Check for
updates

Probabilistic Option Prioritizing in the Graph Model for Conflict Resolution

Leandro Chaves Rêgo [1,2] · **Giannini Italino Alves Vieira** [3]

Abstract

Probabilistic preferences have been proposed in the graph model for conflict resolution (GMCR) to accommodate both situations in which a decision maker (DM) vacillates in which criteria to use when comparing two scenarios and also situations in which there is uncertainty regarding who will act as a DM representative. In this paper, we propose two option prioritizing techniques to obtain probabilistic preferences in the GMCR more efficiently. The crisp preference option prioritizing relies on an ordered sequence of preference statements that determines the crisp preference relation. In the first proposed technique, a probability distribution is associated with a class of ordered sequences of preference statements of the DM, where the probability of state s being preferred to state t by the DM consists of the sum of the probabilities of the ordered sequences of preference statements where s is preferred to t according to the crisp preference based on the corresponding ordered sequence of preference statements. In the second technique proposed, we allow for uncertainty both on the set of preference statements considered by a DM and also on which preference statement within the set is the most important one for him. An application is provided to illustrate the use of these techniques.

✉ Leandro Chaves Rêgo
leandro@dema.ufc.br

Giannini Italino Alves Vieira
italino@ufc.br

[1] Statistics and Applied Math Department, Universidade Federal do Ceará, Fortaleza, CE 60455-760, Brazil

[2] Graduate Programs in Statistics and Management Engineering, Universidade Federal de Pernambuco, Recife, PE 50740-550, Brazil

[3] Universidade Federal do Ceará, Crateús, CE 63700-000, Brazil

1 Introduction

The Graph Model for Conflict Resolution (GMCR) is a mathematical tool developed by Kilgour et al. (1987) and Kilgour and Hipel (2005) which is useful in the modelling and analysis of strategic conflicts. In the GMCR, the agents (players), usually called decision makers (DMs), have some available set of binary options which they must decide which ones they will take in the conflict. Each subset of options which are taken by the DMs in the conflict determines a conflict state. As some combinations of options may be impossible to occur in the conflict, there may be some infeasible states and these are not considered in the GMCR. The DMs have preferences about the feasible conflict states and from these preferences, the stability of the states is analyzed. In the GMCR, there are several stability notions that vary according to how many steps ahead the conflict is analyzed and on what are the credible moves of DMs in a conflict. The most commonly used stability notions are: Nash stability (Nash 1950), metarational stability (Howard 1971), symmetric metarational stability (Howard 1971) and sequential stability (Fraser and Hipel 1979). Newer stability notions have been proposed to model different human behaviors in conflict situations (Rêgo and Vieira 2017; Rêgo and Vieira 2019; Zeng et al. 2006).

There exists a series of works that introduce different preference structures in the GMCR to better capture some real-world conflicts. To handle conflicts in which there is no knowledge regarding what is the preference of some DM between two states, Li et al. (2004) introduced preference uncertainty in the GMCR. Bashar et al. (2012, 2015) introduced fuzzy preferences in the GMCR to model situations in which DMs might not have a clear-cut preference between two states. Grey numbers can also be used to model preference uncertainty within the GMCR (Kuang et al. 2015). There are conflicts in which there might be uncertainty regarding the type of a DM or the DM himself may be uncertain about his own preferences. This situation was modeled by Rêgo and Santos (2015) using a probabilistic preference structure. This work was further extended to handle vagueness in the elicitation of probabilistic preferences by Rêgo and Santos (2018). Two-level of preference strength was introduced in the GMCR by Hamouda et al. (2006), while Xu et al. (2009) extended this model to allow for a multi-level preference ranking structure within the GMCR. Finally, Yu et al. (2017, 2018) introduced a hybrid preference structure which combines fuzzy preferences and preference strength within the GMCR.

Besides choosing the appropriate preference structure, it is important to have a method to obtain the DMs' preferences in order to apply the GMCR in real conflicts. There are three main methods used to determine DMs' preferences in a conflict situation: option weighting, option prioritizing and direct ranking (Peng et al. 1997). All of these methods were developed to obtain crisp preferences (binary relations) in the GMCR, which is the most usual preference structure used in the GMCR. Some papers in the GMCR literature adjust the idea of option prioritizing to elicit other preference structures in the GMCR. More specifically, in Bashar et al. (2014), the authors propose an option prioritizing technique to more efficiently determine fuzzy preferences by using fuzzy truth values for DMs' preference statements, and in Yu et al. (2016), the option prioritizing technique was used to determine unknown preferences in the

GMCR. Hou et al. (2015) and Zhao and Xu (2017) defined an option prioritizing method to obtain three-level preferences and grey preferences, respectively.

In this paper, we are interested in proposing option prioritizing methods to obtain probabilistic preferences in the GMCR, called here GMCRP. The GMCRP is an extension of the GMCR, proposed by Rêgo and Santos (2015), where DMs do not simply prefer one state over another, but prefer one state over another with a certain probability. Probabilistic preferences in decision-theoretical models date back to the work of Luce (1958), in which there is uncertainty regarding the choice criteria used by DMs when choosing between two available objects. In Fishburn and Gehrlein (1977), social choice lottery rules are analyzed for the elections of two candidates with voters who may be uncertain about whom they prefer, where the uncertainty of a voter over a particular candidate is reflected by a probability. For a comparison between some stochastic models for binary choices under risk, see (Wilcox 2008).

Although probabilistic preferences have a long history on decision theory, it was only in the work of Rêgo and Santos that this preference structure was introduced in the GMCR. However, in Rêgo and Santos (2015) no technique is proposed to determine the probabilistic preferences needed to model a conflict situation using the GMCRP. Motivated by this fact and by the ideas employed in the works of Bashar et al. (2014), Yu et al. (2016), Hou et al. (2015) and Zhao and Xu (2017), in this paper, we propose two option prioritizing techniques to obtain probabilistic preferences in the GMCR. In the first one, instead of a single ordered sequence of preference statements, a class of ordered sequences of preference statements is considered, each one of them is associated with a probability which describes the chance with which the criteria chosen by the DM considers the priority of the statements according to the order of the sequence. Thus, the probability of state s being preferred to state t by the DM is given by the sum of the probabilities of the ordered sequences of preference statements that induce a crisp preference relation according to which s is preferred to t by the DM. In the second technique proposed, there exists two sources of uncertainty: one regarding what is the set of preference statements considered by the DM and another regarding what preference statement within a given set is the most important one for the DM.

