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Direct Iterative Procedures for Consensus Building with Additive Preference Relations Based on the Discrete Assessment Scale

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Abstract
Individual consistency and group consensus are both important when seeking reliable and satisfying solutions for group decision making (GDM) problems using additive preference relations (APRs). In this paper, two new algorithms are proposed to facilitate the consensus reaching process, the first of which is used to improve the individual consistency level, and the second of which is designed to assist the group to achieve a predefined consensus level. Unlike previous GDM studies for consistency and consensus building, the proposed algorithms are essentially heuristic, modify only some of the elements in APRs to reduce the number of preference modifications in the consistency and consensus process, and have modified preferences that belong to the original evaluation scale to make the generated suggestions easier to understand. In particular, the consensus algorithm ensures that the individual consistency level is still acceptable when the predefined consensus level is achieved. Finally, classical examples and simulations are given to demonstrate the effectiveness of the proposed approaches.

Keywords Group decision making · Additive preference relation · Consistency · Consensus

1 Introduction

The aim of GDM is to increase the overall satisfaction level of the ultimate decision for a group of decision makers (DMs) (Cabrerizo et al. 2018; Palomares et al. 2013). Before making a decision, it is important to determine what methods are to be used

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for the DMs to assess the alternatives. Generally speaking, there have been three types of preference structures used: preference orderings (Contreras 2010), utility functions (Gong et al. 2016), and preference relations (also referred to as pairwise comparison matrices) (Li et al. 2016). Compared with specific preference sequences and utility values, DMs are more likely to give pairwise comparisons when there are two alternatives (Cavallo and D'Apuzzo 2009).

Various types of preference relations have been proposed in literature (Orlovsky 1978; Saaty 1980), such as additive preference relations, multiplicative preference relations, and linguistic preference relations. As additive preference relations (APRs) have been one of the most commonly used, they have been extensively studied. Further, for consistency and consensus are the key fundamental issues for effective GDM problems with additive preference relations (APRs) (Gupta 2018; Wu and Xu 2018b), this paper develops two new algorithms to solve the recognized consistency and consensus problems when using classical APRs.

Individual consistency ensures that the DMs are not making random or illogical pairwise comparisons as inconsistent preference relations usually result in poor results. To effectively manage individual consistency with APRs, Herrera-Viedma et al. (2007b) proposed a GDM problem model in which the DMs were able to express their incomplete fuzzy preference relations, and developed an iterative procedure guided by the additive consistency properties to estimate the missing information in the DM's preferences. Zhang et al. (2012) developed linear optimization models to address APRs consistency and consensus issues. Zhang et al. (2018b) proposed three new additive consistency indexes and a number of mixed 0-1 linear programming models that minimized the total adjustment and the number of elements to be adjusted. Wu et al. (2019) developed multi-stage optimization models for both individual consistency and group consensus problems to refine the obtained solutions. Other consistency measures and associated consistency improvement processes have been also proposed (Alonso et al. 2008; Wan et al. 2018).

Consensus is when a decision group achieves sufficient agreement before the selection process. Various schemes have been suggested to facilitate consensus (Dong et al. 2019; Hou 2015; Pérez et al. 2014; Rezaei and Ortt 2013). For example, Dong et al. (2016) proposed a new consensus framework to manage non-cooperative behavior that had a self-management mechanism that generated dynamic weights for the DMs. González-Arteaga et al. (2016) constructed a novel consensus measure based on Pearson's correlation coefficient to measure the consistency of the preferences expressed by the DMs for the alternatives. Li et al. (2019) designed an optimization-based consensus rule to determine the adjustment range for each preference value to ensure individual consistency. Zhang et al. (2018a) developed a novel consensus reaching model for large-scale GDM problems with a heterogeneous preference structure that accounted for individual concerns and satisfaction, and Zhang et al. (2019) proposed a GDM consensus model based on an optimized heterogeneous preference structure, that minimized the loss of heterogeneous preference information between the individual DM preference vectors. Although these previous attempts to manage individual consistency and group consensus have made significant progress, there remain limitations and unanswered research questions, as follows.

(1) The consistency and consensus improvement strategies in existing approaches can be categorized into global and local preference modifications. In the global modifications (Xu et al. 2013; Wu and Xu 2012), as all elements from the original preference relations are adjusted after the improvement process, persuading the DMs to change each element of their APRs could result in a significant cost. If the changes to each element results in a unit cost, to reduce this cost, it is necessary to change only a few elements in the original preference relations to achieve the predefined consistency and consensus (Wu et al. 2018a). Further, after the consistency improvement, the modified DM judgements or preferences are continuous preferences that are no longer equal to any of the original terms from the 0–1 scale. Therefore, it is difficult for the DMs to accept a continuous preference modifications that is different from their initial preferences. If all original preferences are represented by a 0–1 discrete scale, the generated suggestions should also belong to the same scale, which means that any new approach should aim to change only a few elements in each decision round, with all obtained suggestions belonging to the original 0–1 scale.

(2) As individual consistency ensures that the preferences have reasonable pairwise comparisons, individual consistency improvements have been usually applied *before* the consensus reaching process (Chu et al. 2016; Wu and Xu 2012), which means that the subsequent consensus improvements processes are based on the assumption that the individual consistencies are acceptable (Xu et al. 2013). However, it has been found that previously achieved consistencies can be easily lost during the consensus improvement process (Li et al. 2019), which intuitively means that without parallel approaches that can jointly control consistency and consensus building, significant time and costs may be needed for further individual consistency improvements. While an optimization-based consensus rule has been developed to guarantee individual consistency when building consensus (Li et al. 2019), simpler approaches are needed to negate the necessity to repeat the consistency improvement process.

In this paper, two new algorithms are proposed to facilitate the GDM consensus reaching process,the first of which improves the individual consistency, and the second of which assists the group to achieve the predefined consensus level. The contributions of these proposed algorithms are that only some of the APRs elements are modified; and all modified preferences belong to the original evaluation scale, which ensures that the individual consistency is acceptable when the predefined consensus is achieved. The importance of the contributions should be further explained. The willingness to modify the individual preferences is clearly a prerequisite in a decision making process concerning consistency and consensus building. When people make pairwise comparisons, their preferences are expected to be rational and reliable. It is imperative to ask those whose consistency level and/or consensus level are lower than some predefined thresholds. The proposed algorithms provide the DMs suggestions on how to modify their preferences when they make decisions. Only a few of the most promising DMs and alternatives are needed to reconsider. If the DMs use these algorithms and follow the suggested revisions, they are likely to achieve the predefined consistency and consensus at a reasonable cost. In addition, they could get convincing supports for the preference revision process. The proposed algorithms essentially guide the DMs towards a satisfactory status described by consistency and consensus.

The remainder of this paper is organized as follows. In Sect. 2, an algorithm to improve the APR is investigated, in Sect. 3, an algorithm is presented to achieve the consensus process while controlling the individual consistency level for GDMs using APRs, in Sect. 4 several simulations are given to demonstrate the practicality of the proposed algorithms, and conclusions are given in Sect. 5.

2 APR Consistency Improvement

In this section, an algorithm to improve the APR individual consistency is presented, and a classical numerical example is given.

2.1 Consistency Control

Definitions for APR and its associated consistency can be found in previous studies (Herrera-Viedma et al. 2004, 2007a; Wu et al. 2018a). For a finite set of alternatives $X = \{x_1, x_2, \ldots, x_n\}$, an APR about X is denoted by $B = (b_{ij})_{n\times n} \subset X \times X$, $b_{ij} \in [0, 1]$, in which the elements b_{ij} of B have additive reciprocity, $b_{ij} + b_{ji} = 1, i, j = 1, 2, \ldots, n$. It should be noted that two discrete scales have been most commonly used for APRs (Wu et al. 2018a), the numerical scale for which are based on [0, 1],

$$S_{[0,1]} = \{0, 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9, 1\},$$

and a numerical scale based on [0.1, 0.9] which excludes the two extreme end points,

$$S_{[0.1,0.9]} = \{0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9\}.$$

The terms in $S_{[0,1]}$ and $S_{[0.1,0.9]}$ are called discrete preferences in the original scale.

