A Core-Partition Ranking Solution to Coalitional Ranking Problems

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Abstract

A coalitional ranking problem is described by a weak order on the set of nonempty coalitions of a given agent set. A social ranking is a weak order on the set of agents. We consider social rankings that are consistent with stable/core partitions. A partition is stable if there is no coalition better ranked in the coalitional ranking than the rank of the cell of each of its members in the partition. The core-partition social ranking solution assigns to each coalitional ranking problem the set of social rankings such that there is a core-partition satisfying the following condition: a first agent gets a higher rank than a second agent if and only if the cell to which the first agent belongs is better ranked in the coalitional ranking than the cell to which the second agent belongs in the partition. We provide an axiomatic characterization of the core-partition social ranking and an algorithm to compute the associated social rankings.

Keywords Coalitional ranking problem · Social ranking · Core partition · Axiomatic characterization · Hedonic games

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1 Introduction

The coalitional ranking problem refers to the problem of finding an ordinal ranking over the set of agents (called social ranking), given an ordinal ranking over its power set (called coalitional ranking). Such problems and their social ranking solutions have been recently investigated by Khani et al. (2019), Bernardi et al. (2019), Algaba et al. (2021) and Béal et al. (2022). These studies consider single-valued solutions, that is, social ranking solutions that assign to each coalitional ranking of the considered domain a unique social ranking. In this article, we are interested in set-valued solutions, that is, social ranking solutions that assign to each coalitional ranking problem of the domain a set, possible empty, of social rankings.

A social ranking solution can be viewed either as the ordinal counterpart of a solution for cooperative games with transferable utility – where for each payoff vector, agents are ranked according to the payoff they receive for their participation in the game – or as the inverse of the well-known problem of ranking groups of objects from a ranking over the individual objects (see, e.g., Barberà et al. 2004).

We propose a set-valued social ranking solution derived from a no-blocking condition inherent to the concept of core in many social environments. Precisely, our solution is constructed from the idea that the population of agents can be socially organized into a partition. Given a coalitional ranking problem, a partition is blocked by a coalition if the latter has a strictly higher rank in the coalitional ranking than the cells of the partition to which its members belong. A partition that is not blocked by any coalition is a stable partition or a core-partition. Any partition induces a natural social ranking: an agent is better ranked than a second agent if the cell to which the first agent belongs is better ranked in the coalitional ranking than the cell to which the second agent belongs. A social ranking is a core-partition social ranking if it is induced by a core-partition. The Core-partition social ranking solution introduced in this article assigns to each coalitional ranking the set of all core-partition social rankings. In other words, each selected social ranking is induced by an organization of the population into a partition that cannot be blocked by any coalition.

We consider the domain of all coalitional ranking problems that can be constructed from any finite agent set, that is, a domain of coalitional ranking problems with a variable set of agents. The Core-partition social ranking solution is non-empty-valued on this domain.

Our main result is an axiomatic characterization of the Core-partition social ranking solution which invokes three axioms.

The first axiom is based on a specific expansion of the population of agents. So consider an initial coalitional ranking problem. Assume now that the population is expanded in such a way that the coalition of all newly added agents wins unanimous support: this coalition is the unique maximal coalition in the larger coalitional ranking, the relative ranking between any two original coalitions remains the same, and each other coalition with both new and original agents has an arbitrary ranking. Then, our axiom imposes that the solution set of this new coalitional ranking on a larger set of agents is computed from the solution set of the initial one by putting the set of newcomers at the top of each social ranking of this solution set. In a sense, if

a set of newcomers wins unanimous support in a coalitional ranking, then they are socially top-ranked. This axiom both has the flavor of some consistency-type axioms (if one starts from the larger problem and if the coalition of top-ranked agents is removed from it) and of the so-called bracing lemma which also relates the solution sets before and after a similar population expansion. On these points, we refer to Thomson (2011) for more details.

The second axiom is an invariance axiom built from a monotonicity condition. Imagine that in a coalitional ranking problem two disjoint coalitions have the same rank and their union lies in their lower counter set, i.e., the union of these coalitions has a lower rank than their constituent. Hence, the two coalitions have no interest in merging. The axiom indicates that the solution set is invariant to any monotonic transformation of the coalitional ranking induced by an improvement of this union, provided that this union remains in the lower contour set of the two original coalitions. This axiom is reasonable because the (equal) individual performance of these two coalitions is always at least as good as that of their union.

The last axiom is based on a decomposition principle. It specifies the conditions under which the solution set of a coalitional ranking problem can be decomposed as the union of the solution sets of variants of this coalitional ranking when some specific top coalitions of the coalitional ranking are each promoted as the only top coalition, ceteris paribus. Such a decomposition is possible according to the axiom if the set of top coalitions is closed/stable by union of disjoint coalitions.

The rest of the article is organized as follows. Section 2 presents the connections between our model, other approaches to the coalitional ranking problems and other literatures. In particular, we explain what distinguish our approach from some results on hedonic games. Section 3 gives basic notation and definitions. Section 4 presents core-partitions and the core-partition social ranking solution and provides some of their properties. The axiomatic study is contained in Sect. 5. Section 6 proves that the three axioms invoked in our characterization are logically independent. Section 7 provides an intuitive algorithm which computes all core-partitions. Section 8 concludes.

2 Related Literature

On the coalitional ranking problem, Khani et al. (2019) introduce and axiomatically characterize a social ranking solution that is inspired from the Banzhaf index for cooperative voting games (Banzhaf 1964). Bernardi et al. (2019) and Algaba et al. (2021) study lexicographic solutions based on the idea that the most influential agents are those which belong to (small) coalitions ranked in the highest position in the coalitional ranking, and provide several axiomatic characterizations of these solutions. Béal et al. (2022) characterize two other lexicographic solutions: the resulting social rankings are computed from the individual performance of the agents, then, when this performance criterion does not allow to decide between two agents, a collective performance criterion is progressively applied to the coalitions of higher size. As we already underlined, the main differences with our approach are

that these solutions are single-valued whereas we consider set-valued solutions and that our framework allows for a variable population of agents.

A coalitional ranking can also be seen as a specific hedonic game. In a hedonic game, each agent is endowed with a preference (a weak order) over the set of coalitions to which it can belong and a popular objective is to find core-partitions. In order to do so, a coalition blocks a partition if all its members prefer this coalition to their respective cell of the partition. A partition is stable if it cannot be blocked by any coalition. A coalitional ranking problem can be considered as a specific hedonic game in which the preferences of any pair of agents agree when they compare two coalitions containing these two agents. This property, introduced in Farrell and Scotchmer (1988),[1] is sufficient for the nonemptiness of the set of core-partitions. The core of a hedonic game has received much attention in the literature. For instance, Banerjee et al. (2001) relax the common ranking property. They introduce two properties of top coalitions, which are sufficient to ensure the nonemptiness of the core of a hedonic game. Iehlé (2007) provides a necessary and sufficient condition under which the core of a hedonic game is nonempty. This condition is based on a concept of pivotal balancedness. Karakaya and Klaus (2017) consider the domain of hedonic games with strict preferences. They offer, among other (im)possibilities results, an axiomatic characterization of the core on the subdomain of hedonic games with a nonempty core in terms on Maskin monotonocity and Coalitional unanimity. In this context, Maskin monotonocity indicates that if a partition is selected by the solution set at some hedonic game, then it is also selected at a hedonic game where this partition improved in the preference ranking of the agents, ceteris paribus. Coalitional unanimity requires that a coalition which is unanimously best for all its members is always part of a partition of the solution set. Even if the literature on hedonic games (with the common ranking property) and our work are build around the concept of core-partitions, the similarities end there. We do not axiomatize the set of core-partitions but the set of social rankings that are consistent with core-partitions in the sense explained in the introduction. As a consequence, even if some axioms from both literatures seem to have the same flavor, they do not deal with the same objects.

The frameworks of hedonic games with the common ranking property and coalitional ranking problems are merged into a new class of coalition formation games by Lucchetti et al. (2022). They focus on core-partitions in various settings in which the agents' preferences over coalitions depend on both the coalitional ranking and the agents' position according to a social ranking within each coalition. In one of these settings, these two criteria are used lexicographically: among two coalitions containing an agent, this agent prefers the first coalition to the second if the former is better ranked than the latter in the coalitional ranking or if the two coalitions have the same coalitional rank and the social ranking solution better ranks the agent when it is applied to the first coalition than to the second. In this setting, core-partitions may not exist. The authors provide algorithms which determine core-partitions when they exist.

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[1] It is is called the common ranking property in Banerjee et al. (2001).

The concept of preference-approval structure introduced by Brams and Sanver (2009) has some distant link with our approach. A preference-approval structure is a pair consisting of a social ranking over a finite set of alternatives and a subset of approved or acceptable alternatives such that any approved alternative should be ranked strictly higher than any disapproved alternative in the social ranking. A parallel with social rankings induced by partitions can be made as follows. Consider the bipartition that distinguishes the approved alternatives from the disapproved alternatives. If a coalitional ranking were to exist on all non-empty subsets of alternatives, then it would be reasonable to assume that the coalition of approved alternatives is better ranked than the coalition of disapproved alternatives. Then, the condition that any approved alternative should be ranked strictly higher than any disapproved alternative in the social ranking is exactly what our (core)-partition social ranking solution would impose from the aforementioned bipartition and the hypothetical coalitional ranking. Dong et al. (2021) axiomatize a distance function between preference-approval structures. This function is then used in a preference aggregation model where each agent is endowed with its own preference-approval structure.

