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On the Likelihood of the Borda Effect: The Overall Probabilities for General Weighted Scoring Rules and Scoring Runoff Rules

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Abstract

The *Borda Effect*, first introduced by Colman and Poutney (Behav Sci 23:15–20, 1978), occurs in a preference aggregation process using the Plurality rule if given the (unique) winner there is at least one loser that is preferred to the winner by a majority of the electorate. Colman and Poutney (1978) distinguished two forms of the Borda Effect: the *Weak Borda Effect*, describing a situation under which the unique winner of the Plurality rule is majority dominated by only one loser; and the *Strong Borda Effect*, under which the Plurality winner is majority dominated by each of the losers. The *Strong Borda Effect* is well documented in the literature as the Strong Borda Paradox. Colman and Poutney (1978) showed that the probability of the *Weak Borda Effect* is not negligible; but they only focused on the Plurality rule. In this note, we extend the work of Colman and Poutney (1978) by providing, for three-candidate elections, representations of the limiting probabilities of the (*Weak*) *Borda Effect* for the whole family of scoring rules and scoring runoff rules. Our analysis leads us to highlight that there is a relation between the (*Weak*) *Borda Effect* and Condorcet efficiency. We perform our analysis under the assumptions of Impartial Culture and Impartial Anonymous Culture, which are two well-known assumptions often used for such a study.

Keywords Borda effect · Rankings · Scoring rules · Probability · Impartial culture · Impartial and anonymous culture

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1 Introduction

de Borda (1781) and de Condorcet (1785) who were both members of the Paris Royal Academy of Sciences, proposed alternative voting rules to the one that was in use in the academy at the time. With $m \geq 3$ candidates, the *Borda rule* gives $m - k$ points to a candidate each time she is ranked kth by a voter and the winner is the candidate with the highest total number of points. This rule belongs to the family of scoring rules containing all the voting systems under which candidates receive points according to the position they have in voters' rankings and the total number of points received by a candidate defines her score; the winner is the candidate with the highest score. The best-known scoring rules are the Plurality rule, the Borda rule and the Antiplurality rule. Under the Antiplurality rule, the winner is the candidate with the fewest number of last places in the voters' rankings. de Condorcet (1785) criticized the Borda rule in that there can exist a candidate who is preferred to the Borda winner by more than half of the electorate. de Condorcet (1785) proposed the *Pairwise Majority Rule* based on pairwise comparisons.[1] According to this rule, a candidate should be declared the winner if she beats all the other candidates in pairwise comparisons; such a candidate is called the *Condorcet winner*. Nonetheless, the Condorcet principle has a drawback: the Condorcet winner does not always exist and calculation of the winner one can end in *majority cycles*.

de Borda (1784) showed that for a given voting rule, the Plurality rule can elect the *Condorcet loser*, a candidate who loses all his pairwise comparisons. The Borda-Condorcet debate thus emphasized the fact that pairwise comparisons may not agree with scoring rules. The possible disagreements gave rise to the definition of the following phenomena or voting paradoxes: (i) the *Strong Borda Paradox*, which occurs when a scoring rule elects the Condorcet loser when she exists; (ii) the *Strict Borda Paradox*, which occurs when the collective rankings of a scoring rule is completely the reverse of that of the pairwise comparisons; and (iii) the *Weak Borda Paradox*, in which a scoring rule reverses the ranking of the pairwise comparisons on some pairs of candidates without necessarily electing the Condorcet loser; in other words, this paradox occurs if given that there is a Condorcet loser, she is not ranked last by the scoring rule. The study of the likelihood of each of these three paradoxes is well addressed in the social choice literature. Without seeking to be exhaustive, the reader may refer to the theoretical works of Diss and Gehrlein (2012), Diss et al. (2018), Diss and Tlidi (2018), Gehrlein and Fishburn (1976, 1978a), Gehrlein and Lepelley (1998, 2010, 2011, 2017), Kamwa and Valognes (2017), Lepelley (1993, 1996), Lepelley et al. (2000a, b), Saari (1994), Saari and Valognes (1999), Tataru and Merlin (1997). We might also mention the empirical works that have investigated these paradoxes using real-world data, notably Bezembinder (1996), Colman and Poutney (1978), Riker (1982), Taylor (1997), Van Newenhizen (1992) and Weber (1978). A summary of the results of these empirical studies can be found in Gehrlein and Lepelley (2011, p 15).

In addition to the variations of the Borda Paradox just listed, there is also the less well known *Borda Effect* first introduced by Colman and Poutney (1978). They

[1] See Young (1988) for a modern interpretation of Condorcet's rule.

distinguished the *Strong Borda Effect* and the *Weak Borda Effect*: the *Strong Borda Effect* describes a situation in which the Plurality rule elects the Condorcet loser, while the *Weak Borda Effect* concerns a situation under which the Plurality winner is majority dominated by only one of the Plurality losers. Evidently, the *Strong Borda Effect* is equivalent to the *Strong Borda Paradox*. The *Weak Borda Effect* is a little bit special and subtle, however, and if one is not careful one can misunderstand this phenomenon and thereby miscalculate its occurrences. This is indeed what happened to Gillett (1984, 1986).

Gillett (1984) criticized Colman (1980) of misusing the *Weak Borda Effect* as an indicator of the likelihood that the Plurality rule would produce an outcome inconsistent with the wishes of the majority. Then he showed that the likelihood of the Weak Borda Effect provides an inadequate, poor and misleading index of the propensity of Plurality/Majority disagreement. Colman (1984) replied that this criticism was based on a misunderstanding, as he "had proposed it [the Weak Borda Effect] not as an index of Plurality-majority disagreement, but rather as an index of the propensity of the Plurality voting procedure to select a unique winner when a majority of a committee or an electorate ... prefer one of the defeated alternatives to the plurality winner". This misunderstanding appears clearly in the introduction to Gillett (1986), where one can read : "The *Weak Borda Effect* refers to a situation which can occur under the plurality voting system whereby *at least one* of the losing candidates is preferred to the winning candidate by a simple majority of the voters ... ". Note that this definition refers to the overall *Borda Effect*. This misunderstanding obviously led Gillet to question the probability of the *Weak Borda Effect* calculated by Colman (1980). Subsequently Colman (1986) responded to all the misunderstandings and criticisms of Gillett (1984, 1986).

For three-candidate elections, Colman and Poutney (1978, p 17) reported the exact probabilities of the *Strong Borda Effect* and the *Weak Borda Effect* for groups of voters ranging in size from 7 to 301. According to their results "the smallest committee size in which the Strong or Weak Borda Effect can occur is seven, probabilities 0.018 and 0.126, respectively. In a committee of eight members it is useful to know that the effect cannot occur, but in groups of nine or more there is a significant probability of its occurrence. The likelihood of the strong and weak effects tends to rise as the number of voters increases until with 301 voters the probabilities are 0.029 and 0.276, respectively, with no obvious asymptote in sight". With the use of survey data regarding voters' preference rankings, Colman and Poutney (1978) found the occurrence of the *Borda Effect* in fifteen instances out of 261 three-cornered contests in the results of the 1966 British General Election. A similar experiment was conducted by Nurmi and Suojanen (2004).

As their analysis was only focused on the Plurality rule, the results of Colman and Poutney (1978, p 17) are quite limited in scope as the Borda Effect can also be observed with all the scoring rules and scoring runoff rules. To our knowledge, apart from Colman and Poutney (1978) no other paper has investigated the *Weak Borda Effect* under other scoring rules nor under scoring runoff rules. The main objective of this paper is to fill this gap in the literature by providing, for three-candidate elections, representations of the overall limiting probabilities for general weighted scoring rules and scoring runoff rules. We show that these representations can be deduced from the well-known results on the likelihood of the Strong Borda Paradox and on Condorcet efficiency. The Condorcet efficiency of a voting procedure is the conditional probability that it will elect the Condorcet winner, given that a Condorcet winner exists. We perform our analysis under the assumptions of Impartial Culture (IC) and Impartial Anonymous Culture (IAC), two well-known assumptions under which such studies are often driven in the social choice literature. These assumptions are defined in Sect. 2.4.

The rest of the paper is structured as follows: Section 2 is devoted to basic notations and definitions. Section 3 presents our results. In Sect. 4, we extend our analysis to scoring runoff rules when we eliminate all the candidates who obtain strictly less than the average score; we also enrich the topic by including an analysis with single-peaked preferences. Section 5 concludes.