The paper is organized as follows. In Sect. 2, we present the background, recalling the definitions of GMCR, GMCRP and option prioritizing. In Sect. 3, we propose two techniques for preference elicitation based on probabilistic option prioritizing in the GMCRP. In Sect. 4, we present an application of the proposed techniques to highlight their usefulness. Finally, in Sect. 5, we finish the paper with the main conclusions found.

2 Theoretical Background

In this section, we briefly recall some concepts in the literature about the GMCR that will be essential for the good understanding of this paper (Fang et al. 1993).

2.1 GMCR

The GMCR is a theoretical model of conflict analysis proposed by Kilgour et al. (1987) that is extremely useful for modeling strategic conflict situations in a simple and effective way. Formally, the GMCR consists of a collection of directed graphs, defined on the same set of states, where each graph represents how some DM in the conflict may change the current conflict state. In this paper, we view a conflict state as being determined by some combination of options that may or not be taken by the DMs in the conflict. The collection of all DMs directed graphs create the integrated graph model for the conflict. Together with this collection of graphs, in order to analyze the states' stability some preference structure is imposed on the set of feasible conflict states. This preference structure can, for example, be an asymmetric binary relation as in the original GMCR, a fuzzy preference relation (Bashar et al. 2012), an uncertain preference relation (Li et al. 2004), a grey preference relation (Kuang et al. 2015), a probabilistic preference relation (Rêgo and Santos 2015) or an upper and lower probabilistic preference (Rêgo and Santos 2018). In this work, we are particularly interested in the probabilistic preference structure.

In the GMCR, the finite set of DMs is represented by $N = \{1, 2, \ldots, n\}$, and the finite set of feasible states is represented by $S = \{s, s_1, \ldots, s_v\}$. In the GMCR, there are several stability concepts which aim to represent the most varied types of DMs' behaviors in conflict situations. Some of these concepts are Nash stability (Nash 1950), metarational stability (GMR) (Howard 1971), symmetric metarational stability (SMR) (Howard 1971), sequential stability (SEQ) (Fraser and Hipel 1979) and symmetric sequential stability (SSEQ) (Rêgo and Vieira 2017). These concepts or variants of them have been proposed for several preference structures that have been used in the GMCR. Thus the stability analysis relies on the information about the preferences which DMs hold about the conflict states. Therefore, in the case of the GMCRP, knowing how to derive probabilistic preferences more efficiently is fundamental for the stability analysis of strategic conflicts modeled by this extension of the GMCR.

2.2 Crisp Preferences

In most of the works in the GMCR literature, it is assumed that DM i's preferences can be expressed by an asymmetric binary relation, denoted by $\succ_i$, and called here by crisp preference (Kreps 1988), where $s_1 \succ_i s_2$ indicates that DM i strictly prefers state s_1 over state s_2. From this definition of strict preference, another preference relation for DM i, called weak preference, is derived and denoted by $\succeq_i$. Thus, DM i weakly prefers state s_1 over s_2 whenever he does not strictly prefer state s_2 over s_1, i.e., $s_1 \succeq_i s_2$ if and only if $s_2 \nsucc_i s_1$. Finally, if none of the states are strictly preferred over the other, then these states are said to be indifferent, which is denoted by $\sim_i$. Formally, $s_1 \sim_i s_2$ if and only if $s_1 \nsucc_i s_2$ and $s_2 \nsucc_i s_1$.

In the next subsection, we recall the technique of crisp option prioritizing, which is used to determine a crisp preference relation for the DMs in a conflict situation. Later on, we adapt this technique in two different ways to determine a probabilistic preference relation for DMs in the GMCRP.

2.3 Crisp Option Prioritizing

The technique of obtaining preferences through option prioritizing was proposed by Peng et al. (1997). This technique relies on specifying preferences by asking the DMs to provide an ordered sequence of preference affirmations (from most to least important) which are boolean combinations of the options of the conflict. The results recalled here can be found in more details in (Peng et al. 1997).

Let $\mathcal{O}$ be the set of all binary options available in the conflict, i.e., $\mathcal{O} = \cup_{i=1}^{n} \mathcal{O}_i$, where $\mathcal{O}_i = \{o_1^i, o_2^i, \ldots, o_{m_i}^i\}$ is the set of options available to DM i in the conflict, where $\mathcal{O}_i \neq \emptyset$ and $\mathcal{O}_i \cap \mathcal{O}_j = \emptyset$ for $i \neq j$, i.e., each DM must have at least one option in the conflict and no two DMs can share the same option, because if these sets had some option in common, then it would not be possible to distinguish which DM took the option.[1] A state is defined in terms of the options as being a combination of all the options in $\mathcal{O}$ that are taken or not by the DMs in the conflict. In option prioritizing, each DM involved in the conflict is required to provide an ordered sequence of preference statements. Such preference statements are expressed in terms of logical propositions involving the options in the set $\mathcal{O}$. For example, a preference statement can be o_1^i iff $(o_2^j \& -o_3^j)$, i.e. if the DM expresses this statement he prefers that option o_1^i for DM i be taken if and only if among options o_2^j and o_3^j only option o_2^j is taken by DM j. Thus, if this is the most important statement for the DM, then the states that satisfy it should be preferred by the DM to those that do not. Thus, if Ω is a preference statement, then for each conflict state s, $\Omega(s)$ can be either true (T) or false (F) depending on whether the options which are taken in s satisfy or not the statement Ω. Thus, Peng et al. (1997) define a crisp preference relation, based on an ordered sequence of preferences statements as follows:

Definition 1 Let $s, s_1 \in S$ and $\mathcal{C}_i = (\Omega_1, \Omega_2, \ldots, \Omega_{w_i})$ be an ordered sequence of all preferences statements given by DM $i \in N$. State $s \in S$ is strictly preferred to state $s_1 \in S$ by DM i if and only if there is positive integer $t \leq w_i$ such that $\Omega_1(s) = \Omega_1(s_1)$, $\Omega_2(s) = \Omega_2(s_1), \ldots, \Omega_{t-1}(s) = \Omega_{t-1}(s_1)$, $\Omega_t(s) = T$ and $\Omega_t(s_1) = F$.

In other words, the above definition establishes that state s is preferable to state s_1 by DM i if according to the ordered sequence of preference statements, state s is the first one to uniquely satisfy a preference statement. Otherwise, state s_1 is weakly preferred to state s by DM i.

Peng et al. (1997) also provide the following scoring scheme to determine the preference relation of DMs over the set of states. This scheme provides an equivalent ranking of preferences to the crisp preference relation method and consider a score under each preference statement.

