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Fuzzy Multichoice Games with Fuzzy Characteristic Functions

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Abstract In this paper, a generalized form of fuzzy multichoice games with fuzzy characteristic functions is proposed, which can be seen as an extension of traditional fuzzy games. Based on the extension Hukuhara difference, fuzzy multichoice games with fuzzy characteristic functions are studied, and a Shapley function is discussed. The notion of fuzzy multichoice population monotonic allocation scheme (FMPMAS) is defined. When the given fuzzy multichoice game with fuzzy characteristic functions is convex, we show that the proposed Shapley function is a FMPMAS. Furthermore, two special kinds of fuzzy multichoice games with fuzzy characteristic functions called fuzzy multichoice games with multilinear extension form and fuzzy characteristic functions and fuzzy multichoice games with Choquet integral form and fuzzy characteristic functions are researched.

Keywords Fuzzy multichoice game · Shapley function · Extension Hukuhara difference · Multilinear extension · Choquet integral

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1 Introduction

A multichoice game, proposed by Hsiao and Raghavan (1993), is a generalization of a traditional game in which each player has several activity levels. There are main four branches of solutions for this class of games that are extensions of the Shapley function (Shapley 1953), which were introduced by philosophers (Derks and Peters 1993; Hsiao and Raghavan 1993; Klijn et al. 1999; Peters and Zank 2005; van den Nouweland et al. 1995). The characterizations of the solutions for multichoice games can be seen in the literature (Borkotokey 2008; Hsiao and Raghavan 1993; Hwang and Liao 2008, 2009). Moreover, van den Nouweland et al. (1995) showed the relationship between core, dominant core and Weber set for multichoice games. Recently, Meng and Zhang (2014) discussed multichoice games with a coalition structure and defined a payoff value, and Meng et al. (2014) presented another multichoice coalition value named the generalized symmetric coalitional Banzhaf value, and two axiomatic systems are established.

There are some situations where some players do not fully participate in a coalition, but to a certain degree. In this situation, a coalition is called a fuzzy coalition, which is formed by some players with partial participations (that is, the player offers a part of resources that he owns). Aubin (1974) first discussed in this area. The solution concepts for fuzzy games have been studied by many researchers: The Shapley function for fuzzy games is studied by philosophers (Butnariu 1980; Butnariu and Kroupa 2008; Li and Zhang 2009; Meng and Zhang 2010; Tsurumi et al. 2001). Specially, Li and Zhang (2009) introduced a simplified expression of the Shapley value for fuzzy games, which can be applied to all kinds of fuzzy games that were introduced by Aubin (1974). The core for fuzzy games is focused by philosophers (Tijs et al. 2004; Yu and Zhang 2009). The lexicographical solution for fuzzy games is discussed by Sakawa and Nishizaki (1994).

As some researchers (Borkotokey 2008; Mares 2000; Mares and Vlach 2001; Yu and Zhang 2010) noticed, there are many uncertain factors during the process of negotiation and coalition forming, so in most situations players can only know imprecise information regarding the real outcome of cooperation. Hence, it is unrealistic that the players know the exacting payoff of every coalition. The crisp games with fuzzy characteristic functions were researched by philosophers (Mares 2000; Mares and Vlach 2001; Yu and Zhang 2010). The fuzzy games with fuzzy characteristic functions were discussed by Borkotokey (2008) and Yu and Zhang (2010).

In this paper we discuss the solution for fuzzy multichoice games with fuzzy characteristic functions. A fuzzy multichoice game with fuzzy characteristic functions is a generalization of a traditional fuzzy game in which each player has several activity levels. Based on the extension Hukuhara difference, a Shapley value for this kind of games is introduced, which is enlightened by van den Nouweland et al. (1995). An axiomatic definition of the given Shapley value is offered, and its explicit form is given. When fuzzy multichoice games with fuzzy characteristic functions are convex, the given Shapley value belongs to the core, and it always derives a FMPMAS. To better understand the given Shapley value, we study two particular kinds of fuzzy multichoice games with fuzzy characteristic functions, which are extensions of fuzzy games introduced by Meng and Zhang (2010) and Tsurumi et al. (2001).

This paper is organized as follows: in Sect. 2, we introduce the concepts of fuzzy numbers and the extension Hukuhara difference on fuzzy numbers. Then, the model of fuzzy multichoice games with fuzzy characteristic functions is proposed. In Sect. 3, a Shapley value for fuzzy multichoice games with fuzzy characteristic functions is proposed, and some properties of the given Shapley value are researched. In Sect. 4, we mainly discuss two special kinds of fuzzy multichoice games with fuzzy characteristic functions including the expressions of the Shapley values, the axiomatic systems, and the numerical examples.

2 Preliminaries

Let us start by recalling the most general definition of a fuzzy number. Let $\mathbb{R}$ be $(-\infty, \infty)$, i.e., the set of all real numbers.

Definition 2.1 (Zadeh 1965) A fuzzy number, denoted by $\tilde{u}$, is a fuzzy subset of $\mathbb{R}$ with membership function $\mu_{\tilde{u}}: \mathbb{R} \to [0, 1]$ satisfying the following conditions:

(1) $\mu_{\tilde{u}}$ is upper semi-continuous;
(2) there exists an interval number $[a, d]$ such that $\mu_{\tilde{u}}(x) = 0$ for any $x \notin [a, d]$;
(3) there exist real numbers b, c such that $a \le b \le c \le d$ and (1) $\mu_{\tilde{u}}(x)$ is nondecreasing on $[a, b]$ and nonincreasing on $[c, d]$; (2) $\mu_{\tilde{u}}(x) = 1$ for any $x \in [b, c]$.

By $\tilde{\mathbb{R}}$, we denote the set of all fuzzy numbers. Note that the definition of fuzzy numbers represents heterogeneous data forms including crisp data, fuzzy numbers, interval values and linguistic variables. They are represented by different membership functions defined on their domains. An important type of fuzzy numbers in common use is the trapezoidal fuzzy number (Dubois et al. 2000) whose membership function has the form

$$\mu_{\tilde{a}}(x) = \begin{cases} \frac{x-l_a}{h_a-l_a} & l_a \le x \le h_a \\ 1 & h_a \le x \le p_a \\ \frac{r_a-x}{r_a-p_a} & p_a \le x \le r_a \\ 0 & otherwise \end{cases},$$

where $l_a, h_a, p_a, r_a \in \mathbb{R}$ with $l_a \le h_a \le p_a \le r_a$.

The set of all trapezoidal fuzzy numbers is denoted by $\tilde{\mathbb{R}}_T$. For any $\tilde{a} \in \tilde{\mathbb{R}}_T$, we use (l_a, h_a, p_a, r_a) to denote $\tilde{a}$, namely, $\tilde{a} = (l_a, h_a, p_a, r_a)$.

For any $\tilde{a} \in \tilde{\mathbb{R}}$, the level set is defined as $\tilde{a}_\lambda = \{x \in \mathbb{R} | \mu_{\tilde{a}}(x) \ge \lambda\}$, $\lambda \in [0, 1]$. It follows from the properties of the membership function of a fuzzy number $\tilde{a}$ that each of its λ-cut $\tilde{a}_\lambda$ is an interval number, denoted by $\tilde{a}_\lambda = [\tilde{a}_\lambda^L, \tilde{a}_\lambda^R]$, $\lambda \in (0, 1]$, where $\tilde{a}_\lambda^L$ and $\tilde{a}_\lambda^R$ mean the lower and upper bounds of $\tilde{a}_\lambda$.

Let $\tilde{a}, \tilde{b} \in \tilde{\mathbb{R}}$, from the extension principle on fuzzy sets proposed by Zadeh (1973), we have

$$(\tilde{a} + \tilde{b})_\lambda = \tilde{a}_\lambda + \tilde{b}_\lambda = \left[\tilde{a}_\lambda^L + \tilde{b}_\lambda^L, \tilde{a}_\lambda^R + \tilde{b}_\lambda^R\right],$$

$$(\tilde{a}-\tilde{b})_\lambda = \tilde{a}_\lambda - \tilde{b}_\lambda = \left[\tilde{a}_\lambda^L - \tilde{b}_\lambda^R, \tilde{a}_\lambda^R - \tilde{b}_\lambda^L\right],$$
$$(m\tilde{a})_\lambda = m\tilde{a}_\lambda = \left[m\tilde{a}_\lambda^L, m\tilde{a}_\lambda^R\right] \qquad m \in \mathbb{R},\ m \geq 0.$$

Definition 2.2 For any $\tilde{a}, \tilde{b} \in \tilde{\mathbb{R}}$, we have

$$\tilde{a} \geq \tilde{b} \text{ if and only if } \tilde{a}_\lambda^L \geq \tilde{b}_\lambda^L \text{ and } \tilde{a}_\lambda^R \geq \tilde{b}_\lambda^R, \quad \forall \lambda \in (0, 1];$$
$$\tilde{a} = \tilde{b} \text{ if and only if } \tilde{a}_\lambda^L = \tilde{b}_\lambda^L \text{ and } \tilde{a}_\lambda^R = \tilde{b}_\lambda^R, \quad \forall \lambda \in (0, 1].$$

From the extension principle on fuzzy sets (Zadeh 1973), in general, we cannot have $\tilde{a} + \tilde{b} - \tilde{b} = \tilde{a}$ for any $\tilde{a}, \tilde{b} \in \tilde{\mathbb{R}}$. The Hukuhara difference on fuzzy sets (Banks and Jacobs 1970) can well deal with this issue, described by Definition 2.3.

Definition 2.3 (Banks and Jacobs 1970) Let $\tilde{a}, \tilde{b} \in \tilde{\mathbb{R}}$, if there exists $\tilde{c} \in \tilde{\mathbb{R}}$ such that $\tilde{a} = \tilde{b} + \tilde{c}$, then $\tilde{c}$ is called the Hukuhara difference between $\tilde{a}$ and $\tilde{b}$, denoted by $\tilde{c} = \tilde{a} -_H \tilde{b}$.

From Definition 2.3, we can obtain $\tilde{a} -_H \tilde{a} = 0$, $\tilde{a} + \tilde{b} -_H \tilde{b} = \tilde{a}$ and $(\tilde{a} -_H \tilde{b})_\lambda = \tilde{a}_\lambda -_H \tilde{b}_\lambda = [\tilde{a}_\lambda^L - \tilde{b}_\lambda^L, \tilde{a}_\lambda^R - \tilde{b}_\lambda^R]$ for any $\lambda \in (0, 1]$. Moreover, the Hukuhara difference between $\tilde{a}$ and $\tilde{b}$ exists if and only if $\tilde{a}_\lambda^L - \tilde{b}_\lambda^L \leq \tilde{a}_\beta^L - \tilde{b}_\beta^L \leq \tilde{a}_\beta^R - \tilde{b}_\beta^R \leq \tilde{a}_\lambda^R - \tilde{b}_\lambda^R$ for any $\lambda, \beta \in (0, 1]$ with $\lambda \leq \beta$. Although the Hukuhara difference has some advantages for the subtract operator on fuzzy sets, the necessary condition restricts its using scope. For this reason, we omit the necessary condition of the Hukuhara difference, and give the definition of the extension Hukuhara difference as follows:

Definition 2.4 For any $\tilde{a}, \tilde{b} \in \tilde{\mathbb{R}}$ and $\lambda \in (0, 1]$, $\tilde{a}_\lambda -_H \tilde{b}_\lambda = [\tilde{a}_\lambda^L - \tilde{b}_\lambda^L, \tilde{a}_\lambda^R - \tilde{b}_\lambda^R]$ is said to the extension Hukuhara difference between $\tilde{a}$ and $\tilde{b}$.

For any $\tilde{a}, \tilde{b} \in \tilde{\mathbb{R}}$, in this paper we adopt the extension Hukuhara difference between $\tilde{a}$ and $\tilde{b}$. If there is no fear of conflict, we still use $\tilde{a} -_H \tilde{b}$ to denote the extension Hukuhara difference between $\tilde{a}$ and $\tilde{b}$. Now, let us consider some desirable properties of the extension Hukuhara difference on fuzzy numbers.

Proposition 1 *Let $\tilde{a}$, $\tilde{b}$ and $\tilde{c}$ be any three fuzzy numbers.*

(I) *Commutativity $\tilde{a} -_H \tilde{b} = -_H\tilde{b} + \tilde{a}$;*
(II) *Associativity $\tilde{a} -_H \tilde{b} -_H \tilde{c} = \tilde{a} -_H (\tilde{b} + \tilde{c})$;*
(III) *Identity $\tilde{a} -_H \tilde{a} = \tilde{0}$;*
(IV) *Monotonicity If $\tilde{b} \leq \tilde{c}$, then $\tilde{a} -_H \tilde{c} \leq \tilde{a} -_H \tilde{b}$, where we consider $\tilde{e} \leq \tilde{f}$ for any two fuzzy numbers when $\tilde{e}_\lambda^L \leq \tilde{f}_\lambda^L$ and $\tilde{e}_\lambda^R \leq \tilde{f}_\lambda^R$ for all $\lambda \in (0, 1]$.*

Proof According to decomposition theorem of fuzzy numbers, for (I), we have

$$\tilde{a} -_H \tilde{b} = \bigcup_{\lambda \in (0,1]} \lambda \left(\tilde{a}_\lambda -_H \tilde{b}_\lambda\right) = \bigcup_{\lambda \in (0,1]} \lambda \left(\left[\tilde{a}_\lambda^L - \tilde{b}_\lambda^L, \tilde{a}_\lambda^R - \tilde{b}_\lambda^R\right]\right)$$

and

$$-_H\tilde{b}+\tilde{a}=\bigcup_{\lambda\in(0,1]}\lambda\left(-_H\tilde{b}_\lambda\right)+\bigcup_{\lambda\in(0,1]}\lambda\tilde{a}_\lambda=\bigcup_{\lambda\in(0,1]}\lambda\left(-_H\tilde{b}_\lambda+\tilde{a}_\lambda\right)$$
$$=\bigcup_{\lambda\in(0,1]}\lambda\left(\left[-\tilde{b}^L_\lambda+\tilde{a}^L_\lambda,-\tilde{b}^R_\lambda+\tilde{a}^R_\lambda\right]\right)=\bigcup_{\lambda\in(0,1]}\lambda\left(\left[\tilde{a}^L_\lambda-\tilde{b}^L_\lambda,\tilde{a}^R_\lambda-\tilde{b}^R_\lambda\right]\right).$$

Thus, $\tilde{a}-_H\tilde{b}=-_H\tilde{b}+\tilde{a}$.

