

Coalition Formation and Stability

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Abstract This paper aims to develop, for any cooperative game, a solution notion that enjoys stability and consists of a coalition structure and an associated payoff vector derived from the Shapley value. To this end, two concepts are combined: those of strong Nash equilibrium and Aumann–Drèze coalitional value. In particular, we are interested in conditions ensuring that the grand coalition is the best preference for all players. Monotonicity, convexity, cohesiveness and other conditions are used to provide several theoretical results that we apply to numerical examples including real-world economic situations.

Keywords Game theory · TU cooperative game · Monotonicity · Superadditivity · Convexity · Cohesiveness · Shapley value · Coalition structure · Aumann–Drèze value · Strong Nash equilibrium · Stability

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1 Introduction

Most of human groups are often faced to the need of making collective decisions. The agents in such a group may be individuals, families, enterprises, political parties, trade unions, towns, regions, countries, and other social organizations. A usual group

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decision-making procedure consists in carrying out a negotiation addressed to reach full or partial agreements among the involved agents. The situation can be studied from many different points of view depending on the decision at stake, the relationships among the agents, their particular interests, and the procedure used to make the decision. There exist in the literature a lot of contributions on the topic. Without trying to be exhaustive, we mention here Gamson (1961), Brams (2003), Brams et al. (2005), Hajdukova (2006), Alcalde and Romero-Medina (2006), Ray (2007), Blockmans and Guerry (2016) and De Almeida and Wachowicz (2017). Many other references are discussed along the text.

Basic tools for any analysis are: (a) a description of the set of agents and the subsets able to arrive at an agreement between their members; (b) an evaluation of the utility obtained by each agent in any possible circumstance; (c) the consequent individual preferences on the set of outcomes; and (d) the additional ingredients that one wishes to take into account when dealing with a given problem. In this paper we propose to adopt the game theory perspective, because cooperative games provide a useful (although not unique) model for discussing many aspects of the topic. With this model, any value allows us to determine the possible payoffs and hence to define all individual preferences. E.g., simple games have been widely used to study political decision-making mechanisms. In this case, the power distribution, rather than a payoff distribution in economic terms, is the relevant issue. Ideological constraints for the agents, commonly present in politics, may be introduced in the model adapting a given value conveniently. As an example, we recall the generalizations of the Banzhaf value given by the symmetric coalitional binomial semivalues (Carreras and Puente 2012) or the multinomial probabilistic values (Carreras and Puente 2015).

Our main objective is to present a new approach to the subject from the game theory viewpoint. To this end, we consider: (a) the most basic model, which is that of TU cooperative game; (b) the best known and accepted allocation method, the Shapley value, from which we obtain the preferences based on maximizing its allocations; and (c) a stability criterion derived from the Nash strong equilibrium notion. Our study is centered on coalition formation using agents' best preferences and obtaining as solution(s) the coalition structure(s) where each agent lies in his preferred coalition and receives therefore the allocation given to him by the Shapley value on the game restricted to this coalition. This ensures the stability of the coalition structure and leads to our solution concept. We obtain several theoretical results on existence and uniqueness and illustrate the possibilities to apply these results to real-world simulated economic problems.

While writing this paper, we have revised many articles related with the topic. We have found minor similarities with our work in some of them and essential differences in some others (details are given along the text). Then, it seems interesting to summarize here the highlights of the article in order to give clear insights into the novelty that it represents and the gap that it fills.

- An applied game theory approach is used
- A new concept of solution for a cooperative game is established
- A noncooperative glance over any cooperative game is adopted
- The idea of stability is based on the Nash strong equilibrium notion

- Our procedure leads to using the Aumann–Drèze value, which is the result of applying the Shapley value to subgames
- The procedure works for any cooperative game
- The grand coalition is not necessarily assumed to form
- All theoretical results are strictly original and useful in practice, and they concern the process relevant to group decision and negotiation
- Our model is simple, but it admits possibilities of sophistication based on introducing additional information not included in the characteristic function of the game and, e.g., modifying, if necessary, the value used for allocating payoffs

Game theory studies conflictive situations that arise when a set of agents (called *players*), which may have different or even opposed interests, must take individual decisions to obtain some kind of individual payoffs as the result of their interaction. Usually, there exists a certain level of competition that, in some cases, is compatible with the possibility of total or partial cooperation.

Frequently, “cooperative game” and “noncooperative game” are considered in the literature antagonistic notions. In a cooperative game, the players are allowed to communicate between them in order to coordinate their actions, looking for a joint profit derived from their agreement. Nothing of this is permitted in the noncooperative case, where each player has a set of strategies and chooses one of them, trying to maximize his payoff and being aware that this payoff may well depend on the choices simultaneously made by the other players.

At this point, it is convenient to specify the relative importance of two concepts: communication possibilities and enforceability of the agreements.¹ We quote from Harsanyi (1982):

This distinction was first proposed by Nash (1950, 1951), who defined cooperative games as games permitting both communication and enforceable agreements between the players, and defined noncooperative games as games permitting neither communication nor enforceable agreements.

[...] it is now commonly agreed that it is preferable to distinguish cooperative games and noncooperative games on the basis of one single criterion. It turns out that enforceability or unenforceability of agreements is a much more important characteristic of a game than presence or absence of communication is.

Thus, we adopt in the sequel two basic assumptions for cooperative games: first, free communication is allowed between players in order to coordinate strategies; second, any agreement to this end, arrived at by some or all players, is completely enforceable and, of course, each player may sign only one agreement at most. So, if the players agree to cooperate then a negotiation is carried out, one or more binding contracts can be established among all players or within subsets of players (called *coalitions*) and, finally, the benefits of the cooperation are to be shared as specified in the contract(s).

A cooperative game merely describes the utility of the coalitions, independently, a priori, of whether they will really form or not. Therefore, we adopt two more additional assumptions: first, all players agree that the Shapley value (Shapley 1953; Roth 1988)

¹ A suggestion for which we are grateful to a reviewer.

is the “universal sharing rule” to be used in all circumstances;² second, in consequence, each player perfectly knows the possibilities of the others, that is, the coalitions they might choose and the payoffs they would receive according to the coalition structure derived from their respective choices.

Often, it is implicitly assumed that, at the end, the grand coalition will form and its utility will be divided among all players. However, one can raise objections to this assumption, since such a full agreement may depend on many factors (for example, the sharing rule used) and it is not unlikely that in certain cases the players prefer other options to organize themselves. We quote Shenoy’s basic ideas (Shenoy 1979) to this respect:

The theory of n -person cooperative games is a mathematical theory of coalition behavior. A fundamental problem posed in game theory is to determine what outcomes are likely to occur if a game is played by “rational players”. I.e. given an n -person cooperative game, it is natural to inquire (1) what will be the final allocation of payoffs to each of the players and (2) which of the possible coalitions can be expected to form. These two aspects of coalition behavior are closely related. The final allocation of payoffs to each of the players depends on the coalitions that finally form, and the coalitions that finally form depend on the available payoffs to each player in each of these coalitions. Since the publication in 1944 of the monumental work *Theory of Games and Economic Behavior* by Von Neumann and Morgenstern (1953), most of the research in n -person game theory has been concerned explicitly with predicting players’ payoff and only implicitly (if at all) with predicting which coalitions shall form. In this paper, the primary emphasis is on the second aspect of coalition behavior, namely the formation of coalitions.

We subscribe Shenoy’s opinion. What is important when analysing a game is to determine: (a) which coalitions—not necessarily the grand coalition—will form, and (b) which are the payoffs that players will subsequently receive. When using the Shapley value on subgames we are in fact following the philosophy of the Aumann–Drèze value (1974) (AD value, for short, in the sequel).

Grounds for our approach may be found, on one hand, in a strong criticism justified in previous works (Amer et al. 2007; Carreras and Owen 2011, 2013) against the use of the proportional rule as universal sharing rule; on the other, in a renewed interest on the AD value, revealed in very recent articles (Wiese 2007; Casajus 2009; Tutić 2010; Alonso-Mejjide et al. 2015; Carreras and Owen 2016).

The Shapley value of any player in any game is a weighted (convex) sum of the marginal contributions of the player to all possible coalitions. Therefore, depending on the game, the payoff assigned to a fixed player by the Shapley value might be even smaller than the utility that this player can obtain alone. In this case, forming the grand coalition is harmful for this player and hence it would be difficult to persuade him to enter this coalition.

² A survey that shows the impact of the Shapley value in several scientific disciplines is due to Moretti and Patrone (2008).

One could argue that this situation is avoided if the game is superadditive, because in this case the Shapley value assigns to each player at least his individual utility. Nevertheless, there are superadditive games where the formation of the grand coalition is not the best option for all players, and even for none of them. We provide an example.

Example 1.1 (Aumann and Drèze 1974) Let us consider the symmetric, monotonic and superadditive 3-person cooperative game u defined by

$$u(\emptyset) = 0, \quad u(\{i\}) = 0, \quad u(\{i, j\}) = 8 \quad \text{for all distinct } i, j \quad \text{and} \quad u(\{1, 2, 3\}) = 9.$$

Superadditivity holds here because $u(\{i\}) + u(\{j\}) \leq u(\{i, j\})$ and $u(\{i\}) + u(\{j, k\}) \leq u(N)$ for all distinct $i, j, k \in N$. The players might choose (a) to remain all alone, (b) to join a partner and leave aside the other player, or (c) to form the grand coalition $\{1, 2, 3\}$. Then, from the symmetric role of all players in this game, it is clear that the payoff to a player would be 0 if he remains alone, 4 if he joins just a partner, and 3 if all form the grand coalition. Therefore, players' preferences as to all these options are, schematically,

- $\{1, 2\} \equiv \{1, 3\} > \{1, 2, 3\} > \{1\}$ for player 1,
- $\{1, 2\} \equiv \{2, 3\} > \{1, 2, 3\} > \{2\}$ for player 2,
- $\{1, 3\} \equiv \{2, 3\} > \{1, 2, 3\} > \{3\}$ for player 3.

We conclude that any organisation of the form $\mathcal{B} = \{\{i, j\}, \{k\}\}$, with i, j, k distinct, would be a “solution” of this game and would be “stable” in the sense that no player or group of players has a strict incentive and the power to modify this structure.

The convenience to simultaneously deal with coalition formation and payoffs allocation inspired the notions of *coalition structure* and *coalitional value*, introduced by Aumann and Drèze (1974), and gave a new impulsion to the development of value theory. These authors extended the Shapley value to this new framework, using the approach of *isolated unions*, and obtained the first coalitional value, the AD value. A second approach, that of *bargaining unions*, was used by Owen (1977), when introducing what is called now Owen value.

Two main differences between these values are: (1) the Owen value satisfies efficiency, whereas the AD value satisfies relative efficiency (that is, in each union); and (2) the payoffs given by the AD value within each union are independent of the organisation of the remaining players, but this is not true for the Owen value. The reason is that the AD value is intended for being applied when the players are assumed to stop the bargaining once they have formed the unions, whereas the Owen value is based on the assumption that they form unions only as a previous step addressed to attain a better bargaining position when forming, at the end, the grand coalition.

The existence of different coalitional values raised the convenience of testing the *stability* of any coalition structure with regard to a given value. This is a great contribution of Hart and Kurz (1984). These authors assume that both a game and a coalitional value are given, define the notion of stability for any coalition structure and try to determine which coalition structures (if any) are stable in the given game with regard to the given coalitional value. They introduced two kinds of stability that they called

γ -stability and δ -stability, both based on the notion of (strong) Nash equilibrium for noncooperative games (Nash 1950; Aumann 1959).³ They gave an axiomatic characterization of a “CS value”, which coincides with the Owen value, and used only this value in their analysis of games with a coalition structure. Hart and Kurz’s experience with the Owen value raises concerns that in general it is not easy to find valuable results on stability.

Many authors have worked on stability from different approaches. E.g., Wiese (2007) and Casajus (2009) define variations of the AD value that take into account, in some way, the players’ outside options (a viewpoint that we will not share here). A Casajus’ nice result shows that any game admits some stable coalition structure for his value, while Tutić (2010) presents a 4-person game that admits no stable coalition structure for either the AD value or the Wiese value.

In general, in any cooperative game, the players are still interested, individually, in obtaining the best possible payoff. This introduces a “noncooperative flavor” in the cooperative game theory. Thus, a cooperative game might be rather viewed just as a tool that defines the strategies available to each player, as well as the payoffs obtained by applying a given general sharing rule to any profile of strategies. And this is our approach.

The notion of strategy is usual in the context of noncooperative games but it is not so common when dealing with cooperative games. A strategy for a player will consist in choosing a coalition to which this player belongs.

Once each player has computed his payoff when he joins any possible coalition, each player chooses one coalition among those that give him the maximum payoff. A coalition is called *optimal* when each of its members has chosen it. The optimal coalitions, jointly with the singletons corresponding to the players not appearing in any of them, constitute a coalition structure in the player set. Such a coalition structure is *stable*, in the sense that there is no reason to change it, so it can be considered, jointly with the payoffs allocated to the players according to the Shapley value in each involved subgame, as a *solution* of the cooperative game.

This stability idea recalls the notion of strong Nash equilibrium for noncooperative games. Any other (unstable) coalition structure, jointly with the attached payoffs to each player, can be interpreted as an “outcome” for the game following Yang (2011), but it represents an inefficient behavior of the players, since at least one of them will feel unsatisfied and, moreover, will have the opportunity to change his choice. Among other questions, we will pay special attention to this: which conditions must satisfy a cooperative game to ensure that the grand coalition is stable in the previously defined sense?