Let w be the total number of preference statements and let $\Psi(s)$ be the score of state s. This score is defined taking into account the set of preference statements provided by the DM. Let $\Psi_t(s)$ be the score of state s based on the statement Ω_t defined as:

---

[1] The assumption that the sets of DMs' options are disjoint for different DMs can always be made without loss of generality by assuming that options are labeled by the DM who takes it.

$$\Psi_t(s) = \begin{cases} 2^{w-t}, & \text{if } \Omega_t(s) = T, \\ 0, & \text{otherwise.} \end{cases}$$

for every $0 < t \leq w$. Then, $\Psi(s)$ is defined as follows:

$$\Psi(s) = \sum_{t=1}^{w} \Psi_t(s).$$

The crisp preference relation is then defined by $s \succ s_1$ if and only if $\Psi(s) > \Psi(s_1)$. It is easy to verify that this definition is equivalent to Definition 1.

In this paper, we adapt this crisp option prioritizing technique to determine a probabilistic preference relation over the set of states for the DMs. For that, let us recall about probabilistic preferences in the GMCR in the next subsection.

2.4 Probabilistic Preferences

In a recent work on the GMCR, Rêgo and Santos (2015) made an extension of the GMCR, denoted by GMCRP, in that a new preference structure was proposed. This preference structure aimed to better capture conflicts in the real world in which DMs do not necessarily prefer one state to another with probability 1. More specifically, Rêgo and Santos proposed to use a probabilistic preference, in which DMs prefer one state over another with a certain probability, i.e., the notation $P_i(s, s_1)$ expresses the chance with which DM i strictly prefers state s over s_1. Such a probability is formally defined on $S \times S$ and satisfies the following properties:

(a) $P_i(s, s_1) \geq 0, \ \forall s, s_1 \in S$,
(b) $P_i(s, s) = 0, \ \forall s \in S$,
(c) $P_i(s, s_1) + P_i(s_1, s) \leq 1, \ \forall s, s_1 \in S$.

In others words, (a) requires that for any two states in S, there is no negative probabilities. Condition (b) requires that no DM i can strictly prefer one state over itself with positive probability and condition (c) requires that the sum of the probabilities that some DM i strictly prefers state s to other state s_1 and strictly prefers s_1 over s is at most equal to 1. This later condition allows DMs to be indifferent between states with positive probability.

Probabilistic preferences in the GMCR are specially useful when DMs can be of different types and one does not know with certainty which type will actively participate in the conflict. These types may represent different criteria that a DM can use while evaluating the states or can model the possibility of different representatives that may act on behalf of the DM. As pointed out in Rêgo and Santos (2015), probabilistic preferences may also be used when a DM does not represent a monolithic party and one can obtain a probability distribution over the set of possible representatives of this DM. Other types of preference structures have been proposed in the GMCR to account for uncertainty. The one that is most similar to probabilistic preferences is fuzzy preferences, as proposed by (Bashar et al. 2012, 2014). As argued in Rêgo and Santos (2015), in spite of probabilistic and fuzzy preferences being expressed as a

number between 0 and 1, they have different interpretations which lead to different stability notions. Moreover, data of previous choices made by DMs may be used to estimate probabilistic preferences and due to its more widely known interpretation, probabilities are easier to elicit when compared to fuzzy numbers. More discussion about differences between fuzzy and probabilistic preferences in the GMCR can be found in (Rêgo and Santos 2015).

In Rêgo and Santos (2015), the probabilistic preferences are assumed as given in the problem. However, in order to perform the stability analysis of the conflict, these probabilistic preferences should be elicited from the DMs. Therefore, techniques that help to determine such probabilistic preferences are very important for the practical application of the GMCRP. Based on this open problem, we propose in the next section two techniques for determining such probabilistic preferences. These techniques rely heavily on the crisp option prioritizing technique mentioned in Sect. 2.3.

3 Probabilistic Option Prioritizing

In this section, we present two techniques to determine probabilistic preferences for the DMs in the GMCRP. All techniques are based on the idea of crisp option prioritizing. The first technique, called *Random Sequence of Preference Statements*, consists in replacing a single ordered sequence of preference statements by a class of ordered sequences of preference statements together with a probability distribution over such class. Intuitively, the probability of a given ordered sequence of preference statements represents the chance with which this ordered sequence will be chosen by the DM to evaluate his preferences over the states. The second technique, called *Random Sets of Random Preference Statements* consists of both a probability distribution on a class of sets of preference statements and probability distributions over each one of these sets. Intuitively, the probability of some set of preference statements represents the chance with which this set will be considered by the DM when evaluating states and the probability of some preference statement given a set is the chance with which this statement is considered the most important within the set.

The probabilities described in the techniques for obtaining probabilistic preferences are assumed as given in this paper. In real-world conflicts, they can be obtained either by estimating the probabilities using past conflict data or by means of an elicitation method to obtain an *a priori* probability from experts, such as one of those that can be found in (O'Hagan et al. 2006; Renooij 2001).

3.1 Random Sequences of Preference Statements

In this first technique, we define a probability distribution over a class of ordered sequences of preference statements. From the probability distribution over the class, we define a technique to determine the probabilistic preference of one state over another for a given DM. It is important to highlight that according to this technique the sequences of preference statements are ordered in terms of priority given by the DM.

Formally, let Φ_i be a collection of ordered sequence of preference statements given by DM i, i.e.,

$$\Phi_i = \{(\Omega_1^1, \Omega_2^1, \ldots, \Omega_{w_1}^1), (\Omega_1^2, \Omega_2^2, \ldots, \Omega_{w_2}^2), \ldots, (\Omega_1^r, \Omega_2^r, \ldots, \Omega_{w_r}^r)\},$$

for some positive integer r. Let $\mathcal{P}_i$ be a probability distribution over Φ_i and let $\mathcal{P}_i((\Omega_1^k, \Omega_2^k, \ldots, \Omega_{w_k}^k)) = \alpha_k$, $1 \le k \le r$ be the probability of the ordered sequence of preference statements $(\Omega_1^k, \Omega_2^k, \ldots, \Omega_{w_k}^k)$. Let $\Phi_i^{s,s_1} = \{(\Omega_1^k, \Omega_2^k, \ldots, \Omega_{w_k}^k) \subset \Phi_i : \Omega_1^k(s) = \Omega_1^k(s_1), \Omega_2^k(s) = \Omega_2^k(s_1), \ldots, \Omega_{u-1}^k(s) = \Omega_{u-1}^k(s_1), \Omega_u^k(s) = T$ and $\Omega_u^k(s_1) = F$, for some positive integer $u \le w_k\}$ be the class of all ordered sequences of preference statements such that according to crisp option prioritizing based on these sequences state s is preferable to state s_1 for DM i, i.e., $s \succ_i s_1$ according to crisp option prioritizing based on any ordered sequence of preference statements in Φ_i^{s,s_1}. Thus the probabilistic preference based on random sequences of preference statements, P_i^b, for DM i, is defined as follows:

Definition 2 Let $s, s_1 \in S$ and $i \in N$. Then, the probabilistic preference of state s over state s_1 for DM i, denoted by $P_i^b(s, s_1)$, is given by

$$P_i^b(s, s_1) = \sum_{(\Omega_1^k, \Omega_2^k, \ldots, \Omega_{w_k}^k) \subseteq \Phi_i^{s,s_1}} \alpha_k.$$

Thus, Definition 2 establishes that the probabilistic preference of state s over state s_1 for DM i is the sum of all probabilities of the ordered sequences of preference statements according to which DM i has a crisp preference for state s over state s_1.