For (II): Similar to (I), we derive

$$\tilde{a}-_H\tilde{b}-_H\tilde{c}=\bigcup_{\lambda\in(0,1]}\lambda\left(\tilde{a}_\lambda-_H\tilde{b}_\lambda-_H\tilde{c}_\lambda\right)$$
$$=\bigcup_{\lambda\in(0,1]}\lambda\left(\left[\tilde{a}^L_\lambda-\tilde{b}^L_\lambda-\tilde{c}^L_\lambda,\tilde{a}^R_\lambda-\tilde{b}^R_\lambda-\tilde{c}^R_\lambda\right]\right)$$

and

$$\tilde{a}-_H(\tilde{b}+\tilde{c})=\bigcup_{\lambda\in(0,1]}\lambda\tilde{a}_\lambda-_H\bigcup_{\lambda\in(0,1]}\lambda\left(\tilde{b}_\lambda+\tilde{c}_\lambda\right)$$
$$=\bigcup_{\lambda\in(0,1]}\lambda\tilde{a}_\lambda-_H\bigcup_{\lambda\in(0,1]}\lambda\left(\left[\tilde{b}^L_\lambda+\tilde{c}^L_\lambda,\tilde{b}^R_\lambda+\tilde{c}^R_\lambda\right]\right)$$
$$=\bigcup_{\lambda\in(0,1]}\lambda\left(\tilde{a}_\lambda-_H\left(\left[\tilde{b}^L_\lambda+\tilde{c}^L_\lambda,\tilde{b}^R_\lambda+\tilde{c}^R_\lambda\right]\right)\right)$$
$$=\bigcup_{\lambda\in(0,1]}\lambda\left(\left[\tilde{a}^L_\lambda-\tilde{b}^L_\lambda-\tilde{c}^L_\lambda,\tilde{a}^R_\lambda-\tilde{b}^R_\lambda-\tilde{c}^R_\lambda\right]\right),$$

by which we get $\tilde{a}-_H\tilde{b}-_H\tilde{c}=\tilde{a}-_H(\tilde{b}+\tilde{c})$.

From Definition 2.4, we can easily show that (III) holds.
For (IV): From $\tilde{b}\le\tilde{c}$, we derive $\tilde{b}^L_\lambda\le\tilde{c}^L_\lambda$ and $\tilde{b}^R_\lambda\le\tilde{c}^R_\lambda$ for all $\lambda\in(0,1]$. Thus, we have $\tilde{a}^L_\lambda-\tilde{c}^L_\lambda\le\tilde{a}^L_\lambda-\tilde{b}^L_\lambda$ and $\tilde{a}^R_\lambda-\tilde{c}^R_\lambda\le\tilde{a}^R_\lambda-\tilde{b}^R_\lambda$ for all $\lambda\in(0,1]$. From $\tilde{a}-_H\tilde{b}=\bigcup_{\lambda\in(0,1]}\lambda([\tilde{a}^L_\lambda-\tilde{b}^L_\lambda,\tilde{a}^R_\lambda-\tilde{b}^R_\lambda])$ and $\tilde{a}-_H\tilde{c}=\bigcup_{\lambda\in(0,1]}\lambda\left([\tilde{a}^L_\lambda-\tilde{c}^L_\lambda,\tilde{a}^R_\lambda-\tilde{c}^R_\lambda]\right)$, we get $\tilde{a}-_H\tilde{c}\le\tilde{a}-_H\tilde{b}$.

Example 2.1 Let $\tilde{v}_0$ be a multichoice game with the trapezoidal fuzzy characteristic function defined on $N=\{1,2\}$, where $m=\{2,2\}$, namely, the players 1 and 2 both have three activity levels. The coalition values are given by $\tilde{v}_0(1,0)=\tilde{v}_0(0,1)=(1,3,4,5)$, $\tilde{v}_0(1,1)=(2,7,10,12)$, $\tilde{v}_0(2,0)=\tilde{v}_0(0,2)=(3,6,9,11)$, $\tilde{v}_0(1,2)\tilde{v}_0(2,1)=(6,12,15,18)$ and $\tilde{v}_0(2,2)=(15,20,32,34)$. By $\tilde{v}_0(2,2)^L_\lambda-\tilde{v}_0(1,2)^L_\lambda=\tilde{v}_0(2,2)^L_\lambda-\tilde{v}_0(2,1)^L_\lambda=9-\lambda$ and $\tilde{v}_0(2,2)^R_\lambda-\tilde{v}_0(1,2)^R_\lambda=\tilde{v}_0(2,2)^R_\lambda-\tilde{v}_0(2,1)^R_\lambda=16+\lambda$, we know the Hukuhara difference between $\tilde{v}_0(2,2)$ and $\tilde{v}_0(1,2)$ as well as $\tilde{v}_0(2,2)$ and $\tilde{v}_0(2,1)$ do not exist. Hence, we cannot use the

Hukuhara difference in this example. If we adopt the extension Hukuhara difference and use the following equation

$$\tilde{\psi}\,(N, m, \tilde{v}_0) = \frac{\Pi_{i\in N}(m_i!)}{\left(\sum_{i\in N} m_i\right)!}\sum_{\sigma}\tilde{v}_0^{\sigma}, \tag{1}$$

which is a Shapley function introduced by van den Nouweland et al. (1995), and σ is an admissible order for $\tilde{v}_0$.

From Eq. (1), we have $\tilde{\psi}_{11}\,(N, m, \tilde{v}_0) = \tilde{\psi}_{21}\,(N, m, \tilde{v}_0) = (1.3, 3.8, 5, 6)$ and $\tilde{\psi}_{12}\,(N, m, \tilde{v}_0) = \tilde{\psi}_{22}\,(N, m, \tilde{v}_0) = (6.2, 6.2, 11, 11)$. It is easy to see that $(\tilde{\psi}_{ij})_{i\in N, j\in\{1,2\}}$ satisfies individual rationality and efficiency. If we use the vector $(\tilde{\psi}_{ij})_{i\in N, j\in\{1,2\}}$ as the players' payoffs, then the players 1 and 2 can both accept. Moreover, the players 1 and 2 are symmetric in this game, and they get the same payoffs, which is consistent with the people's intuition.

Remark 2.1 It is easy to see that the extension Hukuhara difference could deal with more situations than the Hukuhara difference.

Example 2.2 In Example 2.1, if the coalition values are defined by $\tilde{v}_0(1, 0) = \tilde{v}_0(0, 1) = (1, 3, 4, 5)$, $\tilde{v}_0(1, 1) = (2, 7, 10, 12)$, $\tilde{v}_0(2, 0) = \tilde{v}_0(0, 2) = (3, 6, 9, 10)$, $\tilde{v}_0(1, 2) = \tilde{v}_0(2, 1) = (6, 12, 15, 17)$ and $\tilde{v}_0(2, 2) = (15, 20, 32, 33)$. Similar to Example 2.1, we cannot use the Hukuhara difference either. When we adopt the extension Hukuhara difference, by Eq. (1) we have $\tilde{\psi}_{11}\,(N, m, \tilde{v}_0) = \tilde{\psi}_{21}\,(N, m, \tilde{v}_0) = (1.3, 3.8, 5, 6)$ and $\tilde{\psi}_{12}\,(N, m, \tilde{v}_0) = \tilde{\psi}_{22}\,(N, m, \tilde{v}_0) = (6.2, 6.2, 11, 10.5)$.

By $\tilde{\psi}_{12}\,(N, m, \tilde{v}_0) = \tilde{\psi}_{22}\,(N, m, \tilde{v}_0) = (6.2, 6.2, 11, 10.5)$, we know that the extension Hukuhara difference is not suitable in this game. Namely, there exist games with fuzzy characteristic functions that cannot apply the extension Hukuhara difference.

3 The General Form

In traditional multichoice games, we demand every player's activity levels belong to natural numbers set $\mathbb{N}$, and the difference between two adjacent levels is 1, which is somewhat unnatural in some situations. There is a class of cooperative games that every player has several activity levels, and each level belongs to [0, 1]. We call this class of games as fuzzy multichoice games. Calvo and Santos (2000) introduced a solution for continuum fuzzy multichoice games and showed that the Aumann–Shapley value for fuzzy continuum games is a special case of the given solution for continuum fuzzy multichoice games. In this section, we introduce fuzzy multichoice games with fuzzy characteristic functions.

Let $N = \{1, 2, \ldots, n\}$ be a set of players, and suppose that each player $i \in N$ has $m_i + 1 \in \mathbb{N}$ activity options. We set $FM_i = \{0, a_1, \ldots, a_{m_i}\}$ as the action level space of player $i \in N$ such that $a_k \in (0, 1]$ for all $k = 1, 2, \ldots, m_i$ and $a_p < a_{p+1}$ for all $p = 1, 2, \ldots, m_i - 1$. The action 0 means not participating. Let e^S denote the vector in N satisfying $e_i^S = 0$ if $i \notin S$, and $e_i^S = 1$ if $i \in S$ for any $S \subseteq N$.

A function $\tilde{v}: \Pi_{i\in N}FM_i \to \tilde{\mathbb{R}}$ with $\tilde{v}(e^{\varnothing}) = 0$ gives for each fuzzy coalition $\tilde{x} = (x_1, x_2, \ldots, x_n) \in FM = \Pi_{i\in N}FM_i$ the worth that the players can obtain when each player i plays at level $x_i \in FM_i$. The set of all fuzzy multichoice games with fuzzy characteristic functions on player set N is denoted by FMC^N. For all $\tilde{x}, \ \tilde{y} \in FM$, we have $\tilde{x} \vee \tilde{y} = (x_i \vee y_i)_{i\in N}$ and $\tilde{x} \wedge \tilde{y} = (x_i \wedge y_i)_{i\in N}$. Furthermore, we denote $\tilde{x} \le \tilde{y}$ for all $\tilde{x}, \ \tilde{y} \in FM$ if and only if $x_i \le y_i$ for all $i \in N$. Let FM_i^+ denote $FM_i \backslash \{0\}$ for all $i \in N$. For any $\tilde{x} = (x_1, x_2, \ldots, x_n) \in FM$, let $\tilde{x}_{\text{sub}} = (k_{x_1}, k_{x_2}, \ldots, k_{x_n})$, where k_{x_i} denotes the subscript of the activity level of the player i in fuzzy coalition $\tilde{x}$. For all $\tilde{x} \in FM$, let Supp $\tilde{x} = \{i \in N | x_i > 0\}$. $\tilde{m} = (a_{m_1}, a_{m_2}, \ldots, a_{m_n})$ is the "maximum fuzzy coalition" in FM.

Example 2.3 Let $N = \{1, 2\}$, $FM_1^+ = \{a_1 = 0.2, a_2 = 0.5\}$ and $FM_2^+ = \{a_1 = 0.3, a_2 = 0.4\}$. If $\tilde{x} = (0.5, 0.3)$, then $\tilde{x}_{\text{sub}} = (2, 1)$ and Supp $\tilde{x} = \{1, 2\}$, namely, $x_1 = 0.5, \ x_2 = 0.3, \ k_{x_1} = 2$ and $k_{x_2} = 1$.

Definition 2.5 A game $\tilde{v} \in FMC^N$ is said to be convex if it satisfies

$$\tilde{v}(\tilde{x} \vee \tilde{y}) + \tilde{v}(\tilde{x} \wedge \tilde{y}) \ge \tilde{v}(\tilde{x}) + \tilde{v}(\tilde{y}) \qquad \forall \tilde{x}, \tilde{y} \in FM.$$

Definition 2.6 Let $\tilde{v} \in FMC^N$, the core $C(N, \tilde{m}, \tilde{v})$ of $\tilde{v}$ is defined by

$$C(N, \tilde{m}, \tilde{v}) = \left\{ \tilde{w} \in \tilde{\mathbb{R}}^{\sum\limits_{i\in N} m_i} \middle| \sum_{i\in N} \tilde{w}_{ia_{m_i}} = \tilde{v}(\tilde{m}), \sum_{i\in \text{Supp}\tilde{x}} \tilde{w}_{ix_i} \ge \tilde{v}(\tilde{x}), \quad \forall \tilde{x} \in FM \right\}.$$

Obviously, Definitions 2.5 and 2.6 respectively degenerate to be the definitions of the convexity and the core for traditional fuzzy games, when we restrict the domain of $\tilde{v} \in FMC^N$ in the setting of traditional fuzzy games (Aubin 1974).

Definition 2.7 A vector $\tilde{z} = ((\tilde{z}_{1a_{i_1}})_{i_1\in M_1^+}, (\tilde{z}_{2a_{i_2}})_{i_2\in M_2^+}, \ldots, (\tilde{z}_{na_{i_n}})_{i_n\in M_n^+})$ is called an imputation of $\tilde{v} \in FMC^N$ if it satisfies the following conditions:

(1) $\sum_{i\in N} \tilde{z}_{ia_{m_i}} = \tilde{v}(\tilde{m})$,
(2) $\tilde{z}_{ia_j} \ge \tilde{v}(a_j e^i) \quad \forall i \in N, \ a_j \in FM_i^+$.

Note that the definition above can be applicable to traditional (fuzzy) games by restricting the domain of $\tilde{v} \in FMC^N$ in the setting of it.

Next, we dedicate to study a Shapley function for FMC^N . First, we introduce a Shapley value for multichoice games introduced by van den Nouweland et al. (1995):

$$\psi\,(N, m, v) = \frac{\Pi_{i\in N}(m_i!)}{\left(\sum_{i\in N} m_i\right)!} \sum_{\sigma} v^{\sigma}, \tag{2}$$

where σ is an admissible order for v, and v^{σ} means the value of v with respect to σ. Equation (2) is equivalent to the following equation.

$$\psi_{ij}\,(N, m, v) = \sum_{S\in M, S_i=j} h_{ij}(S) \left(v(S) - v(S - e^i) \right) \tag{3}$$

for any $i \in N$ and any $j \in \{1, 2, \ldots, m_i\}$, where S_i denotes the activity level of the player i in coalition S, $h_{ij}(S) = \left(\frac{(\sum_{k:(S|S_i-1)_k \neq 0} S_k)!}{\prod_{k:(S|S_i-1)_k \neq 0}(S_k!)} \frac{(\sum_{k \in N}(m_k - S_k))!}{\prod_{k \in N}((m_k - S_k)!)}\right) / \frac{(\sum_{k \in N} m_k)!}{\prod_{k \in N}(m_k!)}$ and $S|S_i - 1 = (S_1, \ldots, S_{i-1}, S_i - 1, S_{i+1}, \ldots, S_n)$.

$(\sum_{k:(S|S_i-1)_k \neq 0} S_k)! / \prod_{k:(S|S_i-1)_k \neq 0}(S_k!)$ is the number of admissible orders from coalition $\varnothing$ to the coalition S, where S_i is the last step, and $(\sum_{k \in N}(m_k - S_k))! / \prod_{k \in N}((m_k - S_k)!)$ is the number of admissible orders from coalition S to the maximum coalition $m = (m_1, \ldots, m_n)$.

If we adopt Eq. (3) in the framework of fuzzy multichoice games with fuzzy characteristic functions, then we have

$$\tilde{\psi}_{ia_j}(N, \tilde{m}, \tilde{v}) = \sum_{\tilde{x} \in FM, k_{x_i} = j} h_{ij}(\tilde{x}_{\text{sub}})(\tilde{v}(\tilde{x}) -_H \tilde{v}(\tilde{x} - (a_j - a_{j-1})e^i)) \forall i \in N, j \in \{1, 2, \ldots, m_i\}, \quad (4)$$

where $h_{ij}(\tilde{x}_{\text{sub}})$ is the potential weight for $\tilde{x}_{\text{sub}} = (k_{x_1}, k_{x_2}, \ldots, k_{x_n})$, which is denoted by $h_{ij}(\tilde{x}_{\text{sub}}) = \left(\frac{(\sum_{p:(\tilde{x}|k_{x_i}-1)_p \neq 0} k_{x_p})!}{\prod_{p:(\tilde{x}|k_{x_i}-1)_p \neq 0}(k_{x_p}!)} \frac{(\sum_{p \in N}(m_p - k_{x_p}))!}{\prod_{p \in N}(m_p - k_{x_p})!}\right) / \frac{(\sum_{p \in N} m_p)!}{\prod_{p \in N}(m_p!)}$ with $x|k_{x_i} - 1 = (x_1, \ldots, x_{i-1}, x_{k_{x_i}-1}, x_{i+1}, \ldots, x_n)$ and $k_{x_i} = j$.