2 Preliminaries

2.1 Preferences

Let N be a set of n voters ($n \geq 2$) and $A = \{a, b, c\}$ a set of three candidates. Individual preferences are linear orders, these are complete, asymmetric and transitive binary relations on A. With three candidates, there are exactly 6 linear orders $P_1, P_2, \ldots, P_6$ on A. A *voting situation* is a 6-tuple $\pi = (n_1, n_2, \ldots, n_t, \ldots, n_6)$ that indicates the total number n_t of voters casting each of the complete linear orders such that $\sum_{t=1}^{6} n_t = n$. We will simply write abc to denote the linear order on A according to which a is strictly preferred to b, b is strictly preferred to c; and by transitivity a is strictly preferred to c. Table 1 describes a voting situation on $A = \{a, b, c\}$.

Given $a, b \in A$ and a voting situation π, we denote by $n_{ab}(\pi)$ (or simply n_{ab}) the total number of voters who strictly prefer a to b. If $n_{ab} > n_{ba}$, we say that candidate a majority dominates candidate b; or equivalently, a beats b in a pairwise majority voting. In such a case, we will simply write $a\mathbf{M}b$. Candidate a is said to be the *Condorcet winner* (resp. the *Condorcet loser*) if $a\mathbf{M}b$ and $a\mathbf{M}c$ (resp. $b\mathbf{M}a$ and $c\mathbf{M}a$). If for a given voting situation we get $a\mathbf{M}b$, $b\mathbf{M}c$ and $c\mathbf{M}a$, this describes a majority cycle.

2.2 Voting Rules

Scoring rules are voting systems that give points to candidates according to the position they have in voters' ranking. For a given scoring rule, the total number of points received by a candidate defines her score for this rule. The winner is the candidate with the highest score. In general, with three candidates and complete strict rankings, a scoring vector is a 3-tuple $w = (w_1, w_2, w_3)$ of real numbers such that $w_1 \geq w_2 \geq w_3$ and $w_1 > w_3$. Given a voting situation π, each candidate receives w_k each time she is ranked kth ($k = 1, 2, 3$) by a voter. The score of a candidate $a \in A$ is the sum $S(\pi, w, a) = \sum_{t=1}^{6} n_t w_{r(t,a)}$ where $r(t, a)$ is the rank of candidate a according to voters of type t.

A normalized scoring vector has the shape $w_\lambda = (1, \lambda, 0)$ with $0 \leq \lambda \leq 1$. For $\lambda = 0$, we obtain the *Plurality rule*. For $\lambda = 1$, we have the *Antiplurality rule* and for $\lambda = \frac{1}{2}$, we have the *Borda rule*. From now on, we will denote by $S(\pi, \lambda, a)$ the score

Table 1 Possible strict rankings on $A = \{a, b, c\}$

$n_1 : abc$	$n_2 : acb$	$n_3 : bac$
$n_4 : bca$	$n_5 : cab$	$n_6 : cba$

Table 2 Scores of candidates

$S(\pi, \lambda, a) = n_1 + n_2 + \lambda(n_3 + n_5)$
$S(\pi, \lambda, b) = n_3 + n_4 + \lambda(n_1 + n_6)$
$S(\pi, \lambda, c) = n_5 + n_6 + \lambda(n_2 + n_4)$

of candidate a when the scoring vector is $w_\lambda = (1, \lambda, 0)$ and the voting situation is π; without loss of generality, w_λ will be used to refer to the voting rule. Table 2 gives the score of each candidate in $A = \{a, b, c\}$ given the voting situation of Table 1.

If for a given λ, candidate a scores better than candidate b, we denotes it by $a\mathbf{S}_\lambda b$. In one-shot voting, the winner is the candidate with the largest score. Runoff systems involve two rounds of voting: in the *first round*, the candidate with the smallest score is eliminated. In the *second round*, a majority contest determines who is the winner. Without loss of generality, we will denote by w_{λ^r} the runoff rule under which w_λ is used at the first stage. Runoff systems are widely used in the real world: in France, they are used for presidential, legislative and departmental elections; they are used for presidential elections in many other countries (Finland, Argentina, Austria, Egypt, etc.) and organizations such as the International Olympic Committee to designate the host city of the Olympic Games.

2.3 The Borda-Likewise Effects

Consider Tables 1 and 2 and let us assume that candidate a is the winner for the one-shot scoring rule w_λ. This means that $a\mathbf{S}_\lambda b$ and $a\mathbf{S}_\lambda c$. In such a case, we get the *Strong Borda Paradox* or the *Strong Borda Effect* if $b\mathbf{M}a$ and $c\mathbf{M}a$: candidate a is the Condorcet loser and she is elected by w_λ. If $b\mathbf{M}a$, $c\mathbf{M}a$, $b\mathbf{M}c$ and $c\mathbf{S}_\lambda b$, the collective ranking of w_λ is acb while that of the Pairwise Majority rule is bca; this defines the *Strict Borda Paradox*. If there is a Condorcet loser and she is not ranked last by w_λ, we get the *Weak Borda Paradox*. The *Weak Borda effect* happens if only one of the two candidates b and c who majority dominates candidate a. To get an overview on how these paradoxes are connected, let us assume the relations of Table 3. As one can notice in Table 3, the Strict Borda Paradox, the Strong Borda Paradox and the Weak Borda Effect are all subcases of the Weak Borda Paradox; the Strict Borda Paradox is the severe form of the Strong Borda Paradox while these two paradoxes are not connected with the Weak Borda Effect.

With runoff systems, it is obvious that the *Strong Borda Paradox* and the *Strict Borda Paradox* never occur for all λ; but this can be the case for the *Weak Borda Paradox* and the *Weak Borda Effect*. Under these rules, the *Borda Effect* is just equivalent to the *Weak Borda Effect*.

Table 3 Results of the scoring rule, of the pairwise comparisons and the resulting paradoxes

Result of the scoring rule	Results of the pairwise comparisons	Resulting paradoxes
$aS_\lambda b$ and $aS_\lambda c$ and $bS_\lambda c$	bMa, cMa and cMb	Strict Borda paradox
	a is the Condorcet loser (bMa and cMa)	Strong Borda paradox
	bMa and/or cMa and/or cMb	Weak Borda paradox
	only bMa or cMa	Weak Borda effect

2.4 The Probability Models

As stated in Sect. 1, the likelihood of each of the variations of the Borda paradox is well addressed in the social choice literature. Most of the time, the probabilities are obtained by assuming the Impartial Culture hypothesis (IC) or that of Impartial and Anonymous Culture (IAC).

For our framework with three candidates, IC assumes that each voter chooses her preference according to a uniform probability distribution and gives a probability of $\frac{1}{6}$ for each ranking to be chosen independently. The likelihood of a given voting situation $\tilde{n} = (n_1, n_2, n_3, n_4, n_5, n_6)$ is given by $Prob(\tilde{n}) = \frac{n!}{\prod_{i=1}^{6} n_i!} \times (6)^{-n}$. For our analysis under this assumption with an infinite electorate, we follow the same technique as Cervone et al. (2005), David and Mallows (1961), Gehrlein (1979, 2002, 2017).

Under IAC, first introduced by Gehrlein and Fishburn (1976), the likelihood of a given event is calculated with respect to the ratio between the number of voting situations in which the event is likely over the total number of possible voting situations. It is known that the total number of possible voting situations in three-candidate elections is given by the following five-degree polynomial in n: $C^n_{n+3!-1} = \frac{(n+5)!}{n!5!}$. The number of voting situations associated with a given event can be reduced to the solutions of a finite system of linear constraints with rational coefficients. As recently pointed out in the social choice literature, the appropriate mathematical tools to find these solutions are the Ehrhart polynomials. The background of this notion and its connection with the polytope theory can be found in Gehrlein and Lepelley (2011, 2017), Lepelley et al. (2008), and Wilson and Pritchard (2007). This technique has been widely used in numerous studies analyzing the probability of electoral events in the case of three-candidate elections under the IAC assumption. As we deal only with the probability with large electorates, we follow a procedure that was developed in Cervone et al. (2005) and recently used in many research papers such as Diss and Gehrlein (2012, 2015), Diss et al. (2010, 2012), Gehrlein et al. (2015), Moyouwou and Tchantcho (2015) among others. This technique is based on the computation of polytopes' volumes.

3 Likelihood of the Weak Borda Effect in Three-Candidate Elections

Colman and Poutney (1978, p 17) reported for three-candidate elections the exact probabilities of the Weak Borda Effect for groups of voters ranging in size from 7 to

301. Their calculations were performed under the IC hypothesis. For three-candidate elections, we provide representations for the limiting probabilities of the Weak Borda Effect for the whole family of the scoring rules and scoring runoff rules under IC and IAC.