When the preference representation structure is a preference relation, it is necessary to measure the consistency of the preference relation to ensure the intrinsic logic in the individual pairwise comparisons. One of the most frequently used APR consistency definitions has been additive transitivity. Let $B = (b_{ij})_{n\times n} \subset X \times X$ be an APR. Then B is additively consistent if (Herrera-Viedma et al. 2004)

$$b_{ij} = b_{ik} + b_{kj} - 0.5, \quad \forall i, j, k = 1, 2, \ldots, n. \tag{1}$$

Correspondingly, a consistency index to measure the consistency level of B is defined as (Herrera-Viedma et al. 2007a; Wu et al. 2018a):

$$CI(B) = 1 - \frac{4}{n(n-1)(n-2)} \sum_{i=1}^{n-2} \sum_{j=i+1}^{n-1} \sum_{k=j+1}^{n} |b_{ij} + b_{jk} - b_{ik} - 0.5|. \tag{2}$$

For a predefined threshold $\overline{CI}$, if $CI(B) \geq \overline{CI}$, then B is said to have acceptable consistency. It is obvious that when $CI(B) = 1$ (2) holds. A bigger $CI(B)$, indicates that B is more consistent.

As real-world decision-making environments are often complex, it is difficult to obtain an APR B that has perfect consistency $CI(B) = 1$. When the consistency level of a given APR is not satisfactory, individuals are asked to modify their preferences to obtain better consistency and improve the reliability of their judgements. To improve the consistency of B, the basic idea is to determine the position at which the individual needs to modify their preferences.

Let $B = (b_{ij})_{n\times n}$ be an APR and $C = (c_{ij})_{n\times n}$ be a consistent APR corresponding to B. C is used to determine the position that has the largest influence on $CI(B)$. Different approaches can be used to obtain C (Wu et al. 2018a); however, to minimize the deviations between B and C, the following optimization model is constructed,

$$\begin{aligned} &\min \quad d(B,C) = \frac{1}{n\times(n-1)} \sum_{i=1}^{n} \sum_{j=1, j\neq i}^{n} |b_{ij} - c_{ij}| \\ &s.t. \begin{cases} c_{ij} \geq 0, & i,j = 1,2,\cdots,n, \\ c_{ij} + c_{ji} = 1, & i,j = 1,2,\cdots,n, \\ c_{ij} + c_{jk} - c_{ik} = 0.5, & i,j,k = 1,2,\cdots,n. \end{cases} \end{aligned} \tag{3}$$

By solving model (3), a perfectly consistent APR C is obtained. Based on $B = (b_{ij})_{n\times n}$ and $C = (c_{ij})_{n\times n}$, a deviation matrix $D = (d_{ij})_{n\times n}$ is then defined, the elements in which represent the absolute value for the difference between b_{ij} and c_{ij}. where

$$d_{ij} = |b_{ij} - c_{ij}|.$$

The following algorithm is presented to improve the consistency level of B.

Algorithm 1
Input: A given APR $B = (b_{ij})_{n\times n}$, the consistency threshold $\overline{CI}$, and the number of maximum iterations r_{max}.
Output: Modified APR $\overline{B} = (\overline{b}_{ij})_{n\times n}$, $CI(\overline{B})$, and the iterations r and the success flag value $flag$.

Step 1 Set $r = 1$, $B_1 = (b_{ij,1})_{n\times n}$, $B_0 = B$, and calculate the consistency of B_1 . If $CI(B_1) \geq \overline{CI}$, go to Step 6; otherwise go to Step 2.

Step 2 Compute the perfectly consistent APR C that corresponds to B_1 using model (3).

Step 3 Compute the consistency level for $B_r = (b_{ij,r})_{n\times n}$, $CI(B_r)$,

$$CI(B_r) = 1 - \frac{4}{n(n-1)(n-2)} \sum_{i=1}^{n-2} \sum_{j=i+1}^{n-1} \sum_{k=j+1}^{n} |b_{ij,r} + b_{jk,r} - b_{ik,r} - 0.5|. \tag{4}$$

If $CI(B_r) \geq \overline{CI}$, or $r \geq r_{max}$ go to Step 6; otherwise go on to Step 4. Note that when $r = 1$, $CI(B_r)$ has been calculated.

Step 4 Calculate the deviation matrix $D_r = (d_{ij,r})_{n\times n}$, where $d_{ij,r} = |b_{ij,r} - c_{ij}|$.

Step 5 Determine which elements in B_r need to be changed and how to change them. The maximum element for D_r is used to determine the elements to be changed. If $d_{i^*,j^*} = \max\{d_{ij,r}|i < j\}$, then the element for B_r at positions (i^*, j^*) and (j^*, i^*) should be modified. It should be noted that if more than one position is identified in Step 5, either the preferences for all these positions are changed in the same round, or one of the positions is randomly chosen. In this paper, if more than one position is identified, the position with the minimum summation of row number and column number is selected. The guidance rule is described in the following.

Let $O_s = \{0, 0.1, \ldots, 0.5, \ldots, 0.9, 1\}$ be a discrete numerical scale used by participants to express their preferences, with the lth element of O_s being denoted as $O_s(l)$. There are 11 elements in O_s each of which corresponds to a position in the position set $P_s = \{1, 2, \ldots, 10, 11\}$. Let $B_r^l = (b_{ij,r}^l)_{n\times n}$, where

$$b_{ij,r}^l = \begin{cases} b_{ij,r-1}, & i \neq i^*, j \neq j^*, \\ O_s(l), & i = i^*, j = j^*, \\ 1/O_s(l), & i = i^*, j = j^*. \end{cases} \tag{5}$$

Let $Pos_{i^*j^*} = \{l|CI(B_r^l) > CI(B_r)\}$ and $P1_{i^*j^*} = \{l|CI(B_r^l) \geq \overline{CI}\}$. Then there are two possible cases.

Case 1: $Pos_{i^*j^*} \neq \emptyset$, which implies that in this round, the individual consistency level can be improved. There are still two cases needing to be discussed.

(I) $P1_{i^*j^*} \neq \emptyset$, which indicates that the predefined consistency level can be achieved by selecting some l. In this case, an appropriate index $l_{i^*j^*}$ is used to find the modified element in this round. To preserve as much of the initial information as possible, the modified element should be $O_s(l_{i^*j^*})$. It follows that

$$l_{i^*j^*} = \arg\min_l \{d(B_r^l, B)|l \in P1_{i^*j^*}\}, \tag{6}$$

where

$$d(B_r^l, B) = \frac{2}{n \times (n-1)} \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} |b_{ij,r}^l - b_{ij}|. \tag{7}$$

Through the reciprocal property, the element at position (j^*, i^*) is changed to $1 - O_s(l_{i^*j^*})$.

(II) $P1_{i^*j^*} = \emptyset$, that is, $\{l|CI(B_r) < CI(B_r^l) < \overline{CI}\} \neq \emptyset$, which indicates that there is no l such that $CI(B_r^l) \geq \overline{CI}$ but the individual consistency level can be improved. In this case, the modified element at position (i^*, j^*) is also denoted as $O_s(l_{i^*j^*})$, and we have

$$l_{i^*j^*} = \arg\max_l \{CI(B_r^l)|l \in Pos_{i^*j^*}\}. \tag{8}$$

Similarly, the element at position (j^*, i^*) is replaced by $1 - O_s(l_{i^*j^*})$. Note that, if more than one number in $Pos_{i^*j^*}$ meets the above condition, the one which is closest to the original preference should be selected.

In both cases, the modified preference relations in this round are denoted as $B_{r+1} = (b_{ij,r+1})_{n\times n}$, where

$$b_{ij,r+1} = \begin{cases} b_{ij,r}, & i \neq i^*, j \neq j^*, \\ O_s(l_{i^*j^*}), & i = i^*, j = j^*, \\ 1 - O_s(l_{i^*j^*}), & i = j^*, j = i^*. \end{cases} \tag{9}$$

Let $r = r + 1$. If Case 1(I) is met, set *flag* $= 1$ and go to Step 6; otherwise go to Step 3.

Case 2: $Pos_{i^*j^*} = \emptyset$, which means that the selected position (i^*, j^*) is failed. Sort the elements in $\{d_{ij,r} > 0 | i < j\}$ from the biggest to the smallest. Let D_1 be the sorted set of these elements. Further, let S_P be the set of positions, each of which corresponding one element in D_1. Sequentially choose the other positions in S_P and go to the beginning of Step 5. If all the positions have been examined but the given APR does not achieve the predefined consistency level, let $flag = 0$. and go to Step 6.