The use of the random sequences of preference statements is helpful to determine probabilistic preferences when there is more than one possible representative (such as a diplomat) who can be chosen to negotiate on behalf of the DM (such as a country) and there is uncertainty about who will effectively be the negotiator. In this case, the probability $\mathcal{P}_i((\Omega_1^k, \Omega_2^k, \ldots, \Omega_{w_k}^k))$ can be interpreted as the probability of being selected a representative who has the ranking of most important preference statements given by the sequence $(\Omega_1^k, \Omega_2^k, \ldots, \Omega_{w_k}^k)$.

The following example illustrates how we can obtain probabilistic preferences using the technique of random sequences of preference statements.

Example 1 Consider a hypothetical strategic conflict composed of two decision makers, DM i and DM j, and four states, say s, s_1, s_2 and s_3. Suppose that in this conflict DM j has no uncertainty regarding the order of importance of her preference statements, but DM i has uncertainty about his preference statements. Consider that $\Phi_i = \{(\Omega_1^1, \Omega_2^1), (\Omega_1^2, \Omega_2^2, \Omega_3^2)\}$ and suppose that $\alpha_1 = \mathcal{P}_i(\Omega_1^1, \Omega_2^1) = 0.3$ and $\alpha_2 = \mathcal{P}_i(\Omega_1^2, \Omega_2^2, \Omega_3^2) = 0.7$. Also admit that $\Omega_1^1(s) = \Omega_1^1(s_1) = \Omega_2^1(s_1) = \Omega_1^1(s_3) = \Omega_1^2(s_1) = \Omega_2^2(s_2) = \Omega_2^1(s_3) = \Omega_2^2(s_3) = \Omega_3^2(s_1) = \Omega_3^2(s_2) = F$ and $\Omega_1^2(s) = \Omega_2^2(s) = \Omega_2^2(s_1) = \Omega_2^2(s_2) = \Omega_2^1(s) = \Omega_1^1(s_2) = \Omega_2^1(s_2) = \Omega_1^1(s_3) = \Omega_3^2(s) = \Omega_3^2(s_3) = T$. As DM j has no uncertainty regarding the order of importance of her preference statements, her preferences can be obtained by the crisp option

Table 1 Probabilistic preferences of DM i according to the random sequences of preference statements technique

$P_i^b(\cdot, \cdot)$	s	s_1	s_2	s_3
s	0.0	1.0	0.7	0.7
s_1	0.0	0.0	0.0	0.7
s_2	0.3	1.0	0.0	1.0
s_3	0.3	0.3	0.0	0.0

prioritizing technique of Peng et al. (1997). On the other hand, from Definition 2, we can obtain the probabilistic preferences of DM i. Table 1 presents the probabilistic preferences of DM i in this conflict, where each cell contains the probability with which DM i prefers the row state to the column state. Note that, for example:

$$P_i^b(s, s_1) = \alpha_1 + \alpha_2 = \mathcal{P}_i(\Omega_1^1, \Omega_2^1) + \mathcal{P}_i(\Omega_1^2, \Omega_2^2, \Omega_3^2) = 1.0,$$
$$P_i^b(s, s_3) = \alpha_2 = \mathcal{P}_i(\Omega_1^2, \Omega_2^2, \Omega_3^2) = 0.7 \text{ and}$$
$$P_i^b(s_2, s) = \alpha_1 = \mathcal{P}_i(\Omega_1^1, \Omega_2^1) = 0.3.$$

Thus, whenever faced with the problem of choosing between s and s_3, there is a probability of 0.7 that s will be chosen. Other values can be interpreted likewise.

Theorem 1 shows that the crisp option prioritizing technique is a particular case of the random sequence of preference statements technique.

Theorem 1 *Let $(\Omega_1^k, \Omega_2^k, \ldots, \Omega_{w_k}^k) \in \Phi_i$. Assume that $\succ_i$ is the crisp preference relation derived from the crisp option prioritizing based on $\mathcal{C}_i = (\Omega_1^k, \Omega_2^k, \ldots, \Omega_{w_k}^k)$, where the sequence of preferences statements is ordered from the most to the least preferred. If $\mathcal{P}_i(\mathcal{C}_i) = 1$, then*

(a) $s \succ_i s_1$ *if and only if* $P_i^b(s, s_1) = 1$ *and*
(b) $s \sim_i s_1$ *if and only if* $P_i^b(s, s_1) = 0$ *and* $P_i^b(s_1, s) = 0$.

Proof For (a), note that $s \succ_i s_1$ if and only if $(\Omega_1^k, \Omega_2^k, \ldots, \Omega_w^k) \in \Phi_i^{s, s_1}$, which is equivalent to $P_i^b(s, s_1) = 1$. For (b), note that $P_i^b(s, s_1)$ can only assume the values 0 or 1. Thus, $s \sim_i s_1$ if and only if $s \not\succ_i s_1$ and $s_1 \not\succ_i s$, which, by Part (a), is equivalent to $P_i^b(s, s_1) = 0$ and $P_i^b(s_1, s) = 0$. $\qquad\square$

3.2 Random Sets of Random Preference Statements

This second technique mixes features of both previous techniques. Now instead of defining a probability distribution over a class of ordered sequences of preference statements, we define a probability distribution over a class of sets of random preference statements.

Let $\varphi_i = \{\mathbb{C}_i^1, \mathbb{C}_i^2, \ldots, \mathbb{C}_i^r\}$ be a class of sets of preference statements, where $\mathbb{C}_i^k = \{\Omega_1^k, \Omega_2^k, \ldots, \Omega_{w_k}^k\}$, for some k in $\{1, \ldots, r\}$. Suppose that $\mathbf{P_i}$ is a probability

distribution on φ_i representing the chance with which some set of statements, $\mathbb{C}_i^k$, is chosen and that $\mathbb{P}_i^k$ is a probability distribution on $\mathbb{C}_i^k$ representing the uncertainty with which some preference statement is considered the most important one for the DM in the set $\mathbb{C}_i^k$. Let ${}^k\varphi_i^{s,s_1} = \{\Omega_v^k \in \mathbb{C}_i^k : \Omega_v^k(s) = T \text{ and } \Omega_v^k(s_1) = F\}$. Thus, the chance that state s is preferred over state s_1 by DM i is the chance that some preference statement in ${}^k\varphi_i^{s,s_1}$ is considered the most important for some k.