3.1 Representations of the Limiting Probability for One-Shot Scoring Rules

Given a voting situation on $A = \{a, b, c\}$ and $w_\lambda = (1, \lambda, 0)$, we denote by $P(a; b\mathbf{M}a)$ the probability of the situation described by the following inequalities:

$$\begin{cases} a\mathbf{S}_\lambda b \\ a\mathbf{S}_\lambda c \\ b\mathbf{M}a \end{cases} \Leftrightarrow \begin{cases} (1-\lambda)n_1 + n_2 + (\lambda - 1)n_3 - n_4 + \lambda n_5 - \lambda n_6 > 0 \\ n_1 + (1-\lambda)n_2 + \lambda n_3 - \lambda n_4 + (\lambda - 1)n_5 - n_6 > 0 \\ -n_1 - n_2 + n_3 + n_4 - n_5 + n_6 > 0 \end{cases} \quad (1)$$

Also, we denote by $P(a; b\mathbf{M}a; c\mathbf{M}a, \hbar)$ the probability of the situation under which the winner is beaten in pairwise comparisons by the two other candidates under assumption $\hbar$; $\hbar$ stands here for IC or IAC.

It follows that with three candidates, $P^\lambda_{\text{WBE}}(3, \infty, \hbar)$, the limiting probability of the Weak Borda Effect under assumption $\hbar$, is given by:

$$\begin{aligned} P^\lambda_{\text{WBE}}(3, \infty, \hbar) &= 3\Big(P(a; b\mathbf{M}a, \hbar) + P(a; c\mathbf{M}a, \hbar) - P(a; b\mathbf{M}a; c\mathbf{M}a, \hbar)\Big) \\ &= 6P(a; b\mathbf{M}a, \hbar) - 3P(a; b\mathbf{M}a; c\mathbf{M}a, \hbar) \\ &= 6P(a; b\mathbf{M}a, \hbar) - P_c(3, \infty, \hbar) \times P^\lambda_{\text{SgBP}}(3, \infty, \hbar) \end{aligned} \quad (2)$$

with $P^\lambda_{\text{SgBP}}(3, \infty, \hbar)$ the conditional probability of the Strong Borda Paradox and $P_c(3, \infty, \hbar)$ the probability that a Condorcet winner (or Condorcet loser) exists; $P^\lambda_{\text{BE}}(3, \infty, \hbar)$, the limiting probability of the Borda Effect under assumption $\hbar$, is given by:

$$\begin{aligned} P^\lambda_{\text{BE}}(3, \infty, \hbar) &= P^\lambda_{\text{WBE}}(3, \infty, \hbar) + 3P(a; b\mathbf{M}a; c\mathbf{M}a, \hbar) \\ &= 6P(a; b\mathbf{M}a, \hbar) \end{aligned} \quad (3)$$

Representations of P_c are known in the literature both under IC and IAC.

$$P_c(3, \infty, IC) = \frac{3}{4} + \frac{3}{2}\sin^{-1}\left(\frac{1}{3}\right) \quad \text{and} \quad P_c(3, \infty, IAC) = \frac{15}{16}$$

Some representations for $P^\lambda_{\text{SgBP}}(3, \infty, \hbar)$ are provided by Gehrlein and Fishburn (1978a), Tataru and Merlin (1997) and Cervone et al. (2005). Now, all we have to do is to find $P(a; b\mathbf{M}a, \hbar)$.

3.1.1 Representation Under IC

Let us define the quantity $z = 1 - \lambda(1-\lambda)$ that we will use along the paper. The representation of the conditional probability of the Strong Borda Paradox provided by Gehrlein and Fishburn (1978a) under IC is as follows:

$$P^{\lambda}_{\text{SgBP}}(3, \infty, IC) = \frac{3\Phi_4(R)}{P_c(3, \infty, IC)} \tag{4}$$

where

$$\begin{aligned}\Phi_4(R) = \frac{1}{9} - \frac{1}{4\pi}\left(\sin^{-1}\left(\sqrt{\frac{2}{3z}}\right) + \sin^{-1}\left(\sqrt{\frac{1}{6z}}\right)\right) \\ + \frac{1}{4\pi^2}\left\{\left(\sin^{-1}\left(\sqrt{\frac{2}{3z}}\right)\right)^2 - \left(\sin^{-1}\left(\sqrt{\frac{1}{6z}}\right)\right)^2 \right.\\ \left. - \int_0^1 \sqrt{\frac{1}{36-(3-t)^2}}\cos^{-1}\left(\frac{6tz - g(t,z)}{2g(t,z)}\right)dt\right\}\end{aligned}$$

with $g(t,z) = 4(3z-2)^2 - (3z-2-tz)^2 + 6(3z-2)$.
Another representation is provided by Tataru and Merlin (1997) as follows:

$$\begin{aligned}P^{\lambda}_{\text{SgBP}}(3, \infty, IC) = \frac{3}{\pi^2 P_c(3, \infty, IC)}\int_0^{2\lambda-1} \\ \left[\frac{2t\cos^{-1}\left(\frac{\sqrt{9t^2+3}}{\sqrt{(t^2+3)(4t^2+1)}}\right)}{(t^2+3)\sqrt{6t^2+2}} + \frac{t\cos^{-1}\left(\frac{\sqrt{3}(1-t^2)}{\sqrt{(3t^2+1)(t^2+3)(4t^2+1)}}\right)}{(t^2+3)\sqrt{6t^2+14}}\right]dt\end{aligned} \tag{5}$$

Let us find $P(a; b\mathbf{M}a, IC)$ following the technique of Gehrlein and Fishburn (1976). To do so, we consider Eq. 1 and define the following three discrete variables:

$$X_1 = \begin{array}{ll} 1-\lambda & : p_1 \\ 1 & : p_2 \\ -1+\lambda & : p_3 \\ -1 & : p_4 \\ \lambda & : p_5 \\ -\lambda & : p_6 \end{array} \quad X_2 = \begin{array}{ll} 1 & : p_1 \\ 1-\lambda & : p_2 \\ \lambda & : p_3 \\ -\lambda & : p_4 \\ -1+\lambda & : p_5 \\ -1 & : p_6 \end{array} \quad X_3 = \begin{array}{ll} -1 & : p_1 \\ -1 & : p_2 \\ 1 & : p_3 \\ 1 & : p_4 \\ -1 & : p_5 \\ 1 & : p_6 \end{array}$$

where p_i is the probability that a voter who is randomly selected from the electorate is associated with the ith ranking of Table 1. Under IC, $p_i = \frac{1}{6}$. For $X_j > 0$, this indicates that the jth inequality of Eq. 1 is satisfied. With n voters, Eq. 1 fully describes the Borda effect when the average value of each of the X_j are positive. According to Gehrlein and Fishburn (1978b), $P(a; b\mathbf{M}a, IC)$ is equal to the joint probability that $\overline{X}_1 > 0$, $\overline{X}_2 > 0$ and $\overline{X}_3 > 0$; when $n \to \infty$, it is equivalent to the trivariate normal positive orthant probability $\Phi_3(R')$ such that $\overline{X}_j\sqrt{n} \geq E(\overline{X}_j\sqrt{n})$ and R' is a correlation matrix between the variables X_j. Thus $P(a; b\mathbf{M}a) = \Phi_3(R')$. In our case, R' is as follows

$$R' = \begin{bmatrix} 1 & \frac{1}{2} & -\sqrt{\frac{2}{3z}} \\ & 1 & -\sqrt{\frac{1}{6z}} \\ & & 1 \end{bmatrix}$$

Given the form of R', we can easily derive $\Phi_3(R')$ from the work of David and Mallows (1961):

$$\Phi_3(R') = \frac{1}{6} - \frac{1}{4\pi}\left(\sin^{-1}\left(\sqrt{\frac{2}{3z}}\right) + \sin^{-1}\left(\sqrt{\frac{1}{6z}}\right)\right) \tag{6}$$

Following Eqs. 2 and 3, we derive Proposition 1.

Proposition 1 *For three-candidate elections and scoring rule w_λ,*

$$\begin{aligned} P^\lambda_{WBE}(3, \infty, IC) &= 6\Phi_3(R') - 3\Phi_4(R) \\ &= 1 - P_c(3, \infty, IC) \times P^\lambda_{CE}(3, \infty, IC) \\ P^\lambda_{BE}(3, \infty, IC) &= 1 - P_c(3, \infty, IC)\left(P^\lambda_{CE}(3, \infty, IC) - P^\lambda_{SgBP}(3, \infty, IC)\right) \end{aligned}$$

with $P^\lambda_{CE}(3, \infty, IC)$ being the conditional probability that the winner is the Condorcet winner given that a Condorcet winner exists.

According to Proposition 1, the representations of the limiting probability of the Weak Borda and that of the Borda Effect under IC can be deduced from those of Condorcet efficiency and of the Strong Borda Paradox, which are well documented in the literature.