The organization of the paper is as follows. We first provide basic preliminaries in Sect. 2. Next, we propose some numerical examples in Sect. 3 to illustrate the problem. In Sect. 4, the notions of monotonicity, convexity, cohesiveness and others are used to establish five main results. Section 5 includes more examples analysed with these results. In Sect. 6, applications to economic problems are sketched. Section 7 concludes.

³ In a previous work Hart and Kurz (1983), they defined two more notions, α -stability and β -stability, for NTU cooperative games, but we will restrict our study to TU games.

2 Preliminaries and Formal Definitions

We will assume that the reader is familiar with the grounds of the cooperative and noncooperative game theories. We first recall some basic ideas, establish the notation that will be used throughout this work, and introduce the main notions formally. For more details we refer the reader to e.g. (Driessen 1988; González-Díaz et al. 2010; Owen 2013).

Let $N = \{1, 2, \dots, n\}$ be the set of *players* and 2^N be the set of *coalitions* (subsets of N). A *cooperative game* in N is defined by (and identified with) its *characteristic function* $u : 2^N \rightarrow \mathbb{R}$, which assigns to each coalition $S \subseteq N$ a real number $u(S)$, interpreted as the utility that coalition S can obtain if all its members agree, independently of the behavior of the remaining players, i.e. the members of $N \setminus S$. The only restriction is that $u(\emptyset) = 0$ for any game u .

Player $i \in N$ is a *null player* in game u if $u(S \cup \{i\}) = u(S)$ for all $S \subseteq N \setminus \{i\}$. Players $i, j \in N$ are *symmetric players* in game u if $u(S \cup \{i\}) = u(S \cup \{j\})$ for all $S \subseteq N \setminus \{i, j\}$. Endowed with the usual linear operations $u + u'$ and λu for any $\lambda \in \mathbb{R}$, the set of all cooperative games in N becomes a real vector space \mathcal{G}_N of dimension $2^n - 1$.

If $u \in \mathcal{G}_N$ and $\emptyset \neq T \subseteq N$, the *restriction* of u to T is the game $u_T \in \mathcal{G}_T$ defined by $u_T(S) = u(S)$ for all $S \subseteq T$.⁴ We also say that u_T is a *subgame* of u . Obviously, $u_N = u$.

The following conditions, that define special classes of games, are hereditary, in the sense that if one of them holds for a game then it holds for all its subgames.

A game $u \in \mathcal{G}_N$ is *monotonic* if $u(S) \leq u(T)$ whenever $S \subset T$.

A game $u \in \mathcal{G}_N$ is *symmetric* if all $i, j \in N$ are symmetric players in u . This is equivalent to saying that $u(S)$ depends only on the cardinality of coalition S , $s = |S|$, for all $S \subseteq N$.

A game $u \in \mathcal{G}_N$ is *superadditive* if $u(R) + u(S) \leq u(R \cup S)$ when $R \cap S = \emptyset$. It is called *additive* if $u(R) + u(S) = u(R \cup S)$ when $R \cap S = \emptyset$, and *strictly superadditive* if $u(R) + u(S) < u(R \cup S)$ when $R \cap S = \emptyset$.

A *coalition structure* in N is a collection $\mathcal{B} = \{B_1, B_2, \dots, B_m\}$ of pairwise disjoint coalitions (*unions*) such that $B_1 \cup B_2 \cup \dots \cup B_m = N$. The *trivial* coalition structures are $\mathcal{B}^N = \{N\}$ and $\mathcal{B}^n = \{\{1\}, \{2\}, \dots, \{n\}\}$. The second might be understood as a sort of “disagreement point”.

It follows at once that, if u is superadditive and $\mathcal{B} = \{B_1, B_2, \dots, B_m\}$ is a coalition structure in N , then

$$u(B_1) + u(B_2) + \dots + u(B_m) \leq u(N).$$

This fact is interesting: it means that any coalition structure is, in principle, *feasible*, given that the total utility will be able to satisfy the demands of all unions. The inequality becomes strict when the game is strictly superadditive and $\mathcal{B} \neq \mathcal{B}^N$, and an equality under additivity.

⁴ Here u_T will not mean the unanimity game associated to coalition T .

The *Shapley value* (Shapley 1953; Roth 1988) is a map $\Phi : \mathcal{G}_N \longrightarrow \mathbb{R}^n$ that assigns to each game $u \in \mathcal{G}_N$ a vector $\Phi[u] = (\Phi_1[u], \Phi_2[u], \dots, \Phi_n[u])$. The allocation given by the Shapley value to each player $i \in N$ in any game $u \in \mathcal{G}_N$ is

$$\Phi_i[u] = \sum_{S \ni i} \gamma_n(s)[u(S) - u(S \setminus \{i\})], \quad \text{where } s = |S| \text{ and } \gamma_n(s) = \frac{1}{n \binom{n-1}{s-1}}.$$

We will use the Shapley value as universal sharing rule, so it will be applied to any game and also to all its subgames.

The weighting coefficients $\gamma_n(s)$ do not depend on game u . As is well known, for any $i \in N$,

$$\sum_{S \ni i} \gamma_n(s) = 1.$$

Hence $\Phi_i[u] \geq u(\{i\})$ for any player i and any superadditive game u , and $\Phi_i[u] > u(\{i\})$ if, moreover, $u(S) > u(\{i\}) + u(S \setminus \{i\})$ for at least one coalition S containing i . This means that such a player would prefer joining the grand coalition instead of remaining alone.

However, given a coalition structure \mathcal{B} , the same holds for any subgame u_{B_k} . That is, $\Phi_i[u_{B_k}] \geq u(\{i\})$ for all B_k and all $i \in B_k$, and $\Phi_i[u_{B_k}] > u(\{i\})$ if, moreover, $u(S) > u(\{i\}) + u(S \setminus \{i\})$ for at least one coalition $S \subseteq B_k$ containing i , so such players $i \in B_k$ would also prefer joining B_k instead of remaining alone.

Now we introduce the main notions formally.

Let u be a cooperative game in N . Following Hart and Kurz’s γ -model (1984), we first set up an auxiliary noncooperative game $\Gamma^\Phi(u)$.

For each $i \in N$ the strategy set is $\Sigma_i = \{S \subseteq N : i \in S\}$, thus having cardinality 2^{n-1} . The *strategy space* of $\Gamma^\Phi(u)$ is $\Sigma_1 \times \Sigma_2 \times \dots \times \Sigma_n$, so any profile of strategies $\sigma \in \Sigma_1 \times \Sigma_2 \times \dots \times \Sigma_n$ is of the form $\sigma = (S_1, S_2, \dots, S_n)$, with $i \in S_i$ for each $i \in N$.

Given profile $\sigma = (S_1, S_2, \dots, S_n)$, a nonempty coalition S is said to be σ -selected if and only if $S_i = S$ for each $i \in S$. We set $\Omega_\sigma = \{S \subseteq N : S \text{ is } \sigma\text{-selected}\}$. If $S, T \in \Omega_\sigma$ are distinct then $S \cap T = \emptyset$. We then consider the coalition structure

$$\mathcal{B}_\sigma = \{S : S \in \Omega_\sigma\} \cup \{\{j\} : j \notin U_\sigma\}, \quad \text{where } U_\sigma = \bigcup_{S \in \Omega_\sigma} S.$$

The *payoffs* in $\Gamma^\Phi(u)$ are given for each profile σ by

$$k_i(\sigma) = \begin{cases} \Phi_i[u_{S_i}] & \text{if } S_i \in \Omega_\sigma, \\ u(\{i\}) & \text{otherwise.} \end{cases}$$

Thus, in practice, a strategy of any player in a cooperative game will imply to remain alone unless all members of the coalition he chooses make the same choice. If all members of the coalition agree to choose it, then each one of them obtains the payoff

given to him by the application of the Shapley value to the restricted game. Otherwise (and even if only one of these members does not choose it), the payoff obtained by each player that chose the coalition will be just his individual utility.⁵ For example, if player 1 chooses $\{1, 2\}$ but 2 and 3 choose $\{2, 3\}$, then player 1 gets $u(\{1\})$. Of course, if a player chooses to remain alone, he will also obtain his individual utility, because $\Phi_i[u(\{i\})] = u(\{i\})$ for each $i \in N$.

Definition 2.1 $\Gamma^\Phi(u)$, with strategy space $\Sigma_1 \times \Sigma_2 \times \cdots \times \Sigma_n$ and payoff functions k_1, k_2, \dots, k_n , is the *noncooperative game associated to game u with respect to the Shapley value Φ* .

Given a profile $\sigma = (S_1, S_2, \dots, S_n)$, a *defector group* of σ is a nonempty coalition $T \subseteq N$ with strategies $\tilde{\sigma}_i \in \Sigma_i$ for each $i \in T$ such that $k_i((\tilde{\sigma}_i)_{i \in T}, (\sigma_j)_{j \in N \setminus T}) > k_i(\sigma)$ for all $i \in T$.

If profile σ is free from defector groups, then it is a *strong Nash equilibrium* in $\Gamma^\Phi(u)$. We then say that \mathcal{B}_σ is a *stable coalition structure in u* (with regard to the Shapley value).

Now we translate these ideas to the cooperative language. We are assuming that all players try to optimize their individual payoffs. Hence, once each player has computed his payoff when he joins any possible coalition, he chooses one of the coalitions that maximize this payoff. This gives rise to a profile σ^* .

Definition 2.2 A coalition is called *optimal* if it gives to each of its members his best payoff.

When these decisions have been taken, any two different optimal coalitions are disjoint, and the collection of these chosen optimal coalitions, jointly with the singletons corresponding to the players not appearing in any of them, constitutes a coalition structure \mathcal{B}_{σ^*} in the player set. The optimal coalitions form Ω_{σ^*} , the family of σ^* -selected members of \mathcal{B}_{σ^*} .

Definition 2.3 If profile σ^* is a strong Nash equilibrium in $\Gamma^\Phi(u)$, the coalition structure \mathcal{B}_{σ^*} can be considered, jointly with the payoffs allocated to the players, as a *solution* of the cooperative game, since it gives a behavioral pattern and a subsequent payoff vector.

Such a coalition structure is *stable* because it comes from a profile that is a strong Nash equilibrium. This means that there exists no *strict* incentive in payoff terms for any player (and, in fact, neither for any set of players) to change his decision and move from one coalition to another.

Of course, this solution may not be unique (cf. Example 3.2 below) or not exist (cf. Example 3.3). Any other (unstable) coalition structure may be considered as an outcome for the game, but it represents an inefficient behavior of the players because at least one of them will feel unsatisfied (cf. Example 3.2).

⁵ The difference between the γ -model and the δ -model lies here. In the δ -model, Hart and Kurz assume that, if only some members of a coalition choose it, the subcoalition consisting of these members forms, while the others become singletons. Of course, if a coalition forms in the γ -model it also forms in the δ -model, but the converse is not true. Here we prefer using the γ -model solely because the subcoalition might have a utility no longer interesting to its members.

3 Examples

Some numerical examples will illustrate the above notions.

Examples 3.1 (a) Let $n = 3$ and u be the monotonic game defined by

$$\begin{aligned}
 u(\emptyset) &= 0, & u(\{1\}) &= 1, & u(\{2\}) &= 0, & u(\{3\}) &= 0, \\
 u(\{1, 2\}) &= 3, & u(\{1, 3\}) &= 2, & u(\{2, 3\}) &= 1, & u(N) &= 5.
 \end{aligned}$$

The Shapley value is $\Phi[u] = (2.5, 1.5, 1)$. The five coalition structures are $\mathcal{B}^n = \{\{1\}, \{2\}, \{3\}\}$, $\mathcal{B}^{\{1,2\}} = \{\{1, 2\}, \{3\}\}$, $\mathcal{B}^{\{1,3\}} = \{\{1, 3\}, \{2\}\}$, $\mathcal{B}^{\{2,3\}} = \{\{1\}, \{2, 3\}\}$ and $\mathcal{B}^N = \{N\}$. The strategy set of player 1 is, with a simplified notation, $\Sigma_1 = \{1, 12, 13, 123\}$, Σ_2 and Σ_3 being analogous. This gives $4^3 = 64$ profiles in the noncooperative game. The payoffs derived from applying the Shapley value Φ are as follows:

Player	\mathcal{B}^n	$\mathcal{B}^{\{1,2\}}$	$\mathcal{B}^{\{1,3\}}$	$\mathcal{B}^{\{2,3\}}$	\mathcal{B}^N
1	1	2	1.5	1	2.5
2	0	1	0	0.5	1.5
3	0	0	0.5	0.5	1

There are 14 Nash equilibria in $\Gamma^\Phi(u)$. However, only one of them is strong: the equilibrium defined by $\sigma = (123, 123, 123)$ that gives rise to the only stable coalition structure: \mathcal{B}^N .