Formally, the probabilistic preference based on random sets of random preference statements, $P_i^c(s, s_1)$, for DM i, is defined as follows:

Definition 3 Let $s, s_1 \in S$ and $i \in N$. Then, the probabilistic preference of state s over state s_1 for DM i, denoted by $P_i^c(s, s_1)$, is given by

$$P_i^c(s, s_1) = \sum_{k=1}^{r} \mathbf{P_i}(\mathbb{C}_i^k) \cdot \sum_{\Omega_v^k \in {}^k\varphi_i^{s,s_1}} \mathbb{P}_i^k(\Omega_v^k).$$

The use of this second technique is indicated when there is uncertainty regarding who will effectively be the negotiator and also each possible negotiator has uncertainty about which preference statement is the most important for him.

Example 2 illustrates how we can obtain probabilistic preferences using the technique of random sets of random preference statements.

Example 2 Consider once more the hypothetical strategic conflict of Example 1. As before, only DM i has uncertainty about his preferences. Consider that $\varphi_i = \{\mathbb{C}_i^1, \mathbb{C}_i^2\}$ and suppose that $\mathbf{P}_i(\mathbb{C}_i^1) = 0.3$ and that $\mathbf{P}_i(\mathbb{C}_i^2) = 0.7$. Also admit that $\mathbb{C}_i^1 = \{\Omega_1, \Omega_2\}$ and $\mathbb{C}_i^2 = \{\Omega_1, \Omega_3\}$. Moreover, assume that $\mathbb{P}_i^1(\Omega_1) = 0.2$, $\mathbb{P}_i^1(\Omega_2) = 0.8$, $\mathbb{P}_i^2(\Omega_1) = 0.4$ and $\mathbb{P}_i^2(\Omega_3) = 0.6$. Finally, consider that $\Omega_1(s) = \Omega_1(s_1) = \Omega_2(s_1) = \Omega_2(s_3) = \Omega_3(s_3) = F$ and $\Omega_2(s) = \Omega_1(s_2) = \Omega_2(s_2) = \Omega_1(s_3) = \Omega_3(s) = \Omega_3(s_1) = \Omega_3(s_2) = T$. Thus, it follows that:

$$\begin{aligned}
&{}^1\varphi_i^{t,t} = {}^1\varphi_i^{t,s_2} = {}^1\varphi_i^{s_1,t} = \emptyset, \forall t \in S,\\
&{}^1\varphi_i^{s,s_1} = {}^1\varphi_i^{s,s_3} = {}^1\varphi_i^{s_2,s_3} = \{\Omega_2\},\\
&{}^1\varphi_i^{s_2,s} = {}^1\varphi_i^{s_3,s} = {}^1\varphi_i^{s_3,s_1} = \{\Omega_1\},\\
&{}^1\varphi_i^{s_2,s_1} = \{\Omega_1, \Omega_2\},\\
&{}^2\varphi_i^{t,t} = {}^2\varphi_i^{t,s_2} = {}^2\varphi_i^{s,s_1} = {}^2\varphi_i^{s_1,s} = \emptyset, \forall t \in S,\\
&{}^2\varphi_i^{s_2,s_1} = {}^2\varphi_i^{s_2,s} = {}^2\varphi_i^{s_3,s} = {}^2\varphi_i^{s_3,s_1} = \{\Omega_1\} \text{ and}\\
&{}^2\varphi_i^{s,s_3} = {}^2\varphi_i^{s_1,s_3} = {}^2\varphi_i^{s_2,s_3} = \{\Omega_3\}.
\end{aligned}$$

From Definition 3, we can obtain the probabilistic preferences of DM i. Table 2 presents the probabilistic preferences of DM i in this conflict, where each cell contains the probability with which DM i prefers the row state to the column state. Note that, for example:

$P_i^c(\cdot, \cdot)$	s	s_1	s_2	s_3
s	0.0	0.24	0.0	0.66
s_1	0.0	0.0	0.0	0.42
s_2	0.34	0.58	0.0	0.66
s_3	0.34	0.34	0.0	0.0

Table 2 Probabilistic preferences of DM i according to random sets of random preference statements technique

$$P_i^c(s, s_1) = \mathbf{P}_i(\mathbb{C}_i^1) \cdot \mathbb{P}_i^1(\Omega_2) = (0.3) \cdot (0.8) = 0.24,$$
$$P_i^c(s_2, s_1) = \mathbf{P}_i(\mathbb{C}_i^1) \cdot (\mathbb{P}_i^1(\Omega_1) + \mathbb{P}_i^1(\Omega_2)) + \mathbf{P}_i(\mathbb{C}_i^2) \cdot \mathbb{P}_i^2(\Omega_1)$$
$$= (0.3) \cdot (0.2 + 0.8) + (0.7) \cdot (0.4) = 0.58,$$
$$P_i^c(s, s_3) = \mathbf{P}_i(\mathbb{C}_i^1) \cdot \mathbb{P}_i^1(\Omega_2) + \mathbf{P}_i(\mathbb{C}_i^2) \cdot \mathbb{P}_i^2(\Omega_3)$$
$$= (0.3) \cdot (0.8) + (0.7) \cdot (0.6) = 0.66 \text{ and}$$
$$P_i^c(s_2, s) = \mathbf{P}_i(\mathbb{C}_i^1) \cdot \mathbb{P}_i^1(\Omega_1) + \mathbf{P}_i(\mathbb{C}_i^2) \cdot \mathbb{P}_i^2(\Omega_1) = (0.3) \cdot (0.2) + (0.7) \cdot (0.4) = 0.34.$$

It is easy to verify that the three requirements ((a), (b) and (c)) of the definition of probabilistic preferences proposed in Rêgo and Santos (2015) are satisfied if one determines it according to any of the two proposed techniques of this paper. In the next section, we illustrate the use of the two probabilistic option prioritizing techniques described in this section applying it in a conflict regarding water export.

4 Application

To illustrate the two proposed techniques for determining probabilistic preferences, we consider a modification of an actual conflict that has been studied in the GMCR literature, which is called the Gisborne water export conflict. The idea of the Gisborne water export dispute described below was taken from Li et al. (2004) and Rêgo and Santos (2015) and can be found in greater details in these papers.