Finally, the approach in Piccione and Razin (2009) also shares some similarities to ours. The authors investigate social rankings emerging from core-partitions associated with a coalitional ranking. Nonetheless, there are several differences between the two articles. Firstly, Piccione and Razin (2009) consider coalitional rankings which are strict total orders while we consider weak orders. Secondly, they impose a separability condition on their coalitional rankings: for four disjoints coalitions, if a first coalition is ranked higher that a second coalition and if a third coalition is ranked higher than a fourth coalition, then the union of the first and third coalition is ranked higher than the union of the second and fourth coalition. Contrary to this rather narrow class of coalitional rankings, we allow for all weak orders over the set of nonempty coalitions. Thirdly, the ranking induced by a partition used in Piccione and Razin (2009) is consistent with the ranking of the cells of the partition but they further decide between two agents belonging to the same cell by comparing the corresponding singletons. As a consequence, a social ranking in Piccione and Razin (2009) is always a strict total order on the population of agents while we allow for weak orders. Fourthly, the stability concept adopted by Piccione and Razin (2009) is based on a blocking condition that is weaker than the classical one, so that the set of their stable social rankings is larger than the set of core-partitions. The authors' main results are a full description of the (nonempty) set of partitions that are stable under all the coalitional rankings in their specific class and an axiomatic characterization of this set.

3 Notation and Definitions

For any finite set A, denote by Ω_A the set of nonempty subsets of A and by $\mathcal{R}(A)$ the set of weak orders on A, i.e., the set of all reflexive, transitive and complete binary relations on A. Given a weak order $\succsim \in \mathcal{R}(A)$, if A is nonempty, $M(A, \succsim)$ represents its nonempty set of maximal/top elements, i.e., $M(A, \succsim)$ is the greatest/best

equivalence class with respect to the quotient order. For any subset B of A, $\succsim_B$ stands for the restriction of $\succsim$ to the subset B, i.e.,

$$\forall x, y \in B, \quad x \succsim_B y \iff x \succsim y.$$

Let $\mathbb{N}$ be the universe of **agents** and $\mathcal{F}$ be the collection of all finite sets of $\mathbb{N}$. For any set of agents $N \in \mathcal{F}$, any element of Ω_N is a called a **coalition**. A **coalitional ranking problem** is a pair $(\Omega_N, \succsim)$ where $N \in \mathcal{F}$ is the agent set and $\succsim \in \mathcal{R}(\Omega_N)$ is the coalitional ranking. For two coalitions $S, T \in \Omega_N$, $S \succsim T$ means that S is at least as highly ranked as T in $\succsim$. The asymmetric and symmetric part of $\succsim$ are denoted by $\succ$ and $\sim$ respectively. For a coalitional ranking $\succsim \in \mathcal{R}(\Omega_N)$ and a coalition $S \in \Omega_N$, the **lower contour set** of $\succsim$ at S is the set

$$L(\succsim, S) = \{T \in \Omega_N : S \succsim T\}.$$

Denote by

$$\mathcal{R}_\Omega = \bigcup_{N \in \mathcal{F}} \mathcal{R}(\Omega_N),$$

the domain of coalitional rankings. A **social ranking** on N is a weak order $\succcurlyeq$ in $\mathcal{R}(N)$. In a similar way as above, $>$ denote the asymmetric part of $\succcurlyeq$ and $\cdot$ its symmetric part. Denote by

$$\mathcal{R}_\mathbb{N} = \bigcup_{N \in \mathcal{F}} \mathcal{R}(N)$$

the set of social rankings that one can construct from any finite set of agents of $\mathbb{N}$.

For any coalitional ranking problem $(\Omega_N, \succsim) \in \mathcal{R}_\Omega$ and any coalition $S \in \Omega_N$, the **coalitional ranking subproblem** induced by $(\Omega_N, \succsim) \in \mathcal{R}_\Omega$ and S is the restricted coalitional ranking $\succsim_S$ to the set Ω_S.

A **social ranking (set-valued) solution** is a correspondence $f : \mathcal{R}_\Omega \rightrightarrows \mathcal{R}_\mathbb{N}$ which assigns to each coalitional ranking problem $(\Omega_N, \succsim) \in \mathcal{R}_\Omega(N)$ a possibly empty set of social rankings $f(\Omega_N, \succsim) \subseteq \mathcal{R}(N)$. If $N = \emptyset$, then $\Omega_N = \emptyset$ and $\mathcal{R}(N) = \mathcal{R}(\Omega_N) = \{\emptyset\}$. In such a case, one uses the convention that $f(\Omega_\emptyset, \emptyset) = \{\emptyset\}$. The social ranking solution f is **nonempty-valued** on $\mathcal{R}_\Omega$ if it holds that $f(\Omega_N, \succsim) \neq \emptyset$ for each $(\Omega_N, \succsim) \in \mathcal{R}_\Omega(N)$.

4 Core-Partitions and the Core-Partition Ranking Solution

Core partitions of a coalitional ranking problem

Given $N \in \mathcal{F}$, a **partition** on N is a set $P = \{P_1, \dots, P_k\}$ of mutually disjoint nonempty coalitions that covers N, i.e., P is such that

$$\bigcup_{q \in \{1,\dots,k\}} P_q = N \quad \text{and} \quad \forall q, r \in \{1, \dots, k\}, q \neq r, \quad P_q \cap P_r = \emptyset.$$

For a given agent $i \in N$ and a partition P, $P(i)$ stands for the unique coalition/cell in P containing i. Denote by $\mathcal{P}(N)$ the set of partitions of N. For any two partitions $P = \{P_1, \ldots, P_k\}$ and $P' = \{P'_1, \ldots, P'_{k'}\}$ on N, P' is **coarser** than P if, for each $P'_q \in P'$, there is a nonempty subset $A_q \subseteq \{1, \ldots, k\}$ such that

$$P'_q = \bigcup_{r \in A_q} P_r.$$

Given a coalitional ranking problem $(\Omega_N, \succsim) \in \mathcal{R}$ and a partition $P \in \mathcal{P}(N)$, one says that P is **blocked by coalition** $S \in \Omega_N$ if,

$$\forall i \in S, \quad S \succ P(i).$$

In words, P is blocked by S if, for each agent in S, coalition S is ranked higher than the coalition/cell in P containing this agent. The interpretation is that if agents prefer to be assigned to the best possible coalitions, then each one prefers to be in S than in the cell that the partition assigns to each of them. In this sense, P is not stable. A partition P is a **core-partition** if it is stable, i.e., if it cannot be blocked by any nonempty coalition. Equivalently, P is a core-partition if, for each $S \in \Omega_N$, there is $i \in S$ such that $P(i) \succsim S$. Denote by $\mathcal{CP}(\Omega_N, \succsim) \subseteq \mathcal{P}(N)$ the set of core-partitions of the coalitional ranking problem $(\Omega_N, \succsim)$.

Example 1 Assume that $N = \{1, 2, 3, 4, 5\}$. Consider the coalitional ranking problem $(\Omega_N, \succsim)$ such that

$$\{1, 2\} \sim \{2, 3\} \succ \{4, 5\} \sim \{3, 4\} \succ N \sim \{2\} \sim \{3\} \succ \{1\} \succ S \sim T,$$

for each other pair of coalitions $\{S, T\}$. This coalitional ranking admits five equivalence classes and

$$M(\Omega_N, \succsim) = \{\{1, 2\}, \{2, 3\}\}.$$

Its set $\mathcal{CP}(\Omega_N, \succsim)$ of core-partitions contains three elements:

$$P = \{\{1, 2\}, \{4, 5\}, \{3\}\}, \quad P' = \{\{2, 3\}, \{4, 5\}, \{1\}\}, \quad P'' = \{\{1, 2\}, \{3, 4\}, \{5\}\}.$$

Example 2 Let $N = \{1, 2, \ldots, n\}$, and consider the coalitional ranking problem $(\Omega_N, \succsim)$ where coalitions are ranked according to the smallest index they contain:

$$\forall S, T, \in \Omega_N, \quad (S \succsim T) \iff (\min(S) \leq \min(T)).$$

Hence S is ranked at the $\min(S)$-th equivalence class of $(\Omega_N, \succsim)$, where the best equivalence class contains all coalitions $S \ni 1$ and the worst equivalence class is the singleton $\{n\}$. For a nonempty coalition $T \subseteq N$, consider $i = \min(T)$. For any $S \ni i$, $S \succsim T$, from which one concludes that $\mathcal{CP}(\Omega_N, \succsim) = \mathcal{P}(N)$.

Remark 1 The notion of core-partition has been extensively used in coalition formation games. It is known that for $N \neq \emptyset$, $\mathcal{CP}(\Omega_N, \succsim) \neq \emptyset$ (see, e.g., Farrell and Scotchmer, 1988, Banerjee et al. 2001). The computation of $\mathcal{CP}(\Omega_N, \succsim)$ and the discussion

on the nonemptiness of core partitions (already mentioned in the introduction) are postponed to section 7.