Given that z is symmetric about $\lambda = 0.5$, it follows that $P^\lambda_{\text{WBE}}(3, \infty, IC) = P^{1-\lambda}_{\text{WBE}}(3, \infty, IC)$ and $P^\lambda_{\text{BE}}(3, \infty, IC) = P^{1-\lambda}_{\text{BE}}(3, \infty, IC)$. We report in Table 4 the computed values of the limiting probability of the (Weak) Borda Effect for $\lambda = 0(0.1)1$. For $0 \leq \lambda \leq \frac{1}{2}$, the probability tends to decrease, and it increases for $\frac{1}{2} \leq \lambda \leq 1$. We find that the limiting probability is minimized by the Borda rule ($\lambda = \frac{1}{2}$) and it is maximized by the Plurality rule ($\lambda = 0$) and the Antiplurality rule ($\lambda = 1$). The fact that the Borda rule does not exhibit the strong Borda Effect has been pointed out by Fishburn and Gehrlein (1976); they showed that the Borda rule is the only scoring rule that always guarantees that the Condorcet loser, when she exists, is not the unique winner.

Table 4 Computed values of the Borda effect under one-shot and scoring runoff rules

λ	One-shot rules						Runoff rules	
	Strong Borda effect		Weak Borda effect		Borda effect		Borda effect	
	IC	IAC	IC	IAC	IC	IAC	IC	IAC
0	0.0338	0.0277	0.3092	0.1736	0.3431	0.2014	0.1216	0.0920
0.1	0.0217	0.0180	0.2766	0.1582	0.2983	0.1762	0.1095	0.0834
0.2	0.0115	0.0098	0.2432	0.1447	0.2547	0.1545	0.0992	0.0752
0.3	0.0042	0.0039	0.2119	0.1350	0.2161	0.1389	0.0919	0.0683
0.4	0.0006	0.0006	0.1876	0.1328	0.1883	0.1336	0.0884	0.0637
0.5	0.0000	0.0000	0.1779	0.1458	0.1779	0.1458	0.0877	0.0625
0.6	0.0006	0.0012	0.1876	0.1825	0.1882	0.1837	0.0884	0.0632
0.7	0.0042	0.0057	0.2119	0.2335	0.2161	0.2392	0.0919	0.0664
0.8	0.0115	0.0127	0.2432	0.2913	0.2547	0.3040	0.0992	0.0724
0.9	0.0217	0.0209	0.2766	0.3513	0.2983	0.3722	0.1095	0.0805
1	0.0338	0.0295	0.3092	0.4097	0.3431	0.4392	0.1216	0.0903

3.1.2 Representation Under IAC

Under IAC, when $n \to \infty$, we deduce $P_c(3, \infty, IAC) \times P^{\lambda}_{\text{SgBP}}(3, \infty, IAC)$ from the results of Cervone et al. (2005):

$$
\begin{aligned}
&P_c(3, \infty, IAC) \times P^{\lambda}_{\text{SgBP}}(3, \infty, IAC) \\
&= \begin{cases} \frac{(2\lambda-1)^3(12-9\lambda-2\lambda^2)}{432(\lambda-1)^3} & \text{for } 0 \leq \lambda \leq \frac{1}{2} \\ \frac{(2\lambda-1)^3(2-53\lambda+331\lambda^2-88\lambda^3+12\lambda^4)}{1728\lambda^3(3\lambda-1)(\lambda+1)} & \text{for } \frac{1}{2} \leq \lambda \leq 1 \end{cases}
\end{aligned} \tag{7}
$$

In order to find $P(a; b\mathbf{M}a, IAC)$, let us denote by $\mathcal{V}^{\lambda}_{ab}$ the set of all voting situations at which a is the winner given λ and is majority dominated only by b. A profile $\pi \in \mathcal{V}^{\lambda}_{ab}$ implies that the inequalities of Eq. 1 are satisfied. Notice that as $n \to \infty$, $P(a; b\mathbf{M}a, IAC) = vol(P_{ab})$, the 5-dimensional volume of the polytope P_{ab} is obtained from the characterization of $\mathcal{V}^{\lambda}_{ab}$ just by replacing each n_j by $p_j = \frac{n_j}{n}$ in the simplex $\mathfrak{S} = \{(p_1, p_2, \ldots, p_6) : \sum_{t=1}^{6} p_j = 1 \text{ with } p_j \geq 0, j = 1, 2, \ldots, 6\}$. Given $0 \leq \lambda \leq 1$, computing $vol(P_{ab})$ leads the following:[2]

$$
P(a; b\mathbf{M}a, IAC) = \begin{cases} \frac{58-221\lambda+276\lambda^2+29\lambda^3-328\lambda^4+213\lambda^5-20\lambda^6-8\lambda^7}{864(\lambda+1)(\lambda-1)^3(\lambda-2)} & \text{for } 0 \leq \lambda \leq \frac{1}{2} \\ \frac{-2+37\lambda-318\lambda^2+890\lambda^3-910\lambda^4-246\lambda^5+280\lambda^6+16\lambda^7}{1728\lambda^3(\lambda-2)(\lambda+1)} & \text{for } \frac{1}{2} \leq \lambda \leq 1 \end{cases} \tag{8}
$$

Following Eq. 3, we get Proposition 2.

[2] The computer program we used is available upon request.

Proposition 2 *For three-candidate elections and scoring rule w_λ,*

$$P^{\lambda}_{WBE}(3,\infty,IAC)$$
$$= \begin{cases} \frac{150-513\lambda+529\lambda^2+194\lambda^3-692\lambda^4+367\lambda^5-28\lambda^6-8\lambda^7}{432(\lambda+1)(\lambda-2)(\lambda-1)^3} & \textit{for } 0 \le \lambda \le \frac{1}{2} \\ \frac{8-126\lambda+1163\lambda^2-4939\lambda^3+8882\lambda^4-2416\lambda^5-11580\lambda^6+5984\lambda^7+192\lambda^8}{1728\lambda^3(3\lambda-1)(\lambda+1)(\lambda-2)} & \textit{for } \frac{1}{2} \le \lambda \le 1 \end{cases}$$
$$= 1 - P_c(3,\infty,IAC) \times P^{\lambda}_{CE}(3,\infty,IAC)$$
$$P^{\lambda}_{BE}(3,\infty,IAC)$$
$$= 1 - P_c(3,\infty,IAC)\Big(P^{\lambda}_{CE}(3,\infty,IAC) - P^{\lambda}_{SgBP}(3,\infty,IAC)\Big)$$

Proposition 2 tells us that under IAC, representations of the limiting probabilities of the Weak Borda Effect and that of the Borda Effect can also be deduced from those of Condorcet efficiency and the Strong Borda Paradox. We then derive the values provided in Table 4. Notice that under IAC, the likelihood of the Weak Borda Effect is minimized at $\lambda^\star = \frac{16709}{44883} \approx 0.3723$ where the probability is 0.1324; the likelihood of the Borda effect is minimized at $\lambda^\star = \frac{5063}{13009} \approx 0.3892$ where the probability is 0.1335. For both the Weak Borda Effect and the Borda effect, as λ grows from 0 to $\lambda^\star$, the probability of the effect tends to decrease, and it increases when λ grows from $\lambda^\star$ to 1.

3.2 Representations of the Limiting Probability for Scoring Runoff Rules

One can also observe the Borda effect with runoff scoring rules. Nonetheless, notice that only the Weak Borda Effect can be observed; it is obvious that this cannot be the case for the Strong Borda Effect. So, with runoff scoring rules, the Weak Borda Effect is equivalent to the Borda Effect.

Let us now provide representation of the limiting probability of the Borda Effect for all the scoring runoff rules both under IC and IAC. Without loss of generality, the following inequalities characterize a voting situation exhibiting the Weak Borda Effect.

$$\begin{cases} a\mathbf{S}_\lambda c \\ b\mathbf{S}_\lambda c \\ a\mathbf{M}b \\ c\mathbf{M}a \end{cases} \Leftrightarrow \begin{cases} n_1 + (1-\lambda)n_2 + \lambda n_3 - \lambda n_4 + (\lambda-1)n_5 - n_6 > 0 \\ \lambda n_1 - \lambda n_2 + n_3 + (1-\lambda)n_4 - n_5) + (\lambda-1)n_6 > 0 \\ n_1 + n_2 - n_3 - n_4 + n_5 - n_6 > 0 \\ -n_1 - n_2 - n_3 + n_4 + n_5 + n_6 > 0 \end{cases} \quad (9)$$

Remark 1 We notice that under the Borda runoff, the Borda effect can only occur in case of a majority cycle. This is because if the inequalities of Eq. 9 are satisfied and $c\mathbf{M}b$, this indicates that the Condorcet winner c is ranked last by the Borda rule: we know that this is not possible. So, in three-candidate elections, it is only in case of a majority cycle that the Borda runoff can produce the Borda effect.