The strategic conflict in Lake Gisborne's water export began in the mid-1990s, when a Newfoundland and Labrador company, called Canada Wet Incorporated, submitted a project to export water from Lake Gisborne, located in south coast of the province of Newfoundland and Labrador-Canada. Analyzing the benefits to the local economy from water exports, the Newfoundland and Labrador local government initially approved the project at the end of 1996. However, some environmental groups opposed the project's approval, arguing that the consequences of the project would be serious not only for the environment, but also for the local culture. Subsequently, the Federal Government of Canada also opposed the idea of exporting water and introduced its own policy of banning water exports from Canada's major river basins. Faced with the position of the federal government, in late 1999 the local government created a new bill that also prohibited water export from Lake Gisborne, causing Canada Wet Incorporated to abandon the Gisborne project.

Table 3 Gisborne conflict states

F								
o_1^F . continue	N	Y	N	Y	N	Y	N	Y
GP								
o_2^{GP} . lift	N	N	Y	Y	N	N	Y	Y
S								
o_3^S. appeal	N	N	N	N	Y	Y	Y	Y
State	s_1	s_2	s_3	s_4	s_5	s_6	s_7	s_8

Table 4 Preferences statements of DM **GP**

Statements	Interpretation
o_2^{GP}	GP lifts the ban on exportation of water
$-o_1^F$	F does not continue to ban the export of water
$-o_3^S$	S does not appeal for continuing the project

In 2001, aiming at developing the local economy, the new Prime Minister of New-foundland and Labrador decided to revise the Gisborne project, which again led to further criticism from environmentalists. In late 2001, the Justice Minister promised that no legislation aimed at lifting the ban on water exports in Canada's lakes would be introduced during the next session of the legislature.

Note that the Gisborne conflict may be represented by a GMCR with three DMs, i.e., the first DM is the federal government and groups that are opposed to water exports (denoted by **F**), the second DM is the provincial government (denoted by **GP**) and the third DM represents the companies that want to export water from lakes in Canada (denoted by **S**).

Table 3 illustrates the DMs involved in the conflict, their options, and the feasible states of the conflict. In this table, o_1^F (continue) represents the option of DM **F** to continue with the agreement to ban water exports in Canada, o_2^{GP} (lifts) represents the option of DM **GP** to lift the ban on water exports and o_3^S represents the option of DM **S** to appeal for continuation of the Gisborne Project. In this table, *Y* denotes that the option is taken and *N* denotes that the option is not taken.

As in the work of Rêgo and Santos (2015), we assume that the preferences of DMs **F** and **S** are crisp and can be determined through the existing techniques in the GMCR literature as, for example, the crisp option prioritizing technique (Peng et al. 1997). However, the preferences of DM **GP** are uncertain as they depend on whether the current government is economically oriented or has more environmental orientation. We assume that such uncertainty is modeled by a probabilistic preference and illustrate the use of both techniques proposed to obtain such preferences.

To illustrate the techniques, consider that DM **GP** can think of three preference statements to rank the states, denoted by $\mathbb{C}_{GP} = \{o_2^{GP}, -o_1^F, -o_3^S\}$. The meaning of these preferences statements are described in Table 4, where—represents the negation of an option.

Table 5 Truth values of the preferences statements in $\mathbb{C}_{GP}$ at the feasible states

Statements	s_1	s_2	s_3	s_4	s_5	s_6	s_7	s_8
o_2^{GP}	F	F	T	T	F	F	T	T
$-o_1^F$	T	F	T	F	T	F	T	F
$-o_3^S$	T	T	T	T	F	F	F	F

This set of preference statements is used in Hou et al. (2015) to illustrate a technique based on option prioritizing to obtain three-level preferences in the GMCR. Note that at each state these preference statements can be either true (T) or false (F). Table 5 illustrates the truth value of each preference statement of **GP** at each one of the eight feasible states of the conflict.

4.1 Illustration of Random Sequence of Preference Statements

In the work of Rêgo and Santos (2015), they consider that in the Lake Gisborne conflict, DM **GP** can be economically oriented with probability p and environmentally oriented with probability $1 - p$. In Rêgo and Santos (2015), the determination of this probabilistic preference is done directly by pairwise comparison of states. To illustrate the technique of random sequence of preference statements to determine probabilistic preferences, we consider two ordered sequences of preference statements for DM **GP**, as follows: $\Phi_{GP} = \{(o_2^{GP}, -o_1^F, -o_3^S), (-o_2^{GP}, o_1^F, -o_3^S)\}$.

The sequences of preference statements $(o_2^{GP}, -o_1^F, -o_3^S)$ and $(-o_2^{GP}, o_1^F, -o_3^S)$ represent the economically and environmentally oriented types of DM GP, respectively. For example, in the economically oriented type, DM GP gives high priority to lifting the ban on water exports, medium priority to DM F abandoning the agreement to ban water exports from Canada and also prefers that DM S does not appeal for continuation of the Gisborne project due to the legal costs that it may incur.

Suppose that there is uncertainty about whether DM **GP** is economically or environmentally oriented. Such uncertainty is modeled by a probability distribution $\mathcal{P}_{GP}$ on Φ_{GP} such that $\mathcal{P}_{GP}((o_2^{GP}, -o_1^F, -o_3^S)) = p$ and $\mathcal{P}_{GP}(-o_2^{GP}, o_1^F, -o_3^S) = 1 - p$. Table 6 displays the probabilistic preference of DM **GP** derived from $\mathcal{P}_{GP}$ according to Definition 2. Each cell of Table 6 represents the probabilistic preference of DM **GP** for the row state over the column state. Thus, for example, $P_{GP}^b(s_3, s_7)$ is given by the sum of the probabilities of the sequences of preference statements that induce a crisp preference for s_3 over s_7 for DM **GP**. From Table 5, one can see that both sequences satisfy this requirement. Thus, we have that $P_{GP}^b(s_3, s_7) = \mathcal{P}_{GP}((o_2^{GP}, -o_1^F, -o_3^S)) + \mathcal{P}_{GP}(-o_2^{GP}, o_1^F, -o_3^S) = 1$. For another example, since only the sequence $(-o_2^{GP}, o_1^F, -o_3^S)$ induces a crisp preference for s_5 over s_3 for DM **GP**, it follows that

$$P_{GP}^b(s_5, s_3) = \mathcal{P}_{GP}((-o_2^{GP}, o_1^F, -o_3^S)) = 1 - p.$$

The other cells in Table 6 can be obtained in a similar way. Note that the probabilistic preferences obtained in Table 6 are the same as those obtained in (Rêgo and Santos 2015).