For any $i \in N$ and any $j \in \{1, 2, \ldots, m_i\}$, $\tilde{\psi}_{ia_j}(N, \tilde{m}, \tilde{v})$ denotes the player i's increasing fuzzy payoff from participation level a_{j-1} to a_j.

Definition 3.1 Let $\tilde{v} \in FMC^N$, $\tilde{y} \in FM$ is called a carrier for $\tilde{v}$ in FM if $\tilde{v}(\tilde{x} \wedge \tilde{y}) = \tilde{v}(\tilde{x})$ for all $\tilde{x} \in FM$.

From Definition 3.1, we know that a carrier for $\tilde{v} \in FMC^N$ degenerates to be a carrier for traditional fuzzy games when the domain of $\tilde{v} \in FMC^N$ is limited to it.

Definition 3.2 A function $\tilde{f} : FMC^N \to \tilde{\mathbb{R}}^{\sum_{i \in N} m_i}$ is said to be a Shapley function on FMC^N if it satisfies the following axioms:

Axiom 1 Let $\tilde{v} \in FMC^N$, and $\tilde{y} \in FM$ be a carrier of $\tilde{v}$, $\sum_{i \in \text{Supp}\, \tilde{y}} \tilde{f}_{iy_i}(N, \tilde{m}, \tilde{v}) = \tilde{v}(\tilde{y})$.

Axiom 2 Let $\tilde{v} \in FMC^N$, and all $i_1, i_2 \in N$,

$$\sum_{\tilde{y} \le \tilde{x}, k_{x_{i_2}} = k_{y_{i_2}}} h_{i_2 k_{y_{i_2}}}(\tilde{x}_{\text{sub}}) \tilde{f}_{i_1 y_{i_1}}(N, \tilde{m}, u_{\tilde{y}}) = \sum_{\tilde{y} \le \tilde{x}, k_{x_{i_1}} = k_{y_{i_1}}} h_{i_1 k_{y_{i_1}}}(\tilde{x}_{\text{sub}}) \tilde{f}_{i_2 y_{i_2}}(N, \tilde{m}, u_{\tilde{y}}),$$

where $\tilde{y} \in FM \backslash \{e^{\varnothing}\}$, $k_{y_{i_1}} \in FM^+_{i_1}$ and $k_{y_{i_2}} \in FM^+_{i_2}$. $u_{\tilde{y}}(\tilde{x}) = 1$ for any $\tilde{x} \in FM$ with $\tilde{y} \le \tilde{x}$, otherwise, $u_{\tilde{y}}(\tilde{x}) = 0$.

Axiom 3 Let $\tilde{v}, \tilde{w} \in FMC^N$, $\tilde{f}(N, \tilde{m}, \tilde{v} + \tilde{w}) = \tilde{f}(N, \tilde{m}, \tilde{v}) + \tilde{f}(N, \tilde{m}, \tilde{w})$.

It is easy to see that the above axioms can be seen as the extensions on traditional (fuzzy) games. Namely, when the domain of $\tilde{v} \in FMC^N$ is restricted in the setting of traditional (fuzzy) games, then they are the axioms on them.

For any $\tilde{v} \in FMC^N$, in order to eliminate the situation in Example 2.2, we always assume that the fuzzy payoff of every player in each activity level obtained by Eq. (5) is a fuzzy number.

Lemma 3.1 *Let $\tilde{v} \in FMC^N$, define the function $\tilde{\psi}(N, \tilde{m}, \tilde{v})$ as shown in Eq. (4), then we have*

$$\sum_{i \in \text{Supp}\,\tilde{y}} \sum_{j=1}^{k_{y_i}} \tilde{\psi}_{ia_j}(N, \tilde{m}, \tilde{v}) = \tilde{v}(\tilde{y}),$$

where $\tilde{y} \in FM$ is a carrier of $\tilde{v}$.

Proof Since $\tilde{y} \in FM$ is a carrier, for any $\tilde{x} \in \Pi_{k \in N \setminus \{i\}} FM_k$ we have

$$\begin{aligned} \tilde{v}\left(\tilde{x} \vee a_j e^i\right) &= \tilde{v}\left(\left(\tilde{x} \vee a_j e^i\right) \wedge \tilde{y}\right) = \tilde{v}\left((\tilde{x} \wedge \tilde{y}) \vee \left(a_j e^i \wedge \tilde{y}\right)\right) \\ &= \tilde{v}\left((\tilde{x} \wedge \tilde{y}) \vee y_i e^i\right), \end{aligned}$$

where $j > k_{y_i}$.
By Eq. (4), we have $\tilde{\psi}_{ia_j}(N, \tilde{m}, \tilde{v}) = 0$ for any $i \in N$ and any $j \in \{1, 2, \ldots, m_i\}$ with $j > k_{y_i}$.
From efficiency of Eq. (4), we have

$$\begin{aligned} \sum_{i \in \text{Supp}\,\tilde{y}} \sum_{j=1}^{k_{y_i}} \tilde{\psi}_{ia_j}(N, \tilde{m}, \widehat{v}) &= \sum_{i \in \text{Supp}\tilde{y}} \sum_{j=1}^{m_i} \tilde{\psi}_{ia_j}(N, \tilde{m}, \tilde{v}) \\ &= \sum_{i \in N} \sum_{j=1}^{m_i} \tilde{\psi}_{ia_j}(N, \tilde{m}, \tilde{v}) \\ &= \tilde{v}(\tilde{m}) \\ &= \tilde{v}(\tilde{m} \wedge \tilde{y}) \\ &= \tilde{v}(\tilde{y}). \end{aligned}$$

□

Lemma 3.2 *Let $\tilde{v} \in FMC^N$, define the function $\tilde{\psi}(N, \tilde{m}, \tilde{v})$ as shown in Eq. (4), then we have*

$$\begin{aligned} &\sum_{\tilde{y} \le \tilde{x}, k_{x_{i_2}} = k_{y_{i_2}}} h_{i_2 k_{y_{i_2}}}(\tilde{x}_{\text{sub}}) \tilde{\psi}_{i_1 y_{i_1}}\left(N, \tilde{m}, u_{\tilde{y}}\right) \\ &\quad = \sum_{\tilde{y} \le \tilde{x}, k_{x_{i_1}} = k_{y_{i_1}}} h_{i_1 k_{y_{i_1}}}(\tilde{x}_{\text{sub}}) \tilde{\psi}_{i_2 y_{i_2}}\left(N, \tilde{m}, u_{\tilde{y}}\right), \end{aligned}$$

where $\tilde{y} \in FM \setminus \{e^{\varnothing}\}$, $k_{y_{i_1}} \in FM^+_{i_1}$, $k_{y_{i_2}} \in FM^+_{i_2}$, and $u_{\tilde{y}}$ as shown in Definition 3.2.

Proof By Eq. (4), we have

$$\begin{aligned}
&\tilde{\psi}_{i_1 y_{i_1}}(N, \tilde{m}, u_{\tilde{y}}) \\
&= \sum_{\tilde{x}\in FM, k_{x_{i_1}}=k_{y_{i_1}}} h_{i_1 k_{y_{i_1}}}(\tilde{x}_{\text{sub}}) \left(u_{\tilde{y}}(\tilde{x}) \quad - u_{\tilde{y}}(\tilde{x} - (a_{k_{y_{i_1}}} - a_{k_{y_{i_1}}-1})e^i)\right) \\
&= \sum_{\tilde{y}\le\tilde{x}, k_{x_{i_1}}=k_{y_{i_1}}} h_{i_1 k_{y_{i_1}}}(\tilde{x}_{\text{sub}}) \left(u_{\tilde{y}}(\tilde{x}) - u_{\tilde{y}}(\tilde{x} - (a_{k_{y_{i_1}}} - a_{k_{y_{i_1}}-1})e^i)\right) \\
&= \sum_{\tilde{y}\le\tilde{x}, k_{x_{i_1}}=k_{y_{i_1}}} h_{i_1 k_{y_{i_1}}}(\tilde{x}_{\text{sub}}).
\end{aligned}$$

Similarly, we have $\tilde{\psi}_{i_2 y_{i_2}}(N, \tilde{m}, u_{\tilde{y}}) = \sum_{\tilde{y}\le\tilde{x}, k_{x_{i_2}}=k_{y_{i_2}}} h_{i_2 k_{y_{i_2}}}(\tilde{x}_{\text{sub}})$. The proof is finished. □

Theorem 3.1 *Define a function* $\tilde{\varphi}: FMC^N \to \tilde{\mathbb{R}}^{\sum_{i\in N} m_i}$ *by*

$$\tilde{\varphi}_{ia_j}(N, \tilde{m}, \tilde{v}) = \sum_{1\le h\le j} \sum_{\tilde{x}\in FM, k_{x_i}=h} h_{ih}(\tilde{x}_{\text{sub}})(\tilde{v}(\tilde{x}) \\ -_H \tilde{v}(\tilde{x} - (a_h - a_{h-1})e^i)) \forall i \in N,\ j \in \{1, 2, \ldots, m_i\}, \quad (5)$$

where $h_{ih}(\tilde{x}_{\text{sub}})$ *is the potential weight for fuzzy coalition* $\tilde{x}$ *as shown in Eq.* (4). *Then* $\tilde{\varphi}$ *is the unique Shapley function on* FMC^N.

Proof Existence. Axiom 1: From Eqs. (4) and (5), we have $\tilde{\varphi}_{ia_j}(N, \tilde{m}, \tilde{v}) = \sum_{1\le k\le j} \tilde{\psi}_{ia_k}(N, \tilde{m}, \tilde{v})$. From Lemma 3.1, we obtain

$$\sum_{i\in \text{Supp}\,\tilde{y}} \tilde{\varphi}_{iy_i}(N, \tilde{m}, \tilde{v}) = \sum_{i\in \text{Supp}\,\tilde{y}} \sum_{1\le k\le k_{y_j}} \tilde{\psi}_{ia_k}(N, \tilde{m}, \tilde{v}) = \tilde{v}(\tilde{y}).$$

From $\tilde{\varphi}_{iy_i}(N, \tilde{m}, u_{\tilde{y}}) = \sum_{1\le j\le k_{y_i}} \tilde{\psi}_{ia_j}(N, \tilde{m}, u_{\tilde{y}}) = \tilde{\psi}_{iy_i}(N, \tilde{m}, u_{\tilde{y}})$ and Lemma 3.2, one can easily get Axiom 2.
Axiom 3: From Eq. (5), it obviously holds.
Uniqueness. Hypothesis, Eq. (5) satisfies these axioms. According to Hwang and Liao (2009), for any $\tilde{v} \in FMC^N$ it can be expressed by

$$\tilde{v} = \sum_{\tilde{y}\in FM\backslash\{e^{\varnothing}\}} \tilde{c}_{\tilde{y}} u_{\tilde{y}},$$

where $\widehat{c}_{\tilde{y}} = \tilde{v}(\tilde{y}) -_H \sum_{\tilde{x}\le\tilde{y}, \tilde{x}\ne\tilde{y}} \tilde{c}_{\tilde{x}}$, and $u_{\tilde{y}}$ as shown in Definition 3.2.
By Axiom 3, we only need to prove the uniqueness of Eq. (5) for $u_{\tilde{y}}$, where $\tilde{y} \in FM\backslash\{e^{\varnothing}\}$.

By Axiom 2, we have

$$\tilde{\varphi}_{jy_j}(N, \tilde{m}, u_{\tilde{y}}) = \frac{\sum_{\tilde{y} \le \tilde{x}, k_{x_j} = k_{y_j}} h_{jk_{y_j}}(\tilde{x}_{\text{sub}})}{\sum_{\tilde{y} \le \tilde{x}, k_{x_i} = k_{y_i}} h_{ik_{y_i}}(\tilde{x}_{\text{sub}})} \tilde{\varphi}_{iy_i}(N, \tilde{m}, u_{\tilde{y}}).$$

Since $\tilde{y}$ is a carrier for $u_{\tilde{y}}$, by axiom 1 we derive

$$1 = u_{\tilde{y}}(\tilde{y}) = \sum_{j \in \text{Supp } \tilde{y}} \tilde{\varphi}_{jy_j}(N, \tilde{m}, u_{\tilde{y}}).$$

If we fix $i \in \text{Supp } \tilde{y}$, then we have

$$\begin{aligned} 1 &= \sum_{j \in \text{Supp } \tilde{y}} \tilde{\varphi}_{jy_j}(N, \tilde{m}, u_{\tilde{y}}) \\ &= \tilde{\varphi}_{iy_i}(N, \tilde{m}, u_{\tilde{y}}) + \sum_{j \in \text{Supp } \tilde{y} \setminus \{i\}} \frac{\sum_{\tilde{y} \le \tilde{x}, k_{x_j} = k_{y_j}} h_{jk_{y_j}}(\tilde{x}_{\text{sub}})}{\sum_{\tilde{y} \le \tilde{x}, k_{x_i} = k_{y_i}} h_{ik_{y_i}}(\tilde{x}_{\text{sub}})} \tilde{\varphi}_{iy_i}(N, \tilde{m}, u_{\tilde{y}}) \\ &= \frac{\sum_{j \in \text{Supp } \tilde{y}} \sum_{\tilde{y} \le \tilde{x}, k_{x_j} = k_{y_j}} h_{jk_{y_j}}(\tilde{x}_{\text{sub}})}{\sum_{\tilde{y} \le \tilde{x}, k_{x_i} = k_{y_i}} h_{ik_{y_i}}(\tilde{x}_{\text{sub}})} \tilde{\varphi}_{iy_i}(N, \tilde{m}, u_{\tilde{y}}). \end{aligned}$$

Since $\sum_{j \in \text{Supp } \tilde{y}} \sum_{\tilde{y} \le \tilde{x}, k_{x_j} = k_{y_j}} h_{jk_{y_j}}(\tilde{x}_{\text{sub}}) = 1$, we get $\tilde{\varphi}_{iy_i}(N, \tilde{m}, u_{\tilde{y}}) = \sum_{\tilde{y} \le \tilde{x}, k_{x_i} = k_{y_i}} h_{ik_{y_i}}(\tilde{x}_{\text{sub}})$ and $\tilde{\varphi}_{iy_i}(N, \tilde{m}, u_{\tilde{y}}) = 0$, otherwise. The proof is finished. □

Obviously, Eq. (5) degenerates to be the Shapley value for traditional fuzzy games, when we limit the domain of $\tilde{v} \in FMC^N$ in the framework of it.

Sprumont (1990) proposed a PMAS as a reasonable solution concept for traditional games, which specifies not only how to allocate the maximum coalition but also how to allocate the worth of every coalition. Tsurumi et al. (2001) extended PMAS to fuzzy games, and defined a FPMAS, which is as an extension of PMAS. Here, we further extend PMAS to FMC^N and give the following definition for FMPMAS. As will be seen later, FMPMAS is an extension of PMAS and FPMAS.