4.1 The Core-partition social ranking solution

Let us now establish a link between partitions and social rankings. Each coalitional ranking problem $(\Omega_N, \succsim) \in \mathcal{R}_\Omega$ and each partition $P \in \mathcal{P}(N)$ induce a social ranking in $\mathcal{R}(N)$ denoted by $\succ_{(\succsim, P)}$ such that

$$\forall i, j \in N, \quad i \succ_{(\succsim, P)} j \iff P(i) \succsim P(j).$$

In words, the agents are ranked consistently with the rank of their cell. We are interested in social rankings that emerge from core-partitions. Formally, a **core-partition social ranking** for a coalitional ranking problem $(\Omega_N, \succsim) \in \mathcal{R}_\Omega$ is a social ranking $\succ \in \mathcal{R}(N)$ such that there is a core-partition $P \in \mathcal{CP}(\Omega_N, \succsim)$ satisfying $\succ = \succ_{(\succsim, P)}$. Denote by $\mathcal{CSR}$ the social ranking solution which assigns to each coalitional ranking problem $(\Omega_N, \succsim)$ the set of its core-partition social rankings $\mathcal{CSR}(\Omega_N, \succsim)$. By Remark 1, we know that $\mathcal{CSR}$ is nonempty-valued on $\mathcal{R}_\Omega$.

Example 3 Consider again Example 1. Using $\mathcal{CP}(\Omega_N, \succsim)$, the set $\mathcal{CSR}(\Omega_N, \succsim)$ of core-partition social rankings is formed by the following social rankings:

$$1 \cdot_{(\succsim, P)} 2 >_{(\succsim, P)} 4 \cdot_{(\succsim, P)} 5 >_{(\succsim, P)} 3, \quad 2 \cdot_{(\succsim, P')} 3 >_{(\succsim, P')} 4 \cdot_{(\succsim, P')} 5 >_{(\succsim, P')} 1,$$

and

$$1 \cdot_{(\succsim, P'')} 2 >_{(\succsim, P'')} 4 \cdot_{(\succsim, P'')} 3 >_{(\succsim, P'')} 5.$$

For a coalitional ranking problem $(\Omega_N, \succsim) \in \mathcal{R}_\Omega$ and a partition $P \in \mathcal{P}(N)$, define

$$M_P(\Omega_N, \succsim) = P \cap M(\Omega_N, \succsim)$$

as the set of cells of P that belong to the best equivalence class of $(\Omega_N, \succsim)$.

In the following, we establish properties for $M_P(\Omega_N, \succsim)$ and $\mathcal{CP}(\Omega_N, \succsim)$ that will be useful for the rest of the article. The first part of Proposition 1 indicates that $M_P(\Omega_N, \succsim)$ is nonempty if P is a core-partition; the second part of Proposition 1 indicates that a core-partition is consistent in the sense that by removing the cells of $M_P(\Omega_N, \succsim)$ from P, we get a core-partition of the coalitional ranking restricted to the set of remaining agents. Even if Part (ii) follows from Proposition 3 in Karakaya and Klaus (2017), we provide a short proof for completeness.

Proposition 1 *For each* $(\Omega_N, \succsim) \in \mathcal{R}_\Omega$ *and each* $P \in \mathcal{CP}(\Omega_N, \succsim)$, *it holds that*:

(i) *for each* $S \in M(\Omega_N, \succsim)$, *there is* $T \in M_P(\Omega_N, \succsim)$ *such that* $S \cap T \neq \emptyset$. *Thus,* $M_P(\Omega_N, \succsim) \neq \emptyset$;

(ii) $\quad P \backslash M_P(\Omega_N, \gtrsim) \in \mathcal{CP}\big(N \backslash (\cup_{S \in M_P(\Omega_N, \gtrsim)} S), \gtrsim_{N \backslash (\cup_{S \in M_P(\Omega_N, \gtrsim)} S)} \big)$.

Proof Fix any $(N, \gtrsim) \in \mathcal{R}_\Omega$ and any $P \in \mathcal{CP}(\Omega_N, \gtrsim)$.

Part *(i)*. Pick any nonempty $S \in M(\Omega_N, \gtrsim)$ and $P \in \mathcal{CP}(\Omega_N, \gtrsim)$. Assume that, for each $T \in M_P(\Omega_N, \gtrsim)$, $S \cap T = \emptyset$. It results that, for each $i \in S$, $P(i)$ does not belong to $M(\Omega_N, \gtrsim)$ so that $S \succ P(i)$. This implies that $P \notin \mathcal{CP}(\Omega_N, \gtrsim)$, a contradiction. Thus, if $P \in \mathcal{CP}(\Omega_N, \gtrsim)$, then, for each $S \in M(\Omega_N, \gtrsim)$, there is $T \in M_P(\Omega_N, \gtrsim)$ such that $S \cap T \neq \emptyset$. It follows that $M_P(\Omega_N, \gtrsim) \neq \emptyset$ for $P \in \mathcal{CP}(\Omega_N, \gtrsim)$.

Part *(ii)*. First observe that $P \backslash M_P(\Omega_N, \gtrsim)$ is a partition of $N \backslash (\cup_{S \in M_P(\Omega_N, \gtrsim)} S)$, where the set $\cup_{S \in M_P(\Omega_N, \gtrsim)} S \neq \emptyset$ by point (i) of Proposition 1. The case in which $N \backslash (\cup_{S \in M_P(\Omega_N, \gtrsim)} S)$ is empty is trivial. So, assume that there is a nonempty coalition $T \subseteq N \backslash (\cup_{S \in M_P(\omega_N, \gtrsim)} S)$ such that, for each $i \in T$, $T \succ_{N \backslash (\cup_{S \in M_P(\Omega_N, \gtrsim)} S)} P(i)$, where $P(i) \in P \backslash M_P(\Omega_N, \gtrsim)$. Obviously, $T, P(i) \subseteq N$, and, for each $i \in T$, $T \succ P(i)$, which yields that P is not a core-partition of $(\Omega_N, \gtrsim)$, a contradiction. Conclude that $P \backslash M_P(\Omega_N, \gtrsim)$ is a core-partition of $(N \backslash (\cup_{S \in M_P(\Omega_N, \gtrsim)} S), \gtrsim_{N \backslash (\cup_{S \in M_P(\Omega_N, \gtrsim)} S)})$. $\qquad \square$

Remark 2 Given $(\Omega_N, \gtrsim)$ and $\succcurlyeq \in \mathcal{CSR}(\Omega_N, \gtrsim)$, let $P_\succcurlyeq$ be the partition of N into equivalent classes according to $\succcurlyeq$, and consider a partition $P \in \mathcal{CP}(\Omega_N, \gtrsim)$ such that $\succcurlyeq = \succcurlyeq_{(\gtrsim, P)}$. A priori, $P_\succcurlyeq$ and P can be different. For each $S \in P$ and each $i, j \in S$, $P(i) = P(j)$ so that $i \cdot j$. By definition of $P_\succcurlyeq$, there is $T \in P_\succcurlyeq$ such that $i, j \in T$. Therefore, $S \subseteq T$, which implies that $P_\succcurlyeq$ is coarser than any $P \in \mathcal{CP}(\Omega_N, \gtrsim)$ such that $\succcurlyeq = \succcurlyeq_{(\gtrsim, P)}$. In particular, the best equivalence class $M(N, \succcurlyeq)$ of $\succcurlyeq$ is given by:

$$M(N, \succcurlyeq) = \cup_{S \in M_P(\Omega_N, \gtrsim)} S.$$

Example 4 Assume that $N = \{1, 2, 3, 4, 5\}$. Consider the coalitional ranking problem $(\Omega_N, \gtrsim)$ such that

$$\{1, 2\} \sim \{2, 3\} \sim \{4, 5\} \sim \{3, 4\} \succ N \sim \{2\} \sim \{3\} \succ \{1\} \succ S \sim T,$$

for each other pair of coalitions $\{S, T\}$. Obviously, the partition

$$P = \big\{\{1, 2\}, \{4, 5\}, \{3\}\big\} \in \mathcal{CP}(\Omega_N, \gtrsim),$$

and the associated core-partition social ranking $\succcurlyeq_{(\gtrsim, P)}$ is given by

$$1 \cdot_{(\gtrsim, P)} 2 \cdot_{(\gtrsim, P)} 4 \cdot_{(\gtrsim, P)} 5 \succcurlyeq_{(\gtrsim, P)} 3.$$

The partition $P_{\succcurlyeq_{(\gtrsim, P)}}$ of N induced by the equivalence classes of $\succcurlyeq_{(\gtrsim, P)}$ is given by

$$P_{\succcurlyeq_{(\gtrsim, P)}} = \big\{\{1, 2, 4, 5\}, \{3\}\big\},$$

and $P_{\succcurlyeq_{(\gtrsim, P)}}$ is coarser than P.

5 Axiomatic Study

In this section, we introduce three new axioms for a (set-valued) social ranking solution and examine how permissive or restrictive they are by means of examples or by considering a relevant subdomain of the full domain. It turns out that the combination of these axioms characterizes the social ranking solution $\mathcal{CSR}$. As underlined by Thomson (2001), this axiomatic approach is classical and common to many collective choice problems. According to Thomson (2019),

> The axiomatic analysis holds a central place in economic design. It allows us to go beyond current practice and hypothetical examples. It provides explicit arguments for the use of particular allocation rules in terms of meaningful criteria of good behavior (...)