3.2.1 Representation Under IC

One can get a representation of the limiting probability of the Borda effect by following the technique of David and Mallows based on quadrivariate normal positive orthant probabilities, as we did in Sect. 3.1.1. We consider Eq. 9 and define the following four discrete variables:

$$X_1 = \begin{array}{rl} 1 & : p_1 \\ 1-\lambda & : p_2 \\ \lambda & : p_3 \\ -\lambda & : p_4 \\ -1+\lambda & : p_5 \\ -1 & : p_6 \end{array} \quad X_2 = \begin{array}{rl} \lambda & : p_1 \\ -\lambda & : p_2 \\ 1 & : p_3 \\ 1-\lambda & : p_4 \\ -1 & : p_5 \\ -1+\lambda & : p_6 \end{array} \quad X_3 = \begin{array}{rl} 1 & : p_1 \\ 1 & : p_2 \\ -1 & : p_3 \\ -1 & : p_4 \\ 1 & : p_5 \\ -1 & : p_6 \end{array} \quad X_4 = \begin{array}{rl} -1 & : p_1 \\ -1 & : p_2 \\ -1 & : p_3 \\ 1 & : p_4 \\ 1 & : p_5 \\ 1 & : p_6 \end{array}$$

With n voters, Eq. 9 fully describes the Weak Borda effect when the average value of each of the X_j are positive. According to Gehrlein and Fishburn (1978b), $P^{\lambda^r}_{WBE}(3, \infty, IC)$, the limiting probability of the Weak Borda effect, is equal to the joint probability that $\overline{X}_1 > 0$, $\overline{X}_2 > 0$, $\overline{X}_3 > 0$ and $\overline{X}_4 > 0$; when $n \to \infty$, it is equivalent to the quadrivariate normal positive orthant probability $\Phi_4(R'')$ such that $\overline{X}_j\sqrt{n} \geq E(\overline{X}_j\sqrt{n})$ and where R'' is a correlation matrix between the variables X_j. The matrix R'' is as follows

$$R'' = \begin{bmatrix} 1 & \frac{1}{2} & \sqrt{\frac{1}{6z}} & -\sqrt{\frac{2}{3z}} \\ & 1 & -\sqrt{\frac{1}{6z}} & -\sqrt{\frac{1}{6z}} \\ & & 1 & -\frac{1}{3} \\ & & & 1 \end{bmatrix}$$

Comparing this to the results of David and Mallows (1961) and the related literature, the matrix R'' does not appear to be at all close to any special form that we are familiar with; finding a representation for $\Phi_4(R'')$ seems to be a tricky task.[3] Fortunately, Gehrlein (1979) [see also Gehrlein (2017)] developed a general representation to obtain numerical values of $\Phi_4(R'')$ as a function of a series of bounded integrals over a single variable. Using the formula suggested by Gehrlein (1979), we get $\Phi_4(R'')$ and then we derive Proposition 3.

Proposition 3 *For three-candidate elections and a scoring runoff rule w_{λ^r},*

$$\begin{aligned} P^{\lambda^r}_{WBE}(3, \infty, IC) &= 6\Phi_4(R'') \\ &= \frac{1}{2} + \frac{3}{2\pi^2}\Bigg[-\left(\frac{2}{3z-2}\right)^{\frac{1}{2}} \int_0^1 \cos^{-1}\left(\frac{F_1(z,t)}{N_1(z,t) \times N_2(z,t)}\right)dt \\ &\quad -\left(\frac{1}{6z-1}\right)^{\frac{1}{2}} \int_0^1 \cos^{-1}\left(\frac{F_2(z,t)}{N_2(z,t) \times N_3(z,t)}\right)dt \\ &\quad -\frac{\sqrt{2}}{4} \int_0^1 \cos^{-1}\left(\frac{F_3(z,t)}{N_1(z,t) \times N_3(z,t)}\right)dt\Bigg] \\ &= 1 - P_c(3, \infty, IC) \times P^{\lambda^r}_{CE}(3, \infty, IC) \end{aligned}$$

[3] Thanks to Bill Gehrlein for pointing this out and for his help.

where

$$F_1(z,t) = (6z)^{-\frac{3}{2}}(9z - 6t^2); \quad N_1(z,t) = \frac{1}{3}\left(9 - t^2 - \frac{1}{z}\left(4t^2 + \frac{3}{2}\right)\right)^{\frac{1}{2}}$$

$$F_2(z,t) = (6z)^{-\frac{3}{2}}(3zt^2 - 9z - 3t^2); \quad N_2(z,t) = \frac{1}{2}\left(\frac{3z - 2t^2}{z}\right)^{\frac{1}{2}}$$

$$F_3(z,t) = \frac{zt^2 - 9z + 7t^2 - 3}{18z}; \quad N_3(z,t) = \frac{1}{3}\left(\frac{-2zt^2 + 18z - 5t^2 - 3}{2z}\right)^{\frac{1}{2}}$$

According to Proposition 3, one can derive the representation of the Borda Effect for runoff scoring rules from that of Condorcet efficiency. To our knowledge, only the representations of Condorcet efficiency for the Plurality runoff, the Antiplurality runoff and the Borda runoff are provided in the literature [see for example Gehrlein and Lepelley (2011)]. So, from Proposition 3, the reader can get the overall Condorcet efficiency of the scoring runoff rules.

The computed values of $P^{\lambda^r}_{WBE}(3, \infty, IC)$ are provided in Table 4. It comes out that for all λ, we get $8.7\% < P^{\lambda^r}_{WBE}(3, \infty, IC) < 12.2\%$. It tends to decrease for $0 \le \lambda \le \frac{1}{2}$ and it increases for $\frac{1}{2} \le \lambda \le 1$. The probability is minimized by the Borda runoff ($\lambda = \frac{1}{2}$) and maximized by the Plurality runoff ($\lambda = 0$) and the Antiplurality runoff ($\lambda = 1$).

Remark 2 Our formula is in line with Remark 1 since we find for the Borda runoff that $P^{\lambda^r}_{WBE}(3, \infty, IC)$ is equal to the probability of a majority cycle under IC, which is well documented in the literature.

3.2.2 Representation Under IAC

Following the same scheme as in Sect. 3.1.2, we compute the volume and get Proposition 4.

Proposition 4 *For three-candidate elections and a scoring runoff rule w_{λ^r},*

$$\begin{aligned} &P^{\lambda^r}_{WBE}(3, \infty, IAC) \\ &= \begin{cases} \frac{96\lambda^7+176\lambda^6+1028\lambda^5-6420\lambda^4+11138\lambda^3-9157\lambda^2+3777\lambda-636}{1728(\lambda-2)(3\lambda-2)(\lambda-1)^3} & \textit{for } 0 \le \lambda \le \frac{1}{2} \\ \frac{-16\lambda^5+128\lambda^4-133\lambda^3+68\lambda^2-7\lambda-1}{432\lambda^3} & \textit{for } \frac{1}{2} \le \lambda \le 1 \end{cases} \\ &= 1 - P_c(3, \infty, IAC) \times P^{\lambda^r}_{CE}(3, \infty, IAC) \end{aligned}$$

The computed values of $P^{\lambda^r}_{WBE}(3, \infty, IAC)$ are provided in Table 4. We notice that the probabilities are lower that those obtained under IC for all λ with $6.4\% < P^{\lambda^r}_{WBE}(3, \infty, IAC) < 9.3\%$. It tends to decrease for $0 \le \lambda \le \frac{1}{2}$ and it increases for $\frac{1}{2} \le \lambda \le 1$. The probability is minimized by the Borda runoff ($\lambda = \frac{1}{2}$) and maximized by the Plurality runoff ($\lambda = 0$). Remark 2 also holds here.

4 Further Analysis

In this section, we extend our analysis in two ways. There are two main classes of scoring runoff rules: (i) where we eliminate only one alternative on each step, and (ii) when we eliminate all candidates who obtain strictly less than the average score, as in Kim and Roush (1996) or Favardin and Lepelley (2006). In Sect. 3.2, we only considered the former type of this rule. In this section, we want to consider the second type. We expect that it will greatly improve the results, because Favardin and Lepelley (2006), Kim and Roush (1996), Lepelley and Valognes (2003) among others have shown that runoff rules based on average are better than normal ones in terms of manipulability.

It is well established in the literature that single-peakedness, a particular measure of the social homogeneity, considerably influences the likelihood of voting paradoxes [see for instance Lepelley and Valognes (2003)]. It would therefore be interesting to include further analysis with single-peaked preferences. By so doing, one is expecting to measure the impact of a left-right political divide that mimics some voting bodies. So, we will provide probability representations of the Borda Effect for single-peaked preferences in three-candidate elections.