(b) Let $n = 3$ and u be the monotonic game defined by

$$\begin{aligned}
 u(\emptyset) &= 0, & u(\{i\}) &= 0, & u(\{1, 2\}) &= 6, & u(\{1, 3\}) &= 1, \\
 u(\{2, 3\}) &= 1, & u(N) &= 6.
 \end{aligned}$$

Here, $\Phi[u] = (17/6, 17/6, 1/3)$. Only profiles of the form $(12, 12, X)$, where X stands for any coalition containing player 3, are strong Nash equilibria. All of them lead to the only stable coalition structure $\mathcal{B}^{\{1,2\}} = \{\{1, 2\}, \{3\}\} \neq \mathcal{B}^N$, with payoff vector $(3, 3, 0)$.

Example 3.2 Let us take $n = 3$ and consider the game u defined by

$$\begin{aligned}
 u(\emptyset) &= 0, & u(\{1\}) &= 1, & u(\{2\}) &= 3, & u(\{3\}) &= 0, \\
 u(\{1, 2\}) &= 6, & u(\{1, 3\}) &= 1, & u(\{2, 3\}) &= 3, & u(\{1, 2, 3\}) &= 6.
 \end{aligned}$$

The game is superadditive. Player 3 is a null player. The Shapley value of the non-trivial involved games, u , $u_{\{1,2\}}$, $u_{\{1,3\}}$ and $u_{\{2,3\}}$, is as follows:

$$\begin{aligned}
 \Phi_1[u] &= 2, & \Phi_2[u] &= 4, & \Phi_3[u] &= 0, \\
 \Phi_1[u_{\{1,2\}}] &= 2, & \Phi_2[u_{\{1,2\}}] &= 4, & \Phi_3[u_{\{1,3\}}] &= 0, \\
 \Phi_1[u_{\{1,3\}}] &= 1, & \Phi_2[u_{\{2,3\}}] &= 3, & \Phi_3[u_{\{2,3\}}] &= 0.
 \end{aligned}$$

According to these payoffs, players' preferences are the following:

- Player 1 wishes to enter either coalition $\{1, 2, 3\}$ or $\{1, 2\}$ instead of forming a coalition with player 3 or remaining alone. We will simply write

$$\{1, 2, 3\} \equiv \{1, 2\} > \{1, 3\} \equiv \{1\} \text{ for player 1.}$$

- Player 2 has similar preferences:

$$\{1, 2, 3\} \equiv \{1, 2\} > \{2, 3\} \equiv \{2\} \text{ for player 2.}$$

- Player 3 is indifferent with respect to any possible coalition:

$$\{1, 2, 3\} \equiv \{1, 3\} \equiv \{2, 3\} \equiv \{3\} \text{ for player 3.}$$

In view of these preferences, two solutions arise for this game. If all players choose coalition $\{1, 2, 3\}$, this coalition is optimal and a solution of the game is given by the corresponding coalition structure and the subsequent payoffs:

$$\mathcal{B} = \{\{1, 2, 3\}\} = \mathcal{B}^N, \quad \Phi_1[u] = 2, \quad \Phi_2[u] = 4, \quad \Phi_3[u] = 0.$$

If, instead, players 1 and 2 choose coalition $\{1, 2\}$, this coalition is optimal, and another solution of the game is given by the corresponding coalition structure and the subsequent payoffs:

$$\mathcal{B} = \{\{1, 2\}, \{3\}\} = \mathcal{B}^{\{1,2\}}, \quad \Phi_1[u_{\{1,2\}}] = 2, \quad \Phi_2[u_{\{1,2\}}] = 4, \quad u(\{3\}) = 0.$$

Finally, if e.g. player 1 chooses $\{1, 2, 3\}$ but player 2 chooses $\{1, 2\}$, or conversely, then there is no optimal coalition, and the outcome of the game, defined by

$$\mathcal{B} = \{\{1\}, \{2\}, \{3\}\} = \mathcal{B}^n, \quad u(\{1\}) = 1, \quad u(\{2\}) = 3, \quad u(\{3\}) = 0,$$

is not a solution. It lacks stability because players 1 and 2 would like to modify their choices and give rise to one of the previous solutions. Note that, if e.g. two players obtain simultaneously their respective best payoff in more than one coalition containing both, as it happens here, then the contract they will sign should mention, not only the payoff allocated to each player, but also which of these coalitions they must choose in order to coordinate strategies effectively. We also remark that if a player can choose among two optimal coalitions, his payoff will be the same, but this does not extend to non-optimal coalitions.

Example 3.3 Let $n = 3$ and consider the superadditive game u defined by

$$u(\{1, 2\}) = 1, \quad u(\{2, 3\}) = u(\{1, 2, 3\}) = 2 \quad \text{and otherwise} \quad u(S) = 0.$$

The Shapley value of the non-trivial involved games, u , $u_{\{1,2\}}$, $u_{\{1,3\}}$ and $u_{\{2,3\}}$, is as follows:

$$\begin{aligned} \Phi_1[u] &= 1/6, & \Phi_2[u] &= 7/6, & \Phi_3[u] &= 4/6, \\ \Phi_1[u_{\{1,2\}}] &= 1/2, & \Phi_2[u_{\{1,2\}}] &= 1/2, & \Phi_3[u_{\{1,2\}}] &= 0, \\ \Phi_1[u_{\{1,3\}}] &= 0, & \Phi_2[u_{\{1,3\}}] &= 1, & \Phi_3[u_{\{1,3\}}] &= 1. \end{aligned}$$

Notice that, in spite of superadditivity, $\Phi_1[u]$ is not the best payoff for player 1. In this case, players’ preferences do not define any optimal coalition. They are as follows:

- $\{1, 2\} > \{1, 2, 3\} > \{1\} \equiv \{1, 3\}$ for player 1,
- $\{1, 2, 3\} > \{2, 3\} > \{1, 2\} > \{2\}$ for player 2,
- $\{2, 3\} > \{1, 2, 3\} > \{3\} \equiv \{1, 3\}$ for player 3.

We conclude that there is no solution for this game. Maybe a second bargaining round could be expected. Intuitively, it seems quite reasonable that player 2 would convince (or force) the others to accept the grand coalition.⁶

Example 3.4 Let us take $n = 3$ and $\alpha > 0$, and consider the game u defined by

$$\begin{aligned} u(\emptyset) &= 0, & u(\{1\}) &= 5, & u(\{2\}) &= 1, & u(\{3\}) &= 0, \\ u(\{1, 2\}) &= 9, & u(\{1, 3\}) &= 5, & u(\{2, 3\}) &= 4, & u(\{1, 2, 3\}) &= \alpha. \end{aligned}$$

It is clear that game u is superadditive for $\alpha \geq 9$. The Shapley value of the non-trivial involved games, u , $u_{\{1,2\}}$, $u_{\{1,3\}}$ and $u_{\{2,3\}}$, is as follows:

$$\begin{aligned} \Phi_1[u] &= 2.5 + \alpha/3, & \Phi_2[u] &= \alpha/3, & \Phi_3[u] &= \alpha/3 - 2.5, \\ \Phi_1[u_{\{1,2\}}] &= 6.5, & \Phi_2[u_{\{1,2\}}] &= 2.5, & \Phi_3[u_{\{1,2\}}] &= 0, \\ \Phi_1[u_{\{1,3\}}] &= 5, & \Phi_2[u_{\{1,3\}}] &= 2.5, & \Phi_3[u_{\{1,3\}}] &= 1.5. \end{aligned}$$

The allocations given by the Shapley value in game u are the best for players 1 and 3 only if $\alpha \geq 12$, and for player 2 only if $\alpha \geq 7.5$. Thus, the relevant cases are the following.

- (1) $\alpha < 7.5$. The possible optimal coalitions are $\{1, 2\}$ and $\{2, 3\}$, which would give rise to two solutions.
- (2) $\alpha = 7.5$. The possible optimal coalitions are again $\{1, 2\}$ and $\{2, 3\}$, which would give rise to two solutions, but player 2 is indifferent between any of them and $\{1, 2, 3\}$.
- (3) $7.5 < \alpha < 12$. The preferences diverge: player 1 still prefers $\{1, 2\}$, player 2 prefers $\{1, 2, 3\}$ only, and player 3 still prefers $\{2, 3\}$. There is no solution.
- (4) $\alpha = 12$. Player 1 is indifferent between $\{1, 2\}$ and $\{1, 2, 3\}$, player 2 prefers $\{1, 2, 3\}$ only, and player 3 is indifferent between $\{2, 3\}$ and $\{1, 2, 3\}$. The only solution is given by the grand coalition $\{1, 2, 3\}$.

⁶ The lack of solution appears here and in other examples below. Following Segal (2003), we could call *collusion proof* to any game where this occurs. In such a game, players are reduced to form, in principle, the trivial structure \mathcal{B}^n , that we have called the “disagreement point”.

- (5) $\alpha > 12$. All players prefer the grand coalition $\{1, 2, 3\}$, which yields the only solution.

We see that superadditivity does not ensure the existence of a solution but neither it is a necessary condition for this existence.

4 Conditions for Stability in Coalition Formation

This section contains the theoretical results. We begin by the trivial role of null players. A general property when null players arise in a game is that the solution (if any) is never unique. Let $i \in N$ be a null player in game u . If $\mathcal{B} = \{B_1, B_2, \dots, B_m\}$ is a stable coalition structure for game $u_{N \setminus \{i\}}$, then $\mathcal{B}' = \{B_1, B_2, \dots, B_m, \{i\}\}$, as well as any coalition structure of N of the form $\mathcal{B}^k = \{B_1, B_2, \dots, B_k \cup \{i\}, \dots, B_m\}$ for any k , is stable for game u . More generally, we have:

Proposition 4.1 (Null players do not matter) *Let $u \neq 0$ be a game and $Z(u) \subset N$ be the nonempty set of null players in u . Then the stable coalition structures in u are obtained from the stable coalition structures in the restricted game $u_{N \setminus Z(u)}$ by allowing all null players $i \in Z(u)$ to incorporate in any way.*

Proof It is straightforward. □

Examples 4.2 Let us illustrate the arrangement of null players in the stable coalition structures of the restricted game.

- (a) Let $n = 3$ and u be the superadditive monotonic game defined by

$$\begin{aligned} u(\emptyset) = u(\{3\}) = 0, \quad u(\{1\}) = u(\{1, 3\}) = 1, \quad u(\{2\}) \\ = u(\{2, 3\}) = 2, \quad u(\{1, 2\}) = u(N) = 4. \end{aligned}$$

Here $Z(u) = \{3\}$ and the only stable coalition structure in $u_{N \setminus Z(u)}$ is $\mathcal{B}^{N \setminus Z(u)} = \{\{1, 2\}\}$. Hence, the only stable coalition structures in u are $\mathcal{B}^{\{1,2\}} = \{\{1, 2\}, \{3\}\}$ and \mathcal{B}^N , with payoff vector $(1.5, 2.5, 0)$ in both cases.

- (b) Now let $n = 3$ and u be the monotonic game defined by

$$\begin{aligned} u(\emptyset) = u(\{3\}) = 0, \quad u(\{1\}) = u(\{2\}) = u(\{1, 2\}) \\ = u(\{1, 3\}) = u(\{2, 3\}) = u(N) = 4. \end{aligned}$$

Here $Z(u) = \{3\}$ again and the only stable coalition structure in $u_{N \setminus Z(u)}$ is $\mathcal{B}^{n-1} = \{\{1\}, \{2\}\}$. Hence, the only stable coalition structures in u are $\mathcal{B}^n = \{\{1\}, \{2\}, \{3\}\}$, $\mathcal{B}^{\{1,3\}} = \{\{1, 3\}, \{2\}\}$ and $\mathcal{B}^{\{2,3\}} = \{\{1\}, \{2, 3\}\}$, with payoff vector $(4, 4, 0)$ in all cases.

We next present five main results, with a previous lemma for the first and another for the third. In all results, it is implicitly assumed that the Shapley value is the universal sharing rule.

Definition 4.3 (Shapley 1971) A game u is *convex* if, for all $R, S \subseteq N$,

$$u(R \cup S) + u(R \cap S) \geq u(R) + u(S).$$

It can be seen that this condition is equivalent to the following:

$$u(S \cup \{i\}) - u(S) \leq u(T \cup \{i\}) - u(T)$$

for all $S \subseteq T \subseteq N \setminus \{i\}$ and all $i \in N$. In words, the incentives for joining a coalition increase as the coalition grows (the so-called “snowball effect”) (Driessen 1988).

If u is convex then it is clearly superadditive, and convexity is hereditary. There is a close connection between the convexity of a game u and the core of u , introduced by Gillies (1953) as

$$C(u) = \{x = (x_1, x_2, \dots, x_n) : \sum_{i \in N} x_i = u(N) \text{ and } \sum_{i \in S} x_i \geq u(S) \text{ if } \emptyset \neq S \subset N\}.$$

Indeed, the Shapley value $\Phi[u]$ of any convex game u belongs to $C(u)$ and is, in fact, the center of gravity of this core, which in turn is the only stable set in Von Neumann and Morgestern’s sense (1953).

Lemma 4.4 Let u be a game and $i, j \in N$ be distinct players. Then, setting $t = |T|$,⁷

$$\begin{aligned} \Phi_i[u] - \Phi_i[u_{N \setminus \{j\}}] &= \sum_{T \not\ni i, j} \gamma_n(t + 2)[u(T \cup \{i, j\}) \\ &\quad - u(T \cup \{i\}) - u(T \cup \{j\}) + u(T)]. \end{aligned}$$

Proof The proof is straightforward. □

Theorem 4.5 If u is a convex game then \mathcal{B}^N gives a solution for u .