Definition 3.3 A vector $\tilde{z} = ((\tilde{z}_{1a_{i_1}})_{i_1 \in M_1^+}, (\tilde{z}_{2a_{i_2}})_{i_2 \in M_2^+}, \ldots, (\tilde{z}_{na_{i_n}})_{i_n \in M_n^+})$ is said to be a FMPMAS for $\tilde{v} \in FMC^N$ if it satisfies the following conditions:

(1) $\sum_{i \in N} \tilde{z}_{ia_{m_i}}(\tilde{x}) = \tilde{v}(\tilde{x}), \ \forall \tilde{x} \in FM$,
(2) $\tilde{z}_{ia_j}(\tilde{y}) \ge \tilde{z}_{ia_j}(\tilde{x}) \quad \forall i \in \text{Supp } \tilde{x}, \ 0 < j \le k_{x_i}, \ \tilde{x}, \tilde{y} \in FM \ s.t. \ \tilde{x} \le \tilde{y}, \ k_{x_i} = k_{y_i}$,

where $\tilde{z}_{ia_j}(\tilde{x})$ and $\tilde{z}_{ia_j}(\tilde{y})$ respectively denote the player i's payoffs at level a_j for the fuzzy coalitions $\tilde{x}$ and $\tilde{y}$.

Remark 3.1 For any $i \in \text{Supp } \tilde{x}$ and all $\tilde{x}, \tilde{y} \in FM$ satisfying $\tilde{x} \le \tilde{y}$, the condition $k_{x_i} = k_{y_i}$ must be satisfied, otherwise, we cannot guarantee that the conclusion $\tilde{z}_{ia_j}(\tilde{y}) \ge \tilde{z}_{ia_j}(\tilde{x})$ holds for all $\tilde{x}, \tilde{y} \in FM$ with $\tilde{x} \le \tilde{y}$.

From Definition 3.3, we know that the definition of FMPMAS degenerates to be the definition of FPMAS, when we limit the domain of $\tilde{v} \in FMC^N$ in the framework of traditional fuzzy games. Moreover, the definition of FMPMAS degenerates to be the definition of PMAS, when the domain of $\tilde{v} \in FMC^N$ is restricted in the framework of traditional games.

Lemma 3.3 *Let $\tilde{v} \in FMC^N$ be convex, and $\tilde{v}(a_j e^i) \geq \tilde{v}(a_{j-1} e^i)$ for any $i \in N$ and any $j \in \{1, 2, \ldots, m_i\}$, then*

(1) $\sum_{i \in N} \sum_{j=1}^{m_i} \tilde{\psi}_{ia_j}(N, \tilde{m}, \tilde{v}_{\tilde{x}}) = \tilde{v}(\tilde{x}) \quad \forall \tilde{x} \in FM,$
(2) $\tilde{\psi}_{ia_j}(N, \tilde{m}, \tilde{v}_{\tilde{x}}) \leq \tilde{\psi}_{ia_j}(N, \tilde{m}, \tilde{v}_{\tilde{y}}) \forall i \in Supp\tilde{x}, 0 < j \leq k_{x_i}, \tilde{x}, \tilde{y} \in FM \ s.t. \tilde{x} \leq \tilde{y}, k_{x_i} = k_{y_i},$

where $\tilde{\psi}(N, \tilde{m}, \tilde{v})$ as shown in Eq. (4), $\tilde{v}_{\tilde{x}}$ and $\tilde{v}_{\tilde{y}}$ respectively denote the restriction of $\tilde{v}$ in $\tilde{x}$ and $\tilde{y}$.

Proof From Eq. (4), we have

$$\tilde{\psi}_{ia_j}(N, \tilde{m}, \tilde{v}_{\tilde{x}}) = \sum_{\tilde{z} \leq \tilde{x}, k_{z_i} = j} h_{ij}^{\tilde{x}}(\tilde{z}_{\text{sub}}) \left(\tilde{v}(\tilde{z}) -_H \tilde{v}\left(\tilde{z} - (a_j - a_{j-1}) e^i \right) \right),$$

where $h_{ij}^{\tilde{x}}(\tilde{z}_{\text{sub}})$ is the restriction of $h_{ij}(\tilde{z}_{\text{sub}})$ in $\tilde{x}$, namely, $h_{ij}^{\tilde{x}}(\tilde{z}_{\text{sub}}) = \left(\frac{(\sum_{g:(\tilde{z}|k_{z_i}-1)_g \neq 0} k_{z_g})!}{\prod_{g:(\tilde{z}|k_{z_i}-1)_g \neq 0}(k_{z_g}!)} \frac{(\sum_{g \in \text{Supp}\,\tilde{x}}(k_{x_g} - k_{z_g}))!}{\prod_{g \in \text{Supp}\,\tilde{x}}(k_{x_g} - k_{z_g})!} \right) / \frac{(\sum_{g \in \text{Supp}\,\tilde{x}} k_{x_g})!}{\prod_{g \in \text{Supp}\,\tilde{y}}(k_{x_g}!)}$, and $\tilde{z}|k_{z_i} - 1 = (z_{p_1}, \ldots, z_{k_{z_i}-1}, \ldots, z_{p_t})$ such that $\{p_1, \ldots, i, \ldots, p_t\} \subseteq \text{Supp}\,\tilde{x}$.
From Eq. (3), it is easy to get $\sum_{i \in N} \sum_{j=1}^{m_i} \tilde{\psi}_{ia_j}(N, \tilde{m}, \tilde{v}_{\tilde{x}}) = \tilde{v}(\tilde{x})$ for any $\tilde{x} \in FM$.
In the following, we show the second condition in Lemma 3.3.
When $\tilde{x} = \tilde{y}$, the result obviously holds.
When $\tilde{x} \neq \tilde{y}$, there only exist three cases.

Case (I): If Supp $\tilde{x}$ = Supp$\tilde{y}$, and there exists $q \in$ Supp $\tilde{x}$ such that $k_{x_q} < k_{y_q}$, where $q \neq i$. In the following, we prove $\tilde{\psi}_{ia_j}(N, \tilde{m}, \tilde{v}_{\tilde{x}}) \leq \tilde{\psi}_{ia_j}(N, \tilde{m}, \tilde{v}_{\tilde{y}})$ by using induction.
From the convexity of $\tilde{v}$, we have

$$\tilde{v}\left(\tilde{x} \vee a_j e^i\right) -_H \tilde{v}\left(\tilde{x} \vee (a_j - a_{j-1}) e^i\right) \leq \tilde{v}\left(\tilde{y} \vee a_j e^i\right) -_H \tilde{v}\left(\tilde{y} \vee (a_j - a_{j-1}) e^i\right) \tag{6}$$

for all $\tilde{x}, \tilde{y} \in \prod_{k \in N \setminus \{i\}} FM_k$ with $\tilde{x} \leq \tilde{y}$.
① When $\sum_{p \in \text{Supp}\,\tilde{y}} k_{y_p} - \sum_{p \in \text{Supp}\,\tilde{x}} k_{x_p} = 1$, Since

$$\sum_{\tilde{z} \leq \tilde{r} \leq \tilde{x}, k_{r_i} = j} h_{ij}^{\tilde{x}}(\tilde{r}_{\text{sub}}) \leq \sum_{\tilde{z} \leq \tilde{r} \leq \tilde{y}, k_{r_i} = j} h_{ij}^{\tilde{y}}(\tilde{r}_{\text{sub}}) \tag{7}$$

for any $i \in$ Supp $\tilde{x}$, any $0 < j \leq k_{x_i}$ and any $\tilde{z} \leq \tilde{x}$, where $h_{ij}^{\tilde{y}}(\tilde{r}_{\text{sub}})$ denotes the restriction of $h_{ij}(\tilde{r}_{\text{sub}})$ in $\tilde{y}$.

(i) When $\tilde{z} = \tilde{x}$, by Eq. (6) we have

$$\sum_{\tilde{x} \leq \tilde{r} \leq \tilde{y}, k_{r_i}=j} h_{ij}^{\tilde{y}}(\tilde{r}_{\text{sub}}) \left(\tilde{v}(\tilde{r}) -_H \tilde{v}\left((\tilde{r} - a_j e^i) \vee a_{j-1}e^i\right)\right)$$
$$\geq \sum_{\tilde{x} \leq \tilde{r} \leq \tilde{y}, k_{r_i}=j} h_{ij}^{\tilde{y}}(\tilde{r}_{\text{sub}}) \left(\tilde{v}(\tilde{x}) -_H \tilde{v}((\tilde{x} - a_j e^i) \vee a_{j-1}e^i)\right)$$
$$\geq \sum_{\tilde{r}=\tilde{x}, k_{r_i}=j} h_{ij}^{\tilde{x}}(\tilde{x}_{\text{sub}}) \left(\tilde{v}(\tilde{x}) -_H \tilde{v}((\tilde{x} - a_j e^i) \vee a_{j-1}e^i)\right).$$

(ii) When $\sum\limits_{p \in \text{Supp}\, \tilde{x}} k_{x_p} - \sum\limits_{p \in \text{Supp}\, \tilde{z}} k_{z_p} = 1$, by Eq. (5) we have

$$\sum_{\tilde{z} \leq \tilde{r} \leq \tilde{y}, k_{r_i}=j} h_{ij}^{\tilde{y}}(\tilde{r}_{\text{sub}}) \left(\tilde{v}(\tilde{r}) -_H \tilde{v}((\tilde{r} - a_j e^i) \vee a_{j-1}e^i)\right)$$
$$-_H \sum_{\tilde{z} \leq \tilde{r} \leq \tilde{x}, k_{r_i}=j} h_{ij}^{\tilde{x}}(\tilde{r}_{\text{sub}}) \left(\tilde{v}(\tilde{r}) -_H \tilde{v}\left((\tilde{r} - a_j e^i) \vee a_{j-1}e^i\right)\right)$$
$$\geq \left(\sum_{\tilde{z} \leq \tilde{r} \leq \tilde{y}, k_{r_i}=j} h_{ij}^{\tilde{y}}(\tilde{r}_{\text{sub}}) - \sum_{\tilde{z} \leq \tilde{r} \leq \tilde{x}, k_{r_i}=j} h_{ij}^{\tilde{x}}(\tilde{r}_{\text{sub}})\right) (\tilde{v}(\tilde{z})$$
$$-_H \tilde{v}\left((\tilde{z} - a_j e^i) \vee a_{j-1}e^i\right)\Big).$$

(iii) Hypothesis, we have

$$\sum_{\tilde{z} \leq \tilde{r} \leq \tilde{y}, k_{r_i}=j} h_{ij}^{\tilde{y}}(\tilde{r}_{\text{sub}}) \left(\tilde{v}(\tilde{r}) -_H \tilde{v}\left((\tilde{r} - a_j e^i) \vee a_{j-1}e^i\right)\right)$$
$$-_H \sum_{\tilde{z} \leq \tilde{r} \leq \tilde{x}, k_{r_i}=j} h_{ij}^{\tilde{x}}(\tilde{r}_{\text{sub}}) \left(\tilde{v}(\tilde{r}) -_H \tilde{v}\left((\tilde{r} - a_j e^i) \vee a_{j-1}e^i\right)\right)$$
$$\geq \left(\sum_{\tilde{z} \leq \tilde{r} \leq \tilde{y}, k_{r_i}=j} h_{ij}^{\tilde{y}}(\tilde{r}_{\text{sub}}) - \sum_{\tilde{z} \leq \tilde{r} \leq \tilde{x}, k_{r_i}=j} h_{ij}^{\tilde{x}}(\tilde{r}_{\text{sub}})\right) (\tilde{v}(\tilde{z})$$
$$-_H \tilde{v}\left((\tilde{z} - a_j e^i) \vee a_{j-1}e^i\right)\Big), \tag{8}$$

where $\sum_{p \in \text{Supp}\, \tilde{x}} k_{x_p} - \sum_{p \in \text{Supp}\, \tilde{z}} k_{z_p} = h$ and $1 \leq h \leq \sum_{p \in \text{Supp}\, \tilde{x}} k_{x_p} - 1$. In the following, we show Eq. (8) for $\sum_{p \in \text{Supp}\, \tilde{x}} k_{x_p} - \sum_{p \in \text{Supp}\, \tilde{z}} k_{z_p} = h + 1$. Let

$$W = \left\{ w \in \mathbb{N} | \sum_{p \in \text{Supp}\, \tilde{x}} k_{x_p} - \sum_{p \in \text{Supp}\, \tilde{z}^w} k_{z_p^w} = h, \text{Supp}\, \tilde{z}^w \subseteq \text{Supp}\, \tilde{x}, \tilde{z}^w \in FM \right\}.$$

From the convexity of $\tilde{v}$, Eqs. (6) and (8), we have

$$\begin{aligned}&\sum_{\tilde{z}\le\tilde{r}\le\tilde{y},k_{r_i}=j} h_{ij}^{\tilde{y}}(\tilde{r}_{\text{sub}})\left(\tilde{v}(\tilde{r}) -_H \tilde{v}\left(\left(\tilde{r}-a_je^i\right)\vee a_{j-1}e^i\right)\right)\\&\quad -_H \sum_{\tilde{z}\le\tilde{r}\le\tilde{x},k_{r_i}=j} h_{ij}^{\tilde{x}}(\tilde{r}_{\text{sub}})\left(\tilde{v}(\tilde{r}) -_H \tilde{v}\left(\left(\tilde{r}-a_je^i\right)\vee a_{j-1}e^i\right)\right)\\&\ge\left(\sum_{\tilde{z}\le\tilde{r}\le\tilde{y},k_{r_i}=j} h_{ij}^{\tilde{y}}(\tilde{r}_{\text{sub}}) - \sum_{\tilde{z}\le\tilde{r}\le\tilde{x},k_{r_i}=j} h_{ij}^{\tilde{x}}(\tilde{r}_{\text{sub}})\right)(\tilde{v}(\tilde{z})\\&\quad -_H\tilde{v}\left(\left(\tilde{z}-a_je^i\right)\vee a_{j-1}e^i\right)).\end{aligned}$$

Thus,

$$\begin{aligned}&\sum_{\tilde{z}\le\tilde{r}\le\tilde{y},k_{r_i}=j} h_{ij}^{\tilde{y}}(\tilde{r}_{\text{sub}})\left(\tilde{v}(\tilde{r}) -_H \tilde{v}\left(\left(\tilde{r}-a_je^i\right)\vee a_{j-1}e^i\right)\right)\\&\quad -_H \sum_{\tilde{z}\le\tilde{r}\le\tilde{x},k_{r_i}=j} h_{ij}^{\tilde{x}}(\tilde{r}_{\text{sub}})\left(\tilde{v}(\tilde{r}) -_H \tilde{v}\left(\left(\tilde{r}-a_je^i\right)\vee a_{j-1}e^i\right)\right)\\&\ge\left(\sum_{\tilde{z}\le\tilde{r}\le\tilde{y},k_{r_i}=j} h_{ij}^{\tilde{y}}(\tilde{r}_{\text{sub}}) - \sum_{\tilde{z}\le\tilde{r}\le\tilde{x},k_{r_i}=j} h_{ij}^{\tilde{x}}(\tilde{r}_{\text{sub}})\right)(\tilde{v}(\tilde{z})\\&\quad -_H\tilde{v}\left(\left(\tilde{z}-a_je^i\right)\vee a_{j-1}e^i\right))\end{aligned}\tag{9}$$

for any $\tilde{z}\le\tilde{x}$.
From Eqs. (6), (7) and (9), for any $\tilde{z}\le\tilde{x}$ we get

$$\begin{aligned}&\sum_{\tilde{z}\le\tilde{r}\le\tilde{y},k_{r_i}=j} h_{ij}^{\tilde{y}}(\tilde{r}_{\text{sub}})\left(\tilde{v}(\tilde{r}) -_H \tilde{v}\left(\left(\tilde{r}-a_je^i\right)\vee a_{j-1}e^i\right)\right)\\&\quad -_H \sum_{\tilde{z}\le\tilde{r}\le\tilde{x},k_{r_i}=j} h_{ij}^{\tilde{x}}(\tilde{r}_{\text{sub}})\left(\tilde{v}(\tilde{r}) -_H \tilde{v}\left(\left(\tilde{r}-a_je^i\right)\vee a_{j-1}e^i\right)\right)\ge 0\end{aligned}\tag{10}$$