Table 6 Probabilistic preferences of DM **GP** according to the random sequences of preference statements technique

$P_{GP}^b(\cdot,\cdot)$	s_1	s_2	s_3	s_4	s_5	s_6	s_7	s_8
s_1	0.0	p	$1-p$	$1-p$	1.0	p	$1-p$	$1-p$
s_2	$1-p$	0.0	$1-p$	$1-p$	$1-p$	1.0	$1-p$	$1-p$
s_3	p	p	0.0	p	p	p	1.0	p
s_4	p	p	$1-p$	0.0	p	p	$1-p$	1.0
s_5	0.0	p	$1-p$	$1-p$	0.0	p	$1-p$	$1-p$
s_6	$1-p$	0.0	$1-p$	$1-p$	$1-p$	0.0	$1-p$	$1-p$
s_7	p	p	0.0	p	p	p	0.0	p
s_8	p	p	$1-p$	0.0	p	p	$1-p$	0.0

4.2 Illustration of Random Sets of Random Preference Statements

To illustrate the second technique, suppose that both economically and environmentally oriented types of DM **GP** have uncertainty regarding which of the preference statements in their sets is the most important one. Thus, let $\varphi_{GP} = \{\{o_2^{GP}, -o_1^F, -o_3^S\}, \{-o_2^{GP}, o_1^F, -o_3^S\}\}$, $\mathbf{P_{GP}}(\{o_2^{GP}, -o_1^F, -o_3^S\}) = p$, $\mathbf{P_{GP}}(\{-o_2^{GP}, o_1^F, -o_3^S\}) = 1 - p$, $\mathbb{P}_{GP}^1(o_2^{GP}) = 0.6$, $\mathbb{P}_{GP}^1(-o_1^F) = 0.3$, $\mathbb{P}_{GP}^1(-o_3^S) = 0.1$, $\mathbb{P}_{GP}^2(-o_2^{GP}) = 0.5$, $\mathbb{P}_{GP}^2(o_1^F) = 0.3$ and $\mathbb{P}_{GP}^2(-o_3^S) = 0.2$.

Table 7 displays the probabilistic preference of DM **GP** derived from $\mathbf{P}_{GP}$ and $\mathbb{P}_{GP}$ according to Definition 3. Each cell of Table 7 represents the probabilistic preference of DM **GP** for the row state over the column state. Thus, for example, $P_{GP}^c(s_3, s_7)$ is given by the sum of the probabilities of the preference statements which are true at s_3 and false at s_7. From Table 5, one can see that only $-o_3^S$ satisfies this requirement. Thus, we have that

$$
\begin{aligned}
P_{GP}^c&(s_3, s_7) \\
&= \mathbf{P}_{GP}(\{o_2^{GP}, -o_1^F, -o_3^S\})\mathbb{P}_{GP}^1(-o_3^S) + \mathbf{P}_{GP}(\{o_2^{GP}, -o_1^F, -o_3^S\})\mathbb{P}_{GP}^2(-o_3^S) \\
&= p(0.1) + (1-p)(0.2) = 0.2 - 0.1p.
\end{aligned}
$$

For another example, since the only preference statement which is true at s_5 and false at s_3 is $-o_2^{GP}$, it follows that

$$
P_{GP}^c(s_5, s_3) = \mathbf{P}_{GP}(\{-o_2^{GP}, o_1^F, -o_3^S\})\mathbb{P}_{GP}^2(-o_2^{GP}) = 0.5(1 - p).
$$

The other cells in Table 7 can be obtained in a similar way.

5 Conclusion

Inspired by the crisp option prioritizing technique (Peng et al. 1997) used to determine crisp preferences in the GMCR, we proposed two techniques to determine probabilistic preferences in the GMCRP (Rêgo and Santos 2015). Both techniques allow for the

Table 7 Probabilistic preferences of DM **GP** according to the random sets of random preference statements technique

$P_{GP}^{c}(\cdot,\cdot)$	s_1	s_2	s_3	s_4	s_5	s_6	s_7	s_8
s_1	0.0	0.3p	$0.5 - 0.5$p	$0.5 - 0.2$p	$0.2 - 0.1$p	$0.2 + 0.2$p	$0.7 - 0.6$p	$0.7 - 0.3$p
s_2	$0.3 - 0.3$p	0.0	$0.8 - 0.8$p	$0.5 - 0.5$p	$0.5 - 0.4$p	$0.2 - 0.1$p	$1.0 - 0.9$p	$0.7 - 0.6$p
s_3	0.6p	0.9p	0.0	0.3p	$0.2 + 0.5$p	$0.2 + 0.8$p	$0.2 - 0.1$p	$0.2 + 0.2$p
s_4	$0.3 + 0.3$p	0.6p	$0.3 - 0.3$p	0.0	$0.5 + 0.2$p	$0.2 + 0.5$p	$0.5 - 0.4$p	$0.2 - 0.1$p
s_5	0.0	0.3p	$0.5 - 0.5$p	$0.5 - 0.2$p	0.0	0.3p	$0.5 - 0.5$p	$0.5 - 0.2$p
s_6	$0.3 - 0.3$p	0.0	$0.8 - 0.8$p	$0.5 - 0.5$p	$0.3 - 0.3$p	0.0	$0.8 - 0.8$p	$0.5 - 0.5$p
s_7	0.6p	0.9p	0.0	0.3p	0.6p	0.9p	0.0	0.3p
s_8	$0.3 + 0.3$p	0.6p	$0.3 - 0.3$p	0.0	$0.3 + 0.3$p	0.6p	$0.3 - 0.3$p	0.0

application of the GMCRP without requiring the pairwise comparison of states by the DMs. In the technique of random sequences of preference statements, the DM must provide a class of ordered sequences of preference statements. Each sequence may represent a type of a DM and there is uncertainty regarding which type will actually participate in the conflict. Finally, in the second technique, the DM must provide a class of unordered sets of preference statements and both a probability over the class and probability distributions over each one of the sets in the class. This latter case includes uncertainty both regarding a DM type and each type has uncertainty regarding which preference statement is the most important one in his set. We showed that the random sequences of preference statements technique generalizes the crisp option prioritizing technique. Finally, we illustrate the use of Both techniques in a modified version of the Lake Gisborne conflict (Li et al. 2004).

It is important to point out that not all probabilistic preferences as described in Rêgo and Santos (2015) can be obtained by the two techniques proposed in this article. However, using the crisp option prioritizing method one cannot obtain intransitive preference relations. In spite of this limitation of the option prioritizing technique, it is an easy to apply method, that is general enough for a wide range of applications. Likewise, the probabilistic option prioritizing techniques proposed in this work can make the use of probabilistic preferences in the GMCR more widespread among practitioners.