Proof Let $i, j \in N$ be distinct players in u . For any $T \subseteq N \setminus \{i, j\}$ we apply convexity taking $R = T \cup \{i\}$ and $S = T \cup \{j\}$ and obtain from Lemma 4.4 that $\Phi_i[u] - \Phi_i[u_{N \setminus \{j\}}] \geq 0$.⁸

Inductively, it follows that, for any $i \in N$,

$$\Phi_i[u] \geq \Phi_i[u_{N \setminus \{j\}}] \geq \Phi_i[u_{N \setminus \{j, k\}}] \geq \dots \geq \Phi_i[u_{\{i\}}] = u(\{i\}).$$

This means that, for all $i \in S \subset N$, $\Phi_i[u] \geq \Phi_i[u_S]$. Thus, for any $\mathcal{B} = \{B_1, B_2, \dots, B_m\} \neq \mathcal{B}^N$, any B_k and any $i \in B_k$, $\Phi_i[u] \geq \Phi_i[u_{B_k}]$. We then conclude that N is an optimal coalition for all players and hence \mathcal{B}^N is a solution for u . □

⁷ In Segal (2003), the expression $[u(T \cup \{i, j\}) - u(T \cup \{i\}) - u(T \cup \{j\}) + u(T)]$ is denoted as $\Delta_{ij}^2[u](T)$, and Δ_{ij}^2 is called the “second-order difference operator”.

⁸ This inequality is, in fact, equivalent to the convexity of the game (Ichiishi 1993, Theorem 2.1.3).

- Remarks 4.6* (a) In a convex game, other solutions different from \mathcal{B}^N may appear, as in Example 3.2. Instead, in Example 3.3 the game is not convex and does not possess any solution, whereas in Example 3.4 the game is convex if and only if $\alpha \geq 12$, and \mathcal{B}^N is the only solution for these values of α .
- (b) Convexity is not a necessary condition. The game considered in Remark 4.9(a) below is not convex, but \mathcal{B}^N is a solution.
- (c) In the previous literature, different authors have been interested in giving sufficient conditions for games, weaker than convexity, to ensure that the Shapley value lies in the core. Sprumont (1990) adopted a more general viewpoint, that of population monotonic allocation schemes (cf. Remark 4.7). Using a recursive expression of the Shapley value, he proved that if u is a *quasiconvex* game then $\Phi[u] \in C(u)$. Iñarra and Usategui (1993) introduced average convex games and partially average convex games as wider classes of games u such that $\Phi[u] \in C(u)$. Izawa and Takahashi (1998) found a necessary and sufficient condition for $\Phi[u] \in C(u)$ and called *totally convex* to the games satisfying that condition: among them, there are all average convex games.

Sprumont (1990) gave an example of 3-person game to show that the extended Shapley value of a quasiconvex game may not be a population monotonic allocation scheme. It is the quasiconvex game u defined by

$$u(\{1, 3\}) = u(\{2, 3\}) = 2/3, \quad u(N) = 1, \quad \text{or else } u(S) = 0.$$

The Shapley value is $\Phi[u] = (2/9, 2/9, 5/9)$ and clearly belongs to $C(u)$. What is interesting for us is that players' preferences in this game are

- $\{1, 3\} > \{1, 2, 3\} > \{1, 2\} \equiv \{1\}$ for player 1,
- $\{2, 3\} > \{1, 2, 3\} > \{1, 2\} \equiv \{2\}$ for player 2,
- $\{1, 2, 3\} > \{1, 3\} \equiv \{2, 3\} > \{3\}$ for player 3,

so we conclude that \mathcal{B}^N is not a solution for u and hence *Theorem 4.5 cannot be extended to quasiconvex games*.

Izawa and Takahashi (1998) gave another example of 3-person game such that it is totally convex but not average convex. It is defined by

$$u(\{1, 2\}) = 5, \quad u(\{1, 3\}) = u(\{2, 3\}) = 7, \quad u(N) = 10, \quad \text{or else } u(S) = 0.$$

The Shapley value is $\Phi[u] = (3, 3, 4)$ and clearly belongs to $C(u)$. What is interesting for us is again that players' preferences in this game are

- $\{1, 3\} > \{1, 2, 3\} > \{1, 2\} > \{1\}$ for player 1,
- $\{2, 3\} > \{1, 2, 3\} > \{1, 2\} > \{2\}$ for player 2,
- $\{1, 2, 3\} > \{1, 3\} \equiv \{2, 3\} > \{3\}$ for player 3,

so we conclude that \mathcal{B}^N is not a solution for u and hence *Theorem 4.5 cannot be extended to totally convex games*.

These two (counter)examples share a feature that prevents \mathcal{B}^N from being stable: $u(N)$ is not great enough. This highlights the interest of Theorem 4.14.

Remark 4.7 Sprumont (1990) introduced and studied a refinement of the core allocation notion, the so-called *population monotonic allocation schemes* (PMAS, for short). Given an n -person game u in a player set N , such an scheme is defined by an $n2^{n-1}$ -dimensional vector $\mathbf{x} = (x_S^i)_{i \in S, \emptyset \neq S \subseteq N}$ satisfying

- (1) $\sum_{i \in S} x_S^i = u(S)$ for all nonempty $S \subseteq N$, and
- (2) if $i \in S \subset T$ then $x_S^i \leq x_T^i$.

From condition (1) it readily follows that $\mathbf{x}_N = (x_N^i) \in C(u)$ and $\mathbf{x}_S = (x_S^i) \in C(u_S)$ if $\emptyset \neq S \subset N$. Condition (2) guarantees that, once a coalition S has decided upon an allocation of $u(S)$, no player will be tempted to induce the formation of a coalition smaller than S . This general notion of PMAS seems close to our solution concept because in both approaches it is assumed that if a coalition S forms then its members share the worth $u(S)$, and moreover that this sharing is not affected by the behavior of the external players (those of $N \setminus S$).

In principle, any PMAS could also be used to discuss coalition formation in any game u , by considering that, if $\mathcal{B} = \{B_1, B_2, \dots, B_m\}$ is a coalition structure in N , the payoffs obtained by the players are given as follows: if $i \in B_k$, then this player receives $x_{B_k}^i$. It is worthy mentioning that, although Sprumont accepts that players may not achieve full efficiency, with this interpretation \mathcal{B}^N will always be a *stable coalition structure*, i.e. a solution (maybe not unique), because of condition (2).

However, in this paper we are interested only in solutions derived from the application of the Shapley value to games and subgames (thus, as universal sharing rule), that is, taking $x_S^i = \Phi_i[u_S]$. Then, our solution concept is not necessarily a PMAS, since the translation of condition (2) to our framework implies that, if $i \in S \subset T$, then $\Phi_i[u_S] \leq \Phi_i[u_T]$. Nevertheless, this is not always satisfied, as game u of Example 3.1(b) shows. The coalition structure $\mathcal{B}^{\{1,2\}}$, with payoff vector $(3, 3, 0)$, is a solution but $x_{\{1,2\}}^1 = \Phi_1[u_{\{1,2\}}] > \Phi_1[u] = x_N^1$. Summing up, condition (2) establishes a main difference between PMAS and our solutions.

Anyway, Sprumont’s (1990) results are very interesting. In his Proposition 3, a proof is given for a result communicated by Ichiishi (1987): any convex game admits a PMAS, given by any extended vector of marginal contributions. This proof uses permutations and hence is completely different from our existence proof for Theorem 4.5. Also Corollaries 1 and 2 to Proposition 3 deserve being mentioned here. The first states that every core allocation of a convex game can be reached through a PMAS. The second states that, in particular, the *extended Shapley value* of a convex game, defined by $x_S^i = \Phi_i[u_S]$, is a PMAS (and hence \mathcal{B}^N is a solution for the game, as is stated by our Theorem 4.5).

Definition 4.8 Following Yang (2011), a game u is (strictly) *cohesive* if

$$u(N) > \sum_{k=1}^m u(B_k) \quad \text{for all coalition structures } \mathcal{B} \neq \mathcal{B}^N.$$

In words, this means that \mathcal{B}^N is the only coalition structure that maximizes the sum of utilities of all its unions.

Remarks 4.9 (a) Cohesiveness does not imply superadditivity. Let us take $n = 3$ and consider the game u defined by

$$\begin{aligned}
 u(\emptyset) &= 0, & u(\{1\}) &= 2, & u(\{2\}) &= 1, & u(\{3\}) &= 1, \\
 u(\{1, 2\}) &= 3, & u(\{1, 3\}) &= 3, & u(\{2, 3\}) &= 1, & u(\{1, 2, 3\}) &= 5.
 \end{aligned}$$

The game is monotonic and cohesive, but it is not superadditive because

$$u(\{2\}) + u(\{3\}) \not\leq u(\{2, 3\}).$$

- (b) Cohesiveness neither ensures the existence of solution. Let us consider Example 3.4 with $9 < \alpha < 12$. Then it is easy to check that game u is cohesive, but the discussion carried out in that example shows that it has no solution.
- (c) It may be interesting to recall Bell’s (1938) recursive formula, which gives the number b_n of possible coalition structures in a finite set of cardinality n :

$$b_n = \sum_{k=0}^{n-1} \binom{n-1}{k} b_k, \quad \text{with } b_0 = 1.$$

Thus, for $n = 1, 2, 3, 4, 5, 6, 7, \dots$, there exist $b_n = 1, 2, 5, 15, 52, 203, 877, \dots$ coalition structures, respectively.

This formula tells us the amount of work necessary to check whether a game is cohesive or not, although a previous checking of superadditivity already solves some cases for cohesiveness. E.g., for $n = 4$ a total of 25 checks are needed for superadditivity, but 7 of them check also cohesiveness, so only 7 cases remain to check.

Theorem 4.10 *If u is a cohesive game then no coalition structure $\mathcal{B} \neq \mathcal{B}^N$ gives a solution for u .*

Proof Let u be a cohesive game. Then, using efficiency, for any coalition structure $\mathcal{B} \neq \mathcal{B}^N$ we have

$$\begin{aligned}
 \sum_{i \in B_1} \Phi_i[u_{B_1}] + \sum_{i \in B_2} \Phi_i[u_{B_2}] + \dots + \sum_{i \in B_m} \Phi_i[u_{B_m}] &= \sum_{k=1}^m u(B_k) < u(N) \\
 &= \Phi_1[u] + \Phi_2[u] + \dots + \Phi_n[u].
 \end{aligned}$$

This implies that, necessarily, for some player i in some B_k we must have $\Phi_i[u_{B_k}] < \Phi_i[u]$. Hence, i will not choose B_k as his “best” coalition, B_k will not be an optimal coalition, and \mathcal{B} jointly with its corresponding payoffs will not be a solution of the game. □

Remarks 4.11 (a) Dropping cohesiveness, Theorem 4.10 does not hold. Indeed, in Example 3.2, the game is superadditive but not cohesive. And we find that $\mathcal{B}^N = \{1, 2, 3\}$ is a stable coalition structure but $\mathcal{B}^{(1,2)} = \{\{1, 2\}, \{3\}\}$ is also so.

(b) The converse of Theorem 4.10 is not true. In Example 3.3, no coalition structure $\mathcal{B} \neq \mathcal{B}^N$ gives a solution for u , but the game is not cohesive.

Corollary 4.12 *If u is a convex and cohesive game, then the trivial coalition structure \mathcal{B}^N gives the only solution for u .* □

Lemma 4.13 *Let u be a game and $i \in S \subset N$. Then, setting $s = |S|$ and $t = |T|$,*

$$\Phi_i[u] - \Phi_i[u_S] = \Delta'(u, i, S) + \frac{u(N) - u(N \setminus \{i\})}{n} + \Delta''(u, i, S),$$

where $\Delta'(u, i, S) = \sum_{T \ni i : T \subseteq S} [\gamma_n(t) - \gamma_s(t)][u(T) - u(T \setminus \{i\})]$

and $\Delta''(u, i, S) = \sum_{T \ni i : N \neq T \not\subseteq S} \gamma_n(t)[u(T) - u(T \setminus \{i\})]$.