Step 6 Let $\overline{B} = B_r$. Output $\overline{B}$, $CI(\overline{B})$, r, and $flag$.

Step 7 End.

The steps of Algorithm 1 are clarified by further interpretations. Step 1 determines the consistency level of the original APR and if the original APR does or does not meets the consistency requirements. Step 2 is used to obtain the consistent APR that is associated with the initial APR. Step 3 calculates the current consistency level when $r \geq 2$ and checks whether the termination condition is satisfied. Steps 4 and 5 provide reasonable feedback strategy suggestions, with the deviation matrix obtained in Step 4 being used to identify the positions that need revising, and Step 5 giving accurate advice on what is needed to revise the preferences over the identified positions.

2.2 Numerical Example and Comparisons

Example 1 To demonstrate Algorithm 1, a classical example that has been given in several papers is employed. An APR over four alternatives $\{x_1, x_2, x_3, x_4\}$ is given as (Chiclana et al. 1998; Ma et al. 2006),

$$B = \begin{pmatrix} 0.5 & 0.1 & 0.6 & 0.7 \\ 0.9 & 0.5 & 0.8 & 0.4 \\ 0.4 & 0.2 & 0.5 & 0.9 \\ 0.3 & 0.6 & 0.1 & 0.5 \end{pmatrix}.$$

Setting $\overline{CI} = 0.9$, Algorithm 1 is used to examine and improve the consistency level of B.

Round 1

The individual consistency measure from Wu et al. (2018a) is used, and we have $CI(B_1) = CI(B) = 0.6667$, As $CI(B_1) < \overline{CI}$, the consistent APR C is calculated and then Step 4 in Algorithm 1 is applied, from which the deviation matrix D_1 is obtained;

$$D_1 = \begin{pmatrix} 0 & 0.1531 & 0.0469 & 0 \\ 0.1531 & 0 & 0 & 0.5469 \\ 0.0469 & 0 & 0 & 0.2531 \\ & 0.5469 & 0.2531 & 0.0001 \end{pmatrix}$$

As $(i^*, j^*) = (2, 4)$ and

$$CI(B_1^l) = \{0.5333, 0.5667, 0.6000, 0.6333, 0.6667, 0.7000, 0.7333, 0.7667, 0.8000, 0.8333, 0.8667\},$$

$P1_{i^*j^*} = P1_{24} = \emptyset$ and Case 1(II) for Step 5 applies. Therefore, the modified element at position (2, 4) should be 1 and the modified matrix, denoted as B_2, is given as,

$$B_2 = \begin{pmatrix} 0.5 & 0.1 & 0.6 & 0.7 \\ 0.9 & 0.5 & 0.8 & \underline{1} \\ 0.4 & 0.2 & 0.5 & 0.9 \\ 0.3 & 0 & 0.1 & 0.5 \end{pmatrix}$$

The difference between B and B_2 over the upper triangular part has been underlined. Due to $CI(B_2) = 0.8667 < 0.9$, a second round is needed.

Round 2

As it is found that $(i^*, j^*) = (3, 4)$ and

$$CI(B_2^l) = \{0.7333, 0.7667, 0.8000, 0.8333, 0.8667, 0.9000, 0.9333, 0.9333, 0.9000, 0.8667, 0.8333\},$$

the new element for position (3,4) needs to be selected from {0.5, 0.6, 0.7, 0.8}, and the modified element at position (3, 4) should choose 0.8. The modified APR, denoted as B_3, is given as

$$B_3 = \begin{pmatrix} 0.5 & 0.1 & 0.6 & 0.7 \\ 0.9 & 0.5 & 0.8 & \underline{1} \\ 0.4 & 0.2 & 0.5 & \underline{0.8} \\ 0.3 & 0 & 0.2 & 0.5 \end{pmatrix}$$

Because $CI(B_3) = 0.9 \geq \overline{CI}$, Algorithm 1 is terminated and we have $\overline{B} = B_3$. According to B_3, only two elements in the upper triangular part of B need to be modified.

Table 1 Comparisons for Example 1

Method	$NOC(B, \overline{B})$	$AOC(B, \overline{B})$
Proposed method	2	0.7
Method in Ma et al. (2006)	6	0.8980
Method in Zhang et al. (2012) and Li et al. (2019)	4	0.7

The characteristics of the proposed algorithm are highlighted through a comparison with two classical methods (Ma et al. 2006; Zhang et al. 2012). First, in the iteration-based approach (Ma et al. 2006), when $\Delta t = 0.01$ is set, the revised APR is,

$$\overline{B} = \begin{pmatrix} 0.5000 & \underline{0.2616} & \underline{0.5102} & \underline{0.6282} \\ 0.7384 & 0.5000 & \underline{0.6922} & \underline{0.6694} \\ 0.4898 & 0.3078 & 0.5000 & \underline{0.7024} \\ 0.3718 & 0.3306 & 0.2976 & 0.5000 \end{pmatrix}.$$

Using the optimization-based method (Zhang et al. 2012), the modified APR is

$$\overline{B} = \begin{pmatrix} 0.5000 & \underline{0.2059} & \underline{0.5453} & 0.7000 \\ 0.7941 & 0.5000 & 0.8000 & \underline{0.7522} \\ 0.4547 & 0.2000 & 0.5000 & \underline{0.7127} \\ 0.3000 & 0.2478 & 0.2873 & 0.5000 \end{pmatrix}.$$

The distance between B and the modified APR $\overline{B}$ is a commonly used criterion for comparing different approaches. For APRs, the distance between B and $\overline{B}$ is defined by Eq. (7). This distance measures the amount of change and therefore is further denoted as $AOC(B, \overline{B})$. Another criterion is the number of changes needed, which is defined by

$$NOC(B, \overline{B}) = \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \delta_{ij}, \tag{10}$$

where $\delta_{ij} = 1$ if $|b_{ij} - \overline{b}_{ij}| \neq 0$, and $\delta_{ij} = 0$ if $|b_{ij} - \overline{b}_{ij}| = 0$.

The comparison results are shown in Table 1. The consistency improvement model used in Li et al. (2019) was the same as in Zhang et al. (2012); therefore, the comparison results for these two papers are on the same line. Note also that as the APR has a reciprocal property, the two changes in the revised APR are two times the number of changes in the upper triangular part. However, only counting the number of changes in the upper triangular part does not influence the comparison.

Compared with existing methods, the proposed algorithm was observed to have the following characteristics: (1) fewer elements needed to be changed to achieve the same consistency level; and (2) all modified elements belonged to the original discrete scale. Neither of the other two methods simultaneously had these two characteristics. Nonetheless, these features need to be interpreted with caution. Using the proposed method, it is suggested that the preference for position (2,4) change from 0.4 to 1 and the preference for position (3,4) change from 0.9 to 0.8, which indicates that there is a likely trade-off between the number of modifications and the amount of change required; that is less modifications may mean that some elements need to be drastically modified, such as the case from 0.4 to 1. Although the other two methods had more elements that needed to be modified, the extent of the changes were not so drastic. It must be noted that in practice, the revised APR from the algorithm is used as a reference for the DM to consider the inconsistencies and therefore it is up to the DM to decide on whether the trade-offs between the different objectives are acceptable. In some cases, the DMs may prefer a smaller number of modifications on the pairs, which could result in the greatest consistency improvements, however, in other cases, the DMs may consider that more but smaller modifications are more acceptable.

3 APR Consensus Improvement

The consensus process is dynamic and involves several rounds of discussion and revision on the individual APRs until the desired group agreement is attained (Wu and Xu 2012). When consensus schemes are applied, the DMs need to participate in the discussions to find a consensus solution. The process of reaching consensus has been widely studied and several real world consensus-building applications proposed (Kim 2008; Parreiras et al. 2012). In this section, an algorithm to improve the group consensus for APRs is presented, and a classical numerical example is given.