Since

$$\tilde{\psi}_{ia_j}(N,\tilde{m},\tilde{v}_{\tilde{x}}) = \sum_{\tilde{z}\le\tilde{x},k_{z_i}=j} h_{ij}^{\tilde{x}}(\tilde{z}_{\text{sub}})\left(\tilde{v}(\tilde{z}) -_H \tilde{v}\left(\tilde{z}-\left(a_j-a_{j-1}\right)e^i\right)\right)$$

and

$$\tilde{\psi}_{ia_j}\left(N,\tilde{m},v_{\tilde{y}}\right) = \sum_{\tilde{z}\le\tilde{y},k_{z_i}=j} h_{ij}^{\tilde{y}}(\tilde{z}_{\text{sub}})\left(\tilde{v}(\tilde{z}) -_H \tilde{v}\left(\tilde{z}-\left(a_j-a_{j-1}\right)e^i\right)\right),$$

from the arbitrariness of $\tilde{z}$ and Eq. (10) we obtain

$$\tilde{\psi}_{ia_j}(N, \tilde{m}, \tilde{v}_{\tilde{x}}) \leq \tilde{\psi}_{ia_j}(N, \tilde{m}, \tilde{v}_{\tilde{y}})$$

for any $i \in \text{Supp}\,\tilde{x}$ and all $0< j \leq k_{x_i}$.
② For all $\tilde{x}, \tilde{y} \in FM$ with $\tilde{x} \leq \tilde{y}$ and $k_{x_i} = k_{y_i}$, without out loss of generality, let

$$\sum_{p \in \text{Supp}\,\tilde{y}} k_{y_p} - \sum_{p \in \text{Supp}\,\tilde{x}} k_{x_p} = h,$$

where $1 \leq h \leq \sum_{p \in \text{Supp}\,\tilde{y}} m_p - \sum_{p \in \text{Supp}\,\tilde{x}} k_{x_p}$.
From ①, we have

$$\tilde{\psi}_{ia_j}\left(N, m, \tilde{v}_{\tilde{x}^1}\right) \leq \tilde{\psi}_{ia_j}\left(N, m, \tilde{v}_{\tilde{x}^2}\right) \leq \ldots \leq \tilde{\psi}_{ia_j}\left(N, m, \tilde{v}_{\tilde{x}^h}\right),$$

where $\tilde{x}^1 = \tilde{x}$, $\tilde{x}^h = \tilde{y}$, $\sum_{i \in \text{Supp}\,\tilde{x}^l} k_{x_i^l} + 1 = \sum_{i \in \text{Supp}\,\tilde{x}^{l+1}} k_{x_i^{l+1}}$ and $\text{Supp}\,\tilde{x}^l = \text{Supp}\tilde{x}$ for any $l \in \{1, 2, \ldots, h-1\}$.
Thus, for any $i \in \text{Supp}\,\tilde{x}$ and any $0 < j \leq k_{x_i}$, $\tilde{\psi}_{ia_j}(N, \tilde{m}, \tilde{v}_{\tilde{x}}) \leq \tilde{\psi}_{ia_j}(N, \tilde{m}, \tilde{v}_{\tilde{y}})$, where $\sum_{p \in \text{Supp}\,\tilde{y}} k_{y_p} - \sum_{p \in \text{Supp}\,\tilde{x}} k_{x_p} = h$ and $1 \leq h \leq \sum_{p \in \text{Supp}\,\tilde{y}} m_p - \sum_{p \in \text{Supp}\,\tilde{x}} k_{x_p}$.

Case (II): If $\text{Supp}\,\tilde{x} \subset \text{Supp}\,\tilde{y}$ and $k_{x_q} = k_{y_q}$ for any $q \in \text{Supp}\,\tilde{x}$.

(1) When $\text{Supp}\,\tilde{x} \cup \{l\} = \text{Supp}\tilde{y}$ and $k_{y_l} = 1$, where $l \in N \backslash \text{Supp}\,\tilde{x}$. Since

$$h_{ij}^{\tilde{x}}(\tilde{z}) = h_{ij}^{\tilde{y}}(\tilde{z}) + h_{ij}^{\tilde{y}}(\tilde{z} \vee a_1 e^k)$$

for any $i \in \text{Supp}\,\tilde{x}$, any $0 < j \leq k_{x_i}$ and any $\tilde{z} \leq \tilde{x}$, we have

$$\begin{aligned}\tilde{\psi}_{ia_j}\left(N, \tilde{m}, \tilde{v}_{\tilde{y}}\right) &= \sum_{\tilde{z} \leq \tilde{y}, k_{z_i} = j} h_{ij}^{\tilde{y}}\left(\tilde{z}_{\text{sub}}\right)\left(\tilde{v}\left(\tilde{z}\right) -_H \tilde{v}\left(\tilde{z} - \left(a_j - a_{j-1}\right) e^i\right)\right) \\ &\geq \sum_{\tilde{z} \leq \tilde{x}, k_{z_i} = j} h_{ij}^{\tilde{x}}\left(\tilde{z}_{\text{sub}}\right)\left(\tilde{v}\left(\tilde{z}\right) -_H \tilde{v}\left(\tilde{z} - \left(a_j - a_{j-1}\right) e^i\right)\right) \\ &= \tilde{\psi}_{ia_j}\left(N, \tilde{m}, \tilde{v}_{\tilde{x}}\right)\end{aligned}$$

for any $i \in \text{Supp}\,\tilde{x}$ and any $0 < j \leq k_{x_i}$.

(2) When $\text{Supp}\,\tilde{x} \cup \{l\} = \text{Supp}\tilde{y}$ and $k_{y_l} = h$, where $l \in N \backslash \text{Supp}\,\tilde{x}$ and $2 \leq h \leq m_k$. From case (I), it gets $\tilde{\psi}_{ia_j}(N, \tilde{m}, \tilde{v}_{\tilde{x}}) \leq \tilde{\psi}_{ia_j}(N, \tilde{m}, \tilde{v}_{\tilde{y}})$ for any$i \in \text{Supp}\,\tilde{x}$ and any $0 < j \leq k_{x_i}$.

(3) When $\text{Supp}\,\tilde{x} \cup \{l_p\}_{p \in P} = \text{Supp}\,\tilde{y}$, where $1 \leq k_{y_{l_p}} \leq m_{l_p}$, $P \subseteq \{1, 2, \ldots, n\text{-}\text{Supp}\,\tilde{x}\}$ and $l_p \in N \backslash \text{Supp}\,\tilde{x}$ for any $p \in P$. $|\text{Supp}\,\tilde{x}|$ is the cardinality of Supp $\tilde{x}$. From case (I), (1) and (2), for any $i \in$Supp$\tilde{x}$ and any $0 < j \leq k_{x_i}$, we have

$$\tilde{\psi}_{ia_j}(N, \tilde{m}, \tilde{v}_{\tilde{x}}) \leq \tilde{\psi}_{ia_j}(N, \tilde{m}, \tilde{v}_{\tilde{y}}),$$

where Supp $\tilde{x} \subset$ Supp $\tilde{y}$ and $k_{x_q} = k_{y_q}$ for any $q \in$ Supp $\tilde{x}$.

Case (III): If Supp $\tilde{x} \subset$ Supp $\tilde{y}$, and there exists $q \in$ Supp $\tilde{x}$ such that $k_{x_q} < k_{y_q}$, where $q \neq i$. From cases (I) and (II), one can easily get the result. The proof is finished. □

Theorem 3.2 *Let $\tilde{v} \in FMC^N$ be convex, and $\tilde{v}(a_j e^i) \geq \tilde{v}(a_{j-1} e^i)$ for any $i \in N$ and any $j \in \{1, 2, \ldots, m_i\}$, then the vector $\left(\tilde{\varphi}_{ia_j}(N, \tilde{m}, \tilde{v})\right)_{i \in N, j \in \{1,2,\ldots,m_i\}}$ is a FMPMAS.*

Proof Since $\tilde{\varphi}_{ia_j}(N, \tilde{m}, \tilde{v}) = \sum_{1 \leq k \leq j} \tilde{\psi}_{ia_k}(N, \tilde{m}, \tilde{v})$ for any $i \in N$ and any $j \in \{1, 2, \ldots, m_i\}$. From Lemma 3.3, the conclusion is obtained. □

From Theorem 3.2, we know Eq. (5) degenerates to be a FPMAS, when we limit the domain of $\tilde{v} \in FMC^N$ in the framework of traditional fuzzy games. Moreover, Eq. (5) degenerates to be a PMAS, when the domain of $\tilde{v} \in FMC^N$ is restricted in the framework of traditional games.

Theorem 3.3 *Let $\tilde{v} \in FMC^N$ be convex, then the vector $\left(\tilde{\varphi}_{ia_j}(N, \tilde{m}, \tilde{v})\right)_{i \in M, j \in \{1,2,\ldots,m_i\}} \in C(N, \tilde{m}, \tilde{v})$, where $\tilde{\varphi}(N, \tilde{m}, \tilde{v})$ as shown in Eq. (5).*

Proof From the relationship between the Shapley value and the core for multichoice games discussed by van den Nouweland et al. (1995), for any $\tilde{x} \in FM$, we have

$$\sum_{i \in N} \sum_{j=1}^{m_i} \tilde{\psi}_{ia_j}(N, \tilde{m}, \tilde{v}) = \tilde{v}(\tilde{m})$$

and

$$\sum_{i \in \text{Supp}\, \tilde{x}} \sum_{j=1}^{k_{x_i}} \tilde{\psi}_{ia_j}(N, \tilde{m}, \tilde{v}) \geq \tilde{v}(\tilde{x}),$$

where $\tilde{\psi}(N, \tilde{m}, \tilde{v})$ as shown in Eq. (4).
For any $i \in N$ and any $j \in \{1, 2, \ldots, m_i\}$, by $\tilde{\varphi}_{ia_j}(N, \tilde{m}, \tilde{v}) = \sum_{1 \leq k \leq j} \tilde{\psi}_{ia_k}(N, \tilde{m}, \tilde{v})$ we get

$$\sum_{i \in N} \tilde{\varphi}_{ia_{m_i}}(N, \tilde{m}, \tilde{v}) = \tilde{v}(\tilde{m})$$

and

$$\sum_{i \in \text{Supp}\, \tilde{x}} \tilde{\varphi}_{ix_i}(N, \tilde{m}, \tilde{v}) \geq \tilde{v}(\tilde{x})$$

for any $\tilde{x} \in FM$.
Namely, $\left(\tilde{\varphi}_{ia_j}(N, \tilde{m}, \tilde{v})\right)_{i \in M, j \in \{1,2,\ldots,m_i\}} \in C(N, \tilde{m}, \tilde{v})$. □

Corollary 3.1 *Let* $\tilde{v} \in FMC^N$ *be convex, then the vector* $\left(\tilde{\varphi}_{ia_j}(N, \tilde{m}, \tilde{v})\right)_{i \in M, j \in \{1,2,\ldots,m_i\}}$ *is an imputation of* $\tilde{v} \in FMC^N$.

Corollary 3.2 *Let* $\tilde{v} \in FMC^N$ *be convex, then* $C(N, \tilde{m}, \tilde{v}) \neq \varnothing$.

4 Two Special Cases

In this section, we mainly discuss two kinds of FMC^N named fuzzy multichoice games with multilinear extension form and fuzzy characteristic functions and fuzzy multichoice games with Choquet integral form and fuzzy characteristic functions. These two classes of FMC^N are extensions of fuzzy games proposed by Meng and Zhang (2010) and Tsurumi et al. (2001), respectively. The fuzzy coalition values for these two kinds of fuzzy games (Meng and Zhang 2010; Tsurumi et al. 2001) are written as:

$$v(U) = \sum_{T_0 \subseteq \mathrm{Supp} U} \left\{\Pi_{i \in T_0} U(i) \Pi_{i \in \mathrm{Supp} U \setminus T_0} (1 - U(i))\right\} v_0(T_0),$$

$$v(U) = \sum_{l=1}^{q(U)} v_0([U]_{h_l})(h_l - h_{l-1}),$$

where U is a fuzzy coalition given in (Aubin 1974), T_0 is a crisp coalition as usual. $Q(U) = \{U(i) \,|U(i) > 0, i \in N\}$ and $q(U) = |Q(U)|$, The elements in $Q(U)$ are written in the increasing order as $0 = h_0 \leq h_1 \leq \cdots \leq h_{q(U)}$ and $[U]_{h_l} = \{i|U(i) \geq h_l,\ i \in N,\ l = 1, 2, \ldots, q(U)\}$, and v_0 is a crisp game defined in N.

4.1 Fuzzy Multichoice Games with Multilinear Extension Form and Fuzzy Characteristic Functions

In the following, we discuss fuzzy multichoice games with multilinear extension form and fuzzy characteristic functions. By $OFMC^N$, we denote this class of games. According to Meng and Zhang (2010), the fuzzy coalition value for $OFMC^N$ is given as follows:

$$\tilde{v}(\tilde{x}) = \sum_{T_0 \subseteq \mathrm{Supp}\, \tilde{x}} (\Pi_{i \in T_0} x_i \Pi_{i \in \mathrm{Supp}\, \tilde{x} \setminus T_0} (1 - x_i))\tilde{v}_0(T_0), \tag{11}$$

where $\tilde{x} = (x_1, x_2, \ldots, x_n) \in FM$ and $\tilde{v} \in OFMC^N$.

Theorem 4.1 *Define a function* $\tilde{\varphi}^O : OFMC^N \to \tilde{\mathbb{R}}^{\sum_{i \in N} m_i}$ *by*

$$\begin{aligned}
&\tilde{\varphi}^O_{ia_j}(N, \tilde{m}, \tilde{v}) \\
&= \sum_{1 \leq p \leq j} \sum_{\tilde{x} \in FM, k_{x_i} = p} h_{ip}(\tilde{x}_{\mathrm{sub}}) \left(\sum_{T_0 \subseteq \mathrm{Supp}\, \tilde{x}} (\Pi_{k \in T_0} x_k \Pi_{k \in \mathrm{Supp}\, \tilde{x} \setminus T_0} (1 - x_k))\tilde{v}_0(T_0) \right. \\
&\left. -_H \sum_{T_0 \subseteq \mathrm{Supp}\, \tilde{x}'} (\Pi_{k \in T_0} x'_k \Pi_{k \in \mathrm{Supp}\, \tilde{x}' \setminus T_0} (1 - x'_k))\tilde{v}_0(T_0) \right) \ \forall i \in N,\ j \in \{1, 2, \ldots, m_i\},
\end{aligned} \tag{12}$$

where $\tilde{x}' = \tilde{x} - (a_p - a_{p-1})e^i$, x'_k *is the participation level of player* $k \in N$ *in fuzzy coalition* $\tilde{x}'$, $h_{ip}(\tilde{x}_{\text{sub}})$ *is the potential weight for fuzzy coalition* $\tilde{x}$ *as shown in Eq.* (4), *and* $\tilde{v}_0$ *is the associated game with fuzzy characteristic functions defined in* N. *Then* $\tilde{\varphi}^O$ *is the unique Shapley function on* $OFMC^N$.