Proof The proof is straightforward, starting with

$$\Phi_i[u] - \Phi_i[u_S] = \sum_{T \ni i} \gamma_n(t)[u(T) - u(T \setminus \{i\})] - \sum_{T \ni i : T \subseteq S} \gamma_s(t)[u(T) - u(T \setminus \{i\})].$$

□

Theorem 4.14 *If u is any game, and $u(N)$ is great enough, or—equivalently—the marginal contributions $u(N) - u(N \setminus \{i\})$ are great enough for all $i \in N$, then the trivial coalition structure \mathcal{B}^N gives a solution for u , and it is likely to be the only solution.*

Proof Let u be any game and $\mathcal{B} = \{B_1, B_2, \dots, B_m\}$ be a coalition structure different from \mathcal{B}^N . Let $i \in B_k$ for some k . Then, using Lemma 4.13, with $S = B_k$ (and hence $s = b_k = |B_k|$),

$$\Phi_i[u] - \Phi_i[u_{B_k}] = \Delta'(u, i, B_k) + \frac{u(N) - u(N \setminus \{i\})}{n} + \Delta''(u, i, B_k).$$

Clearly, $u(N)$ does not intervene in either $\Delta'(u, i, B_k)$ or $\Delta''(u, i, B_k)$. It follows that, if the total utility $u(N)$ is great enough, in the sense that, for all $i \in N$,

$$u(N) \geq u(N \setminus \{i\}) - n[\Delta'(u, i, B_k) + \Delta''(u, i, B_k)],$$

or, equivalently, if the marginal contributions $u(N) - u(N \setminus \{i\})$ are great enough for all $i \in N$, that is,

$$u(N) - u(N \setminus \{i\}) \geq -n[\Delta'(u, i, B_k) + \Delta''(u, i, B_k)],$$

then we obtain

$$\Phi_i[u] - \Phi_i[u_{B_k}] \geq 0$$

for all $i \in B_k$ and all $B_k \in \mathcal{B}$, that is, for all $i \in N$ and all $\mathcal{B} \neq \mathcal{B}^N$. This implies that all players $i \in N$ will prefer (maybe not uniquely) the grand coalition to any other subcoalition. In other words, the trivial coalition structure \mathcal{B}^N will be a solution of the game, and the only solution if the above inequalities are strict. \square

Remarks 4.15 (a) The expression “great enough” is suggested by the following fact. Let u be any game. If $u(N)$ is increased by an amount $\delta > 0$ and the other utilities remain invariant, a new game u' is obtained, defined by

$$u'(S) = u(S) \text{ if } S \neq N \text{ and } u'(N) = u(N) + \delta.$$

The effect of δ on the Shapley value of any player $i \in N$ is

$$\Phi_i[u'] = \Phi_i[u] + \frac{\delta}{n},$$

whereas, for any $\mathcal{B} \neq \mathcal{B}^N$, any $B_k \in \mathcal{B}$ and any $i \in B_k$, $\Phi_i[u'_{B_k}] = \Phi_i[u_{B_k}]$. In fact, this is the basis for an alternative proof of Theorem 4.14 of a qualitative nature —i.e. without specifying “how much great” should $u(N)$ be.

- (b) Notice that if u is monotonic then $\Delta'(u, i, B_k) \leq 0$ whereas $\Delta''(u, i, B_k) \geq 0$, and also that both expressions vanish if i is a null player in u .
- (c) The condition provided by Theorem 4.14 is not especially useful in practice, but it guarantees that, by increasing the total utility $u(N)$ sufficiently, we will always find that \mathcal{B}^N is a solution, and it will be the only one if $u(N)$ increases still a bit more.

Theorem 4.16 *If $u \neq 0$ is monotonic then:*

- (a) *If $u(S) > \sum_{i \in S} \Phi_i[u]$ for some nonempty $S \subset N$, then \mathcal{B}^N is not stable in u .*
- (b) *If, moreover, u is superadditive, then neither \mathcal{B}^n is stable in u .*
- (c) *Instead, if $u(S) < \sum_{i \in S} \Phi_i[u]$ for all nonempty $S \subset N$, then \mathcal{B}^N is the only stable coalition structure in u .*

Proof By Proposition 4.1, we assume $Z(u) = \emptyset$. It is easy to see that, for any $S \subset N$,

$$\Phi_i[u_S] \geq \Phi_i[u] \text{ for all } i \in S \text{ if and only if } u(S) \geq \sum_{i \in S} \Phi_i[u].$$

Then:

- (a) In case $>$, the members of S will prefer S to N , so \mathcal{B}^N will not be a solution for u .
- (b) Superadditivity implies that $\Phi_i[u] \geq u(\{i\})$ for all $i \in N$. So the members of S will neither accept \mathcal{B}^n .
- (c) In case $<$ it follows that, for any coalition structure $\mathcal{B} \neq \mathcal{B}^N$,

$$\Phi_i[u_{B_k}] < \Phi_i[u]$$

for all $i \in B_k$ and all k and all $\mathcal{B} \neq \mathcal{B}^N$, that is, for all $i \in N$. This implies that \mathcal{B}^N is the only stable coalition structure in u . □

Remarks 4.17 (a) The interest of this result lies in its easy application in practice.

Once we have computed the Shapley value $\Phi[u]$, it is not especially difficult to compare, for each $S \subset N$, $u(S)$ with $\sum_{i \in S} \Phi_i[u]$.

(b) A reviewer suggested that one might get from Theorem 4.16(c) an appealing result: \mathcal{B}^N is stable for game u if and only if the Shapley value $\Phi[u]$ lies in the core $C(u)$. The “only if” part is true. We will prove an equivalent statement: if $\Phi[u] \notin C(u)$ then \mathcal{B}^N is not stable. Indeed, if $\Phi[u] \notin C(u)$ then there exists a nonempty $S \subset N$ such that $\sum_{i \in S} \Phi_i[u] < u(S)$. Since $u(S) = \sum_{i \in S} \Phi_i[u_S]$ we

obtain $\sum_{i \in S} \Phi_i[u] < \sum_{i \in S} \Phi_i[u_S]$, so $\Phi_i[u] < \Phi_i[u_S]$ for some $i \in S$. Thus, all such

$i \in S$ are not satisfied with \mathcal{B}^N and hence this coalition structure, with the payoffs given by the Shapley value $\Phi[u]$, is not stable. Maybe there exists another stable coalition structure, maybe there is no solution: among the examples provided in Sections 3, 5 and 6 that do not admit \mathcal{B}^N as a solution, there are cases where other solution(s) exist(s), and cases where no solution exists.

However, the “if” part does not hold. The counterexamples given by Sprumont (1990) and Izawa and Takahashi (1998) and detailed in Remark 4.7 are games u with $\Phi[u] \in C(u)$ that do not admit \mathcal{B}^N as a solution. Both games are monotonic but not convex.

Our next result refers to any game u such that $\sum_{k=1}^m u(B_k) = u(N)$ for some $\mathcal{B} \neq \mathcal{B}^N$. Such a game may be called *weakly additive game with respect to \mathcal{B}* , since this condition follows from additivity but is not equivalent to it. Of course, if this happens for all $\mathcal{B} \neq \mathcal{B}^N$ then the game is additive and all coalition structures are solutions of the game, with a common payoff given by $u(\{i\})$ for each $i \in N$.

We will use here a standard notation: if \mathcal{B} is a coalition structure in N then, for each $i \in N$, $B_{(i)}$ will denote the union B_k that contains i . Moreover, given a coalition structure \mathcal{B} , we will distinguish between *singletons* and *larger unions* (unions of cardinality > 1).

Theorem 4.18 (Weakly additive games) *Let u be a weakly additive game with respect to a coalition structure $\mathcal{B} \neq \mathcal{B}^N$. Then there are only two possibilities:*

- (a) *If $\Phi_i[u_{B_{(i)}}] = \Phi_i[u]$ for all $i \in N$, then \mathcal{B}^N is a solution for u if and only if \mathcal{B} is also a solution.*
- (b) *Otherwise, there exists some player i such that $\Phi_i[u_{B_{(i)}}] < \Phi_i[u]$ and hence also some player j such that $\Phi_j[u_{B_{(j)}}] > \Phi_j[u]$. Then four cases arise:*
 - (b.1) *If there exists some player of type i in some larger union, then \mathcal{B} is not a solution for u .*
 - (b.2) *If there exists some player of type j in some union, then \mathcal{B}^N is not a solution for u .*

(b.3) If all members of larger unions are of type j , then \mathcal{B} is a solution for u .

(b.4) If all players are of type i , then \mathcal{B}^N is a solution for u .

Proof Weak additivity immediately yields

$$\sum_{i \in N} \Phi_i[u_{B(i)}] = \sum_{i \in N} \Phi_i[u].$$

Thus, a first possibility is that $\Phi_i[u_{B(i)}] = \Phi_i[u]$ for all $i \in N$. In this case, if all players prefer N then all of them equally prefer their respective union, and conversely. This proves part (a).

Otherwise, there will be some $i \in N$ such that $\Phi_i[u_{B(i)}] < \Phi_i[u]$ or some player j such that $\Phi_j[u_{B(j)}] > \Phi_j[u]$, but the existence of one of them implies the existence of the other in order to keep the equality of sums of values. The rest of the proof of part (b) is straightforward, provided that a distinction is made between singletons and larger unions. \square

5 More Examples

The following examples illustrate the above theorems.

Example 5.1 Let us consider again the 3-player game of Example 3.2, given by

$$\begin{aligned} u(\emptyset) &= 0, & u(\{1\}) &= 1, & u(\{2\}) &= 3, & u(\{3\}) &= 0, \\ u(\{1, 2\}) &= 6, & u(\{1, 3\}) &= 1, & u(\{2, 3\}) &= 3, & u(\{1, 2, 3\}) &= 6. \end{aligned}$$

Theorems 4.5 and 4.10 (or, equivalently, Corollary 4.12), 4.14 and 4.16(c) apply and the conclusion is that \mathcal{B}^N is the only solution of the game as we found in Example 3.2.

Example 5.2 Let us consider the 3-player game introduced in Remark 4.9(a) and defined by

$$\begin{aligned} u(\emptyset) &= 0, & u(\{1\}) &= 2, & u(\{2\}) &= 1, & u(\{3\}) &= 1, \\ u(\{1, 2\}) &= 3, & u(\{1, 3\}) &= 3, & u(\{2, 3\}) &= 1, & u(\{1, 2, 3\}) &= 5. \end{aligned}$$

In this case, Theorems 4.10, 4.14 and 4.16(c) apply and, again, we conclude that \mathcal{B}^N is the only solution of the game.

Example 5.3 Let us take $n = 3$ and consider the game u defined by

$$\begin{aligned} u(\emptyset) &= 0, & u(\{1\}) &= 0, & u(\{2\}) &= 0, & u(\{3\}) &= 5, \\ u(\{1, 2\}) &= 20, & u(\{1, 3\}) &= 7, & u(\{2, 3\}) &= 7, & u(\{1, 2, 3\}) &= 25. \end{aligned}$$

Here, parts (a) and (b) of Theorem 4.16 apply and imply that neither \mathcal{B}^N nor \mathcal{B}^n are solutions. However, $\mathcal{B}^{\{1,2\}}$ is a solution. And, indeed, the Shapley value of the non-trivial involved games, u , $u_{\{1,2\}}$, $u_{\{1,3\}}$ and $u_{\{2,3\}}$, is as follows:

$$\begin{aligned} \Phi_1[u] &= 9.67, & \Phi_2[u] &= 9.67, & \Phi_3[u] &= 5.67, \\ \Phi_1[u_{\{1,2\}}] &= 10, & \Phi_2[u_{\{1,2\}}] &= 10, & \Phi_3[u_{\{1,3\}}] &= 6, \\ \Phi_1[u_{\{1,3\}}] &= 1, & \Phi_2[u_{\{2,3\}}] &= 1, & \Phi_3[u_{\{2,3\}}] &= 6. \end{aligned}$$

According to these payoffs, players' preferences are the following:

- $\{1, 2\} > \{1, 2, 3\} > \{1, 3\} > \{1\}$ for player 1,
- $\{1, 2\} > \{1, 2, 3\} > \{2, 3\} > \{2\}$ for player 2,
- $\{1, 3\} \equiv \{2, 3\} > \{1, 2, 3\} > \{3\}$ for player 3.

The only optimal coalition would be $\{1, 2\}$, and the only solution of the game is given by the corresponding coalition structure and the subsequent payoffs:

$$\mathcal{B}^{\{1,2\}} = \{\{1, 2\}, \{3\}\}, \quad \Phi_1[u_{\{1,2\}}] = 10, \quad \Phi_2[u_{\{1,2\}}] = 10, \quad u(\{3\}) = 5.$$

Example 5.4 Let us consider the 3-player game u introduced in Example 3.3 and given by

$$u(\{1, 2\}) = 1, \quad u(\{2, 3\}) = u(\{1, 2, 3\}) = 2 \quad \text{and otherwise} \quad u(S) = 0.$$

This game is weakly additive with respect to $\mathcal{B}^{\{2,3\}}$. Parts (b.1) and (b.2) of Theorem 4.18 apply and say that \mathcal{B}^N and $\mathcal{B}^{\{2,3\}}$ are not solutions. As we saw in Example 3.3, there is no solution for this game.

Examples 5.5 (a) Any additive game illustrates Theorem 4.18(a). All coalition structures are solutions of the game, with a common payoff given by $u(\{i\})$ for each $i \in N$.

(b) As an illustration of Theorem 4.18(b.3), let $n = 3$ and consider the game u defined by

$$\begin{aligned} u(\emptyset) &= 0, & u(\{i\}) &= 0 \quad \text{for all } i, \\ u(\{1, 2\}) &= u(N) = 4, & u(\{1, 3\}) &= u(\{2, 3\}) = 2, \end{aligned}$$

and the coalition structure $\mathcal{B}^{\{1,2\}} = \{\{1, 2\}, \{3\}\}$, for which weak additivity, the hypothesis of Theorem 4.18, is satisfied. With respect to $\mathcal{B}^{\{1,2\}}$, players 1 and 2 are j players. According to Theorem 4.18(b.2) and (b.3), $\mathcal{B}^{\{1,2\}}$ is a solution for u and \mathcal{B}^N is not. And, in effect, the Shapley value of the non-trivial involved games, $u, u_{\{1,2\}}, u_{\{1,3\}}$ and $u_{\{2,3\}}$, is as follows:

$$\begin{aligned} \Phi_1[u] &= 5/3, & \Phi_2[u] &= 5/3, & \Phi_3[u] &= 2/3, \\ \Phi_1[u_{\{1,2\}}] &= 2, & \Phi_2[u_{\{1,2\}}] &= 2, & \Phi_3[u_{\{1,3\}}] &= 1, \\ \Phi_1[u_{\{1,3\}}] &= 1, & \Phi_2[u_{\{2,3\}}] &= 1, & \Phi_3[u_{\{2,3\}}] &= 1, \end{aligned}$$

and players' preferences are

- $\{1, 2\} > \{1, 2, 3\} > \{1, 3\} > \{1\}$ for player 1,
- $\{1, 2\} > \{1, 2, 3\} > \{2, 3\} > \{2\}$ for player 2,
- $\{1, 3\} \equiv \{2, 3\} > \{1, 2, 3\} > \{3\}$ for player 3.