Recently, Xu et al. (2018) investigated degrees of stabilities using an option-oriented attitude analysis in the GMCR, as a future work, one may try to extend this work to a probabilistic setting using one of the probabilistic option prioritizing techniques proposed in this work.

References

Bashar MA, Kilgour DM, Hipel KW (2012) Fuzzy preferences in the graph model for conflict resolution. IEEE Trans Fuzzy Syst 20(4):760–770. https://doi.org/10.1109/TFUZZ.2012.2183603

Bashar MA, Kilgour DM, Hipel KW (2014) Fuzzy option prioritization for the graph model for conflict resolution. Fuzzy Sets Syst 246:34–48. https://doi.org/10.1016/j.fss.2014.02.011

Bashar MA, Obeidi A, Kilgour DM, Hipel KW (2015) Modeling fuzzy and interval fuzzy preferences within a graph model framework. IEEE Trans Fuzzy Syst 24(4):765–778. https://doi.org/10.1109/TFUZZ.2015.2446536

Fang L, Hipel KW, Kilgour DM (1993) Interactive decision making: the graph model for conflict resolution, vol 11. Wiley, New York

Fishburn PC, Gehrlein WV (1977) Towards a theory of elections with probabilistic preferences. Econometr J Econ Soc 45(8):1907–1924. https://doi.org/10.2307/1914118

Fraser NM, Hipel KW (1979) Solving complex conflicts. IEEE Trans Syst Man Cybern 9(12):805–816. https://doi.org/10.1109/TSMC.1979.4310131

Hamouda L, Kilgour DM, Hipel KW (2006) Strength of preference in graph models for multiple-decision-maker conflicts. Appl Math Comput 179(1):314–327. https://doi.org/10.1016/j.amc.2005.11.109

Hou Y, Jiang Y, Xu H (2015) Option prioritization for three-level preference in the graph model for conflict resolution. In: International conference on group decision and negotiation. Springer, pp 269–280. https://doi.org/10.1007/978-3-319-19515-5-21

Howard N (1971) Paradoxes of rationality: games, metagames, and political behavior. MIT press, Cambridge

Kilgour DM, Hipel KW (2005) The graph model for conflict resolution: past, present, and future. Group Decis Negot 14(6):441–460. https://doi.org/10.1007/s10726-005-9002-x

Kilgour DM, Hipel KW, Fang L (1987) The graph model for conflicts. Automatica 23(1):41–55. https://doi.org/10.1016/0005-1098(87)90117-8

Kreps D (1988) Notes on the theory of choice. Westview, Boulder

Kuang H, Bashar MA, Hipel KW, Kilgour DM (2015) Grey-based preference in a graph model for conflict resolution with multiple decision makers. IEEE Trans Syst Man Cybern Syst 45(9):1254–1267. https://doi.org/10.1109/TSMC.2014.2387096

Li KW, Hipel KW, Kilgour DM, Fang L (2004) Preference uncertainty in the graph model for conflict resolution. IEEE Trans Syst Man Cyber Part A Syst Hum 34(4):507–520. https://doi.org/10.1109/TSMCA.2004.826282

Luce RD (1958) A probabilistic theory of utility. Econometr J Econometr Soc 26(2):193–224. https://doi.org/10.2307/1907587

Nash JF (1950) Equilibrium points in n-person games. Proc Nat Acad Sci 36(1):48–49

O'Hagan A, Buck CE, Daneshkhah A, Eiser JR, Garthwaite PH, Jenkinson DJ, Oakley JE, Rakow T (2006) Uncertain judgements: eliciting experts' probabilities. Wiley, New York

Peng X, Hipel KW, Kilgour DM, Fang L (1997) Representing ordinal preferences in the decision support system GMCR II. In: 1997 IEEE international conference on systems, man, and cybernetics. Computational cybernetics and simulation, vol 1. IEEE, pp 809–814. https://doi.org/10.1109/ICSMC.1997.626196

Rêgo LC, Santos AM (2015) Probabilistic preferences in the graph model for conflict resolution. IEEE Trans Syst Man Cybern Syst 45(4):595–608. https://doi.org/10.1109/TSMC.2014.2379626

Rêgo LC, Santos AM (2018) Upper and lower probabilistic preferences in the graph model for conflict resolution. Int J Approx Reason 98:96–111. https://doi.org/10.1016/j.ijar.2018.04.008

Rêgo LC, Vieira GIA (2017) Symmetric sequential stability in the graph model for conflict resolution with multiple decision makers. Group Decis Negot 26(4):775–792. https://doi.org/10.1007/s10726-016-9520-8

Rêgo LC, Vieira GIA (2019) $Maximin_h$ stability in the graph model for conflict resolution for bilateral conflicts. IEEE Trans Syst Man Cybern Syst. https://doi.org/10.1109/TSMC.2019.2917824

Renooij S (2001) Probability elicitation for belief networks: issues to consider. Knowl Eng Rev 16(3):255–269. https://doi.org/10.1017/S0269888901000145

Wilcox NT (2008) Stochastic models for binary discrete choice under risk: a critical primer and econometric comparison. In: Risk aversion in experiments. Emerald Group Publishing Limited, pp 197–292

Xu H, Hipel KW, Kilgour DM (2009) Multiple levels of preference in interactive strategic decisions. Discrete Appl Math 157(15):3300–3313. https://doi.org/10.1016/j.dam.2009.06.032

Xu P, Xu H, Ke GY (2018) Integrating an option-oriented attitude analysis into investigating the degree of stabilities in conflict resolution. Group Decis Negot 27(6):981–1010. https://doi.org/10.1007/s10726-018-9585-7

Yu J, Hipel KW, Kilgour DM, Zhao M (2016) Option prioritization for unknown preference. J Syst Sci Syst Eng 25(1):39–61. https://doi.org/10.1007/s11518-015-5282-0

Yu J, Hipel KW, Kilgour DM, Fang L (2017) Fuzzy strength of preference in the graph model for conflict resolution with two decision makers. In: 2017 IEEE International Conference on Systems, Man, and Cybernetics (SMC), IEEE, pp 3574–3577, https://doi.org/10.1109/SMC.2017.8123186

Yu J, Hipel KW, Kilgour DM, Fang L (2018) Fuzzy levels of preference strength in a graph model with multiple decision makers. Fuzzy Sets Syst. https://doi.org/10.1016/j.fss.2018.12.016

Zeng DZ, Fang L, Hipel KW, Kilgour DM (2006) Generalized metarationalities in the graph model for conflict resolution. Discrete Appl Math 154(16):2430–2443. https://doi.org/10.1016/j.dam.2006.04.021

Zhao S, Xu H (2017) Grey option prioritization for the graph model for conflict resolution. J Grey Syst 29(3):14–26