Proof The proof Theorem 4.1 is similar to that of Theorem 3.1. □

Obviously, Eq. (12) degenerates to be the Shapley value for fuzzy games with multilinear extension form when the domain of $\tilde{v} \in OFMC^N$ is restricted in setting of it, namely,

$$\begin{aligned} Sh_i(U, v) = &\sum_{S \subseteq U, i \in \text{Supp}S} \frac{(|\text{Supp}S| - 1)!(|\text{Supp}U| - |\text{Supp}S|)!}{|\text{Supp}U|!} \\ &\times \left(\sum_{T_0 \subseteq \text{Supp}S} \left\{ \Pi_{j \in T_0} U(j) \Pi_{j \in \text{Supp}S \setminus T_0} (1 - U(j)) \right\} \right. \\ &\left. \times\ v_0(T_0) - \sum_{T_0 \subseteq \text{Supp}S \setminus \{i\}} \left\{ \Pi_{j \in T_0} U(j) \Pi_{j \in \text{Supp}S \setminus \{T_0 \cup i\}} (1 - U(j)) \right\} v_0(T_0) \right) \\ &\forall i \in \text{Supp } U, \end{aligned} \tag{13}$$

where U is a fuzzy coalition introduced by Aubin (1974), v is a fuzzy games with multilinear extension form. $S \subseteq U$ if and only if $S(i) = U(i)$ or $S(i) = 0$ for any $i \in \text{Supp}U = \{i \in N | U(i) \quad > 0\}$. $|\text{Supp}U|$ and $|\text{Supp}S|$ denote the cardinalities of $\text{Supp}U$ and $\text{Supp}S$, respectively. Similar to Definition 3.2, one can get the associated axiomatic system for Eq. (13), and show its existence and uniqueness.

It is worth pointing out that the Shapley value for fuzzy games with multilinear extension form, given by Eq. (13), is different to that proposed by Meng and Zhang (2010).

Theorem 4.2 *If the associated game* $\tilde{v}_0$ *of* $\tilde{v} \in OFMC^N$ *is convex, then the vector* $\left(\tilde{\varphi}^O_{ia_j}(N, \tilde{m}, \tilde{v})\right)_{i \in N, j \in \{1,2,\ldots,m_i\}}$ *is an imputation of* $\tilde{v} \in OFMC^N$, *where* $\tilde{\varphi}^O(N, \tilde{m}, \tilde{v})$ *as shown in Eq.* (12).

Proof From Definition 2.7 and Theorem 4.1, we only need to show $\tilde{\varphi}^O_{ia_j}(N, \tilde{m}, \tilde{v}) \geq \tilde{v}(a_j e^i)$ for any $i \in N$ and any $j \in \{1, 2, \ldots, m_i\}$. From the convexity of the associated game $\tilde{v}_0$ and Eq. (12), we have

$$\begin{aligned} \tilde{\varphi}^O_{ia_j}(N, \tilde{m}, \tilde{v}) &\geq \sum_{1 \leq p \leq j} \sum_{\tilde{x} \in FM, k_{x_i} = p} h_{ip}(\tilde{x}_{\text{sub}}) \tilde{v}_0(i)(x_i - x_i^{h-1}) \\ &= \sum_{1 \leq p \leq j} \sum_{\tilde{x} \in FM, k_{x_i} = p} h_{ip}(\tilde{x}_{\text{sub}}) \tilde{v}_0(i)(a_p - a_{p-1}) \\ &= \sum_{1 \leq d \leq j} \tilde{v}_0(i)(a_d - a_{d-1}) \end{aligned}$$

$$= \tilde{v}_0(i)(a_j - a_0)$$
$$= \tilde{v}(a_j e^i),$$

where $a_0 = h_0 = 0$. □

In the following, we discuss FMPMAS in $OFMC^N$. Let us show the next lemmas preliminary to the following contents.

Lemma 4.1 *If the associated game $\tilde{v}_0$ of $\tilde{v} \in OFMC^N$ is convex, then we have $\tilde{v}(\tilde{x}) -_H \tilde{v}(\tilde{x}') \geq 0$ for any $\tilde{x} \in FM$, any $i \in N$ and any $j \in \{1, 2, \ldots, m_i\}$, where $\tilde{x}' = \tilde{x} - (a_j - a_{j-1})e^i$ and $k_{x_i} = j$.*

Proof From Eq. (11), we have

$$\tilde{v}(\tilde{x}) -_H \tilde{v}(\tilde{x}') = \sum_{T_0 \subseteq \mathrm{Supp}\,\tilde{x}} \left(\Pi_{k\in T_0} x_k \Pi_{k\in \mathrm{Supp}\,\tilde{x}\setminus T_0} (1 - x_k)\right)\tilde{v}_0(T_0)$$
$$-_H \sum_{T_0 \subseteq \mathrm{Supp}\,\tilde{x}'} \left(\Pi_{k\in T_0} x'_k \Pi_{i\in \mathrm{Supp}\,\tilde{x}'\setminus T_0} \left(1 - x'_k\right)\right)\tilde{v}_0(T_0)$$
$$= \sum_{T_0 \subseteq \mathrm{Supp}\,\tilde{x}\setminus\{i\}} x_i \left(\Pi_{k\in T_0} x_k \Pi_{k\in \mathrm{Supp}\,\tilde{x}\setminus\{T_0\cup i\}} (1 - x_k)\right)\tilde{v}_0(T_0 \cup i)$$
$$+ \sum_{T_0 \subseteq \mathrm{Supp}\,\tilde{x}\setminus\{i\}} (1 - x_i)\left(\Pi_{k\in T_0} x_k \quad \Pi_{k\in \mathrm{Supp}\,\tilde{x}\setminus\{T_0\cup i\}} (1 - x_k)\right)\tilde{v}_0(T_0)$$
$$-_H \sum_{T_0 \subseteq \mathrm{Supp}\,\tilde{x}'\setminus\{i\}} x'_i \left(\Pi_{k\in T_0} x'_k \Pi_{k\in \mathrm{Supp}\,\tilde{x}'\setminus\{T_0\cup i\}} \left(1 - x'_k\right)\right)$$
$$\tilde{v}_0(T_0 \cup i) -_H \sum_{T_0 \subseteq \mathrm{Supp}\,\tilde{x}'\setminus\{i\}} \left(1 - x'_i\right)\left(\Pi_{k\in T_0} x'_k \Pi_k \in \mathrm{Supp}\,\tilde{x}' \setminus \{T_0 \cup i\} \left(1 - x'_k\right)\right)\tilde{v}_0(T_0)$$
$$= \sum_{T_0 \subseteq \mathrm{Supp}\,\tilde{x}\setminus\{i\}} x_i \left(\Pi_{k\in T_0} x_k \Pi_{k\in \mathrm{Supp}\,\tilde{x}\setminus\{T_0\cup i\}} (1 - x_k)\right)\tilde{v}_0(T_0 \cup i)$$
$$+ \sum_{T_0 \subseteq \mathrm{Supp}\,\tilde{x}\setminus\{i\}} (1 - x_i)\left(\Pi_{k\in T_0} x_k \quad \Pi_{k\in \mathrm{Supp}\,\tilde{x}\setminus\{T_0\cup i\}} (1 - x_k)\right)\tilde{v}_0(T_0)$$
$$-_H \sum_{T_0 \subseteq \mathrm{Supp}\,\tilde{x}\setminus\{i\}} x'_i \left(\Pi_{k\in T_0} x_k \Pi_{k\in \mathrm{Supp}\,\tilde{x}\setminus\{T_0\cup i\}} (1 - x_k)\right)$$
$$\times \tilde{v}_0(T_0 \cup i) -_H \sum_{T_0 \subseteq \mathrm{Supp}\,\tilde{x}\setminus\{i\}} \left(1 - x'_i\right)\left(\Pi_{k\in T_0} x_k \Pi_{k\in \mathrm{Supp}\,\tilde{x}\setminus\{T_0\cup i\}} (1 - x_k)\right)\tilde{v}_0(T_0)$$
$$= \sum_{T_0 \subseteq \mathrm{Supp}\setminus\{i\}} \left(x_i - x'_i\right)\left(\Pi_{k\in T_0} x_k \Pi_{k\in \mathrm{Supp}\,\tilde{x}\setminus\{T_0\cup i\}} (1 - x_k)\right)\tilde{v}_0(T_0 \cup i)$$
$$+ \sum_{T_0 \subseteq \mathrm{Supp}\,\tilde{x}\setminus\{i\}} \left(x'_i - x_i\right)\left(\Pi_{k\in T_0} x_k \Pi_{k\in \mathrm{Supp}\,\tilde{x}\setminus\{T_0\cup i\}} (1 - x_k)\right)\tilde{v}_0(T_0)$$
$$= \sum_{T_0 \subseteq \mathrm{Supp}\,\tilde{x}\setminus\{i\}} \left(x_i - x'_i\right)\left(\Pi_{k\in T_0} x_k \Pi_{k\in \mathrm{Supp}\,\tilde{x}\setminus\{T_0\cup i\}} (1 - x_k)\right)\left(\tilde{v}_0(T_0 \cup i) -_H \tilde{v}_0(T_0)\right).$$

From $x_i - x'_i = (a_j - a_{j-1})e^i > 0$ and the convexity of the associated game $\tilde{v}_0$, we get $\tilde{v}(\tilde{x}) -_H \tilde{v}(\tilde{x}') \geq 0$. □

Lemma 4.2 *Let* $\tilde{x}, \tilde{y} \in FM$ *such that* $k_{x_q} + 1 = k_{y_q}$ *for some* $q \in Supp\tilde{x}$ *and* $k_{x_i} = k_{y_i}$ *for any* $i \in Supp\tilde{x}\backslash\{q\}$, *if the associated game* $\tilde{v}_0$ *of* $\tilde{v} \in OFMC^N$ *is convex, then we have*

$$\tilde{v}(\tilde{y}) -_H \tilde{v}(\tilde{y}') \geq \tilde{v}(\tilde{x}) -_H \tilde{v}(\tilde{x}'),$$

where $Supp\tilde{x} = Supp\tilde{y}$, $k_{x_i} = k_{y_i} = j$, $\tilde{x}' = \tilde{x} - (a_j - a_{j-1})e^i$, $\tilde{y}' = \tilde{y} - (a_j - a_{j-1})e^i$ *and* $j \in \{1, 2, \ldots, m_i\}$.

Proof From Lemma 4.1, we obtain

$$\begin{aligned}
\tilde{v}(\tilde{y}) -_H \tilde{v}(\tilde{y}') &= \sum_{T_0 \subseteq \text{Supp}\,\tilde{y}\backslash\{i\}} \left(y_i - y_i'\right) \left(\Pi_{k \in T_0} y_k \Pi_{k \in \text{Supp}\,\tilde{y}\backslash\{T_0 \cup i\}}\right.\\
&\quad \left.\times (1 - y_k)\right) \left(\tilde{v}_0 (T_0 \cup i) -_H \tilde{v}_0 (T_0)\right)\\
&= \sum_{T_0 \subseteq \text{Supp}\,\tilde{y}\backslash\{i,q\}} y_q \left(y_i - y_i'\right) \left(\Pi_{k \in T_0} y_k \Pi_{k \in \text{Supp}\,\tilde{y}\backslash\{T_0 \cup \{i,q\}\}}\right.\\
&\quad \left.\times (1 - y_k)\right) \left(\tilde{v}_0 (T_0 \cup \{i, q\}) -_H \tilde{v}_0 (T_0 \cup q)\right)\\
&\quad + \sum_{T_0 \subseteq \text{Supp}\,\tilde{y}\backslash\{i,q\}} \left(1 - y_q\right) \left(y_i - y_i'\right) \left(\Pi_{k \in T_0} y_k \Pi_{k \in \text{Supp}\,\tilde{y}\backslash\{T_0 \cup \{i,q\}\}}\right.\\
&\quad \left.\times (1 - y_k)\right) \left(\tilde{v}_0 (T_0 \cup i) -_H \tilde{v}_0 (T_0)\right)\\
&= \sum_{T_0 \subseteq \text{Supp}\,\tilde{x}\backslash\{i,q\}} y_q \left(x_i - x_i'\right) \left(\Pi_{k \in T_0} x_k \Pi_{k \in \text{Supp}\,\tilde{x}\backslash\{T_0 \cup \{i,q\}\}}\right.\\
&\quad \left.\times (1 - x_k)\right) \left(\tilde{v}_0 (T_0 \cup \{i, q\}) -_H \tilde{v}_0 (T_0 \cup q)\right)\\
&\quad + \sum_{T_0 \subseteq \text{Supp}\,\tilde{x}\backslash\{i,q\}} \left(1 - y_q\right) \left(x_i - x_i'\right) \left(\Pi_{k \in T_0} x_k \Pi_{k \in \text{Supp}\,\tilde{x}\backslash\{T_0 \cup \{i,q\}\}}\right.\\
&\quad \left.\times (1 - x_k)\right) \left(\tilde{v}_0 (T_0 \cup i) -_H \tilde{v}_0 (T_0)\right)
\end{aligned}$$

and

$$\begin{aligned}
&\tilde{v}(\tilde{x}) -_H \tilde{v}(\tilde{x}')\\
&\quad = \sum_{T_0 \subseteq \text{Supp}\,\tilde{x}\backslash\{i\}} \left(x_i - x_i'\right) \left(\Pi_{k \in T_0} x_k \Pi_{k \in \text{Supp}\,\tilde{x}\backslash\{T_0 \cup i\}}\right.\\
&\qquad \left.\times (1 - x_k)\right) \left(\tilde{v}_0 (T_0 \cup i) -_H \tilde{v}_0 (T_0)\right)\\
&\quad = \sum_{T_0 \subseteq \text{Supp}\,\tilde{x}\backslash\{i,q\}} x_q \left(x_i - x_i'\right) \left(\Pi_{k \in T_0} x_k \Pi_{k \in \text{Supp}\,\tilde{x}\backslash\{T_0 \cup \{i,q\}\}}\right.\\
&\qquad \left.\times (1 - x_k)\right) \left(\tilde{v}_0 (T_0 \cup \{i, q\}) -_H \tilde{v}_0 (T_0 \cup q)\right)\\
&\qquad + \sum_{T_0 \subseteq \text{Supp}\,\tilde{x}\backslash\{i,q\}} \left(1 - x_q\right) \left(x_i - x_i'\right) \left(\Pi_{k \in T_0} x_k \Pi_{k \in \text{Supp}\,\tilde{x}\backslash\{T_0 \cup \{i,q\}\}}\right.\\
&\qquad \left.\times (1 - x_k)\right) \left(\tilde{v}_0 (T_0 \cup i) -_H \tilde{v}_0 (T_0)\right).
\end{aligned}$$