6 Applications to Real-World Economic Problems

In this section we present a miscellaneous of simulated real life situations where our theory can apply. An inspiring source has been the work by Fiestras-Janeiro et al. (2011). Following Remark 6.1, in cost games we denote utilities by negative numbers.

Remark 6.1 (Cost games and savings) In a cooperative game, it is generally assumed that $u(S) > 0$ represents a profit or saving for coalition S . Superadditivity and cohesiveness conditions are therefore full of sense. However, when dealing with a *cost game* c , where $c(S)$ represents the amount that coalition S will have to pay, such cost is often given by a positive number, in which case superadditivity is the *worst* condition in order to promote coalition formation, and recourse has to be made to *subadditivity* conditions.

To avoid this duplicity, we only need to represent costs by negative numbers. If we do so, superadditivity and cohesiveness are convenient conditions also for cost games. Then, using this (natural) representation convention, given a cost game c the *associated saving game* u is defined by

$$u(S) = c(S) - \sum_{i \in S} c(\{i\}) \quad \text{for each } S \subseteq N,$$

and it is easy to check that players' preferences, and hence the optimal coalitions for each player and the stable coalition structures (if any), are the same in u as in c .

Example 6.2 Let $n = 3$ and consider a cost game c defined by

$$\begin{aligned} c(\emptyset) &= 0, & c(\{1\}) &= -4, & c(\{2\}) &= -6, & c(\{3\}) &= -5, \\ c(\{1, 2\}) &= -6, & c(\{1, 3\}) &= -6, & c(\{2, 3\}) &= -8, & c(\{1, 2, 3\}) &= -8. \end{aligned}$$

This cost game is cohesive and convex, and $c(N)$ is great enough. The application of the Shapley value yields

$$\begin{aligned} \Phi_1[c] &= -1.5, & \Phi_2[c] &= -3.5, & \Phi_3[c] &= -3, \\ \Phi_1[c_{\{1,2\}}] &= -2, & \Phi_2[c_{\{1,2\}}] &= -4, & \Phi_3[c_{\{1,3\}}] &= -3.5, \\ \Phi_1[c_{\{1,3\}}] &= -2.5, & \Phi_2[c_{\{2,3\}}] &= -4.5, & \Phi_3[c_{\{2,3\}}] &= -3.5. \end{aligned}$$

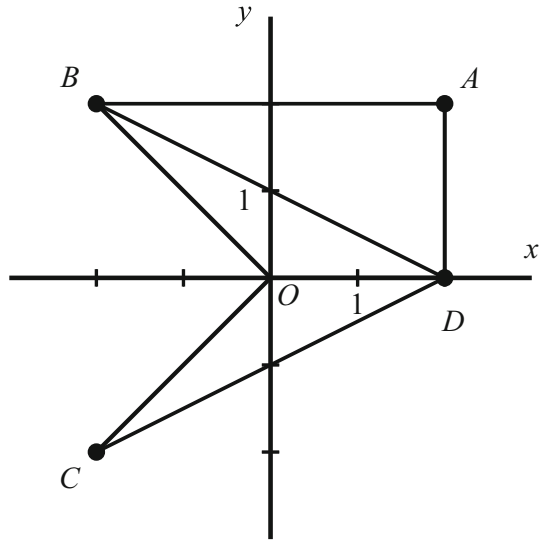
Players' preferences are as follows:

- $\{1, 2, 3\} > \{1, 2\} > \{1, 3\} > \{1\}$ for player 1,
- $\{1, 2, 3\} > \{1, 2\} > \{2, 3\} > \{2\}$ for player 2,
- $\{1, 2, 3\} > \{1, 3\} \equiv \{2, 3\} > \{3\}$ for player 3.

The trivial coalition structure is the only solution of the game.

Example 6.3 (A collective supply) Three towns located at points $A(2, 2)$, $B(-2, 2)$ and $C(-2, -2)$ (distances given in km) are interested in being supplied with a fluid (say, gas) from a production or distribution center located at $D(2, 0)$ (see Fig. 1). We are interested only in the connection costs. The cost in monetary units of establishing

Fig. 1 A collective supply



any channel is of 1000/km, so the cost of a link to each town defined by a straight line (DA , DB and DC , respectively) is given by

$$c(\{A\}) = -2000, \quad c(\{B\}) = -4480, \quad c(\{C\}) = -4480.$$

Taking profit of the good orographic conditions, the supplier offers common links to any two towns simultaneously ($DA + AB$, $DA + DC$ and $DO + OB + OC$, respectively) and even a full link to the three towns ($DA + DO + OB + OC$) with costs

$$\begin{aligned} c(\{A, B\}) &= -6000, & c(\{A, C\}) &= -6480, & c(\{B, C\}) &= -7660, \\ c(\{A, B, C\}) &= -9660. \end{aligned}$$

Let us discuss coalition formation. Notice that the game is not convex, is not cohesive and neither $c(N)$ is great enough because $c(\{A\}) + c(\{B, C\}) = c(N)$. We will assume that all towns accept the Shapley value as the sharing rule in all possible cases. The cost game c played by the towns has been described above. If each town signs an individual contract, then the corresponding costs will be

$$c(\{A\}) = -2000, \quad c(\{B\}) = -4480 \quad \text{and} \quad c(\{C\}) = -4480.$$

If all agree to sign a full joint contract, the sharing is

$$\Phi_A[c] = -1920, \quad \Phi_B[c] = -3750 \quad \text{and} \quad \Phi_C[c] = -3990.$$

Finally, if $\{X, Y\}$ is any two-player coalition that signs a joint contract, and $c_{\{X, Y\}}$ denotes the restriction of game c to this coalition, then we have

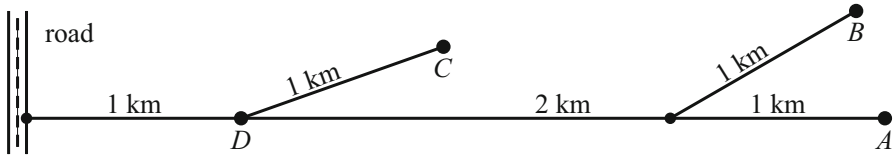


Fig. 2 The four cottages

$$\begin{aligned} \Phi_A[c_{\{A,B\}}] &= -1760, & \Phi_B[c_{\{A,B\}}] &= -4240, & \Phi_C[c_{\{A,C\}}] &= -4480, \\ \Phi_A[c_{\{A,C\}}] &= -2000, & \Phi_B[c_{\{B,C\}}] &= -3830, & \Phi_C[c_{\{B,C\}}] &= -3830. \end{aligned}$$

So the preferences of each town on coalitions containing it are

- $\{A, B\} > \{A, B, C\} > \{A\} \equiv \{A, C\}$ for town A,
- $\{A, B, C\} > \{B, C\} > \{A, B\} > \{B\}$ for town B,
- $\{B, C\} > \{A, B, C\} > \{C\} \equiv \{A, C\}$ for town C.

We conclude that, strictly speaking, there is no solution for the problem. However, intuitively, it seems that, in a further negotiation, town B might convince (or force) the others to form the grand coalition, as its position in the bargaining seems to be the strongest.

Example 6.4 (The four cottages) A dirt track connects four cottages with a main road, as is shown in Fig. 2. Asphaltting the track costs 60,000 euros/km.

If S is any subset of cottages, $c(S)$ will represent the cost (in thousands of euros) of asphaltting the fraction of the track that connects the cottages of S with the road. This gives a cost game c in the player set $N = \{1, 2, 3, 4\}$ of the respective owners of A, B, C and D, defined by

$$\begin{aligned} c(\emptyset) &= 0, & c(\{1\}) &= -240, & c(\{2\}) &= -240, & c(\{3\}) &= -120, \\ c(\{4\}) &= -60, & c(\{1, 2\}) &= -300, & c(\{1, 3\}) &= -300, & c(\{1, 4\}) &= -240, \\ c(\{2, 3\}) &= -300, & c(\{2, 4\}) &= -240, & c(\{3, 4\}) &= -120, & c(\{1, 2, 3\}) &= -360, \\ c(\{1, 2, 4\}) &= -300, & c(\{1, 3, 4\}) &= -300, & c(\{2, 3, 4\}) &= -300, & c(\{1, 2, 3, 4\}) &= -360. \end{aligned}$$

The game is cohesive and $c(N)$ is great enough. We omit the details about the Shapley value on subgames. By applying the standard technique proposed in previous sections, we find that the preferences of cottage owners are as follows:

- $\{1, 2, 3, 4\} > \{1, 2, 3\} \equiv \{1, 2, 4\} > \{1, 2\} > \{1, 3, 4\} > \{1, 3\} \equiv \{1, 4\} > \{1\}$ for owner 1,
- $\{1, 2, 3, 4\} > \{1, 2, 3\} \equiv \{1, 2, 4\} > \{1, 2\} > \{2, 3, 4\} > \{2, 3\} \equiv \{2, 4\} > \{2\}$ for owner 2,
- $\{1, 2, 3, 4\} > \{1, 2, 3\} \equiv \{1, 3, 4\} \equiv \{2, 3, 4\} > \{1, 3\} \equiv \{2, 3\} \equiv \{3, 4\} > \{3\}$ for owner 3,
- $\{1, 2, 3, 4\} > \{1, 2, 4\} \equiv \{1, 3, 4\} \equiv \{2, 3, 4\} > \{1, 4\} \equiv \{2, 4\} \equiv \{3, 4\} > \{4\}$ for owner 4.

Thus, in this problem all players strictly prefer forming the grand coalition and share the total cost (in thousand euros) in this way:

$$\Phi_1[c] = 135, \quad \Phi_2[c] = 135, \quad \Phi_3[c] = 75 \quad \text{and} \quad \Phi_4[c] = 15.$$

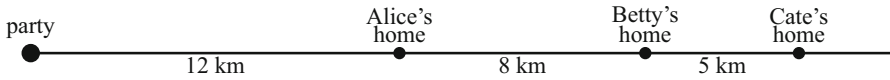


Fig. 3 A taxi trip

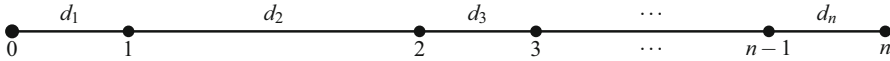


Fig. 4 Logistics itinerary

This fits well the common sense standard rule applied by most councils in problems of this kind.

Examples 6.5 (A taxi trip and a logistics problem) (a) A relatively similar problem is the following. Three friends, Alice, Betty and Cate, respectively denoted as A , B and C , leave a party, call a taxi, and ask the driver to carry them home together. The trip is represented in Fig. 3 and the fee is one euro/km.

The cost game c played by the friends is given by

$$\begin{aligned}
 c(\emptyset) &= 0, & c(\{A\}) &= -12, & c(\{B\}) &= -20, & c(\{C\}) &= -25, \\
 c(\{A, B\}) &= -20, & c(\{A, C\}) &= -25, & c(\{B, C\}) &= -25, & c(\{A, B, C\}) &= -25.
 \end{aligned}$$

This game is convex and cohesive, so the only solution is \mathcal{B}^N with the payoffs given by the Shapley value of the game $\Phi[c] = (-4, -8, -13)$. Again, this solution fits well common sense, according to which friends travelling in the taxi in a given interval should share equitably the cost of that interval.

(b) We generalize this example to a transportation problem. To fix ideas, let $N = \{1, 2, \dots, n\}$ be a set of consumers that want to be supplied with diverse goods from a logistic centre located at the origin 0. The single itinerary followed by the carrier (using, e.g., truck, train, ship or plane) can be described as in Fig. 4.

The transportation costs in each stretch of the route are given by negative numbers d_1, d_2, \dots, d_n , e.g. proportional to the respective lengths. Of course, if only a subset S of consumers decided to make use of this service, the trip would finish at point $\max\{S\}$ (the consumer of S furthest from 0), with the subsequent cost reduction. If costs are assumed to be allocated in all cases by the Shapley value, the problem is whether the consumers prefer to form the grand coalition or splitting into cheaper unions.

The cost game c that describes this problem is defined by

$$c(S) = \sum_{k=1}^{\max\{S\}} d_k \quad \text{for each } S \subseteq N.$$

It is not difficult to check that this is a convex and cohesive game, so the only solution is the coalition structure \mathcal{B}^N with associated payoffs given by the Shapley value of the game:

$$\Phi_i[c] = \sum_{k=1}^i \frac{d_k}{n + 1 - k} \quad \text{for each } i \in N.$$

Once more, this fits common sense, which would recommend sharing equitably the cost of each stretch among the consumers whose service requires the carrier to go through this stretch.