3.1 Consensus Measure

Let $E = \{e_h | h = 1, 2, \ldots, m\}$ be a set of DMs, and e_h denote the hth DM. Suppose $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_m)^T$ is a weight vector for E, where $\sum_{h=1}^{m} \lambda_h = 1$ and $\lambda_h \in [0, 1]$. Let $B_h = (a_{ij})_{n\times n} (h = 1, 2, \ldots, m)$ be m APRs over a set of alternatives $X = \{x_1, x_2, \ldots, x_n\}$. Suppose B_c is the group APR, $B_c = (b^c_{ij})_{n\times n}$, and

$$b^c_{ij} = \sum_{h=1}^{m} \lambda_h \times b^h_{ij}. \tag{11}$$

Then the group consensus index for B_h is defined as follows (Wu and Xu 2012),

$$GCI(B_h) = 1-d(B_h, B_c) = 1-\frac{2}{n \times (n-1)} \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} |b^h_{ij} - b^c_{ij}|, h = 1, 2, \ldots, m. \tag{12}$$

A larger $\overline{GCI}$ value indicates a higher degree of consensus, and the smaller the value of $\overline{GCI}$, the greater the differences between the individual DM opinions in the group. If $GCI(B_h) = 1$, then the hth DM is in full consensus with the group preference. However, perfect consensus is difficult to achieve in practice.Depending on the actual situation, the DMs establish a threshold $\overline{GCI}$. If $GCI(B_h) \geq \overline{GCI}$, $h = 1, 2, \ldots, m$, an acceptable level of consensus is achieved. Otherwise, another discussion round starts with feedback to bring the individual APRs closer together.

3.2 Consensus Reaching Process

In this subsection, the complete procedure for the achievement of a predefined consensus level is given, an essential step in which is the feedback, which seeks to give the DMs reasonable advice when they have no ideas as to how to change their preferences. An algorithm is proposed to elucidate the consensus reaching process.

Algorithm 2

Input: Individual APRs $\{B_1, B_2, \ldots, B_m\}$, the weight vector $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_m)^T$, the predefined threshold $\overline{CI}$, $\overline{GCI}$, and the maximum iterations r_{max}.

Output: Modified APRs $\{\overline{B}_1, \overline{B}_2, \ldots, \overline{B}_m\}$, $GCI(\overline{B}_h)$, $CI(\overline{B}_h)$, $h = 1, 2, \ldots, m$, the number of iterations r, and the success flag value $flag$.

Step 1 Set $r = 1$, $B_{h,1} = (b^h_{ij,1})_{n\times n}$, $B_{h,1} = B_h$. Use Algorithm 1 to ensure each $B_{h,r}$ has an acceptable individual consistency with $\overline{CI}$, with the modified preference relations being still denoted as $B_{h,r}$.

Step 2 Calculate the group preference relation $B^c_r = (b^c_{ij,r})_{n\times n}$.

Step 3 Compute the consistency index $CI(B_{h,r})$ and the group consensus index $GCI(B_{h,r})$, if
$CI(B_{h,r}) \geq \overline{CI} \wedge GCI(B_{h,r}) \geq \overline{GCI}$, $h = 1, 2, \ldots, m$ or $r \geq r_{max}$, go to step 6; otherwise continue to the next Step.

Step 4 Identify which DMs need to reconsider their preferences and which pair of alternatives have to be checked, for which the following recognition rules are used:

(1) Recognize the DMs e_{h^-} that are contributing the least to the current consensus process. The corresponding index h^- is determined by

$$h^- = \arg\min_h GCI(B_{h,r}). \tag{13}$$

The minimum GCI in the rth round is denoted as GCI_r. If more than one DM is identified in this step, then all are checked for the next identification rule, or one is selected randomly for the next identification rule.

(2) Recognize the places that have to be modified for DM e_{h^-}. Suppose for e_{h^-}, the preference over (i^*, j^*) is the farthest compared to the group preference over (i^*, j^*), then $b^{h^-}_{i^*j^*}$ needs to be modified. Following the above notations, the place (i^*, j^*) is given by,

$$(i^*, j^*) = \arg\max_{i,j} |b^{h^-}_{ij,r} - b^c_{ij,r}|. \tag{14}$$

From (13)–(14), e_{h^-} needs to modify the APR for the pair (x_{i^*}, x_{j^*}).

Step 5 Generate suggestions. This step is critical and is used to modify the preference for the identified DMs. Let $O_s = \{0, 0.1, \ldots, 0.6, \ldots, 0.9, 1\}$, be the set of eleven original terms with the lth element of O_s being denoted as $O_s(l)$. Let $B^l_{h^-,r} = (b^{h^-,l}_{ij,r})_{n\times n}$, where,

$$b^{h^-,l}_{ij,r} = \begin{cases} b^{h^-}_{ij,r-1}, & i \neq i^*, j \neq j^*, \\ O_s(l), & i = i^*, j = j^*, \\ 1/O_s(l), & i = i^*, j = j^*. \end{cases} \tag{15}$$

Let $Pos_{i^*j^*}(e_{h^-}) = \{l | CI(B^l_{h^-,r}) \geq \overline{CI} \wedge GCI(B^l_{h^-,r}) > GCI_r\}$ and $P1_{i^*j^*}(e_{h^-}) = \{l | CI(B^l_{h^-,r}) \geq \overline{CI} \wedge GCI(B^l_{h^-,r}) \geq \overline{GCI}\}$.

Therefore, there are two cases need to to be checked.

Case 1: $Pos_{i^*j^*}(e_{h^-}) \neq \emptyset$, which implies that in the current round, the consensus level can be improved. The following two cases are examined.

(I) $P1_{i^*j^*}(e_{h^-}) \neq \emptyset$ which indicates that both the predefined consistency and consensus can be satisfied by choosing some l. To preserve as much initial information as possible, the modified element should be $O_s(l_{i^*j^*})$, where

$$l_{i^*j^*} = \arg\min_l \{d(B^l_{h^-,r}, B_{h^-}) | l \in P1_{i^*j^*}(e_{h^-})\}. \quad (16)$$

The element on position (j^*, i^*) should be $1 - O_s(l_{i^*j^*})$.

(II) $P1_{i^*j^*}(e_{h^-}) = \emptyset$, that is, $\{l | CI(B^l_{h^-,r}) \geq \overline{CI} \wedge GCI_r < GCI(B^l_{h^-,r}) < \overline{GCI}\} \neq \emptyset$. In this case, only an appropriate index $l_{i^*j^*}$ needs to be chosen to determine the element that needs to be modified in this round. It follows that

$$l_{i^*j^*} = \arg\max_l \{GCI(B^l_{h^-,r}) | l \in Pos_{i^*j^*}(e_{h^-})\}. \quad (17)$$

The element at position (j^*, i^*) is then obtained by $1 - O_s(l_{i^*j^*})$. Note that, if more than one number in $Pos_{i^*j^*}(e_{h^-})$ meets the above condition, the one which is closest to the original preference should be selected.

The modified preference relation for e_{h^-} in this round is denoted as $B_{h^-,r+1} = (b^{h^-}_{ij,r+1})_{n\times n}$, where

$$b^{h^-}_{ij,r+1} = \begin{cases} b^{h^-}_{ij,r}, & i \neq i^*, j \neq j^*, \\ O_s(l_{i^*j^*}), & i = i^*, j = j^*, \\ 1 - O_s(l_{i^*j^*}), & i = j^*, j = i^*. \end{cases} \quad (18)$$

Let $r = r + 1$. If Case 1(I) is met, set $flag = 1$ and go to Step 6; otherwise, go to Step 2.

Case 2: $Pos_{i^*j^*}(e_{h^-}) = \emptyset$, which implies that in the current round, the consensus level can not be improved. Choose the second position for the identified DM e_{h^-} and go to the beginning of Step 5. If all the positions for e_{h^-} can not be improved, choose the DM who has the second minimum GCI, and go to 2) of Step 4. If all the DMs are examined but the consensus level is not satisfactory, let $flag = 0$ and go to Step 6.

Step 6 Let $\overline{B}_h = B_{h,r}$. Output the modified $\overline{B}_h, h = 1, 2, \cdots, m$, $GCI(\overline{B}_h)$, $h = 1, 2, \cdots, m$, r, and $flag$.

Step 7 End.

To fully understand the above algorithm, more explanations are given for some of the steps. Step 1 is used to improve the individual consistency levels to ensure that the APRs satisfy the desired consistency at the beginning of the consensus process. Step 2 computes the group APR using Eq. (11), which is then used to calculate the group consensus index for each APR. Step 3 determines whether the termination conditions have been met. Steps 4 and 5 are central to the feedback as they guide the DMs in changing their preferences; that is, Step 4 sequentially identifies the DMs and the positions that need to be revised. Step 5 assists the DMs to precisely revise their preferences.