4.1 The Borda Effect Under Another Version of Iterative Scoring Rules

The iterative scoring rules eliminate all the candidates who obtain strictly less than the average score at each stage of the elimination process. Given the notation of Sect. 2.2, if the iterative scoring rule is associated with $\lambda = 1$, this defines the *Kim-Roush voting rule* (Kim and Roush 1996); we get the *Nanson rule* (Nanson 1883) if the iterative scoring rule is associated with $\lambda = \frac{1}{2}$.

For our framework with three candidates, when we eliminate all the candidates who obtain strictly less than the average score, the following scenarios are conceivable:

Case 1: *at the first run, two candidates (assume b and c) are eliminated and candidate a wins.*

This means that given $\overline{S}(\pi,\lambda) = \frac{S(\pi,\lambda,a)+S(\pi,\lambda,b)+S(\pi,\lambda,c)}{3}$ the average score, we get $S(\pi,\lambda,a) \geq \overline{S}(\pi,\lambda)$, $S(\pi,\lambda,b) < \overline{S}(\pi,\lambda)$ and $S(\pi,\lambda,c) < \overline{S}(\pi,\lambda)$.

If in addition,

- $b\mathbf{M}a$ and $c\mathbf{M}a$, we get the Strong Borda Effect. Without loss of generality, this situation is fully characterized by the following inequalities:

$$\begin{cases} S(\pi,\lambda,b) < \overline{S}(\pi,\lambda) \\ S(\pi,\lambda,c) < \overline{S}(\pi,\lambda) \\ b\mathbf{M}a \\ c\mathbf{M}a \end{cases}$$
$$\Leftrightarrow \quad \begin{cases} (1-2\lambda)n_1 + (1+\lambda)n_2 + (\lambda-2)n_3 + (\lambda-2)n_4 + (1+\lambda)n_5 + (1-2\lambda)n_6 > 0 \\ (1+\lambda)n_1 + (1-2\lambda)n_2 + (1+\lambda)n_3 + (1-2\lambda)n_4 + (\lambda-2)n_5) + (\lambda-2)n_6 > 0 \\ -n_1 - n_2 + n_3 + n_4 - n_5 + n_6 > 0 \\ -n_1 - n_2 - n_3 + n_4 + n_5 + n_6 > 0 \end{cases} \tag{10}$$

Let us denote by $\overline{P}^{\lambda^{r2}}_{\text{SBE}}(3, \infty, \hbar)$ the limiting probability of the Strong Borda Effect associated with this situation given $\hbar$.

– $b\mathbf{M}a$ or $c\mathbf{M}a$ but not both, we get the Weak Borda Effect. In this situation, it follows that the probability of the Weak Borda Effect is given by

$$\overline{P}^{\lambda^{r2}}_{\text{WBE}}(3, \infty, \hbar) = 6\overline{P}(a; bMa, \hbar) - \overline{P}^{\lambda^{r2}}_{\text{SBE}}(3, \infty, \hbar)$$

Case 2: *at the first run, only one candidate (assume c) is eliminated and candidate a wins the majority contest versus b.*

In this case, we get $S(\pi, \lambda, a) \geq \overline{S}(\pi, \lambda)$, $S(\pi, \lambda, b) \geq \overline{S}(\pi, \lambda)$, $S(\pi, \lambda, c) < \overline{S}(\pi, \lambda)$ and aMb. Only the Weak Borda Effect is possible if cMa. Let us denote by $\overline{P}^{\lambda^{r1}}_{\text{WBE}}(3, \infty, \hbar)$ the limiting probability of the Weak Borda Effect associated with this situation, which is fully characterized by the following inequalities:

$$\begin{cases} S(\pi, \lambda, a) > \overline{S}(\pi, \lambda) \\ S(\pi, \lambda, b) > \overline{S}(\pi, \lambda) \\ a\mathbf{M}b \\ c\mathbf{M}a \end{cases}$$
$$\Leftrightarrow \quad \begin{cases} (2-\lambda)n_1 + (2-\lambda)n_2 + (2\lambda-1)n_3 - (1+\lambda)n_4 + (2\lambda-1)n_5) - (1+\lambda)n_6 > 0 \\ -(1-2\lambda)n_1 - (1+\lambda)n_2 - (\lambda-2)n_3 - (\lambda-2)n_4 - (1+\lambda)n_5 - (1-2\lambda)n_6 > 0 \\ n_1 + n_2 - n_3 - n_4 + n_5 - n_6 > 0 \\ -n_1 - n_2 - n_3 + n_4 + n_5 + n_6 > 0 \end{cases} \tag{11}$$

From the above, we deduce that in three-candidate elections with iterative scoring rules under which we eliminate all the candidates who score less than the average score, the limiting probability of the Strong Borda Paradox ($P^{\overline{\lambda}^r}_{\text{SBE}}(3, \infty, \hbar)$) and the limiting probability of the Weak Borda Paradox ($P^{\overline{\lambda}^r}_{\text{WBE}}(3, \infty, \hbar)$) are as follows:

$$\overline{P}^{\lambda^{r}}_{\text{WBE}}(3, \infty, \hbar) = \overline{P}^{\lambda^{r2}}_{\text{WBE}}(3, \infty, \hbar) + \overline{P}^{\lambda^{r1}}_{\text{WBE}}(3, \infty, \hbar)$$
$$\overline{P}^{\lambda^{r}}_{\text{BE}}(3, \infty, \hbar) = 6\overline{P}(a; bMa, \hbar) + \overline{P}^{\lambda^{r1}}_{\text{WBE}}(3, \infty, \hbar)$$

Using the same computation techniques as in the previous sections, we derive the representations of the limiting probabilities as stated in Proposition 5 for IC and Proposition 6 for IAC.

Proposition 5 *For three-candidate elections and a scoring runoff rule w_{λ^r} eliminating all the candidates who score less than the average score, we get under IC:*

$$\overline{P}^{\lambda^r}_{WBE}(3,\infty,IC) = \frac{1}{8} - \frac{9}{8\pi}\sin^{-1}\left(\frac{\sqrt{2z}}{2z}\right) + \frac{3\sqrt{2z}}{8z\pi^2}\left(\int_0^1 \frac{\cos^{-1}\left(\frac{G_1(z,t)+\frac{(3-2t^2)(2z)^{-\frac{1}{2}}}{6}}{\sqrt{M_2(z,t)\times\left(M_3(z,t)+\frac{1-t^2}{2z}\right)}}\right)}{\sqrt{1-\frac{t^2}{2z}}}dt - \int_0^1 \frac{\cos^{-1}\left(\frac{G_1(z,t)}{\sqrt{M_1(z,t)\times M_2(z,t)}}\right)}{\sqrt{1-\frac{t^2}{2z}}}dt\right)$$

$$+\frac{1}{4\pi^2}\left(\int_0^1 \frac{\cos^{-1}\left(\frac{G_3(z,t)+\frac{3-2t^2}{6z}}{\sqrt{\left(M_1(z,t)-\frac{t}{6z}\right)\times\left(M_3(z,t)+\frac{1-t^2}{2z}\right)}}\right)}{\sqrt{1-\frac{t^2}{9}}}dt - \int_0^1 \frac{\cos^{-1}\left(\frac{G_3(z,t)}{\sqrt{M_1(z,t)\times M_3(z,t)}}\right)}{\sqrt{1-\frac{t^2}{9}}}dt\right)$$

$$\overline{P}^{\lambda^r}_{BE}(3,\infty,IC) = \frac{1}{4} - \frac{3}{2\pi}\sin^{-1}\left(\frac{\sqrt{2z}}{2z}\right) - \frac{3\sqrt{2z}}{8z\pi^2}\int_0^1 \frac{\cos^{-1}\left(\frac{G_1(z,t)}{\sqrt{M_1(z,t)\times M_2(z,t)}}\right)}{\sqrt{1-\frac{t^2}{2z}}}dt - \frac{1}{4\pi^2}\int_0^1 \frac{\cos^{-1}\left(\frac{G_3(z,t)}{\sqrt{M_1(z,t)\times M_3(z,t)}}\right)}{\sqrt{1-\frac{t^2}{9}}}dt$$

where

$$G_1(z,t) = \frac{(3z + (z-3)t^2)\sqrt{2}}{12z^{\frac{2}{3}}}; \qquad G_3(z,t) = \frac{1}{2} + \frac{(3-z)t^2 - 9}{18z};$$
$$M_1(z,t) = 1 - \frac{9 + (2z+3)t^2}{18z}; \qquad M_2(z,t) = \frac{3}{4} - \frac{t^2}{2z};$$
$$M_3(z,t) = 1 - \frac{9 + 2zt^2}{18z}$$