Thus,

$$\begin{aligned}
&\tilde{v}\left(\tilde{y}\right)-_{H}\tilde{v}\left(\tilde{y}'\right)-_{H}\left(\tilde{v}\left(\tilde{x}\right)-_{H}\tilde{v}\left(\tilde{x}'\right)\right)\\
&=\sum_{T_0\subseteq \mathrm{Supp}\,\tilde{x}\backslash\{i,q\}}\left(y_q-x_q\right)\left(x_i-x_i'\right)\left(\Pi_{k\in T_0}x_k\Pi_{k\in \mathrm{Supp}\,\tilde{x}\backslash\{T_0\cup\{i,q\}\}}\right.\\
&\quad\times(1-x_k))\left(\tilde{v}_0\left(T_0\cup\{i,q\}\right)-_{H}\tilde{v}_0\left(T_0\cup q\right)\right)\\
&\quad+\sum_{T_0\subseteq \mathrm{Supp}\,\tilde{x}\backslash\{i,q\}}\left(x_q-y_q\right)\left(x_i-x_i'\right)\left(\Pi_{k\in T_0}x_k\Pi_{k\in \mathrm{Supp}\,\tilde{x}\backslash\{T_0\cup\{i,q\}\}}\right.\\
&\quad\times(1-x_k))\left(\tilde{v}_0\left(T_0\cup i\right)-_{H}\tilde{v}_0\left(T_0\right)\right)\\
&=\sum_{T_0\subseteq \mathrm{Supp}\,\tilde{x}\backslash\{i,q\}}\left(y_q-x_q\right)\left(x_i-x_i'\right)\left(\Pi_{k\in T_0}x_k\Pi_{k\in \mathrm{Supp}\,\tilde{x}\backslash\{T_0\cup\{i,q\}\}}\right.\\
&\quad\times(1-x_k))\left(\tilde{v}_0\left(T_0\cup\{i,q\}\right)+\tilde{v}_0\left(T_0\right)-_{H}\tilde{v}_0\left(T_0\cup i\right)-_{H}\tilde{v}_0\left(T_0\cup q\right)\right).
\end{aligned}$$

From $k_{x_q} < k_{y_q}$, we have $y_q - x_q > 0$. From $x_i - x_i' = (a_j - a_{j-1})e^i > 0$ and the convexity of the associated game $\tilde{v}_0$, we get $\tilde{v}(\tilde{y}) -_H \tilde{v}(\tilde{y}') \geq \tilde{v}(\tilde{x}) -_H \tilde{v}(\tilde{x}')$. □

Lemma 4.3 *Let $\tilde{x}, \tilde{y} \in FM$ such that $k_{x_q} = k_{y_q}$ for any $q \in Supp\tilde{x}$ and $Supp\tilde{x}\cup\{l\} = Supp\tilde{y}$ with $k_{y_l} = 1$, if the associated game $\tilde{v}_0$ of $\tilde{v} \in OFMC^N$ is convex, then we have*

$$\tilde{v}(\tilde{y}) -_H \tilde{v}(\tilde{y}') \geq \tilde{v}(\tilde{x}) -_H \tilde{v}(\tilde{x}'),$$

where $k_{x_i} = k_{y_i} = j, \tilde{x}' = \tilde{x} - (a_j - a_{j-1})e^i, \tilde{y}' = \tilde{y} - (a_j - a_{j-1})e^i$ and $j \in \{1, 2, \ldots, m_i\}$.

Proof From Lemma 4.1, we have

$$\begin{aligned}
&\tilde{v}\left(\tilde{y}\right)-_{H}\tilde{v}\left(\tilde{y}'\right)\\
&=\sum_{T_0\subseteq \mathrm{Supp}\,\tilde{y}\backslash\{i\}}\left(y_i-y_i'\right)\left(\Pi_{k\in T_0}y_k\Pi_{k\in \mathrm{Supp}\,\tilde{y}\backslash\{T_0\cup i\}}\right.\\
&\quad\times(1-y_k))\left(\tilde{v}_0\left(T_0\cup i\right)-_{H}\tilde{v}_0\left(T_0\right)\right)\\
&=\sum_{T_0\subseteq \mathrm{Supp}\,\tilde{y}\backslash\{i,l\}}y_l\left(y_i-y_i'\right)\left(\Pi_{k\in T_0}y_k\Pi_{k\in \mathrm{Supp}\,\tilde{y}\backslash\{T_0\cup\{i,l\}\}}\right.\\
&\quad\times(1-y_k))\left(\tilde{v}_0\left(T_0\cup\{i,l\}\right)-_{H}\tilde{v}_0\left(T_0\cup l\right)\right)\\
&\quad+\sum_{T_0\subseteq \mathrm{Supp}\,\tilde{y}\backslash\{i,l\}}\left(1-y_l\right)\left(y_i-y_i'\right)\left(\Pi_{k\in T_0}y_k\Pi_{k\in \mathrm{Supp}\,\tilde{y}\backslash\{T_0\cup\{i,l\}\}}\right.\\
&\quad\times(1-y_k))\left(\tilde{v}_0\left(T_0\cup i\right)-_{H}\tilde{v}_0\left(T_0\right)\right)\\
&=\sum_{T_0\subseteq \mathrm{Supp}\,\tilde{y}\backslash\{i,l\}}y_l\left(x_i-x_i'\right)\left(\Pi_{k\in T_0}x_k\Pi_{k\in \mathrm{Supp}\,\tilde{y}\backslash\{T_0\cup\{i,l\}\}}\right.\\
&\quad\times(1-x_k))\left(\tilde{v}_0\left(T_0\cup\{i,l\}\right)-_{H}\tilde{v}_0\left(T_0\cup l\right)\right)\\
&\quad+\sum_{T_0\subseteq \mathrm{Supp}\,\tilde{y}\backslash\{i,l\}}\left(1-y_l\right)\left(x_i-x_i'\right)\left(\Pi_{k\in T_0}x_k\Pi_{k\in \mathrm{Supp}\,\tilde{y}\backslash\{T_0\cup\{i,l\}\}}\right.
\end{aligned}$$

$$\times\left(1-x_k\right)\right)\left(\tilde{v}_0\left(T_0\cup i\right)-_H\tilde{v}_0\left(T_0\right)\right)$$
$$=\sum_{T_0\subseteq \operatorname{Supp}\tilde{x}\backslash\{i\}} y_l\left(x_i-x_i'\right)\left(\Pi_{k\in T_0}x_k\Pi_{k\in \operatorname{Supp}\tilde{x}\backslash\{T_0\cup i\}}\right.$$
$$\times\left(1-x_k\right)\right)\left(\tilde{v}_0\left(T_0\cup\{i,l\}\right)-_H\tilde{v}_0\left(T_0\cup l\right)\right)$$
$$+\sum_{T_0\subseteq \operatorname{Supp}\tilde{x}\backslash\{i\}}\left(1-y_l\right)\left(x_i-x_i'\right)\left(\Pi_{k\in T_0}x_k\Pi_{k\in \operatorname{Supp}\tilde{x}\backslash\{T_0\cup i\}}\right.$$
$$\times\left(1-x_k\right)\right)\left(\tilde{v}_0\left(T_0\cup i\right)-_H\tilde{v}_0\left(T_0\right)\right)$$
$$\geq\sum_{T_0\subseteq \operatorname{Supp}\tilde{x}\backslash\{i\}} y_l\left(x_i-x_i'\right)\left(\Pi_{k\in T_0}x_k\Pi_{k\in \operatorname{Supp}\tilde{x}\backslash\{T_0\cup i\}}\right.$$
$$\times\left(1-x_k\right)\right)\left(\tilde{v}_0\left(T_0\cup i\right)-_H\tilde{v}_0\left(T_0\right)\right)$$
$$+\sum_{T_0\subseteq \operatorname{Supp}\tilde{x}\backslash\{i\}}\left(1-y_l\right)\left(x_i-x_i'\right)\left(\Pi_{k\in T_0}x_k\Pi_{k\in \operatorname{Supp}\tilde{x}\backslash\{T_0\cup i\}}\right.$$
$$\times\left(1-x_k\right)\right)\left(\tilde{v}_0\left(T_0\cup i\right)-_H\tilde{v}_0\left(T_0\right)\right)$$
$$=\sum_{T_0\subseteq \operatorname{Supp}\tilde{x}\backslash\{i\}}\left(x_i-x_i'\right)\left(\Pi_{k\in T_0}x_k\Pi_{k\in \operatorname{Supp}\tilde{x}\backslash\{T_0\cup i\}}\right.$$
$$\times\left(1-x_k\right)\right)\left(\tilde{v}_0\left(T_0\cup i\right)-_H\tilde{v}_0\left(T_0\right)\right)$$
$$=\tilde{v}\left(\tilde{x}\right)-_H\tilde{v}\left(\tilde{x}'\right).$$

□

Theorem 4.3 *If the associated game $\tilde{v}_0$ of $\tilde{v}\in OFMC^N$ is convex, then the vector $\left(\tilde{\varphi}^O_{ia_j}(N,\tilde{m},\tilde{v})\right)_{i\in N, j\in\{1,2,\ldots,m_i\}}$ is a FMPMAS for $\tilde{v}\in OFMC^N$, where $\tilde{\varphi}^O(N,\tilde{m},\tilde{v})$ as shown in Eq. (12).*

Proof From Theorem 3.2, Lemmas 4.1, 4.2 and 4.3, the conclusion is obtained. □

Obviously, Eq. (12) degenerates to be a FPMAS for fuzzy games with multilinear extension form when we limit the domain of $\tilde{v}\in OFMC^N$ in the framework of it, and its associated crisp game is convex.

In the following we show that Eq. (12) is an element in the core for $OFMC^N$ when the associated game $\tilde{v}_0$ of $\tilde{v}\in OFMC^N$ is convex.

Definition 4.1 Let $\tilde{v}\in OFMC^N$, the core $C_O(N,\tilde{m},\tilde{v})$ of $\tilde{v}$ is denoted by

$$C_O(N,\tilde{m},\tilde{v})=\left\{\tilde{w}\in\tilde{\mathbb{R}}^{\sum_{i\in N}m_i}\mid\sum_{i\in N}\tilde{w}_{ia_{m_i}}\right.$$
$$=\sum_{T_0\subseteq \operatorname{Supp}\tilde{m}}(\Pi_{i\in T_0}a_{m_i}\Pi_{i\in \operatorname{Supp}\tilde{m}\backslash T_0}(1-a_{m_i}))\tilde{v}_0(T_0),$$
$$\left.\sum_{i\in \operatorname{Supp}\tilde{x}}\tilde{w}_{ix_i}\geq\sum_{T_0\subseteq \operatorname{Supp}\tilde{x}}(\Pi_{i\in T_0}x_i\Pi_{i\in \operatorname{Supp}\tilde{x}\backslash T_0}(1-x_i))\tilde{v}_0(T_0),\forall\tilde{x}\in FM\right\}.$$

Theorem 4.4 *If the associated game $\tilde{v}_0$ of $\tilde{v} \in OFMC^N$ is convex, then we have $(\tilde{\varphi}^O_{ia_j}(N, \tilde{m}, \tilde{v}))_{i\in N, j\in\{1,\ldots,m_i\}} \in C_O(N, \tilde{m}, \tilde{v})$, where $\tilde{\varphi}^O(N, \tilde{m}, \tilde{v})$ as shown in Eq.* (12).

Proof Similar to Shapley (1971), let

$$\begin{aligned}
\tilde{w}'_{1a_1} &= \tilde{v}\left(a_1 e^1\right), \ldots, \tilde{w}'_{1a_{m_1}} = \tilde{v}\left(a_{m_1} e^1\right) -_H \tilde{v}\left(a_{m_1-1} e^1\right), \ldots, \\
\tilde{w}'_{na_1} &= \tilde{v}\left(\tilde{m} - \left(a_{m_n} e^n - a_1 e^n\right)\right) -_H \tilde{v}\left(\tilde{m} - a_{m_n} e^n\right), \ldots, \\
\tilde{w}'_{na_{m_n}} &= \tilde{v}\left(\tilde{m}\right) -_H \tilde{v}\left(\tilde{m} - \left(a_{m_n} e^n - a_{m_n-1} e^n\right)\right).
\end{aligned}$$

For any $i \in N$ and any $j \in \{1, 2, \ldots, m_i\}$, let $\tilde{w}_{ia_j} = \sum_{1\le h\le j} \tilde{w}'_{ia_h}$.
Obviously, we have $\sum_{i\in N} \tilde{w}_{ia_{m_i}} = \tilde{v}(\tilde{m}) = \sum_{T_0 \subseteq \operatorname{Supp} \tilde{m}} (\Pi_{i\in T_0} a_{m_i} \Pi_{i\in \operatorname{Supp} \tilde{m}\setminus T_0} (1 - a_{m_i}))\tilde{v}_0(T_0)$.
For any $\tilde{x} \in FM\setminus\{e^{\varnothing}\}$, without loss of generality, suppose

$$FM\setminus\tilde{x} = \left\{\left\{a_{p_1+1}, \ldots, a_{m_{j_1}}\right\}, \left\{a_{p_2+1}, \ldots, a_{m_{j_2}}\right\}, \ldots, \left\{a_{p_t+1}, \ldots, a_{m_{j_t}}\right\}\right\},$$

where $j_1 < j_2 < \cdots < j_t,\ p_k + 1 \le m_{j_k}$ and $k_{\tilde{x}_{j_k}} = p_k$ for any $k \in \{1, 2, \ldots, t\}$.
Let $\tilde{r} = (a_{m_1} e^1, a_{m_2} e^2, \ldots, a_{m_{j_1-1}} e^{j_1-1}, a_{p_1+1} e^{j_1})$, then we have

$$\tilde{r} \vee \tilde{x} = \tilde{x} \vee a_{p_1+1} e^{j_1} \text{ and } \tilde{r} \wedge \tilde{x} = \tilde{r} - (a_{p_1+1} - a_{p_1}) e^{j_1}.$$

From Lemmas 4.2, 4.3 and induction, we have

$$\tilde{w}'_{j_1 a_{p_1+1}} = \tilde{v}(\tilde{r}) -_H \tilde{v}(\tilde{r} - (a_{p_1+1} - a_{p_1}) e^{j_1}) \le \tilde{v}(\tilde{x} \vee a_{p_1+1} e^{j_1}) -_H \tilde{v}(\tilde{x})$$

and

$$\begin{aligned}
&\left(\sum_{i\in \operatorname{Supp} \tilde{x}} \sum_{1\le h\le k_{\tilde{x}_i}} \tilde{w}'_{ia_{k_{\tilde{x}_i}}} + \tilde{w}'_{j_1 a_{p_1+1}}\right) -_H \sum_{i\in \operatorname{Supp} \tilde{x}} \sum_{1\le h\le k_{\tilde{x}_i}} \tilde{w}'_{ia_{k_{\tilde{x}_i}}} \\
&\le \tilde{v}(\tilde{x} \vee a_{p_1+1} e^{j_1}) -_H \tilde{v}(\tilde{x}).
\end{aligned}$$