Several approaches have been developed to simultaneously address consistency and consensus for preference relations. The most pertinent are examined to clarify the similarities and differences. Dong and Xu discussed an iteration-based consensus model for both multiplicative preference relations and APRs [see Chapter 3 in Dong and Xu (2016)]. Li et al. presented a novel consensus process that had an individual consistency control that allowed for the adjustment range for each preference value to be determined [see Algorithm 2 in Li et al. (2019)]. The similarity between our proposed approach and these two is in the consensus framework. The proposed approach adopts the same framework as Dong and Xu (2016); that is, all individual preference relations are adjusted to an acceptable consistency before the consensus reaching process. After carefully examination, the ideas in the first three steps of the proposed algorithm and that in Li et al. (2019) are almost the same; that is, to improve the individual consistency level and check the termination condition. As they both adopt the same iteration-based consensus process principles, they have some common procedures. However, there are differences in steps 4 and 5. Li et al. (2019) sought to obtain adjustable ranges based on an optimization-based consensus rule (OCR), and in the feedback strategy, it is suggested that all DMs revise their preferences with the assistance of the adjustable ranges. However, the proposed approach has a local feedback strategy and provides suggestions using the original discrete scale.

Quite recently, multi-stage optimization models are proposed to deal with the same problem (Wu et al. 2019). It is worth emphasizing the difference between this paper and Wu et al. (2019). Although the ideas behind the two papers are inconsistent, a common feature of the two papers is using the original discrete scale in the feedback process. An optimization-based approach is utilized in Wu et al. (2019) while this paper adopts a novel iteration-based approach. As the multi-stage optimization model in Wu et al. (2019) is an NP-hard problem, it may be time-consuming to solve it. By contrast, it is computationally easy to apply the proposed approach. Another difference is that Wu et al. (2019) employs a global strategy while this paper adopts a local strategy when updating the references. With the aid of the proposed approach, the consensus problem is boiled down to step-by-step preference changes. Overall, this paper adds to the relevant literature that it provides a quite different way to solve the same problem (i.e., consistency and consensus building).

As mentioned, a core issue in the interactive consensus reaching process is to guide the DMs to change their preferences to attain a higher consensus level. Generally, however, only the final output from iteration-based models or optimization-based models is shown to DMs to reduce the time needed for the interactive decision-making. The above algorithm is an iteration-based model and has distinct features when compared with optimization consensus models. Optimization-based models provide global information on diverse objects such as the amount of change or the number of adjusted preferences, which are then used to calculate the consensus process costs. If the DMs accept the recommendations given by the optimization models, the total feedback strategy costs can be easily estimated. However, the optimal solution may requires all DMs to change their preferences and an identified DM may be asked to change all elements in their preference relations. However, as the DMs may want to gradually change their preferences, a step-by-step approach could be more acceptable.

Although there is no specific goal in iteration-based consensus processes, an implicit objective is pursued; for example, in Algorithm 2, the implicit objective is to minimize the number of modifications. Let $B_h = (b_{ij}^h)_{n\times n}$ and $F_h = (f_{ij}^h)_{n\times n}$ be the initial and modified APRs. Then $|b_{ij}^h - f_{ij}^h| > 0$ indicates that the preference on position for DM e_h has been changed. The optimization model which aims to minimize the number of modifications, is established as follows:

$$\min \sum_{h=1}^{m}\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\delta_{ij}^h$$
$$s.t.\begin{cases} CI(F_h) \geq \overline{CI}, & h=1,2,\ldots,m, & (19-1)\\ GCI(F_h) \geq \overline{GCI}, & h=1,2,\ldots,m, & (19-2)\\ |b_{ij}^h - f_{ij}^h| \leq M\delta_{ij}^h, & i<j, h=1,2,\ldots,m, & (19-3)\\ \delta_{ij}^h \in \{0,1\}, & i<j, h=1,2,\ldots,m, & (19-4)\\ f_{ij}^h \in S_{[0,1]}, & i<j, h=1,2,\ldots,m. & (19-5) \end{cases} \tag{19}$$

where M is a sufficiently large number. Constraints (19-1) and (19-2) require that the modified APRs satisfy both the consistency and consensus levels. As model (19) is equivalent to an integer programming model and is an NP-hard problem, heuristic approaches have been used to determine satisfactory solutions. A heuristic approach may be a rule of thumb to guide individual actions (Pearl 1984). Steps 4 and 5 in Algorithm 2 indicate such a rule. As noted by Pearl (1984), "the heuristics are simple to calculate relative to the complexity of finding a solution and, although they do not necessarily always guide the search in the correct direction, they quite often do." Cao et al. (2008) presented a method to derive a consistency matrix (multiplicative preference relations) from an inconsistent matrix, which they called a heuristic approach. Based on the above analysis, Algorithm 2 could be considered a heuristic approach to solving model (19) as it provides an effective method for finding the positions at which the preferences need revising, but does not guarantee an optimal solution.

In Step 4 of Algorithm 2, if more than one position is identified, then either the preferences for all these positions are changed in the same round, or one of the positions is randomly chosen. In general, the latter approach requires less implementation time. Consider a case in which two positions need to be revised in the same round. If these two positions are modified in the same round, 121 iterations would be needed; however, if only one position is modified in each of the two rounds, only 22, which would save considerable time. Therefore, in this paper, only one of the identified positions is changed in changes in each round. Changing preferences of two positions is used only when it is failed to modify the preference of one position in a round.

In Algorithm 2, the proposed consensus process guarantees that all the adjusted preference relations are of acceptable consistency and that the modified elements still belong to the original discrete scale. While it is difficult to prove the convergence of

Algorithm 2, the simulations in Sect. 4 confirm that the algorithm is able to successfully achieve a predefined consensus level.

3.3 Numerical Example and Comparisons

The following example is given to demonstrate the viability of the proposed consensus algorithm.

Example 2 Consider the example discussed in Chiclana et al. (2008). Suppose there are four DMs $\{e_1, e_2, e_3, e_4\}$ who provide their APRs over $X = \{x_1, x_2, x_3, x_4\}$. Let $\lambda = (\lambda_1, \lambda_2, \lambda_3, \lambda_4)^T = (\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4})^T$ be the individual weight vector for $B_h, h = 1, 2, \cdots, 4$.

$$B_1 = \begin{pmatrix} 0.5 & 0.2 & 0.6 & 0.4 \\ 0.8 & 0.5 & 0.9 & 0.7 \\ 0.4 & 0.1 & 0.5 & 0.3 \\ 0.6 & 0.3 & 0.7 & 0.5 \end{pmatrix}, \quad B_2 = \begin{pmatrix} 0.5 & 0.7 & 0.9 & 0.5 \\ 0.3 & 0.5 & 0.6 & 0.7 \\ 0.1 & 0.4 & 0.5 & 0.8 \\ 0.5 & 0.3 & 0.2 & 0.5 \end{pmatrix},$$

$$B_3 = \begin{pmatrix} 0.5 & 0.3 & 0.5 & 0.7 \\ 0.7 & 0.5 & 0.1 & 0.3 \\ 0.5 & 0.9 & 0.5 & 0.25 \\ 0.3 & 0.7 & 0.75 & 0.5 \end{pmatrix}, \quad B_4 = \begin{pmatrix} 0.5 & 0.25 & 0.15 & 0.65 \\ 0.75 & 0.5 & 0.6 & 0.8 \\ 0.85 & 0.4 & 0.5 & 0.5 \\ 0.35 & 0.2 & 0.5 & 0.5 \end{pmatrix}.$$