Proposition 6 *For three-candidate elections and a scoring runoff rule w_{λ^r} eliminating all the candidates who score less than the average score, we get under IAC:*

$$\overline{P}^{\lambda^r}_{WBE}(3,\infty,IAC) = \begin{cases} \frac{-445\lambda+80-791\lambda^3+913\lambda^2+315\lambda^4+48\lambda^7+48\lambda^6-165\lambda^5}{2592(\lambda-1)^4} & \textit{for } 0 \le \lambda \le \frac{1}{2} \\ \frac{27+96\lambda-615\lambda^2+833\lambda^3+112\lambda^4-64\lambda^5}{2592\lambda^2} & \textit{for } \frac{1}{2} \le \lambda \le 1 \end{cases}$$

$$\overline{P}^{\lambda^r}_{BE}(3,\infty,IAC) = \begin{cases} \frac{-1321\lambda+244-2837\lambda^3+2829\lambda^2+1205\lambda^4+16\lambda^7-16\lambda^6-119\lambda^5}{2592(\lambda-1)^4} & \textit{for } 0 \le \lambda \le \frac{1}{2} \\ -\frac{\lambda^2-354\lambda^3+1335\lambda^4-1585\lambda^5+144\lambda^6-4\lambda+32\lambda^7+2}{2592\lambda^4} & \textit{for } \frac{1}{2} \le \lambda \le 1 \end{cases}$$

After all computations, we notice that when candidates are eliminated according to the average score, the representation of the (Weak) Borda Effect for this family of scoring runoff rules cannot be deduced from those of Condorcet efficiency and the Strong Borda Paradox, since we get:

$$\overline{P}^{\lambda^r}_{\text{WBE}}(3,\infty,\hbar) \neq 1 - P_c(3,\infty,\hbar) \times P^{\lambda}_{\text{CE}}(3,\infty,\hbar)$$
$$\overline{P}^{\lambda^r}_{\text{BE}}(3,\infty,\hbar) \neq 1 - P_c(3,\infty,\hbar)\left(\overline{P}^{\lambda^r}_{\text{CE}}(3,\infty,\hbar) - \overline{P}^{\lambda^r}_{\text{SgBP}}(3,\infty,\hbar)\right)$$

We provide some computed values of the probabilities in Table 5.

We notice from Table 5 that under IC, $\overline{P}^{\lambda^r}_{\text{SBE}}(3,\infty,\hbar) = \overline{P}^{(1-\lambda)^r}_{\text{SBE}}(3,\infty,\hbar)$ and $\overline{P}^{\lambda^r}_{\text{WBE}}(3,\infty,\hbar) = \overline{P}^{(1-\lambda)^r}_{\text{WBE}}(3,\infty,\hbar)$. For $0 \le \lambda \le \frac{1}{2}$, the probability tends to decrease, and it increases for $\frac{1}{2} \le \lambda \le 1$. We find that the limiting probability is minimized by the Nanson rule ($\lambda = \frac{1}{2}$) and it is maximized at $\lambda = 0$ and by the Kim-Roush rule ($\lambda = 1$). Comparing these results to those of Table 4, it seems that for $\lambda \in [0.3; 0.7]$, the scoring runoff rules with the eliminations according to the average score exhibit the Borda Effect less than the rules under which candidates are eliminated one by one; we get the reverse out of this range.

Under IAC, the likelihood of the Weak Borda Effect is minimized at $\lambda^\star = \frac{27521}{58801} \approx 0.4680$ where the probability is 0.046390; the likelihood of the Borda Effect is minimized at $\lambda^\star = \frac{15031}{32051} \approx 0.4689$ where the probability is 0.046399. For both the Weak

Table 5 Computed values of the Borda effect for scoring runoff rules under which candidates are eliminated according to the average score

λ	Weak Borda effect		Strong Borda effect		Borda effect	
	IC	IAC	IC	IAC	IC	IAC
0	0.1402	0.0849	0.0135	0.0093	0.1537	0.0942
0.1	0.1191	0.0748	0.0084	0.006	0.1275	0.0808
0.2	0.0991	0.0646	0.0043	0.0033	0.1034	0.0679
0.3	0.0819	0.0552	0.0015	0.0013	0.0834	0.0565
0.4	0.0702	0.0481	0.0002	0.0002	0.0704	0.0483
0.5	0.0657	0.0469	0	0	0.0657	0.0469
0.6	0.0702	0.0564	0.0002	0.0005	0.0704	0.0569
0.7	0.0819	0.0745	0.0015	0.0026	0.0834	0.0771
0.8	0.0991	0.0974	0.0043	0.0061	0.1034	0.1035
0.9	0.1191	0.123	0.0084	0.0105	0.1275	0.1335
1	0.1402	0.1501	0.0135	0.0154	0.1537	0.1655

Borda Effect and the Borda effect, as λ grows from 0 to $\lambda^\star$, the probability of the effect tends to decrease, and it increases when λ grows from $\lambda^\star$ to 1. We find that the limiting probability is maximized by the Kim-Roush rule ($\lambda = 1$). Comparing these results to those of Table 4, it seems that for $\lambda \in [0.1; 0.6]$, the scoring runoff rules with the eliminations according to the average score exhibit the Borda Effect less than the rules under which candidates are eliminated one by one; we get the reverse out of this range.

4.2 Representations for the Limiting Probability with Single-Peaked Preferences

Single-peakedness describes situations where it can be appropriate to represent policy options on a one-dimensional axis, such as ideological positions or the possible values of a tax rate on the Left-Right axis. On this axis, a voter will be inclined to vote for an option if it is closer to his preferred position (his bliss point). In our framework with three candidates, single-peaked preferences implies that there is one candidate who is never ranked last by any of the voters. According to Gehrlein (2004), such a candidate appears as a positively unifying candidate since no voter is against her possible election; voters agree at least that such a candidate is not the worst. In this section, we will focus our analysis only on the IAC assumption.

4.2.1 The Case of One-Shot Scoring Rules

On $A = \{a, b, c\}$, let us assume without loss of generality that a is the never-bottom-ranked candidate and b one of the other candidates; this implies that $n_4 = n_6 = 0$. To make the computations, we have to distinguish the situations where a is the winner from those where she is not. On this basis, we derive that with single-peaked preferences, $P^{\lambda}_{\mathrm{WBE}}(3, \infty, IAC)^\star$, the limiting probability of the Weak Borda Effect, is as follows:

$$
\begin{aligned}
&P^{\lambda}_{\mathrm{WBE}}(3,\infty,IAC)^{\star}\\
&\quad=\Big(P(a;b\mathbf{M}a,IAC)^{\star}+P(a;c\mathbf{M}a,IAC)^{\star}-P(a;b\mathbf{M}a;c\mathbf{M}a,IAC)^{\star}\Big)\\
&\qquad+2\Big(P(b;a\mathbf{M}b,IAC)^{\star}+P(b;c\mathbf{M}b,IAC)^{\star}-P(b;a\mathbf{M}b;c\mathbf{M}b,IAC)^{\star}\Big)\\
&P^{\lambda}_{\mathrm{SBE}}(3,\infty,IAC)^{\star}\\
&\quad=P(a;b\mathbf{M}a;c\mathbf{M}a,IAC)^{\star}+2P(b;a\mathbf{M}b;c\mathbf{M}b,IAC)^{\star}
\end{aligned}
$$

Using the same technique as before, we obtain

$$
P(a;b\mathbf{M}a,IAC)^{\star}=P(a;c\mathbf{M}a,IAC)^{\star}=\begin{cases}\frac{\lambda^2}{8(\lambda-2)(\lambda-1)} & \text{for } 0\le\lambda\le\frac{1}{2}\\ \frac{4\lambda^2-7\lambda+2}{8(\lambda-2)} & \text{for } \frac{1}{2}\le\lambda\le 1\end{cases}
$$

$$
P(b;a\mathbf{M}b;c\mathbf{M}b,IAC)^{\star}=P(b;c\mathbf{M}b,IAC)^{\star}=\begin{cases}-\frac{(2\lambda-1)^3}{72(\lambda-1)^2} & \text{for } 0\le\lambda\le\frac{1}{2}\\ 0 & \text{for } \frac{1}{2}\le\lambda\le 1\end{cases}
$$

$$
P(b;a\mathbf{M}b,IAC)^{\star}=\begin{cases}\frac{(5-\lambda)(2\lambda-1)^2}{72(\lambda-1)^2} & \text{for } 0\le\lambda\le\frac{1}{2}\\ 0 & \text{for } \frac{1}{2}\le\lambda\le 1\end{cases}
$$

$$
P(a;b\mathbf{M}a;c\mathbf{M}a,IAC)^{\star}=0
$$

Then, we get Proposition 7.