Our axioms are naturally part of this approach. They discuss the consequences of modifications of the two components of the coalitional ranking problem: an expansion of the population of agents and relevant changes in the considered coalitional ranking. The agents have frequently to face such modifications in group decision making situations. So consider the full domain $\mathcal{R}_\Omega$ and let $f : \mathcal{R}_\Omega \rightrightarrows \mathcal{R}_\mathbb{N}$ be any such social ranking solution.

The first axiom is based on a specific expansion of the population of agents. So consider an initial coalitional ranking problem, say $(N\backslash S, \succsim')$. Assume now that the population is expanded in such a way that the coalition S of all newly added agents wins unanimous support: S is the unique maximal coalition in the larger coalitional ranking $(N, \succsim)$, the relative ranking between any two original coalitions remains the same, and the other coalitions have an arbitrary ranking (except that they do not belong to the best equivalent class). Hence, the restriction $\succsim_{N\backslash S}$ of $\succsim$ to the original population $N\backslash S$ coincides with $\succsim'$. Then, our axiom imposes that the solution set of this new coalitional ranking problem $(N, \succsim)$ is computed from the solution set of the initial problem $(N\backslash S, \succsim')$ by adding to each selected social ranking for $(N\backslash S, \succsim')$ the members of S as the only maximal elements.

To state formally this axiom, a definition is needed. For each $(\Omega_N, \succsim) \in \mathcal{R}_\Omega$, each $S \in \Omega_N$ and each $\succ \in \mathcal{R}(N\backslash S)$, denote by $\succ^{+S} \in \mathcal{R}(N)$ the social ranking on N obtained from $\succ$ by adding S as the set of maximal elements, i.e. $M(N, \succ^{+S}) = \{S\}$, and, for each $i, j \in N\backslash S$, $i \succ^{+S} j$ if and only if $i \succ j$.

Unanimous extension (UE) For each $(\Omega_N, \succsim) \in \mathcal{R}_\Omega$ such that $M(\Omega_N, \succsim) = \{S\}$, it holds that

$$f(\Omega_N, \succsim) = \left\{ \succ^{+S} : \succ \in f(\Omega_{N\backslash S}, \succsim_{N\backslash S}) \right\}.$$

In order to state the next axiom, we introduce a new operation on a coalitional ranking problem $(\Omega_N, \succsim) \in \mathcal{R}_\Omega$. For a given coalition $S \in \Omega_N$, we say that an alternative coalitional ranking problem $(\Omega_N, \succsim') \in \mathcal{R}_\Omega$ is induced from $(\Omega_N, \succsim)$ by a S **-improvement** if the two following conditions hold:

- $L(\succsim, S) \subseteq L(\succsim', S)$;
- $\forall R, T \in \Omega_N\backslash\{S\}, R \succsim' T \Longleftrightarrow R \succsim T$.

In words, the position of S (weakly) improves while the relative ranking of any other pair of coalitions is unchanged.

The next axiom relies on a principle of invariance of the solution set to some improving move. Consider a situation where two disjoint coalitions have the same rank and their union has a lower rank. Then, this principle of invariance indicates that improving the rank of this union, *ceteris paribus*, does not alter the solution set, provided than the new rank of the union has still a rank not higher than that of the two original coalitions. For social choice functions, this principle is related in a certain way to the Maskin monotonicity principle (Maskin 1999). Loosely speaking, this Maskin monotonicity indicates that if an alternative is selected in a certain problem, then it is also selected in a related problem obtained when some elements from which the alternative is constructed improved, *ceteris paribus*. In our case, this principle applies to specific problems where two disjoint coalitions have the same rank, their union is lower ranked, and the improving move is bounded from above by the equivalence class to which these two disjoint coalitions belong.

Invariance to merger upgrading (IMU) For each $(\Omega_N, \succsim) \in \mathcal{R}_{\Omega_N}$, each pair of coalitions $\{S, T\} \subseteq \Omega_N$ such that $S \cap T = \emptyset$, $S \sim T$ and $(S \cup T) \in L(\succsim, S)$, and each $(\Omega_N, \succsim') \in \mathcal{R}_{\Omega_N}$ induced from $(\Omega_N, \succsim)$ by a $(S \cup T)$-improvement where $(S \cup T) \in L(\succsim', S)$, it holds that $f(\Omega_N, \succsim) = f(\Omega_N, \succsim')$.

Example 5 Consider Example 4. If one takes $S = \{1, 2\}$ and $T = \{3, 4\}$ then $S \cap T = \emptyset$, $S \sim T$ and $(S \cup T) \in L(\succsim, S)$. Note that $S, T \in M(\Omega_N, \succsim)$. Consider $(\Omega_N, \succsim')$ induced from $(\Omega_N, \succsim)$ by improving $S \cup T = \{1, 2, 3, 4\}$ up to the best equivalence class so that

$$\{1, 2\} \sim' \{1, 2, 3, 4\} \sim' \{2, 3\} \sim' \{4, 5\} \sim' \{3, 4\} \succ' N \sim' \{2\} \sim' \{3\} \succ' \{1\} \succ' S \sim' T,$$

for each other pair of coalitions $\{S, T\}$. Now if f satisfies (IMU), then $f(\Omega_N, \succsim) = f(\Omega_N, \succsim')$.

Note also that one can also improve $\{2, 3, 4, 5\}$ and $\{1, 2, 4, 5\}$ in the same way by considering respectively $S = \{2, 3\} \sim T = \{4, 5\}$, and $S = \{1, 2\} \sim T = \{4, 5\}$.

The last axiom implements a decomposition property. It specifies situations where the solution set of a coalitional ranking problem can be expressed as the union of the solution sets of coalitional ranking problems built from the original one and containing a unique maximal element. The situations where such a decomposition is possible is related to the structure of the set of maximal elements of the coalitional ranking problem. A subset of coalitions $\mathcal{C} \subseteq \Omega_N$ is **stable by union of disjoint elements** if for any pair of coalitions $\{S, T\} \subseteq \mathcal{C}$ such that $S \cap T = \emptyset$, then $S \cup T \in \mathcal{C}$.[2] Assume that the set of maximal coalitions of a coalitional ranking problem is stable by union of disjoint elements. In such a situation, the axiom indicates that the solution set

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[2] The difference with union stable set systems as studied in Algaba et al. (2021) is that only the union of pairs of intersecting coalitions is considered; and the difference with union closed systems as studied in van den Brink et al. (2011) is that the union of all pairs of coalitions is considered. Thus, each union closed system is union stable and stable by union of disjoint elements.

coincides with the union of the solution sets of variants of the original coalitional ranking problem obtained, for each maximal coalition intersecting all other maximal coalitions, by putting this maximal coalition as the unique maximal coalition, *ceteris paribus.*

For each $(\Omega_N, \succsim) \in \mathcal{R}_\Omega$ define

$$I_M(\Omega_N, \succsim) = \{S \in M(\Omega_N, \succsim) : S \cap T \neq \emptyset, \forall T \in M(\Omega_N, \succsim)\},$$

as the **set of intersecting coalitions** in $M(\Omega_N, \succsim)$. Now, for each $(\Omega_N, \succsim) \in \mathcal{R}_\Omega$ and each $S \in I_M(\Omega_N, \succsim)$, denote by $(\Omega_N, \succsim^S)$ the unique coalitional ranking problem induced from $(\Omega_N, \succsim)$ by a S-improvement such that $M(\Omega_N, \succsim^S) = \{S\}$.

Remark 3 If $M(\Omega_N, \succsim)$ is stable by union of disjoint elements, then $I_M(\Omega_N, \succsim) \neq \emptyset$.

Decomposition by top upgrading (DTU) Consider any $(\Omega_N, \succsim) \in \mathcal{R}_\Omega$ such that $M(\Omega_N, \succsim)$ is stable by union of disjoint elements. Then, it holds that

$$f(\Omega_N, \succsim) = \bigcup_{T \in I_M(\Omega_N, \succsim)} f(\Omega_N, \succsim^T).$$

Example 6 Consider again Example 4. Applying the three improvements presented in Example 5 brings the following coalitional ranking:

$$\{1, 2\} \sim^* \{1, 2, 3, 4\} \sim^* \{1, 2, 4, 5\} \sim^* \{2, 3, 4, 5\} \sim^* \{2, 3\} \sim^* \{4, 5\} \sim^* \{3, 4\}$$
$$\succ^* N \sim^* \{2\} \sim^* \{3\} \succ^* \{1\},$$

and $\{1\} \succ^* S \sim^* T$ for each other pair of coalitions $\{S, T\}$. Remark that $M(\Omega_N, \succsim^*)$ is stable by union of disjoint elements and that

$$I_M(\Omega_N, \succsim^*) = \{\{1, 2, 3, 4\}, \{1, 2, 4, 5\}, \{2, 3, 4, 5\}\}.$$

Hence if f satisfies (DTU), then $f(\Omega_N, \succsim^*) = f(\Omega_N, (\succsim^*)^{\{1,2,3,4\}}) \cup f(\Omega_N, (\succsim^*)^{\{1,2,4,5\}}) \cup f(\Omega_N, (\succsim^*)^{\{2,3,4,5\}})$.

The next result states that the combination of Unanimous extension, Invariance to merger upgrading and Decomposition by top upgrading characterizes the core-partition social ranking solution $\mathcal{CSR}$.

Proposition 2 *The social ranking solution $\mathcal{CSR}$ is the unique solution on $\mathcal{R}_\Omega$ satisfying Unanimous extension (UE), Invariance to merger upgrading (IMU), and Decomposition by top upgrading (DTU).*

Proof The proof is divided into two parts (a) and (b).