Stage 1: Consistency control process
The predefined consistency level is set at $\overline{CI} = 0.9$. Using (2),

$$CI(B_1) = 1, CI(B_2) = 0.7667, CI(B_3) = 0.65, CI(B_4) = 0.8333.$$

As $CI(B_h) < \overline{CI}$, $h = 2, 3, 4$, Algorithm 1 is used to improve the consistency level of these identified APRs. $\overline{B}_1 = B_1$ and the modified APRs are obtained as,

$$\overline{B}_2 = \begin{pmatrix} 0.5 & 0.7 & 0.9 & \underline{0.9} \\ 0.3 & 0.5 & 0.6 & 0.7 \\ 0.1 & 0.4 & 0.5 & 0.8 \\ 0.1 & 0.3 & 0.2 & 0.5 \end{pmatrix}, \overline{B}_3 = \begin{pmatrix} 0.5 & \underline{0.9} & 0.5 & 0.7 \\ 0.1 & 0.5 & 0.1 & 0.3 \\ 0.5 & 0.9 & 0.5 & \underline{0.4} \\ 0.3 & 0.7 & 0.6 & 0.5 \end{pmatrix},$$

$$\overline{B}_4 = \begin{pmatrix} 0.5 & 0.25 & \underline{0.4} & 0.65 \\ 0.75 & 0.5 & 0.6 & 0.8 \\ 0.6 & 0.4 & 0.5 & 0.5 \\ 0.35 & 0.2 & 0.5 & 0.5 \end{pmatrix}.$$

The corresponding consistency indexes are $CI(B_2) = 0.9$, $CI(B_3) = 0.9$, $CI(B_4) = 0.9$. Stage 1 (that is, Step 1 of Algorithm 2 in the proposed paper) is the same as Step 1 of Algorithm 2 in Li et al. (2019). Although both steps aim to obtain adjusted APRs that have acceptable consistencies, the methods used to improve the consistency are different. Therefore, as expected, the modified APRs that have acceptable consistency under the two methods may be different.

Stage 2: Consensus reaching process

Using (11), the group APR is

$$B = \begin{pmatrix} 0.5 & 0.5125 & 0.6 & 0.6625 \\ 0.4875 & 0.5 & 0.55 & 0.625 \\ 0.4 & 0.35 & 0.5 & 0.5 \\ 0.3375 & 0.3750 & 0.5 & 0.5 \end{pmatrix}.$$

Using (12), the consensus indexes are determined as follows,

$$GCI(B_1) = 0.8000, GCI(B_2) = 0.8083, GCI(B_3) = 0.7667, GCI(B_4) = 0.8833.$$

Situation 1

Consider the case used in Li et al. (2019) where $\overline{GCI} = 0.84$. It is known that all DMs are unable to reach the consensus threshold. Using Algorithm 2, the process terminates after 4 rounds.

Round 1

As $GCI(B_{h,1}) < \overline{GCI}$, $h = 1, 2, 3$, move on to Step 3 of Algorithm 2. It is found that $e_{h^-} = e_3$, $(i^*, j^*) = (2, 3)$ and

$$CI(B^l_{2,1}) = \{0.8667, 0.9000, 0.9000, 0.9000, 0.9000, 0.8667, 0.8333, 0.8000, 0.7667, 0.7333, 0.7000\}.$$
$$GCI(B^l_{2,1}) = \{0.7542, 0.7667, 0.7792, 0.7917, 0.8042, 0.8167, 0.8292, 0.8417, 0.8292, 0.8167, 0.8042\}.$$

Because the current consensus level is far from the $\overline{GCI}$ in the first few rounds, more than one position can be chosen; however, when the current consensus level is close to $\overline{GCI}$, only one position is suggested for the identified DMs.

Then $P1_{i^*j^*}(e_2) = \emptyset$ and Case 1(I) in Step 5 of Algorithm 2 applies. Therefore, the modified element in position (2, 3) should be 0.4. As

$$GCI(B_{1,2}) = 0.8125, GCI(B_{2,2}) = 0.8125, GCI(B_{3,2}) = 0.8042,$$
$$GCI(B_{4,2}) = 0.8875,$$

therefore, it follows that $GCI(B_{h,2}) < \overline{GCI}$, $h = 1, 2, 3$, and a second round is needed.

Round 2

It is found that $e_{h^-} = e_3$, $(i^*, j^*) = (1, 2)$ and

$$CI(B^l_{2,2}) = \{0.7000, 0.7333, 0.7667, 0.8000, 0.8333, 0.8667, 0.9000, 0.9000, 0.9000, 0.9000, 0.8667\}.$$
$$GCI(B^l_{2,2}) = \{0.8208, 0.8333, 0.8458, 0.8583, 0.8667, 0.8542, 0.8417, 0.8292, 0.8167, 0.8042, 0.7917\}.$$

The modified element in position (1, 2) should choose 0.6. As

$$GCI(B_{1,3}) = 0.8250, GCI(B_{2,3}) = 0.8000, GCI(B_{3,3}) = 0.8417,$$
$$GCI(B_{4,3}) = 0.9000,$$

it follows that $GCI(B_{h,3}) < \overline{GCI}$, $h = 1, 2$, and a third round is needed.

Round 3

It is found that $e_{h^-} = e_2$, $(i^*, j^*) = (1, 3)$ and

$$\begin{aligned} CI(B^l_{3,3}) &= \{0.7333, 0.7667, 0.8000, 0.8333, 0.8667, 0.9000, 0.9333, 0.9333,\\ &\quad 0.9333, 0.9000, 0.8667\}.\\ GCI(B^l_{3,3}) &= \{0.7875, 0.8000, 0.8125, 0.8250, 0.8375, 0.8500, 0.8375, 0.8250,\\ &\quad 0.8125, 0.8000, 0.7875\}. \end{aligned}$$

The modified element in position (2, 4) should choose 0.5. As

$$GCI(B_{1,4}) = 0.8083, GCI(B_{2,4}) = 0.8500, GCI(B_{3,4}) = 0.8583,$$
$$GCI(B_{4,4}) = 0.9167,$$

it follows that $GCI(B_{h,4}) < \overline{GCI}$, $h = 1$, and a fourth round is needed.

Round 4

It is found that $e_{h^-} = e_1$, $(i^*, j^*) = (2, 3)$ and

$$\begin{aligned} CI(B^l_{1,4}) &= \{0.7000, 0.7333, 0.7667, 0.8000, 0.8333, 0.8667, 0.9000, 0.9333,\\ &\quad 0.9667, 1.0000, 0.9667\}.\\ GCI(B^l_{1,4}) &= \{0.7875, 0.8000, 0.8125, 0.8250, 0.8375, 0.8500, 0.8458, 0.8333,\\ &\quad 0.8208, 0.8083, 0.7958\}. \end{aligned}$$

The modified element in position (2, 3) should choose 0.6. As

$$GCI(B_{1,5}) = 0.8458, GCI(B_{2,5}) = 0.8458, GCI(B_{3,5}) = 0.8708,$$
$$GCI(B_{4,5}) = 0.9125,$$

it follows that as $GCI(B_{h,5}) > \overline{GCI}$, $h = 1, 2, 3, 4$, the predefined consensus level is achieved. The modified preference relations are as follows. $\overline{\overline{B}}_3 = \overline{B}_4$ and the difference between B_h and $\overline{\overline{B}}_h$ over the upper triangular part have been underlined.

Table 2 Consistency and consensus indexes for Example 2

Round	DM	Position	Consistency index	Consensus index
$r = 1$	e_3	(2, 3)	{1.0000, 0.9000, 0.9000, 0.9000}	{0.8125, 0.8125, 0.8042, 0.8875}
$r = 2$	e_3	(1, 2)	{1.0000, 0.9000, 0.9000, 0.9000}	{0.8250, 0.8000, 0.8417, 0.9000}
$r = 3$	e_2	(1, 3)	{1.0000, 1.0000, 0.9000, 0.9000}	{0.8083, 0.8500, 0.8583, 0.9167}
$r = 4$	e_1	(2, 3)	{0.9000, 1.0000, 0.9000, 0.9000}	{0.8458, 0.8458, 0.8708, 0.9125}
$r = 5$	e_1	(1, 2)	{0.9000, 1.0000, 0.9000, 0.9000}	{0.8833, 0.8583, 0.8833, 0.9000}
$r = 6$	e_2	(3, 4)	{0.9000, 0.9000, 0.9000, 0.9000}	{0.8917, 0.8833, 0.8917, 0.8917}
$r = 7$	e_2	(1, 4)	{0.9000, 0.9000, 0.9000, 0.9000}	{0.9000, 0.9083, 0.8833, 0.8875}
$r = 8$	e_3	(2, 4)	{0.9000, 0.9000, 0.9000, 0.9000}	{0.9000, 0.9250, 0.9000, 0.8708}
$r = 9$	e_4	(2, 4)	{0.9000, 0.9000, 0.9000, 0.9000}	{0.9083, 0.9167, 0.9083, 0.8958}
$r = 10$	e_4	(1, 2)	{0.9000, 0.9000, 0.9000, 0.9167}	{0.9063, 0.9188, 0.9104, 0.9021}