Example 6.6 (The bankruptcy problem) The Talmud is an ancient Jewish document where comments on Moses’ law and teachings of the rabbinical school are collected. Towards 1140 AD, Rabbi Ibn Ezra proposed in the Talmud the following problem. Jacob is dead, and each of his sons, Reuben, Simeon, Levi and Judah, presents a brief document in which Jacob recognizes him as heir and bequeaths, respectively, 1/4, 1/3, 1/2, and all of its assets, valued at 120 m.u. (monetary units of that time). All documents bear the same date and, therefore, none has priority over others. The problem is how to divide Jacob’s inheritance.

Let $N = \{1, 2, 3, 4\}$ be the set of Jacob’s sons in the order mentioned above, $E = 120$ be the estate and $d_1 = 30, d_2 = 40, d_3 = 60, d_4 = 120$ be the creditors’ demands. In this example, the estate is the total of Jacob’s assets and the creditors are Jacob’s sons. O’Neill (1982) defined a cooperative game to deal with this kind of problems, now known as “bankruptcy problems”:

$$u(S) = \max \left\{ 0, E - \sum_{j \notin S} d_j \right\} \quad \text{for each } S \subseteq N.$$

This game is given by

$$\begin{aligned} u(\emptyset) &= 0, & u(\{1\}) &= 0, & u(\{2\}) &= 0, & u(\{3\}) &= 0, \\ u(\{4\}) &= 0, & u(\{1, 2\}) &= 0, & u(\{1, 3\}) &= 0, & u(\{1, 4\}) &= 20, \\ u(\{2, 3\}) &= 0, & u(\{2, 4\}) &= 30, & u(\{3, 4\}) &= 50, & u(\{1, 2, 3\}) &= 0, \\ u(\{1, 2, 4\}) &= 60, & u(\{1, 3, 4\}) &= 80, & u(\{2, 3, 4\}) &= 90, & u(\{1, 2, 3, 4\}) &= 120. \end{aligned}$$

The game is convex and cohesive and $u(N)$ is great enough. We omit the details about the Shapley value on subgames. The preferences of Jacob’s sons are as follows:

- $\{1, 2, 3, 4\} > \{1, 2, 4\} \equiv \{1, 3, 4\} > \{1, 4\} > \{1, 2, 3\} \equiv \{1, 2\} \equiv \{1, 3\} \equiv \{1\}$ for Reuben,
- $\{1, 2, 3, 4\} > \{1, 2, 4\} \equiv \{2, 3, 4\} > \{2, 4\} > \{1, 2, 3\} \equiv \{1, 2\} \equiv \{2, 3\} \equiv \{2\}$ for Simeon,
- $\{1, 2, 3, 4\} > \{1, 3, 4\} \equiv \{2, 3, 4\} > \{3, 4\} > \{1, 2, 3\} \equiv \{1, 3\} \equiv \{2, 3\} \equiv \{3\}$ for Levi,
- $\{1, 2, 3, 4\} > \{2, 3, 4\} > \{1, 2, 4\} \equiv \{1, 3, 4\} > \{3, 4\} > \{2, 4\} > \{1, 4\} > \{1\}$ for Judah.

Again, we find that, in accordance with Corollary 4.12, the only solution of the problem is given by the grand coalition, with the payoffs allocated by the Shapley value:

$$\Phi_1[u] = 14.17, \quad \Phi_2[u] = 19.17, \quad \Phi_3[u] = 29.17 \quad \text{and} \quad \Phi_4[u] = 57.50.$$

This solution coincides with the sharing proposed in the Talmud.

More generally, it is not difficult to see that any n -person bankruptcy game is convex and cohesive, provided that $d_1, d_2, \dots, d_n > 0$ and $0 < E < \sum_{i \in N} d_i$. Thus, Corollary 4.12 ensures that forming the grand coalition is the only solution for any such game.

Example 6.7 (An oligopoly market) Three friends, Alan, Burt and Cynthia, have one company each. They dominate the market of a certain product and obtain annual benefits of 100, 200 and 300 monetary units (say, thousands of euros), respectively. A market propection predicts that, by merging companies, the increase of the joint benefit would be 10% for Alan and Burt, 20% for Alan and Cynthia, and 30% for all together. Finally, Burt and Cynthia might try to achieve a risk operation that would give them an increase of 50% on their joint benefit, but only with a success probability p (of course, $0 \leq p \leq 1$). Otherwise, with probability $1 - p$, they would fail and get an increase of 5% only.

We first determine the cooperative game u played by the friends. Taking Alan = 1, Burt = 2 and Cynthia = 3, we find as utility of coalition $\{2, 3\}$ the *expected* joint benefit in terms of p , i.e. $u(\{2, 3\}) = 750p + 525(1 - p)$, and therefore the game is given (in thousands of euros) by

$$u(\emptyset) = 0, \quad u(\{1\}) = 100, \quad u(\{2\}) = 200, \quad u(\{3\}) = 300, \\ u(\{1, 2\}) = 330, \quad u(\{1, 3\}) = 480, \quad u(\{2, 3\}) = 525 + 225p, \quad u(\{1, 2, 3\}) = 780.$$

A complete analysis of coalition formation is as follows. By applying the Shapley value to the involved games $u, u_{\{1,2\}}, u_{\{1,3\}}$ and $u_{\{2,3\}}$, we have all the necessary information. First,

$$\Phi_1[u] = 170 - 75p, \quad \Phi_2[u] = 242.50 + 37.50p \quad \text{and} \quad \Phi_3[u] = 367.50 + 37.50p,$$

and, moreover,

$$\Phi_1[u_{\{1,2\}}] = 115, \quad \Phi_2[u_{\{1,2\}}] = 215, \quad \Phi_3[u_{\{1,3\}}] = 340, \\ \Phi_1[u_{\{1,3\}}] = 140, \quad \Phi_2[u_{\{2,3\}}] = 212.50 + 112.50p, \quad \Phi_3[u_{\{2,3\}}] = 312.50 + 112.50p.$$

Then, the preferred options for each player in terms of p are given by the following table:

If	Alan prefers	Burt prefers	Cynthia prefers
$0 \leq p < 0.40$	{1, 2, 3}	{1, 2, 3}	{1, 2, 3}
$p = 0.40$	{1, 2, 3} \equiv {1, 3}	{1, 2, 3} \equiv {2, 3}	{1, 2, 3}
$0.40 < p < 0.7333$	{1, 3}	{2, 3}	{1, 2, 3}
$p = 0.7333$	{1, 3}	{2, 3}	{1, 2, 3} \equiv {2, 3}
$0.7333 < p \leq 1$	{1, 3}	{2, 3}	{2, 3}

Players' behavior is clear when $0 \leq p \leq 0.40$ (forming the grand coalition) and when $0.7333 \leq p \leq 1$ (forming {1} and {2, 3}), but not when $0.40 < p < 0.7333$. In

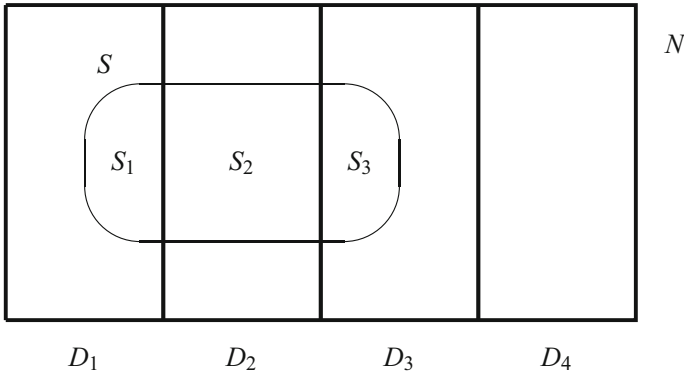


Fig. 5 Decomposition of a coalition

this case, it might be conceivable that Cynthia (player 3) convince or force the others to form the grand coalition. Or either that she accepts to form $\{2, 3\}$, with a loss, with respect to $\{1, 2, 3\}$, going from 0 (for $p = 0.7333$) to 25 (for $p = 0.40$), that is, with a maximum loss of 6.5%.

Decomposable games were introduced by Shapley (1971). Assume that several companies have their headquarters in different countries or work in different industrial sectors. In both cases, there exists a coalition structure that reflects the different locations. All companies may, in principle, joint to others. However, let us assume that, due to strict regulations of the involved countries in the first case, or to the lack of relation between goods produced in different sectors in the second, synergies (if any) may appear only between companies lying in the same union (country or sector, resp.). Then, the game that describes the utility of any coalition of companies is a decomposable game.

Example 6.8 (Decomposable games) Let N be a player set ($n \geq 2$) and $\mathcal{D} = \{D_1, D_2, \dots, D_m\}$ be a coalition structure in N , with $m \geq 2$. The unions of \mathcal{D} will be called here *districts*. If for any $S \subseteq N$ we set $S_k = S \cap D_k$ for $k = 1, 2, \dots, m$, it follows that S can be uniquely written as $S = S_1 \cup S_2 \cup \dots \cup S_m$ (see Fig. 5 for $m = 4$). Of course, $N_k = D_k$ for all k .

Let u be a game in N . According to Shapley (1971), game u is said to be *decomposable* (with respect to \mathcal{D}) if

$$u(S) = u(S_1) + u(S_2) + \dots + u(S_m) \quad \text{for every } S \subseteq N.$$

(If $\mathcal{D} = \mathcal{B}^n$ and hence $m = n$, the condition merely means that u is an additive game.)

For each k , let u^k denote the restriction of u to D_k , and \bar{u}^k denote the *null extension* of u^k to N , defined by $\bar{u}^k(S) = u^k(S_k)$ for all $S \subseteq N$. All $i \notin D_k$ are null players in \bar{u}^k . Games u^k , and even games \bar{u}^k , are called the *components* of u with respect to \mathcal{D} . Clearly,

$$u = \bar{u}^1 + \bar{u}^2 + \dots + \bar{u}^m,$$

and hence, if $i \in D_k$,

$$\Phi_i[u] = \Phi_i[\bar{u}^k] = \Phi_i[u^k].$$

Game u is what we called in Section 4 a weakly additive game (here with regard to \mathcal{D}). Then it follows from Theorem 4.18(a) that \mathcal{B}^N is a solution for u if and only if so is \mathcal{D} . For example, it is easy to check that if all components are convex games then so is u , and the converse is true because the condition is hereditary (this equivalence as to convexity was already stated in Shapley (1971)); in this case these two coalition structures are solution. However, the components of u are in general arbitrary games, and other solutions may exist: if e.g. all u^k are additive, then so is u , whence $\Phi_i[u] = u(\{i\})$ for all $i \in N$, and any coalition structure \mathcal{B} (included \mathcal{B}^N) is a solution of u with these trivial payoffs.

A special property of any decomposable game u is that, if $S = S_1 \cup S_2 \cup \dots \cup S_m$ is any nonempty coalition, then the restriction of u to S is

$$u_S = \bar{u}_{S_1}^1 + \bar{u}_{S_2}^2 + \dots + \bar{u}_{S_m}^m = u_{S_1}^1 + u_{S_2}^2 + \dots + u_{S_m}^m.$$

As a consequence, if $i \in S \cap D_k = S_k$ then

$$\Phi_i[u_S] = \Phi_i[\bar{u}_{S_k}^k] = \Phi_i[u_{S_k}^k].$$

This is a key point for the analysis of coalition formation. It means that the payoff received by i in u_S does not depend on the members of S belonging to other districts, but solely on his partners in S_k . Its implications are crucial in the study of the relation between solutions of u and solutions of all u^k . We complete our analysis with the four following statements.

1. If $\mathcal{B}_1, \mathcal{B}_2, \dots, \mathcal{B}_m$ are solutions for the component games u^1, u^2, \dots, u^m , respectively, with their corresponding payoffs, then $\mathcal{B} = \mathcal{B}_1 \cup \mathcal{B}_2 \cup \dots \cup \mathcal{B}_m$ is a solution for u with the same payoffs.

For each $k = 1, 2, \dots, m$, let $\mathcal{B}_k = \{B_{k1}, B_{k2}, \dots, B_{kp_k}\}$ be a stable coalition structure in district D_k for game u^k . Then $\mathcal{B} = \{B_{11}, \dots, B_{21}, \dots, B_{mp_m}\}$ is a coalition structure in N (see Fig. 6(a) for $m = 4$).

We shall prove that \mathcal{B} is stable for u in N . Let us consider e.g. player $1 \in D_1$ and assume, without loss of generality, that $1 \in B_{11}$, the first union of \mathcal{B}_1 . There are two possibilities. If $B_{11} = \{1\}$ is a (forced or not) singleton, the payoff for player 1 is $u^1(\{1\})$. Otherwise, $|B_{11}| \geq 2$ and B_{11} is an optimal coalition for player 1 in u^1 , so $\Phi_1[u_{B_{11}}^1]$ is the best payoff for player 1 in u^1 .

In the first case (singleton), there is no optimal coalition $S \subseteq D_1$, with $1 \in S$ and $|S| \geq 2$, better than B_{11} for 1 in u^1 . Since $u = u^1$ in D_1 , neither there exists an analogous coalition $S \subseteq D_1$ for u . Thus, $B_{11} = \{1\}$ is a (forced or not) singleton also for u . In the second case (optimal coalition), and again because $u = u^1$ in D_1 , B_{11} is an optimal coalition for player 1 also in u . And, in both cases, joining any coalition $S \not\subseteq D_1$ cannot increase the payoff of player 1 in u because his payoff is independent of the members of $S \setminus D_1$.