(a) One shows that $\mathcal{CSR}$ satisfies the above three axioms. Pick any $(\Omega_N, \succsim) \in \mathcal{R}_\Omega$. *Unanimous extension* Assume that $M(\Omega_N, \succsim) = \{S\}$ for some $S \in \Omega_N$. To show:

$$\mathcal{CSR}(\Omega_N, \succsim) = \left\{ \vartriangleright^{+S} : \vartriangleright \in \mathcal{CSR}(\Omega_{N\backslash S}, \succsim_{N\backslash S}) \right\} \tag{1}$$

To this end, it is sufficient to show that $P \in \mathcal{CP}(\Omega_N, \succsim)$ if and only if P is of the form $P = \{S\} \cup P'$ where $P' \in \mathcal{CP}(\Omega_{N\backslash S}, \succsim_{N\backslash S})$. For the "if part", consider such a partition $P \in \mathcal{P}(N)$ and any $T \in \Omega_N$. Two cases can be distinguished:

- if $T \cap S \neq \emptyset$, pick $i \in T \cap S$. One obtains $P(i) = S \succsim T$ because $S \in M(\Omega_N, \succsim)$;
- if $T \cap S = \emptyset$, then $T \in \Omega_{N\backslash S}$. Because P' is a core-partition of $(\Omega_{N\backslash S}, \succsim_{N\backslash S})$, there is $i \in T$ such that $P'(i) \succsim_{N\backslash S} T$. Hence, $P(i) = P'(i) \succsim T$.

From the above two cases, conclude that P cannot be blocked by T, and so $P \in \mathcal{CP}(\Omega_N, \succsim)$.

For the "only if part", by part (i) of Proposition 1, $M_P(\Omega_N, \succsim) \cap M(\Omega_N, \succsim) \neq \emptyset$ so that $S \in P$ and $M_P(\Omega, \succsim) = \{S\}$. Part (ii) of Proposition 1 implies that $P' = P\backslash\{S\} \in \mathcal{CP}(\Omega_{N\backslash S}, \succsim_{N\backslash S})$, as desired.

Now, if $P = \{S\} \cup P' \in \mathcal{CP}(\Omega_N, \succsim)$ for some $P' \in \mathcal{CP}(\Omega_{N\backslash S}, \succsim_{N\backslash S})$, then its associated social ranking $\vartriangleright_{(\succsim, P)}$ belongs to $\mathcal{CSR}(\Omega_N, \succsim)$ and obviously $\vartriangleright_{(\succsim, P)} = \vartriangleright^{+S}_{(\succsim_{N\backslash S}, P')}$. Reciprocally, $P' \in \mathcal{CP}(N\backslash S, \succsim_{N\backslash S})$ gives rise to a unique social ranking $\vartriangleright_{(\succsim_{N\backslash S}, P')} \in \mathcal{CSR}(N\backslash S, \succsim_{N\backslash S})$ from which the social ranking $\vartriangleright^{+S}_{(\succsim_{N\backslash S}, P')} = \vartriangleright_{(\succsim, P)}$ is a core-partition social ranking of $\mathcal{CSR}(\Omega_N, \succsim)$. Thus, (1) holds, as desired.

Invariance to merger upgrading Consider a pair $\{S, T\} \subseteq \Omega_N$ such that $S \cap T = \emptyset$, $S \sim T$ and $(S \cup T) \in L(\succsim, S)$, and any coalitional ranking problem $(\Omega_N, \succsim') \in \mathcal{R}_\Omega$ induced from $\succsim$ by a $(S \cup T)$-improvement with $(S \cup T) \in L(\succsim', S)$. To show: $\mathcal{CSR}(\Omega_N, \succsim) = \mathcal{CSR}(\Omega_N, \succsim')$. One proceeds by double inclusion.

- $\mathcal{CSR}(\Omega_N, \succsim) \subseteq \mathcal{CSR}(\Omega_N, \succsim')$. By definition of $\mathcal{CSR}$, it suffices to show that $\mathcal{CP}(\Omega_N, \succsim) \subseteq \mathcal{CP}(\Omega_N, \succsim')$. Pick any $P \in \mathcal{CP}(N, \succsim)$, which means that, for each $R \in \Omega_N$, there is $i \in R$ such that $P(i) \succsim R$. For each $R \in \Omega_N \backslash \{(S \cup T)\}$, note that $P(i) \succsim R$ implies $P(i) \succsim' R$ because $S \cup T$ is the only improved coalition in the coalitional ranking $\succsim'$. If $R = S \cup T$, then, since there is $i \in S$ such that $P(i) \succsim S$ and $(S \cup T) \in L(\succsim, S) \cap L(\succsim', S)$, one obtains $P(i) \succsim' S \succsim' (S \cup T)$, whether $P(i) = S \cup T$ or not. Hence, $P \in \mathcal{CP}(N, \succsim')$.

- $\mathcal{CSR}(\Omega_N, \succsim) \supseteq \mathcal{CSR}(\Omega_N, \succsim')$. Pick any $\vartriangleright \in \mathcal{CSR}(\Omega, \succsim')$ and any $P' \in \mathcal{CP}(\Omega_N, \succsim')$ such that $\vartriangleright = \vartriangleright_{(\succsim', P')}$. If $S \cup T \notin P'$, then define $P = P'$; and if $S \cup T \in P'$, then define $P = (P'\backslash\{S \cup T\}) \cup \{S\} \cup \{T\}$. Because $S \cup T \in P'$ only happens when $S \sim' T \sim' S \cup T$, one necessarily has $\vartriangleright_{(\succsim, P)} = \vartriangleright_{(\succsim', P')}$. Let us show that $P \in \mathcal{CP}(\Omega_N, \succsim)$ so that $\vartriangleright \in \mathcal{CSR}(\Omega_N, \succsim)$. For each $R \in \Omega_N$, there is $i \in R$ such that $P'(i) \succsim' R$. If $P'(i) \neq S \cup T$, then $P(i) = P'(i) \succsim R$, whether $R = S \cup T$ or not. If $P'(i) = S \cup T$, then either $i \in S$ or $i \in T$ so that $P(i) = S$ or $P(i) = T$. By definition of $\succsim'$, $S \cup T \in L(\succsim', S) = L(\succsim', T)$ so that, in both cases, $P(i) \succsim' S \cup T = P'(i) \succsim' R$, which implies that $P(i) \succsim R$. Hence, for each $R \in \Omega_N$, there is $i \in R$ such that $P(i) \succsim R$. Conclude that $P \in \mathcal{CP}(\Omega_N, \succsim)$ so that $\vartriangleright = \vartriangleright_{(\succsim', P')} = \vartriangleright_{(\succsim, P)} \in \mathcal{CSR}(\Omega_N, \succsim)$.

Decomposition by top upgrading Assume that $M(\Omega_N, \succsim)$ is stable by union of disjoint elements. To show:

$$\mathcal{CSR}(\Omega_N, \succsim) = \bigcup_{T \in I_M(\Omega_N, \succsim)} \mathcal{CSR}(\Omega_N, \succsim^T). \tag{2}$$

One proceeds by double inclusion.

- $\mathcal{CSR}(\Omega_N, \succsim) \subseteq \cup_{T \in I_M(\Omega_N, \succsim)} \mathcal{CSR}(\Omega_N, \succsim^T)$. Pick any $\succ \in \mathcal{CSR}(\Omega_N, \succsim)$ and any $P \in \mathcal{CP}(\Omega_N, \succsim)$ associated with $\succ \in \mathcal{CSR}(\Omega_N, \succsim)$, i.e., $\succ = \succ_{(\succsim, P)}$. It suffices to show that there is $T \in I_M(\Omega_N, \succsim)$ and $P' \in \mathcal{CP}(\Omega_N, \succsim^T)$ such that $\succ_{(\succsim^T, P')} = \succ_{(\succsim, P)}$. Denote by T the union of the possibly several elements in $M_P(\Omega_N, \succsim)$ and define P' as the partition obtained from P by replacing the elements of $M_P(\Omega_N, \succsim)$ by their union T. Because $M(\Omega_N, \succsim)$ is stable by union of disjoint elements, $T \in M(\Omega_N, \succsim)$. Clearly, $M_{P'}(\Omega_N, \succsim) = \{T\}$, $\succ_{(\succsim, P')} = \succ_{(\succsim, P)}$ and $P' \in \mathcal{CP}(\Omega_N, \succsim)$. It remains to show that $T \in I_M(\Omega, \succsim)$ and that $P' \in \mathcal{CP}(\Omega_N, \succsim^T)$. The first claim results from part (i) of Proposition 1 and the definition of $I_M(\Omega, \succsim)$. For the second claim, pick any $S \in \Omega_N$. Because $P \in \mathcal{CP}(\Omega_N, \succsim)$, there is $i \in S$ such that $P(i) \succsim S$. If $i \in T$, then $T = P'(i) \succsim^T S$. If $i \notin T$, then $P'(i) = P(i) \notin M(\Omega_N, \succsim)$. By definition of $\succsim^T$ and the fact that $P' \in \mathcal{CP}(\Omega_N, \succsim)$, one obtains $P'(i) \succsim^T S$. Hence, $P' \in \mathcal{CP}(\Omega_N, \succsim^T)$.

