at the ‘tip’—of the running procedure, the emerging figure, or the evolving table of numerical values.

References

- Abramovich, S., Brantlinger, A., & Norton, A. (1999). Exploring quadratic-like sequences through a tool kit approach. In G. Goodell (Ed.), *Proceedings of the 11th Annual International Conference on Technology in Collegiate Mathematics*. Reading, MA: Addison Wesley Longman.
- Balacheff, N. (1994). La transposition informatique. Note sur un nouveau problème pour la didactique. In M. Artigue, et al. (Eds.), *Vingt ans de didactique des mathématiques en France* (pp. 364–370). Grenoble: La Pensée Sauvage.
- Cornu, B. (1986). Les Principaux Obstacles à l’Apprentissage de la Notion de Limite. *Bulletin IREM-APMEP de Grenoble*.
- Cornu, B. (1991). Limits. In D. O. Tall (Ed.), *Advanced mathematical thinking* (pp. 153–166). Dordrecht: Kluwer Academic Publishers.
- Cuoco, A., & Goldenberg, P. (1992). Mathematical induction in a visual context. *Interactive Learning Environments*, 2(3–4), 181–203.
- diSessa, A. (2000). *Changing minds: Computers, learning, and literacy*. Cambridge, MA.: MIT Press.

- Dreyfus, T., & Eisenberg, T. (1990). On difficulties with diagrams: Theoretical issues. In G. Booker, P. Cobb, & T. N. De Mendicuti (Eds.), *Proceedings of the 14th International Conference on the Psychology of Mathematics Education* (Vol. 1, pp. 27–31). Mexico.
- Dugdale, S. (1998). A spreadsheet investigation of sequences and series for middle grades through pre-calculus. *Journal of Computers in Mathematics and Science Teaching*, 17(2–3), 203–222.
- Duval, R. (1999). Representations, vision and visualization: Cognitive functions in mathematical thinking. Basic issues for learning. In F. Hitt & M. Santos (Eds.), *Proceedings 21st Conference of the North American Chapter of the International Group for the Psychology of Mathematics Education* (Vol. 1, pp. 3–26). Columbus, Ohio: ERIC Clearinghouse for Science, Mathematics, & Environmental Education.
- Edwards, L. (1998). Embodying mathematics and science: Microworlds as representations. *The Journal of Mathematical Behavior*, 17, 53–78.
- Espinoza, L., & Azcárate, C. (1995). A study on the secondary teaching system about the concept of limit. In *Proceedings of the 19th International Conference for the Psychology of Mathematics Education* (Vol. 2, pp. 11–17). Recife, Brazil.
- Ferrari, E., Laganà, A., Luzi, E., & Trovini, E. (1995). Il Concetto di Infinito nell' Intuizione Matematica. *L'Insegnamento della Matematica e delle Scienze Integrate*, Paderno, Italy, 18B(3), 211–235.
- Fischbein, E. (1987). *Intuition in science and mathematics*. Dordrecht, Holland: Reidel.
- Fischbein, E., Tirosh, D., & Hess, P. (1979). The intuition of infinity. *Educational Studies in Mathematics*, 10, 3–40.
- Galileo Galilei (1954). *Dialogues concerning two new sciences* (trans: Crew, H., & de Salvio, A.). New York: Dover Publications (Original work published 1638).
- Garbin, S., & Azcárate, C. (2002). Infinito Actual e Inconsistencias: Acerca de las incoherencias en los esquemas conceptuales de alumnos de 16–17 años. *Enseñanza De Las Ciencias*, 20(1), 87–113.
- Harel, I., & Papert, S. (Eds.) (1991). *Constructionism*. Norwood, NJ: Ablex Publishing Corporation.
- Harvey, B. (1997). *Computer science logo style, Vols. 1–3*. Cambridge, MA: MIT Press.
- Imaz, C. (1991). Infinitesimal models for real analysis. *International Journal of Mathematical Education in Science and Technology*, 22, 2.
- Jones, C. V. (1987). Las paradojas de Zenón y los primeros fundamentos de las matemáticas. *Mathesis*, 3(1), 3–14.
- Keisler, J. (1986). *Elementary calculus: An approach using infinitesimals* (2nd ed.). Boston, MA: Prindle, Weber & Schmidt. Retrieved 31 October, 2004, from <http://www.math.wisc.edu/~keisler/calc.html>.
- Kissane, B. (2003). The calculator and the curriculum: The case of sequences and series. In W.-C. Yang, S.-C. Chu, T. de Alwis, & M.-G. Lee (Eds.), *Proceedings of the 8th Asian Technology Conference in Mathematics: Technology Connecting Mathematics* (pp. 357–366). Hsinchu, Taiwan: ATCM Inc.
- Lakoff, G., & Núñez, R. (2000). *Where mathematics comes from: How the embodied mind brings mathematics into being*. New York: Basic Books.
- Mamona-Downs, J. (1987). *Students interpretations of some concepts of mathematical analysis*. Doctoral Dissertation, Faculty of Mathematical Studies, University of Southampton.
- Monaghan, J. (2001). Young people's ideas of infinity. *Educational Studies in Mathematics*, 48, 239–257.
- Monaghan, J., Sun, S., & Tall, D. O. (1994). Construction of the limit concept with a computer algebra system. In J. P. da Ponte & J. F. Matos (Eds.), *Proceedings of the 18th International Conference on the Psychology of Mathematics Education* (pp. 279–286). Lisboa.
- Moreno, L., & Waldegg, G. (1991). The conceptual evolution of actual mathematical infinity. *Educational Studies in Mathematics*, 22(5), 211–231.
- Nelson, E. (1977). Internal set theory: A new approach to nonstandard analysis. *Bulletin of the American Mathematical Society*, 83(6), 1165–1198.
- Noss, R., & Hoyles, C. (1996). *Windows on mathematical meanings. Learning cultures and computers*. Dordrecht: Kluwer Academic Press.
- Noss R., & Hoyles, C. (2006). Exploring mathematics through construction and collaboration. In K. R. Sawyer (Ed.), *Cambridge handbook of the learning sciences* (pp. 389–405). Cambridge: Cambridge University Press.
- Núñez E., R. (1993). *En Deçà du Transfinito. Aspects psychocognitifs sous-jacents au concept d'infini en mathématiques* (Vol. 4. Fribourg, Suisse: Éditions Universitaires).
- Papert, S. (1980). *Mindstorms: Children, computers, and powerful ideas*. New York: Basic Books.
- Robinson, A. (1974/1996). *Non-standard analysis*. Revised edition. Princeton, NJ: Princeton University Press. (Original work published 1974).
- Sacristán, A. I. (1991). Los Obstáculos de la intuición en el aprendizaje de procesos infinitos. *Educación Matemática*, 3(1), 5–18.