$$\overline{\overline{B}}_1 = \begin{pmatrix} 0.5 & 0.2 & 0.6 & 0.4 \\ 0.8 & 0.5 & \underline{0.6} & 0.7 \\ 0.4 & 0.4 & 0.5 & 0.3 \\ 0.6 & 0.3 & 0.7 & 0.5 \end{pmatrix}, \overline{\overline{B}}_2 = \begin{pmatrix} 0.5 & \underline{0.7} & 0.5 & 0.9 \\ 0.3 & 0.5 & 0.6 & 0.7 \\ 0.5 & 0.4 & 0.5 & 0.8 \\ 0.1 & 0.3 & 0.2 & 0.5 \end{pmatrix},$$

$$\overline{\overline{B}}_3 = \begin{pmatrix} 0.5 & \underline{0.6} & 0.5 & 0.7 \\ 0.4 & 0.5 & \underline{0.4} & 0.3 \\ 0.5 & 0.6 & 0.5 & 0.4 \\ 0.3 & 0.5 & 0.6 & 0.5 \end{pmatrix} .\overline{\overline{B}}_4 = \begin{pmatrix} 0.5 & 0.25 & \underline{0.4} & 0.65 \\ 0.75 & 0.5 & 0.6 & 0.8 \\ 0.6 & 0.4 & 0.5 & 0.5 \\ 0.35 & 0.2 & 0.5 & 0.5 \end{pmatrix}.$$

The corresponding consistency indexes for the modified APRs are

$$CI(\overline{\overline{B}}_1) = 0.9000, CI(\overline{\overline{B}}_2) = 0.9000, CI(\overline{\overline{B}}_3) = 0.9000, CI(\overline{\overline{B}}_4) = 0.9000.$$

Note that the revised matrices obtained using Li et al.'s approach are very different from those obtained here. Please see the matrices on Page 328 of Li et al. (2019) for details of the revised matrices.

Situation 2

When $\overline{GCI} = 0.9$, all APRs do not reach the consensus level. Using Algorithm 2, the algorithm terminates after 10 rounds. The consistency and consensus indexes are shown in Table 2 and the modified APRs are

$$\overline{\overline{B}}_1 = \begin{pmatrix} 0.5 & \underline{0.5} & 0.6 & 0.4 \\ 0.5 & 0.5 & \underline{0.6} & 0.7 \\ 0.4 & 0.4 & 0.5 & 0.3 \\ 0.6 & 0.3 & 0.7 & 0.5 \end{pmatrix}, \overline{\overline{B}}_2 = \begin{pmatrix} 0.5 & 0.7 & \underline{0.5} & \underline{0.7} \\ 0.3 & 0.5 & 0.6 & 0.7 \\ 0.5 & 0.4 & 0.5 & \underline{0.6} \\ 0.3 & 0.3 & 0.4 & 0.5 \end{pmatrix},$$

$$\overline{\overline{B}}_3 = \begin{pmatrix} 0.5 & \underline{0.6} & 0.5 & 0.7 \\ 0.4 & 0.5 & \underline{0.4} & \underline{0.5} \\ 0.5 & 0.6 & 0.5 & \underline{0.4} \\ 0.4 & 0.5 & 0.6 & 0.5 \end{pmatrix}, \overline{\overline{B}}_4 = \begin{pmatrix} 0.5 & \underline{0.3} & \underline{0.4} & 0.65 \\ 0.7 & 0.5 & 0.6 & 0.8 \\ 0.6 & 0.4 & 0.5 & 0.5 \\ 0.35 & 0.2 & 0.5 & 0.5 \end{pmatrix}.$$

Table 3 Comparisons for Example 2

Method	$\overline{GCI}$	$NOC(B, \overline{\overline{B}})$
Method in Wu and Xu (2012)	0.9	22
Proposed method	0.9	11
Method in Li et al. (2019)	0.84	23
Proposed method	0.84	4

The difference between B_h and $\overline{\overline{B}}_h$ over the upper triangular part are underlined. The corresponding consistency indexes for the modified APRs are

$$CI(\overline{\overline{B}}_1) = 0.9000, CI(\overline{\overline{B}}_2) = 0.9000, CI(\overline{\overline{B}}_3) = 0.9000, CI(\overline{\overline{B}}_4) = 0.9167.$$

The corresponding consensus indexes for the modified APRs are

$$GCI(\overline{\overline{B}}_1) = 0.9063, GCI(\overline{\overline{B}}_2) = 0.9188, GCI(\overline{\overline{B}}_3) = 0.9104, GCI(\overline{\overline{B}}_4) = 0.9021.$$

The comparison results are shown in Table 3.

Compared with existing approaches (Wu and Xu 2012; Li et al. 2019), the proposed algorithm was observed to have the following characteristics: (1) fewer elements needed to be adjusted to arrive at the same consensus level; and (2) all modified elements belonged to the original discrete scale. However, neither of the two methods being compared simultaneously had these two characteristics.

4 Simulation Experiment

In this section, an algorithm is given to conduct consensus simulations with individual consistency control. In the simulation experiment, the initial APRs were randomly generated. Then five factors are considered: the consistency level, the consensus level, the number of changes (NOC), the amount of changes (AOC) and the ratio of change (ROC).

Algorithm 3

Intput: The number of alternatives n, the number of simulations N, the number of DMs m, $\overline{GCI}$, and $\overline{CI}$.

Output: The average number of changes $\overline{NOC}$, the average amount of change $\overline{AOC}$, the average ratio of change $\overline{ROC}$.

Step 1 For each m and fixed N, N groups are respectively generated, each of which have m matrixes. Let $\{B_{1,r}, B_{2,r}, \ldots, B_{m,r}\}$, where $B_{h,r} = (b^h_{ij,r})_{n\times n}$ be the matrix for the hth DM in the rth simulation.

Step 2 For a given $\overline{CI}$ and $\overline{GCI}$, Algorithm 2 is used to achieve consensus for each group, and let $\overline{B}_{h,r} = (\overline{b}^h_{ij,r})_{n\times n}$ be the modified APRs corresponding to $B_{h,r}$. Let NOC^h_r, AOC^h_r and ROC^h_r be the number of changes, the amount of changes and the ratio of changes for the hth DM in the rth simulation, respectively. For the rth

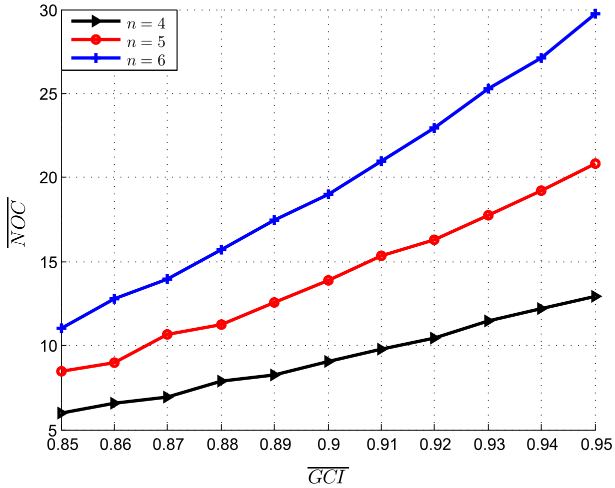

Fig. 1 $\overline{NOC}$ $(m = 4, N = 500)$

simulation, the number of changes NOC_r is calculated by

$$NOC_r = \sum_{h=1}^{m} NOC_r^h = \sum_{h=1}^{m}\sum_{i=1}^{n-1}\sum_{j=i+1}^{n} \delta_{ij,r}^h, \tag{20}$$

where $\delta_{ij,r}^h = 1$ if $|b_{ij,r}^h - \overline{b}_{ij,r}^h| \neq 0$, and $\delta_{ij,r}^h = 0$ if $|b_{ij,r}^h - \overline{b}_{ij,r}^h| = 0$. For the rth simulation, the amount of changes AOC_r is obtained by

$$AOC_r = \sum_{h=1}^{m} AOC_r^h = \sum_{h=1}^{m} \frac{2}{n \times (n-1)} \sum_{i=1}^{n-1}\sum_{j=i+1}^{n} |b_{ij,r}^h - \overline{b}_{ij,r}^h|. \tag{21}$$

For the rth simulation, the ratio of changes ROC_r is obtained by

$$ROC_r = \sum_{h=1}^{m} ROC_r^h = \sum_{h=1}^{m} \frac{NOC_r^h}{n(n-1)/2}. \tag{22}$$

Step 3 Calculate the average number of changes $\overline{NOC} = \frac{\sum_{r=1}^{N} NOC_r}{N}$, the amount of changes $\overline{AOC} = \frac{\sum_{r=1}^{N} AOC_r}{N}$, and the ratio of change $\overline{ROC} = \frac{\sum_{r=1}^{N} ROC_r}{N}$.
Step 4 End.