- $\mathcal{CSR}(\Omega_N, \succsim) \supseteq \cup_{T \in I_M(\Omega_N, \succsim)} \mathcal{CSR}(\Omega_N, \succsim^T)$. Precisely, one establishes that, for each $T \in I_M(\Omega_N, \succsim)$, $\mathcal{CP}(\Omega_N, \succsim^T) \subseteq \mathcal{CP}(\Omega_N, \succsim)$ so that $\mathcal{CSR}(\Omega_N, \succsim^T) \subseteq \mathcal{CSR}(\Omega_N, \succsim)$. For each $P' \in \mathcal{CP}(\Omega_N, \succsim^T)$, $M_{P'}(\Omega_N, \succsim^T) = \{T\} \subseteq M(\Omega_N, \succsim)$ where the inclusion follows from the fact that $T \in I_M(\Omega_N, \succsim)$. Furthermore, $T \in I_M(\Omega_N, \succsim)$ implies $M_{P'}(\Omega_N, \succsim) = \{T\}$. Because, $P' \in \mathcal{CP}(\Omega_N, \succsim^T)$, for each $S \in \Omega_N$, there is $i \in S$ such that $P'(i) \succsim^T S$. If $S \cap T \neq \emptyset$, one picks $i \in S \cap T$ and $P'(i) = T \in M(\Omega_N, \succsim)$ so that $P'(i) \succsim S$. If $S \cap T = \emptyset$, then $P'(i) \neq T$, so that $P'(i) \succsim S$ by definition of $\succsim^T$. This yields that $P' \in \mathcal{CP}(\Omega, \succsim)$, as desired.

(b) One shows that if a social ranking solution f satisfies Unanimous extension, Invariance to merger upgrading and Decomposition by top upgrading, then $f = \mathcal{CSR}$. So, consider such a social ranking solution f. To prove that $f = \mathcal{CSR}$, one proceeds by induction on the number $n \geq 0$ of agents in a coalitional ranking problem.

INITIALIZATION. For $N = \emptyset$, by convention $f(\Omega_\emptyset, \emptyset) = \mathcal{CSR}(\Omega_\emptyset, \emptyset) = \{\emptyset\}$. For $N = \{i\}$ for some $i \in \mathbb{N}$, and for the (unique) coalitional ranking $\succsim$ on $\Omega_{\{i\}}$, $\{i\} \succsim \{i\}$, one applies Unanimous extension to f:

$$f(\Omega_{\{i\}}, \succsim) = \left\{ \emptyset^{+\{i\}} : \{\emptyset\} = f(\Omega_\emptyset, \emptyset) \right\} = \{\cdot\},$$

meaning that $i \cdot i$. Obviously, $\mathcal{CSR}(\Omega_{\{i\}}, \succsim) = \{\cdot\}$, so that $f(\Omega_{\{i\}}, \succsim) = \mathcal{CSR}(\Omega_{\{i\}}, \succsim)$.

INDUCTION HYPOTHESIS. Assume that $f(\Omega_N, \succsim) = \mathcal{CSR}(\Omega_N, \succsim)$ holds for any $(\Omega, \succsim) \in \mathcal{R}_{\Omega_N}$ such that $n < k$ for some integer $k \geq 1$.

INDUCTION STEP. Consider any coalitional ranking problem $(\Omega_N, \succsim) \in \mathcal{R}_\Omega$ such that $n = k$. If $M(\Omega_N, \succsim)$ is not stable by union of disjoints elements, $M(\Omega_N, \succsim)$ contains disjoint coalitions S and T such that $S \cup T$ is not in $M(\Omega_N, \succsim)$. Then, construct the coalitional ranking problem $(\Omega_N, \succsim')$ induced from $(\Omega_N, \succsim)$ by the $(S \cup T)$

-improvement such that $S \cup T \in M(\Omega_N, \succsim')$. Because f and $\mathcal{CSR}$ satisfy Invariance to merger upgrading, $f(\Omega_N, \succsim') = f(\Omega_N, \succsim)$ and $\mathcal{CSR}(\Omega_N, \succsim') = \mathcal{CSR}(\Omega_N, \succsim)$. Continue in this fashion to eventually construct a coalitional ranking problem $(\Omega_N, \succsim^*)$ such that, for each $S, T \in M(\Omega_N, \succsim^*)$ where $S \cap T = \emptyset$, $S \cup T \in M(\Omega_N, \succsim^*)$ so that $M(\Omega_N, \succsim^*)$ is stable by union of disjoint elements. Because Ω_N is a finite set, the (unique) coalitional ranking problem $(\Omega_N, \succsim^*)$ is reached after a finite number of steps. If $M(\Omega_N, \succsim)$ is stable by union of disjoint elements, no operation is required and $\succsim = \succsim^*$. Then, by successive applications of Invariance to merger upgrading, one obtains:

$$f(\Omega_N, \succsim^*) = f(\Omega_N, \succsim) \quad \text{and} \quad \mathcal{CSR}(\Omega_N, \succsim^*) = \mathcal{CSR}(\Omega_N, \succsim). \tag{3}$$

Furthermore, by Remark 3, $I_M(\Omega_N, \succsim^*) \neq \emptyset$. Thus, the conditions underlying Decomposition by top upgrading are met in $(\Omega_N, \succsim^*)$. Now, consider any $S \in I_M(\Omega_N, \succsim^*)$ and construct the coalitional ranking problem $(\Omega_N, \succsim^{*S})$ induced from $(\Omega_N, \succsim^*)$ by the S-improvement such that $M(\Omega_N, \succsim^{*S}) = \{S\}$. Applying Unanimous extension to both f and $\mathcal{CSR}$ and the induction hypothesis, one gets:

$$
\begin{aligned}
f(\Omega_N, \succsim^{*S}) &\overset{\text{UE}}{=} \left\{ \gg^{+S} : \gg \in f(\Omega_{N\setminus S}, \succsim^{*S}_{N\setminus S}) \right\} \\
&\overset{\text{Ind. Hypo.}}{=} \left\{ \gg^{+S} : \gg \in \mathcal{CSR}(\Omega_{N\setminus S}, \succsim^{*S}_{N\setminus S}) \right\} \\
&\overset{(1)}{=} \mathcal{CSR}(\Omega_N, \succsim^{*S})
\end{aligned}
\tag{4}
$$

Because S was arbitrarily chosen in $I_M(\Omega_N, \succsim^*)$, applying Decomposition by top upgrading to both f and $\mathcal{CSR}$ and using the equalities (3)-(4) yield that

$$
\begin{aligned}
f(\Omega_N, \succsim) &\overset{(3)}{=} f(\Omega_N, \succsim^*) \\
&\overset{\text{DTU}}{=} \bigcup_{S \in I_M(\Omega_N, \succsim^*)} f(\Omega_N, \succsim^{*S}) \\
&\overset{(4)}{=} \bigcup_{S \in I_M(\Omega_N, \succsim^*)} \mathcal{CSR}(\Omega_N, \succsim^{*S}) \\
&\overset{(2)}{=} \mathcal{CSR}(\Omega_N, \succsim^*) \\
&\overset{(3)}{=} \mathcal{CSR}(\Omega_N, \succsim),
\end{aligned}
$$

which completes the induction step.

The statement of Proposition 2 follows from (a) and (b). $\qquad\square$

The steps used in the uniqueness part (b) of the previous proof are illustrated in the following example.

Example 7 Consider Example 4. As in Example 5, by successive merger upgrading, we get the following coalitional ranking:

$$\{1,2\} \sim^* \{1,2,3,4\} \sim^* \{1,2,4,5\} \sim^* \{2,3,4,5\} \sim^* \{2,3\} \sim^* \{4,5\} \sim^* \{3,4\}$$
$$\succ^* N \sim^* \{2\} \sim^* \{3\} \succ^* \{1\},$$

and $\{1\} \succ^* S \sim^* T$ for each other pair of coalitions $\{S,T\}$. Hence, by successive applications of Invariance to merger upgrading, one obtains that $f(\Omega_N, \succsim) = f(\Omega_N, \succsim^*)$.

As seen in Example 6, $M(\Omega_N, \succsim^*)$ is stable by union of disjoint elements so that

$$f(\Omega_N, \succsim^*) = f(\Omega_N, (\succsim^*)^{\{1,2,3,4\}}) \cup f(\Omega_N, (\succsim^*)^{\{1,2,4,5\}}) \cup f(\Omega_N, (\succsim^*)^{\{2,3,4,5\}}).$$

Each member of this union gives rise to a unique social ranking by applying Unanimous extension, because $M(\Omega_N, \succsim^S) = \{S\}$ and S contains all but one agent. For instance:

$$f(\Omega_N, (\succsim^*)^{\{1,2,3,4\}}) = \left\{ \succ^{+\{1,2,3,4\}} : \succ \in f(\Omega_{\{5\}}, \succsim_{\{5\}}) \right\} = \{1 \cdot 2 \cdot 3 \cdot 4 > 5\}.$$

The three social rankings composing $f(\Omega_N, \succsim)$ correspond to the following three core-partitions of $(\Omega_N, \succsim)$: $\mathcal{CP}(\Omega_N, \succsim) = \{P^1, P^2, P^3\}$ where $P^1 = \{\{1,2\}, \{3,4\}, \{5\}\}$, $P^2 = \{\{1,2\}, \{4,5\}, \{3\}\}$ and $P^3 = \{\{2,3\}, \{4,5\}, \{1\}\}$.