- Sacristán, A. I. (1997). *Windows on the infinite: Creating meanings in a logo-based microworld*. Doctoral Dissertation, Institute of Education, University of London, UK. Retrieved 8 December, 2007, from <http://www.matedu.cinvestav.mx/~asacristan/PhDSacristan97.pdf>.
- Schwarzenberger, R. L. E., & Tall, D. O. (1978). Conflicts in the learning of real numbers and limits. *Mathematics Teaching*, 82, 44–49.
- Sierpinska, A. (1987). Humanities students and epistemological obstacles related to limits. *Educational Studies in Mathematics*, 18, 371–397.
- Sierpinska, A., & Viwegier, M. (1989). How and when attitudes towards mathematics and infinity become constituted into obstacles in students? In *Proceedings of the 13th International Conference on the Psychology of Mathematics Education* (Vol. 3, pp. 166–173). Paris, France.
- Struik, D. J. (1967). *A concise history of mathematics*. New York: Dover Publications.
- Tall, D. O. (1980). The notion of infinite measuring number and its relevance in the intuition of infinity. *Educational Studies in Mathematics*, 11, 271–284.
- Tall, D. O. (1986). Building and testing a cognitive approach to the calculus using interactive computer graphics. Doctoral dissertation. University of Warwick, UK.
- Tall, D. (2001). Natural and formal infinities. *Educational Studies in Mathematics*, 48, 199–238.
- Tall, D., & Vinner, S. (1981). Concept image and concept definition in mathematics with particular reference to limits and continuity. *Educational Studies in Mathematics*, 12, 151–169.
- Tirosh, D., & Tsamir, P. (1996). The role of representations in students' intuitive thinking about infinity. *International Journal of Mathematics Education in Science and Technology*, 27(1), 33–40.
- Tsamir, P. (1999). The transition from the comparison of finite to the comparison of infinite sets: Teaching prospective teachers. *Educational Studies in Mathematics*, 38(1–3), 209–234.
- Tsamir, P. (2001). When 'The Same' is not perceived as such: The case of infinite sets. *Educational Studies in Mathematics*, 48, 289–307.
- Waldegg, G. (1987). *Esquemas de Respuesta ante el Infinito Matemático. Transferencia de la Operatividad de lo Finito a lo Infinito*. Doctoral dissertation. Centre for Research and Advanced Studies, Mexico.
- Waldegg, G. (1993). La comparaison des ensembles infinis: un cas de résistance à l'instruction. *Annales de Didactique et de Sciences cognitives*, 5, 19–36.
- Wilensky, U. (1991). Abstract meditations on the concrete, and concrete implications for mathematics education. In I. Harel & S. Papert (Eds.), *Constructionism* (pp. 193–204). Norwood, NJ: Ablex Publishing Corporation.