Proposition 7 *For three-candidate elections with single-peaked preferences and scoring rule w_λ,*

$$
\begin{aligned}
P^{\lambda}_{WBE}(3,\infty,IAC)^{\star}&=\begin{cases}\frac{4\lambda^4-47\lambda^3+78\lambda^2-47\lambda+10}{36(2-\lambda)(\lambda-1)^2} & \textit{for } 0\le\lambda\le\frac{1}{2}\\ \frac{4\lambda^2-7\lambda+2}{4(\lambda-2)} & \textit{for } \frac{1}{2}\le\lambda\le 1\end{cases}\\
&=1-P_c(3,\infty,IAC)^{\star}\times P^{\lambda}_{CE}(3,\infty,IAC)^{\star}\\
P^{\lambda}_{BE}(3,\infty,IAC)^{\star}&=\begin{cases}\frac{4\lambda^4-25\lambda^3+36\lambda^2-20\lambda+4}{12(2-\lambda)(\lambda-1)^2} & \textit{for } 0\le\lambda\le\frac{1}{2}\\ \frac{4\lambda^2-7\lambda+2}{4(\lambda-2)} & \textit{for } \frac{1}{2}\le\lambda\le 1\end{cases}\\
&=1-P_c(3,\infty,IAC)^{\star}\\
&\qquad\Big(P^{\lambda}_{CE}(3,\infty,IAC)^{\star}-P^{\lambda}_{SgBP}(3,\infty,IAC)^{\star}\Big)
\end{aligned}
$$

Table 6 provides some values of the limiting probabilities of the Borda Effect under the IAC assumption.

From this table, we notice that the Weak Borda Effect is minimized at $\lambda=\frac{14830}{39291}\approx 0.3774$ with a probability equal to 0.0552, and is maximized at $\lambda=1$. The Strong Borda Effect only occurs for $\lambda\in[0;\frac{1}{2}[$; over this range the likelihood tends to decrease as λ increases. Comparing the figures of Table 6 with those of Table 4, it appears that the likelihood of the Strong Borda Effect is substantially the same in both tables for

Table 6 Computed values of the Borda effect for scoring runoff rules when preferences are single-peaked under the IAC assumption

λ	One-shot rules			Runoff rules
	Weak Borda effect	Strong Borda effect	Borda effect	Borda effect
0	0.1389	0.0278	0.1667	0.0486
0.1	0.1090	0.0176	0.1266	0.0328
0.2	0.0819	0.0094	0.0913	0.0188
0.3	0.0615	0.0036	0.0651	0.0078
0.4	0.0558	0.0006	0.0564	0.0014
0.5	0.0833	0	0.0833	0
0.6	0.1357	0	0.1357	0
0.7	0.1808	0	0.1808	0
0.8	0.2167	0	0.2167	0
0.9	0.2409	0	0.2409	0
1	0.2500	0	0.2500	0

$\lambda \in [0; \frac{1}{2}]$; for $\lambda \in [\frac{1}{2}; 1]$ the paradox vanishes when preferences are single-peaked, while this is not the case when there are not. The likelihood of the (Weak) Borda Effect is clearly lower for all λ when preferences are single-peaked. Thus, the Borda effect tends to be less likely when preferences are single-peaked.

4.2.2 The Case of Runoff Scoring Rules

Recall that with runoff rules only the Weak Borda Effect can be observed, since this cannot be the case for the Strong Borda Effect. It follows that the Weak Borda Effect is equivalent to the Borda Effect. With single-peaked preferences, if we assume that a is the never-bottom-ranked candidate, the Weak Borda Effect occurs if one of the following situations holds:

- (i) $aS_\lambda c$, $bS_\lambda c$, $a\mathbf{M}b$ and $c\mathbf{M}a$ (or $aS_\lambda b$, $cS_\lambda b$, $a\mathbf{M}c$ and $b\mathbf{M}a$)
- (ii) $aS_\lambda c$, $bS_\lambda c$, $b\mathbf{M}a$ and $c\mathbf{M}b$ (or $aS_\lambda b$, $cS_\lambda b$, $c\mathbf{M}a$ and $b\mathbf{M}c$)
- (iii) $bS_\lambda a$, $cS_\lambda a$, $b\mathbf{M}c$ and $a\mathbf{M}b$ (or $bS_\lambda a$, $cS_\lambda a$, $c\mathbf{M}b$ and $a\mathbf{M}c$)

Notice that all the situations (i) to (iii) are disjoints. It comes out that $P^{\lambda^r}_{WBE}(3, \infty, \hbar)$, the limiting probability of the Weak Borda effect is computed as follows

$$P^{\lambda^r}_{WBE}(3, \infty, IAC)^\star = 2\Big(P(a; c\mathbf{M}a, IAC)^\star + P(b; a\mathbf{M}b, IAC)^\star + P(b; c\mathbf{M}b, IAC)^\star\Big)$$

Using the same technique as before, we get:

$$P(a; c\mathbf{M}a, IAC)^\star = P(b; c\mathbf{M}b, IAC)^\star = 0 \text{ for } 0 \leq \lambda \leq 1$$

$$P(b; a\mathbf{M}b, IAC)^\star = \begin{cases} \frac{(\lambda+7)(2\lambda-1)^3}{144(\lambda-2)(\lambda-1)^2} & \text{for } 0 \leq \lambda \leq \frac{1}{2} \\ 0 & \text{for } \frac{1}{2} \leq \lambda \leq 1 \end{cases}$$

Then, we derive Proposition 8.

Proposition 8 *For three-candidate elections with single-peaked preferences and scoring runoff rule* w_{λ^r}

$$
\begin{aligned}
P^{\lambda^r}_{WBE}(3,\infty,IAC)^\star &= P^{\lambda^r}_{BE}(3,\infty,IAC)^\star \\
&= \begin{cases} \frac{(\lambda+7)(2\lambda-1)^3}{72(\lambda-2)(\lambda-1)^2} & for\ \ 0 \leq \lambda \leq \frac{1}{2} \\ 0 & for\ \ \frac{1}{2} \leq \lambda \leq 1 \end{cases} \\
&= 1 - P_c(3,\infty,IAC)^\star \times P^{\lambda^r}_{CE}(3,\infty,IAC)^\star
\end{aligned}
$$

Table 6 reports some computed values of the Borda Effect for runoff scoring rules when preferences are single-peaked. We notice that the paradox is maximized at $\lambda = 0$ and it vanishes for $\lambda \in [\frac{1}{2}; 1]$. Comparing the results of Tables 4 and 6, we can conclude that the Borda effect is less likely when preferences are single-peaked.

5 Concluding Remarks

The *Borda Effect* is one of the ramifications of the declensions of the Borda Paradox, and it was first introduced and defined by Colman and Poutney (1978). Colman and Poutney (1978) distinguished the *Strong Borda Effect* and the *Weak Borda Effect*: the *Strong Borda Effect* describes a situation in which the Plurality rule elects the Condorcet loser, while the *Weak Borda Effect* (WBE) concerns a situation under which the Plurality winner is majority dominated by only one of the Plurality losers. The results of Colman and Poutney (CoP, p 17) are quite limited in scope as they only dealt with the Plurality rule: in fact this phenomenon can also affect all the scoring rules and scoring runoffs. In this paper we found that for three-candidate elections, the representation of the (Weak) Borda Effect for general weighted scoring rules and scoring runoff rules can be deduced from those of Condorcet efficiency and the Strong Borda Paradox. For one-shot rules, we found under assumption $\hbar$ (IC or IAC) that

$$
\begin{aligned}
P^{\lambda}_{\mathrm{WBE}}(3,\infty,\hbar) &= 1 - P_c(3,\infty,\hbar) \times P^{\lambda}_{\mathrm{CE}}(3,\infty,\hbar) \\
P^{\lambda}_{\mathrm{BE}}(3,\infty,\hbar) &= 1 - P_c(3,\infty,\hbar)\left(P^{\lambda}_{\mathrm{CE}}(3,\infty,\hbar) - P^{\lambda}_{\mathrm{SgBP}}(3,\infty,\hbar)\right)
\end{aligned}
$$

These relations, which also hold when preferences are single-peaked, teach us that the Condorcet efficiency of a scoring rule impacts its vulnerability to the Borda Effect: the more it is likely to select the Condorcet winner when it exists, the less it is susceptible to producing the Borda Effect. On the contrary, the more a scoring rule is likely to select the Condorcet loser when it exists, the more it is likely to exhibit the Borda Effect. The first relation holds for scoring runoff rules. We also noted that the likelihood of the Weak Borda Effect is quite low under runoff rules as opposed to one-shot rules. We found that the relations above do not hold for scoring runoff rules eliminating candidates who score less than the average score.

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