Remark 4 Note that on the subdomain of coalitional rankings that are strict total orders, Unanimous extension fully characterizes $\mathcal{CSR}$. On this subdomain, one easily verifies that $\mathcal{CSR}$ is single-valued. This means that the two other axioms invoked in Proposition 2 are key to dealing with ties in the equivalent classes that often arise in less specific coalitional rankings.

The logical independence of the axioms used in Proposition 2 is shown in the following section.

6 Logical Independence of the Axioms

Unanimous extension is not satisfied Consider the constant solution f^C which assigns to each coalitional ranking problem the equivalence social relation $\cdot$, that is all agents belong to the same equivalence class:

$$\forall(\Omega_N, \succsim) \in \mathcal{R}_\Omega, \quad f^C(\Omega_N, \succsim) = \{\cdot\}.$$

Because the image $f^C(\mathcal{R}_\Omega)$ is the equivalence relation $\cdot$, it obviously satisfies Invariance to merger upgrading, and Decomposition by top upgrading. But f^C violates Unanimous extension for the following reason: the social ranking $\succ^{+S}$ used to define the solution set $f^C(\Omega_N, \succsim)$ whenever $M(\Omega_N, \succsim)$ is a singleton, contains at least two distinct equivalence classes when $S \neq N$.

Invariance to merger upgrading is not satisfied Pick any $(\Omega_N, \succsim) \in \mathcal{R}_\Omega$ and define $I(\Omega_N, \succsim)$ as:

$$I(\Omega_N, \succsim) = \big\{ S \in \Omega_N : \forall T \in \Omega_N, (S \sim T \Rightarrow S \cap T \neq \emptyset) \wedge (T \succ S \Rightarrow (\exists R \in \Omega_N ,$$
$$(R \sim T) \wedge (R \cap T) = \emptyset)) \big\}.$$

Note that $I(\Omega_N, \succsim) \neq \emptyset$. Consider the social ranking solution f^I defined recursively as follows:

$$\forall(\Omega_N, \succsim) \in \mathcal{R}_\Omega, \quad f^I(\Omega_N, \succsim) = \bigcup_{S \in I(\Omega_N, \succsim)} \{\succ^{+S} : \succ \in f^I(\Omega_{N \setminus S}, \succsim_{N \setminus S})\}.$$

In case $M(\Omega_N, \succsim) = \{S\}$, then $I(\Omega_N, \succsim) = \{S\}$ and so

$$f^I(\Omega_N, \succsim) = \{\succ^{+S} : \succ \in f^I(\Omega_{N \setminus S}, \succsim_{N \setminus S})\},$$

which is the definition of Unanimous extension. Thus, f^I indeed satisfies this axiom. In case $M(\Omega_N, \succsim)$ is stable by union of disjoint elements, $I(\Omega_N, \succsim) = I_M(\Omega_N, \succsim)$. For each $S \in I_M(\Omega_N, \succsim)$, $(\Omega_N, \succsim^S)$ is such that $M(\Omega_N, \succsim^S) = \{S\} = I(\Omega_N, \succsim^S)$. Note also that $\succsim^S_{N \setminus S} = \succsim_{N \setminus S}$, so that

$$f^I(\Omega_N, \succsim^S) = \big\{\succ^{+S} : \succ \in f^I(\Omega_{N \setminus S}, \succsim^S_{N \setminus S})\big\}$$
$$= \big\{\succ^{+S} : \succ \in f^I(\Omega_{N \setminus S}, \succsim_{N \setminus S})\big\}.$$

Consequently,

$$f^I(\Omega_N, \succsim) = \bigcup_{S \in I_M(\Omega_N, \succsim)} f^I(\Omega_N, \succsim^S),$$

which shows that f^I satisfies Decomposition by top upgrading. To show that f^I violates Invariance to merger upgrading, consider the coalitional ranking problem $(\Omega_N, \succsim)$ with $N = \{1, 2, 3\}$ and defined as:

$$\{2\} \sim \{3\} \succ \{2, 3\} \sim \{1\} \succ \{1, 2, 3\} \succ \{1, 3\} \sim \{1, 2\}.$$

It follows that $I(\Omega_N, \succsim) = \{\{1, 2, 3\}\}$, and so $f^I(\Omega_N, \succsim) = \{\cdot\}$, i.e., $1 \cdot 2 \cdot 3$. Now, consider the coalitional ranking problem $(\Omega_N, \succsim')$ defined as:

$$\{2\} \sim' \{3\} \sim' \{2, 3\} \succ' \{1\} \succ' \{1, 2, 3\} \succ' \{1, 3\} \sim' \{1, 2\}.$$

The coalitional ranking problem $(\Omega_N, \succsim')$ is an improvement from $(\Omega_N, \succsim)$ induced by $\{2, 3\}$ that satisfies the conditions of Invariance to merger upgrading. One has $I(\Omega_N, \succsim') = \{\{2, 3\}\}$. By definition of f^I and the fact that it satisfies Unanimous extension,

$$f^I(\Omega_N, \succsim') = \{2 \cdot 3 > 1\} \neq f^I(\Omega_N, \succsim),$$

a violation of Invariance to merger upgrading.

Decomposition by top upgrading is not satisfied Consider the following solution

$$\forall(\Omega_N, \succsim) \in \mathcal{R}, \quad f^\ell(\Omega_N, \succsim) = \mathcal{CSR}(N, \ell(\succsim)),$$

where $\ell : \mathcal{R}(\Omega_N) \to \mathcal{R}(\Omega_N)$ is defined as

$$S\ell(\succsim)T \iff \big((S \succ T) \text{ or } (S \sim T \text{ and } |S| \leqslant |T|)\big).$$

The solution f^ℓ satisfies Unanimous extension and Invariance to merger upgrading but not Decomposition by top upgrading. Indeed, if $M(\Omega_N, \succsim) = \{S\}$, then $M(\Omega_N, \ell(\succsim)) = \{S\}$ and

$$
\begin{aligned}
f^\ell(\Omega_N, \succsim) &= \mathcal{CSR}(\Omega_N, \ell(\succsim)) \\
&= \{\succ^{+S} : \succ \in \mathcal{CSR}(\Omega_{N\setminus S}, \ell(\succsim)_{N\setminus S})\} \\
&= \{\succ^{+S} : \succ \in \mathcal{CSR}(\Omega_{N\setminus S}, \ell(\succsim_{N\setminus S}))\} \\
&= \{\succ^{+S} : \succ \in f^\ell(\Omega_{N\setminus S}, \succsim_{N\setminus S})\},
\end{aligned}
$$

which shows that f^ℓ satisfies Unanimous extension. Next, if $S, T \in \Omega_N$ are such that $S \cap T = \emptyset$, $S \sim T$ and $(S \cup T) \in L(\succsim, S)$, then for any $(\Omega_N, \succsim') \in \mathcal{R}(\Omega_N)$ induced from $\succsim$ by a $(S \cup T)$-improvement with $(S \cup T) \in L(\succsim', S)$, it holds that $\ell(\succsim')$ is induced from $\ell(\succsim)$ by a $(S \cup T)$-improvement with $(S \cup T) \in L(\ell(\succsim'), S)$. Thus, f^ℓ satisfies Invariance to merger upgrading due to the fact that $\mathcal{CSR}$ satisfies it. Finally, consider the coalitional ranking problem $(\Omega_N, \succsim)$ with $N = \{1, 2\}$ and defined as $\{1\} \sim \{1, 2\} \succ \{2\}$, then $\succsim' = \ell(\succsim)$ is as follows: $\{1\} \succ' \{1, 2\} \succ' \{2\}$. On the one hand, by definition of f^ℓ, one has:

$$f^\ell(\Omega_N, \succsim) = \mathcal{CSR}(\Omega_N, \succsim') = \{\succ\},$$

where $M(N, \succ) = \{1\}$. On the other hand, $M(\Omega_N, \succsim) = \{\{1\}, \{1, 2\}\}$ is stable by union of disjoint elements and $I_M(\Omega_N, \succsim) = M(\Omega_N, \succsim)$, so that $\cup_{S \in I_M(\Omega_N, \succsim)} f^\ell(\Omega_N, \succsim^S) = \{\succ, \succ'\}$, where $\succ'$ is such that $M(N, \succ') = \{1, 2\}$. This contradicts Decomposition by top upgrading.

7 Computing All Core-Partitions

This section is devoted to the computation of the core-partitions of a coalitional ranking problem $(\Omega_N, \succsim)$ from which the set of social rankings $\mathcal{CSR}(\Omega_N, \succsim)$ is obtained. A non-deterministic algorithm is proposed, which computes all core-partitions of a coalitional ranking problem. In particular, this algorithm permits to conclude that the set of all core-partitions of a coalitional ranking problem is nonempty. As noted in Remark 1, this nonemptiness property has already been shown by Farrell and Scotchmer (1988) in another context. The idea behind the forthcoming algorithm is described informally in the proof of Theorem 1 in Farrell and Scotchmer (1988) and in the proof of Theorem 1 in Banerjee et al. (2001) in a slightly more general context. It is also given in Lucchetti et al. (2022). Hence, we only give a proof of Proposition 3 below for completeness. In

turn, this property implies that $\mathcal{CSR}$ is nonempty-valued on $\mathcal{R}_\Omega$. The algorithm is as follows.

```
1   INITIALIZATION:  η ⟵ N
2   WHILE  η ≠ ∅  DO:
3       INITIALIZATION:  PICK ANY  T ∈ M(Ω_η, ≿_η) ,  P ⟵ {{T}}
4       WHILE  ∃T ∈ M(Ω_η, ≿_η) SUCH THAT  ∀S ∈ P, S ∩ T = ∅  DO:
5           P ⟵ P ∪ {{T}}
6   η ⟵ η\(∪_{T∈P} T)
7   OUTCOME:  P ∈ CP(Ω_N, ≿)
```

Proposition 3 *For each* $(\Omega_N, \succsim) \in \mathcal{R}_\Omega$, *the set of outcomes of the above non-deterministic algorithm is exactly* $\mathcal{CP}(\Omega_N, \succsim)$.