As this argument holds for all players of all districts, the proof is complete.

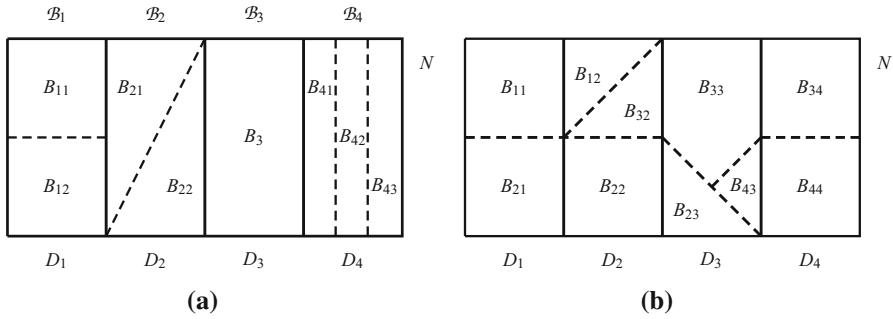


Fig. 6 **a** Construction of \mathcal{B} ; **b** construction of $\mathcal{B}_1, \mathcal{B}_2, \dots, \mathcal{B}_m$

2. If some u^k lacks solution in D_k then u lacks solution in N .
The proof is based on an analogous reasoning.
3. If u has a solution in N then each u^k has a solution in D_k .

This statement is logically equivalent to point 2, but we wish to detail here the way by which a stable coalition structure \mathcal{B} for u in N gives rise to stable coalition structures \mathcal{B}_k for u^k in D_k for $k = 1, 2, \dots, m$. The procedure is illustrated by Fig. 6(b) for $m = 4$. The unions of \mathcal{B} need not be compatible with the districts (unions of \mathcal{D}), but the intersections of all of them with each district define a coalition structure in this district. Thus, for example, in Fig. 6(b) B_1 gives B_{11} and B_{12} , B_2 gives B_{21} , B_{22} and B_{23} , and similar intersections are given by B_3 and B_4 . Collecting these intersections, we obtain $\mathcal{B}_1 = \{B_{11}, B_{21}\}$ in D_1 , $\mathcal{B}_2 = \{B_{12}, B_{22}, B_{32}\}$ in D_2 , $\mathcal{B}_3 = \{B_{23}, B_{33}, B_{43}\}$ in D_3 and $\mathcal{B}_4 = \{B_{34}, B_{44}\}$ in D_4 . These coalition structures are stable by the argument exposed in point 1 and they keep the payoffs allocated by \mathcal{B} .

4. When compounding solutions of the components u^k to obtain a solution for u , optimal coalitions and forced singletons of any district can join optimal coalitions and forced singletons of any other district without modifying the payoffs to the involved players.

For example, let $N = \{1, 2, 3, 4, 5\}$, $D_1 = \{1, 2\}$ and $D_2 = \{3, 4, 5\}$. If $\mathcal{B}_1 = \{\{1\}, \{2\}\}$ is the only solution for u^1 and $\mathcal{B}'_2 = \{\{3, 4\}, \{5\}\}$ and $\mathcal{B}''_2 = \{\{3, 4, 5\}\}$ are the solutions for u^2 , then eight solutions appear for game u , namely:

$$\{\{1\}, \{2\}, \{3, 4\}, \{5\}\}, \{\{1\}, \{2\}, \{3, 4, 5\}\}, \{\{1, 3, 4\}, \{2\}, \{5\}\}, \{\{1, 3, 4, 5\}, \{2\}\}, \{\{1\}, \{2, 3, 4\}, \{5\}\}, \{\{1\}, \{2, 3, 4, 5\}\}, \{\{1, 5\}, \{2\}, \{3, 4\}\}, \{\{1\}, \{2, 5\}, \{3, 4\}\}.$$

As a preface for our final example, we recall an important class of cooperative games. A monotonic game u in N is *simple* if $u(S) = 0$ or 1 for each $S \subseteq N$.⁹ The family of *winning coalitions*, which determines the game, is

$$W(u) = \{S \subseteq N : u(S) = 1\}.$$

⁹ As we do not impose $u(N) = 1$, we accept as simple the null game $u = 0$, because the restriction of a simple game may well be a null subgame.

The family of *minimal winning coalitions*, which also determines the game and is, in general, quite smaller, is

$$W^m(u) = \{S \in W(u) : R \subset S \Rightarrow R \notin W(u)\}.$$

A simple game u is a *weighted majority game* if there exist weights $w_1, w_2, \dots, w_n \geq 0$ attached to the players and a *quota* $q > 0$ such that

$$S \in W(u) \text{ if and only if } \sum_{i \in S} w_i \geq q.$$

In this case we say that $[q; w_1, w_2, \dots, w_n]$ is a *representation* of u . Such a representation, if it exists, is never unique, so we write $u \equiv [q; w_1, w_2, \dots, w_n]$.

If $T \subset N$ then the restricted game of a simple game u is $u_T = 0$ if $T \notin W(u)$, the unanimity game in T if $T \in W^m(u)$, or a more complicated game if $T \in W(u) \setminus W^m(u)$. If $u \equiv [q; w_1, w_2, \dots, w_n]$ and, say, $T = \{1, 2, \dots, t\}$, then the restricted game is $u_T \equiv [q; w_1, w_2, \dots, w_t]$.

Simple games, and in particular weighted majority games, are often used in Political Science to describe binary decision-making mechanisms when proposals are submitted to the approval of a set of agents. They are also useful to represent access structures in Cryptography (cf. Carreras et al. 2004) or semi-coherent structures in Reliability (cf. Carreras et al. 2006). In all these cases, the Shapley value becomes an interesting individual measure of “power” (in a generic sense).

However, these alternative uses of simple games do not prevent to study them as cooperative games. We present below an interesting case where there exists a main player (without veto power) and the others are symmetric. Utilities are taken, quite conventionally, equal to 0 or 1.

Example 6.9 (Apex games) An *apex game* u in N is a weighted majority game of the form

$$u \equiv [n - 1; n - 2, 1, 1, \dots, 1] \text{ with } n \geq 4.$$

These games were introduced by Von Neumann and Morgenstern (1953). The quota q is the least integer greater than half the total of weights, $\sum_{i \in N} w_i = 2n - 3$, so the family of minimal winning coalitions is

$$W^m(u) = \{\{1, 2\}, \{1, 3\}, \dots, \{1, n\}, \{2, 3, \dots, n\}\}.$$

All apex games are clearly superadditive. However, none of them is convex and neither it is cohesive. There exist an *apex player* (player 1) and $n - 1$ peripheral players (players 2 to n). Correspondingly, there are $n - 1$ radial minimal winning coalitions and a peripheral one which prevents player 1 to enjoy veto power (see Fig. 7). The application of the Shapley value yields

$$\Phi_1[u] = 1 - \frac{2}{n} \text{ and } \Phi_i[u] = \frac{2}{n(n - 1)} \text{ for each } i \neq 1.$$

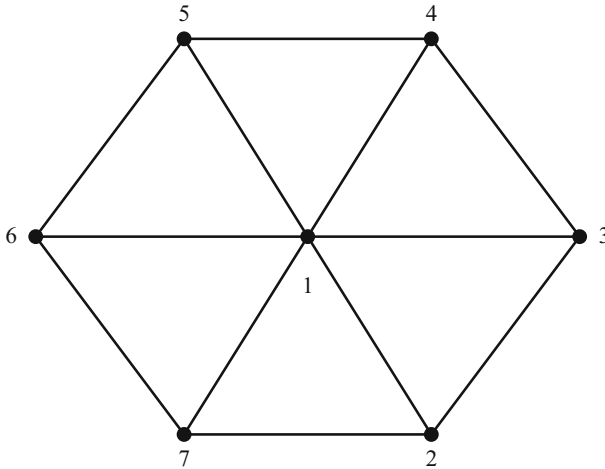


Fig. 7 Apex configuration for $n = 7$

Thus, if $S = N \setminus \{1\}$ then $u(S) = 1$ and $\sum_{i \in S} \Phi_i[u] = 2/n$, so, as $n \geq 4$, we find $u(S) > \sum_{i \in S} \Phi_i[u]$. Hence, Theorem 4.16(a) applies and we conclude that \mathcal{B}^N is never a solution for an apex game.

As to subgames, there are different possibilities. Let $T \subset N$. If $T \subset N \setminus \{1\}$ then $u_T = 0$ and $\Phi_i[u_T] = 0$ for all $i \in T$. If $T = N \setminus \{1\}$ then $t = |T| = n - 1$, u_T is the unanimity game in T , and $\Phi_i[u_T] = 1/(n - 1)$ for all $i \in T$. Otherwise, that is, if $1 \in T$, we find two cases. If $t = |T| = 1$ then $u_T = 0$ and $\Phi_1[u_T] = 0$. If $2 \leq t \leq n - 1$ then $u \equiv [n - 1; n - 2, 1, 1, \overset{t-1}{\dots}, 1]$, and hence $W^m(u_T) = \{\{1, 2\}, \{1, 3\}, \dots, \{1, t\}\}$; thus,

$$\Phi_1[u_T] = 1 - \frac{1}{t} \quad \text{and} \quad \Phi_i[u_T] = \frac{1}{t(t - 1)} \quad \text{for each } i \in T \setminus \{1\}.$$

Now the analysis of coalition formation can be completed. Omitting the details and having in mind that all minor players are symmetric and hence interchangeable, player 1's preferences are as follows:

- If $n = 4$ then $\{1, 2, 3\} > \{1, 2\} \equiv N > \{1\}$.
- If $n \geq 5$ then $\{1, 2, \dots, n - 1\} > \dots > N > \dots > \{1, 2\} > \{1\}$.

Instead, for any other player (player 2, say, to ease the description) the preferences are:

- If $n = 4$ then $\{1, 2\} > N \setminus \{1\} > \{1, 2, 3\} \equiv N > \{2\}$.
- If $n = 5$ then $\{1, 2\} > N \setminus \{1\} > \{1, 2, 3\} > N > \{1, 2, 3, 4\} > \{2\}$.
- If $n = 6$ then $\{1, 2\} > N \setminus \{1\} > \{1, 2, 3\} > \dots > N > \dots > \{2\}$.
- If $n = 7$ then $\{1, 2\} > N \setminus \{1\} \equiv \{1, 2, 3\} > \dots > N > \dots > \{2\}$.
- If $n \geq 8$ then $\{1, 2\} > \{1, 2, 3\} > N \setminus \{1\} > \dots > N > \dots > \{2\}$.

Then, player 1 strictly prefers a payoff of $1 - \frac{1}{n-1}$ forming $\{1, 2, \dots, n-1\}$ or, in words, joining any $n-2$ minor players. Instead, each minor player i strictly prefers a payoff of $1/2$ joining player 1 and forming $\{1, i\}$. The final conclusion is clear: *Any apex game lacks solution*. This is a general statement. It differs from the results found by Hart and Kurz (1984) because these authors studied these games from Owen's approach (1977), whereas ours is Aumann–Drèze's approach (1974).

7 Conclusions and Future Work

In this work we have discussed coalition formation in any cooperative game, assuming that the Shapley value is the universal sharing rule in the sense that it applies to the game and also to all its subgames. A solution notion has been proposed, consisting of a stable coalition structure, which determines a strategy for each one of the players, and the corresponding payoffs they will receive. Essentially, what we have in mind is the analogue of the stability of the AD value. We have been especially interested in the grand coalition as a solution.

Five main results have been provided. The first (Theorem 4.5) establishes convexity as a sufficient condition for the grand coalition to be a solution (existence). The second (Theorem 4.10) establishes cohesiveness as a sufficient condition for the grand coalition to be the only solution if any (uniqueness). The third (Theorem 4.14) establishes a rather qualitative sufficient condition for the grand coalition to be a solution (existence) or even the only solution (existence and uniqueness). The fourth (Theorem 4.16) essentially establishes, under monotonicity, a practical sufficient condition for the grand coalition to be the only solution (existence and uniqueness). The fifth (Theorem 4.18) discusses weakly additive games. Different examples and counterexamples have been included, and examples related to economic problems have been also studied and generalized when possible.

As to future work in this research line, we suggest, among others, the following points:

- The search of *necessary* conditions for existence and/or uniqueness of the grand coalition as a solution.
- The possibility of disregarding the use of the Shapley value for subgames and to allow every coalition to use its own sharing rule (cf. Example 4.2 in Carreras and Magaña 1997).
- The enlargement of the scope of the analysis by considering special relationships among players, due to ideological or strategic affinities and/or incompatibilities, which can influence coalition formation strongly. The introduction in the evaluation of games of additional information not stored in the characteristic function goes back to Myerson (1977, 1980) (affinities) and others (Carreras 1991; Bergantiños et al. 1993) (incompatibilities) and was widely generalised by using *cooperation indices* (Amer and Carreras 1995a, b, 1997, 2001, 2013).
- The enlargement of the scope of the analysis by considering games with *level* coalition structures, a natural extension of the domain of coalitional values already

mentioned by Owen (1977) and developed by other authors (Winter 1989; Alonso-Mejide and Carreras 2011; Alonso-Mejide et al. 2015).

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