In the following, various simulations were conducted based on Algorithm 3. The predefined consistency threshold was set at $\overline{CI} = 0.9$, and the predefined consensus threshold took a value from $\{0.85, 0.86, \ldots, 0.95\}$, and $N = 500$. The results for the

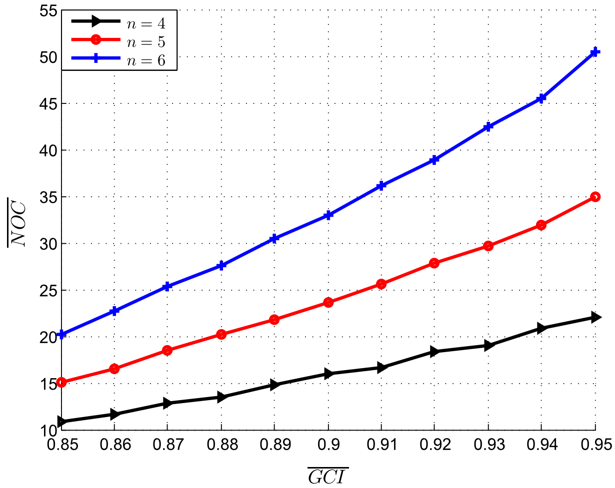

Fig. 2 $\overline{NOC}(m = 6, N = 500)$

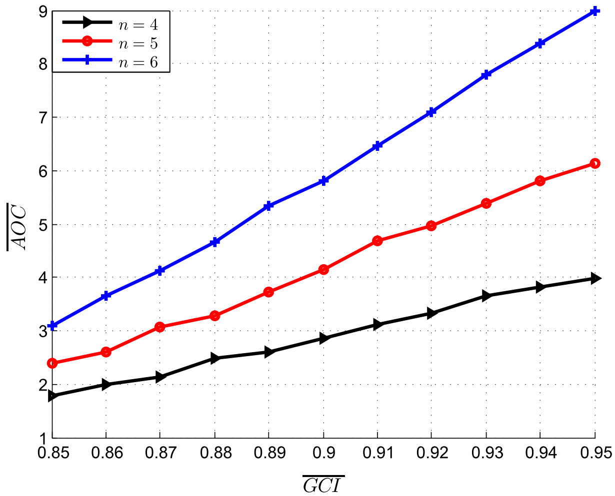

Fig. 3 $\overline{AOC}$ $(m = 4, N = 500)$

average number of changes $\overline{NOC}$ are shown in Figs. 1 and 2, the average amount of changes $\overline{AOC}$ are shown in Figs. 3 and 4, and the average ratio of changes $\overline{ROC}$ are shown in Figs. 5 and 6.

The observations from these simulation experiments were as follows:

(1) From Figs. 1, 2, 3, 4, 5 and 6, it can be seen that with an increase in $\overline{GCI}$, the $\overline{NOC}$, $\overline{AOC}$ and $\overline{ROC}$ of the DMs had increasing trends. It was also observed that with an increase in m, it was more difficult for the groups to reach consensus. These figures show that our proposed approach provided an effective way to build GDM consensus with APRs.

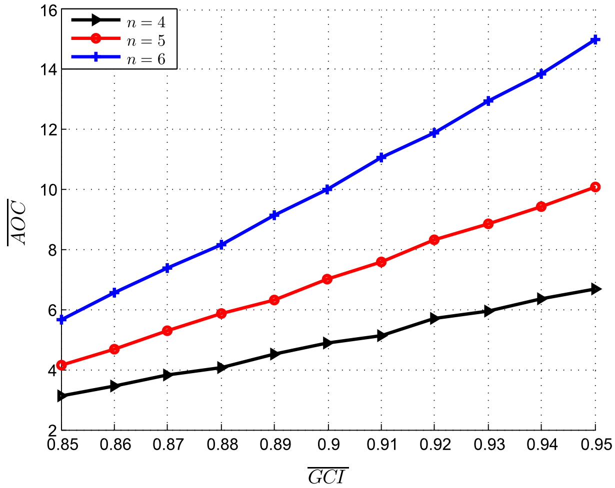

Fig. 4 $\overline{AOC}$ $(m = 6, N = 500)$

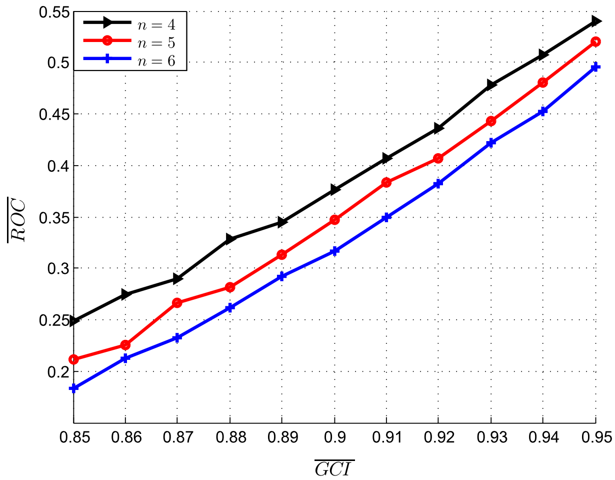

Fig. 5 $\overline{ROC}$ $(m = 4, N = 500)$

(2) For a fixed $\overline{GCI}$, Figs. 1, 2, 3 and 4 show that the bigger the number of alternatives, the bigger the values for NOC and AOC; Figs. 5 and 6 show that the smaller the number of alternatives, the bigger the values for ROC.

The simulation experiments demonstrated that the proposed algorithms successfully achieved the goals in all cases; that is, a predefined consensus level was achieved using Algorithm 2 and the individual consistency was effectively controlled.

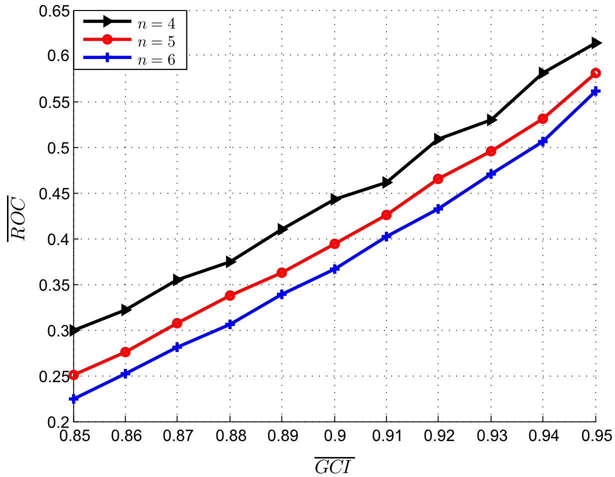

Fig. 6 $\overline{ROC}$ $(m = 6, N = 500)$

5 Conclusions

Consistency and consensus based approaches are powerful techniques for dealing with GDM problems with preference relations. This paper developed corresponding algorithms for GDM with APRs to resolve the problems associated with the difficulties DMs have with traditional element modification and to reduce the number of modifications for each APR. Therefore, the main contributions of this paper are:

(1) A consistency improving algorithm was proposed to ensure that the APRs were of acceptable consistency and that the revised preference relation judgements belonged to the original scale used by participants.
(2) A novel consensus reaching algorithm for APRs was proposed that ensured that each individual APR was of acceptable consistency when the predefined consensus threshold was achieved, and that the revised judgements belonged to the original scale.
(3) Finally, the simulation experiments and comparative studies demonstrated the effectiveness of the proposed approaches.

Note that the proposed approaches are applicable to large-scale GDMs since they are computationally easy. Future research will consider a comparative study of various consensus models for large-scale GDMs.

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