Proof Firstly, remark that the execution of the algorithm always terminates after a finite number of steps since N is a finite set.

Secondly, we prove that each $P = \{P_1, \ldots, P_k\} \in \mathcal{CP}(\Omega_N, \succsim)$ is obtained as some output of the algorithm. Denote by t the number of equivalent classes of $\succsim$ and order them according to the quotient order: $M(\Omega_N, \succsim)$ is the equivalent class 1 and so on. Define the mapping $\pi : \{1, \ldots, k\} \longrightarrow \{1, \ldots, t\}$ such that P_i belongs to equivalent class $\pi(i)$ according to $\succsim$. Note that $P_i \succsim P_j$ if and only if $\pi(i) \leq \pi(j)$. By Remark 2, the pre-image $\pi^{-1}(1)$ is nonempty and each cell P_i such that $i \in \pi^{-1}(1)$ can be chosen in the inner loop $3 - 5$ of the algorithm to form the output P while $\eta = N$: these cells define the set $\cup_{i \in \pi^{-1}(1)} P_i = \cup_{S \in M_P(\Omega_N, \succsim,)} S$. Then, according to the second part of Proposition 1, we have

$$P \backslash M_P(\Omega_N, \succsim) \in \mathcal{CP}\big(N \backslash (\cup_{S \in M_P(\Omega_N, \succsim)} S), \succsim_{N \backslash (\cup_{S \in M_P(\Omega_N, \succsim)} S)}\big).$$

As a consequence, starting from $\eta = N \backslash (\cup_{S \in M_P(\Omega_N, \succsim)} S)$, each coalition $P_i \in P \backslash M_P(\Omega_N, \succsim)$ is such that $\pi(i) > 1$ and may be chosen recursively at step 3 or 4 by the algorithm.

Thirdly, we prove that each output of the algorithm belongs to $\mathcal{CP}(\Omega_N, \succsim)$. So, assume $P = \{P_1, \ldots, P_k\} \in \mathcal{P}(N)$ is some output of the algorithm. By construction, P is a partition of N, whatever the choices made in steps 3 and 4. As above, consider $\pi : \{1, \ldots, k\} \to \{1, \ldots, t\}$ so that P_i belongs to equivalent class $\pi(i)$ according to $\succsim$. Pick any $S \in \Omega_N$. Consider $i_0 \in \{1, \ldots, k\}$ such that $\pi(i_0) = \min\{\pi(i) : P_i \cap S \neq \emptyset\}$; the minimizer i_0 is possibly not unique. Let us show that $P_{i_0} \succsim S$. By construction, P_{i_0} enters P at step 3 or 4 for a given $\eta \subseteq N$. By definition of i_0, for each $i \in \{1, \ldots, k\}$ such that $P_i \cap S \neq \emptyset$, P_i enters P later in the algorithm. Hence, each $j \in S$ belongs to a $P_i = P(j)$ such that $\pi(i) \geq \pi(i_0)$ so that $S \in \Omega_\eta$. By definition of the algorithm, we have $P_{i_0} \in M(\Omega_\eta, \succsim_\eta)$ in $(\Omega_\eta, \succsim_\eta)$. Together with $S \in \Omega_\eta$, this implies that $P_{i_0} \succsim_\eta S$, and so $P_{i_0} \succsim S$. Conclude that each $j \in P_{i_0} \cap S$ satisfies $P(j) \succsim S$, which means that S cannot block P, as desired. □
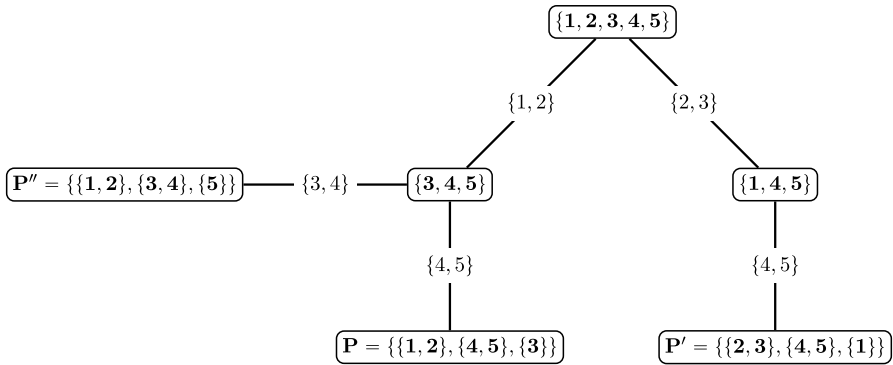
Fig. 1 Decision tree of the algorithm applied to Example 1

Example 8 The key steps of the algorithm can be represented schematically by its decision tree. Each node, other than a leaf, is labeled by the set η corresponding to the steps 1 and 6 from which a choice is made in step 3 and 4. These choices correspond to the label of each arrow. Each leaf computes the output of the algorithm. Consider Example 1. Recall that $N = \{1, 2, 3, 4, 5\}$ and

$$\{1, 2\} \sim \{2, 3\} \succ \{4, 5\} \sim \{3, 4\} \succ N \sim \{2\} \sim \{3\} \succ \{1\} \succ S \sim T,$$

for each other pair of coalitions $\{S, T\}$. The set $\mathcal{CP}(\Omega_N, \succsim)$ of core-partitions contains three elements:

$$P = \big\{\{1, 2\}, \{4, 5\}, \{3\}\big\}, \quad P' = \big\{\{2, 3\}, \{4, 5\}, \{1\}\big\}, \quad P'' = \big\{\{1, 2\}, \{3, 4\}, \{5\}\big\}.$$

Fig. 1 draws the associated decision tree.

8 Conclusion

Other stability concepts identifying subsets of "stable" partitions have been designed for hedonic games (see Aziz and Savani 2016). A possible research agenda would be to replicate our approach for these alternative stability concepts, i.e. adapting the concept to coalitional ranking problems and finding an axiomatic characterization of the social ranking induced by the corresponding stable partitions.

References

Algaba E, Moretti S, Rémila E, Solal P (2021) Lexicographic solutions for coalitional rankings. Soc Choice Welf 57:817–849
Aziz H, Savani R (2016) Hedonic Games. Handb Comput Soc Choice, Chapter 15:356–376

Banerjee S, Konishi H, Sönmez T (2001) Core in a simple coalition formation game. Soc Choice Welf 18:135–153

Banzhaf J III (1964) Weighted voting doesn't work: a mathematical analysis. Rutgers Law Rev 19:317–343

Barberà S, Bossert W, Pattanaik P (2004) Ranking sets of objects. Handb Utility Theory: Ext 2:893–977

Béal S, Rémila E, Solal P (2022) Lexicographic solutions for coalitional rankings based on individual and collective performances. J Math Econ 102:102738

Bernardi G, Lucchetti R, Moretti S (2019) Ranking objects from a preference relation over their subsets. Soc Choice Welf 52:589–606

Brams SJ, Sanver MR (2009) Voting systems that combine approval and preference. In: The mathematics of preference, choice and order: essays in honor of Peter C. Fishburn, Brams SJ, Gehrlein WV, Roberts FS, eds, Springer, Cham 215-237

Dong Y, Li Y, He Y, Chen X (2021) Preference-approval structures in group decision making: axiomatic distance and aggregation. Decis Anal 18:273–295

Farrell J, Scotchmer S (1988) Partnerships. Q J Econ 103:279–297

Iehlé V (2007) The core-partition of a hedonic game. Math Soc Sci 54:176–185

Karakaya M, Klaus B (2017) Hedonic coalition formation games with variable populations: core characterizations and (im)possibilities. Internat J Game Theory 46:435–455

Khani H, Moretti S, Öztürk M (2019) An ordinal Banzhaf index for social ranking. In: Proceedings of the Twenty-Eighth International Joint Conference on Artificial Intelligence (IJCAI-19), pp 378-384

Lucchetti R, Moretti S, Rea T (2022) Coalition formation games and social ranking solutions. In: Proceedings of the 21st International Conference on Autonomous Agents and Multiagent Systems (AAMAS 2022), 862-870

Maskin E (1999) Nash equilibrium and welfare optimality. Rev Econ Stud 66:23–38

Piccione M, Razin R (2009) Coalition formation under power relation. Theor Econ 4:1–15

Thomson W (2001) On the axiomatic method and its recent applications to game theory and resource allocation. Soc Choice Welf 18:327–386

Thomson W (2011) Consistency and its converse: an introduction. Rev Econ Des 15:257–291

Thomson W (2019) On the axiomatics of resource allocation: classifying axioms and mapping out promising directions. In: Laslier JF, Moulin H, Sanver M, Zwicker W (eds) The future of economic design. Springer, Cham