Namely,

$$\begin{aligned}
&\left(\sum_{i\in \operatorname{Supp} \tilde{x}} \sum_{1\le h\le k_{\tilde{x}_i}} \tilde{w}'_{ia_{k_{\tilde{x}_i}}} + \tilde{w}'_{j_1 a_{p_1+1}}\right) -_H \tilde{v}(\tilde{x} \vee a_{p_1+1} e^{j_1}) \\
&\le \sum_{i\in \operatorname{Supp} \tilde{x}} \sum_{1\le h\le k_{\tilde{x}_i}} \tilde{w}'_{ia_{k_{\tilde{x}_i}}} -_H \tilde{v}(\tilde{x})
\end{aligned}$$

and

$$\left(\sum_{i\in \text{Supp}\,\tilde{x}\setminus\{j_1\}} \tilde{w}_{ia_{k_{\tilde{x}_i}}} + \tilde{w}_{j_1 a_{p_1+1}}\right) -_H \tilde{v}(\tilde{x} \vee a_{j_1+1}e^{j_1})$$
$$\leq \sum_{i\in \text{Supp}\,\tilde{x}} \tilde{w}_{ia_{k_{\tilde{x}_i}}} -_H \tilde{v}(\tilde{x}).$$

Repeat the above process $\sum_{i\in N} m_i - \sum_{i\in N} k_{\tilde{x}_i}$ times, we have $\sum_{i\in \text{Supp}\,\tilde{x}} \tilde{w}_{ia_{k_{\tilde{x}_i}}} -_H \tilde{v}(\tilde{x}) \geq 0$. Namely,

$$\sum_{i\in \text{Supp}\,\tilde{x}} \tilde{w}_{ix_i} \geq \sum_{T_0\subseteq \text{Supp}\,\tilde{x}} (\Pi_{i\in T_0} x_i \Pi_{i\in \text{Supp}\,\tilde{x}\setminus T_0}(1-x_i))\tilde{v}_0(T_0).$$

From the construction of Eq. (12), we derive $(\tilde{\varphi}^O_{ia_j}(N, \tilde{m}, \tilde{v}))_{i\in N, j\in\{1,\ldots,m_i\}} \in C_O(N, \tilde{m}, \tilde{v})$. □

Corollary 4.1 *If the associated game $\tilde{v}_0$ of $\tilde{v} \in OFMC^N$ is convex, then $C_O(N, \tilde{m}, \tilde{v}) \neq \varnothing$.*

Example 4.1 Consider a joint production model in which two decision makers, named 1 and 2, pool two resources to make some product. As is in the real life, each decision maker is not willing to supply all its resources to a particular cooperation, and their participation levels are usually not unique.

It is natural for the two decision makers to try to evaluate their venue of the joint project in the early period of the project in order to decide whether the project can be realized or not. However, the profit is dependent on a number of actors such as product market price, product cost, consumer demand, the relation of commodity supply and demand, etc. Hence, the profit of each coalition is an approximate evaluation, which is represented by trapezoidal fuzzy numbers. Thus, we have to consider a fuzzy multichoice game with a trapezoidal fuzzy characteristic function. Let $N = \{1, 2\}$.

If the decision makers 1 and 2 have four and three action levels, respectively, where $FM_1 = \{a_0 = 0, a_1 = 0.2, a_2 = 0.5, a_3 = 0.6\}$ and $FM_2 = \{a_0 = 0, a_1 = 0.3, a_2 = 0.8\}$. The coalition fuzzy values of the associated game $\tilde{v}_0$ of $\tilde{v} \in FMC^N$ are given as follows:

$$\tilde{v}_0(1) = (2, 3, 4, 6), \tilde{v}_0(2) = (2, 4, 5, 7), \tilde{v}_0(1, 2) = (6, 10, 14, 15).$$

When the fuzzy coalition values for $\tilde{v} \in FMC^N$ can be expressed by Eq. (11), namely, $\tilde{v} \in OFMC^N$. From Eq. (12), we have

$$\tilde{\varphi}^O_{1a_1}(N, \tilde{m}, \tilde{v}) = (0.468, 0.702, 0.97, 1.268),$$
$$\tilde{\varphi}^O_{1a_2}(N, \tilde{m}, \tilde{v}) = (1.284, 1.926, 2.71, 3.284),$$

$$\tilde{\varphi}^O_{1a_3}(N, \tilde{m}, \tilde{v}) = (1.598, 2.394, 3.395, 3.998),$$
$$\tilde{\varphi}^O_{2a_1}(N, \tilde{m}, \tilde{v}) = (0.732, 1.398, 1.832, 2.232),$$
$$\tilde{\varphi}^O_{2a_2}(N, \tilde{m}, \tilde{v}) = (2.162, 4.043, 5.405, 6.162).$$

Since the associated game $\tilde{v}_0$ is convex, we get $(\tilde{\varphi}^O_{ia_j}(N, \tilde{m}, \tilde{v}))_{i\in\{1,2\}, j\in\{1,\dots,m_i\}} \in C_O(N, \tilde{m}, \tilde{v})$, and $(\tilde{\varphi}^O_{ia_j}(N, \tilde{m}, \tilde{v}))_{i\in\{1,2\}, j\in\{1,\dots,m_i\}}$ is a FMPMAS for$\tilde{v} \in OFMC^N$ in this example.
If we restrict Eq. (4) in the setting of $OFMC^N$, then we have

$$\tilde{\psi}_{1a_1}(N, \tilde{m}, \tilde{v}) = \tilde{\varphi}^O_{1a_1}(N, \tilde{m}, \tilde{v}),$$
$$\tilde{\psi}_{1a_2}(N, \tilde{m}, \tilde{v}) = (0.816, 1.224, 1.74, 2.016),$$
$$\tilde{\psi}_{1a_3}(N, \tilde{m}, \tilde{v}) = (0.314, 0.468, 0.685, 0.714),$$
$$\tilde{\psi}_{2a_1}(N, \tilde{m}, \tilde{v}) = \tilde{\varphi}^O_{2a_1}(N, \tilde{m}, \tilde{v}),$$
$$\tilde{\psi}_{2a_2}(N, \tilde{m}, \tilde{v}) = (1.43, 2.645, 3.575, 3.93).$$

From Eq. (11), we know the game $\tilde{v} \in OFMC^N$ given by Example 4.1 is strictly monotone increasing corresponding to the players' participations. But we have

$$\tilde{\psi}_{1a_3}(N, \tilde{m}, \tilde{v}) < \tilde{\psi}_{1a_1}(N, \tilde{m}, \tilde{v}) \text{ and } \tilde{\psi}_{1a_3}(N, \tilde{m}, \tilde{v}) < \tilde{\psi}_{1a_2}(N, \tilde{m}, \tilde{v}),$$

which contradicts with the people's intrusion. This is also the season that we use Eq. (5) to denote the Shapley value for FMC^N. Furthermore, the Hukuhara difference cannot be used in this example since it does not satisfy the necessary condition of the Hukuhara difference.

4.2 Fuzzy Multichoice Games with Choquet Integral Form and Fuzzy Characteristic Functions

In this section, we discuss fuzzy multichoice games with Choquet integral form and fuzzy characteristic functions. By $CFMC^N$, we denote this kind of games. According to Tsurumi et al. (2001), the fuzzy coalition value for $CFMC^N$ is written as:

$$\tilde{v}(\tilde{x}) = \sum_{l=1}^{q(\tilde{x})} \tilde{v}_0([\tilde{x}]_{h_l})(h_l - h_{l-1}), \tag{14}$$

where $\tilde{x} = (x_1, x_2, \dots, x_n) \in FM$, $Q(\tilde{x}) = \{x_j | x_j > 0, j \in N\}$, $q(\tilde{x}) = |Q(\tilde{x})|$ and $[\tilde{x}]_{h_l} = \{i | x_i \geq h_l, i \in N, l = 1, 2, \dots, q(\tilde{x})\}$, the elements in $Q(\tilde{x})$ are written in the increasing order as 0= $h_0 \leq h_1 \leq \cdots \leq h_{q(\tilde{x})}$, and $[\tilde{x}]_{h_l}$ is a crisp coalition as usual.

Theorem 4.5 *Define a function* $\tilde{\varphi}^C : CFMC^N \to \tilde{\mathbb{R}}^{\sum_{i\in N} m_i}$ *by*

$$\tilde{\varphi}^C_{ia_j}(N, \tilde{m}, \tilde{v}) = \sum_{1\le g\le j} \sum_{l=1}^{q(FM)} \sum_{\tilde{x}\in FM, k_{x_i}=g} h_{ig}(\tilde{x}_{\text{sub}}) \left(\tilde{v}_0\left([\tilde{x}]_{h_l}\right) -_H \tilde{v}_0\left([\tilde{x}']_{h_l}\right)\right)(h_l - h_{l-1}) \quad \forall i \in N,\ j \in \{1, 2, \ldots, m_i\}, \tag{15}$$

where $q(FM) = |Q(FM)|$, $Q(FM) = FM = \{h_1, \ldots, h_{q(FM)}\}$, $0 = h_0 \le h_1 \le \cdots \le h_{q(FM)}$ *and* $\tilde{x}' = \tilde{x} - (a_g - a_{g-1})e^i$, *and* $h_{ig}(\tilde{x}_{sub})$ *is the potential weight for fuzzy coalition* $\tilde{x}$ *as shown in Eq. (4). Then* $\tilde{\varphi}^C$ *is the unique Shapley function on* $CFMC^N$.

Proof The proof Theorem 4.5 is similar to that of Theorem 3.1. □

Obviously, Eq. (15) degenerates to be the Shapley value for fuzzy games with Choquet integral form when we restrict the domain of $\tilde{v} \in CFMC^N$ in the setting of it.

Theorem 4.6 *If the associated game* $\tilde{v}_0$ *of* $\tilde{v} \in CFMC^N$ *is convex, then the vector* $\left(\tilde{\varphi}^C_{ia_j}(N, \tilde{m}, v)\right)_{i\in N, j\in\{1,2,\ldots,m_i\}}$ *is an imputation for* $\tilde{v} \in CFMC^N$, *where* $\tilde{\varphi}^C(N, \tilde{m}, v)$ *as shown in Eq. (15).*

Proof From Definition 2.7 and Theorem 4.5, we only need to show $\tilde{\varphi}^C_{ia_j}(N, \tilde{m}, \tilde{v}) \ge \tilde{v}(a_j e^i)$ for any $i \in N$ and any $j \in \{1, 2, \ldots, m_i\}$. Since $[\tilde{x}]_{h_l} \ne [\tilde{x}']_{h_l}$ for any $\tilde{x} \in FM$ and any $l \in \{1, 2, \ldots, q(FM)\}$ if and only if $x_i' < h_l \le x_i$. Without loss of generality, suppose $x_i' = h_{l_1}$ and $x_i = h_{l_2}$.
From the convexity of the associated game $\tilde{v}_0$ and Eq. (15), we have

$$\begin{aligned}
\tilde{\varphi}^C_{ia_j}(N, \tilde{m}, \tilde{v}) &= \sum_{1\le g\le j} \sum_{l=1}^{q(FM)} \sum_{\tilde{x}\in FM, k_{x_i}=g} h_{ig}(\tilde{x}_{\text{sub}}) \left(\tilde{v}_0\left([\tilde{x}]_{h_l}\right) -_H \tilde{v}_0\left([\tilde{x}']_{h_l}\right)\right)(h_l - h_{l-1}) \\
&= \sum_{1\le g\le j} \sum_{l_1+1\le l\le l_2} \sum_{\tilde{x}\in FM, k_{x_i}=g} h_{ig}(\tilde{x}_{\text{sub}}) \left(\tilde{v}_0\left([\tilde{x}]_{h_l}\right) -_H \tilde{v}_0\left([\tilde{x}']_{h_l}\right)\right)(h_l - h_{l-1}) \\
&\ge \sum_{1\le g\le j} \sum_{l_1+1\le l\le l_2} \sum_{\tilde{x}\in FM, k_{x_i}=g} h_{ig}(\tilde{x}_{\text{sub}}) \tilde{v}_0(i)(h_l - h_{l-1}) \\
&= \sum_{1\le g\le j} \sum_{\tilde{x}\in FM, k_{x_i}=g} h_{ig}(\tilde{x}_{\text{sub}}) \left(\sum_{l_1+1\le l\le l_2} \tilde{v}_0(i)(h_l - h_{l-1})\right) \\
&= \sum_{1\le g\le j} \sum_{\tilde{x}\in FM, k_{x_i}=g} h_{ig}(\tilde{x}_{\text{sub}}) \left(\tilde{v}_0(i)\left(h_{l_2} - h_{l_1}\right)\right)
\end{aligned}$$

$$= \sum_{1\le g\le j} \sum_{\tilde{x}\in FM, k_{x_i}=g} h_{ig}(\tilde{x}_{\text{sub}}) \left(\tilde{v}_0(i)\left(a_g - a_{g-1}\right)\right)$$
$$= \sum_{1\le g\le j} \tilde{v}_0(i)\left(a_g - a_{g-1}\right)$$
$$= \tilde{v}_0(i)\, a_j$$
$$= \tilde{v}\left(a_j e^i\right),$$

where $a_0 = h_0 = 0$ and $\tilde{x}'$ as shown in Theorem 4.5. □

Theorem 4.7 *If the associated game $\tilde{v}_0$ of $\tilde{v} \in CFMC^N$ is convex, then the vector $(\tilde{\varphi}^C_{ia_j}(N, \tilde{m}, \tilde{v}))_{i\in N, j\in\{1,\ldots,m_i\}}$ is a FMPMAS for $\tilde{v} \in CFMC^N$, where $\tilde{\varphi}^C(N, \tilde{m}, v)$ as shown in Eq. (15).*

Proof From Eq. (15), we know that the first condition in Definition 3.3 holds. In the following, we show the second condition in Definition 3.3. Let

$$\tilde{\eta}_{ia_j}(N, \tilde{m}, \tilde{v}) = \sum_{1\le g\le j} \sum_{\tilde{x}\in FM, k_{x_i}=g} h_{ig}(\tilde{x}_{\text{sub}}) \left(\tilde{v}_0\left([\tilde{x}]_{h_l}\right) -_H \tilde{v}_0([\tilde{x}']_{h_l})\right)$$

for any $i \in N$ and any $j \in \{1, 2, \ldots, m_i\}$.
Then, we have

$$\tilde{\varphi}^C_{ia_j}(N, \tilde{m}, \tilde{v}) = \sum_{l=1}^{q(FM)} \tilde{\eta}_{ia_j}(N, \tilde{m}, \tilde{v})(h_l - h_{l-1}) \quad \forall i \in N,\ j \in \{1, 2, \ldots, m_i\}. \tag{16}$$

From Eq. (16), for any $l \in \{1, 2, \ldots, q(FM)\}$, it is sufficient to show

$$\tilde{\eta}_{ia_j}\left(N, \tilde{m}, \tilde{v}_{\tilde{y}}\right) \ge \tilde{\eta}_{ia_j}\left(N, \tilde{m}, \tilde{v}_{\tilde{x}}\right),$$

where $i \in \text{Supp}\,\tilde{x}$ and $0 < j \le k_{x_i}$.
For all $\tilde{x}, \tilde{y} \in FM$ with $\tilde{x} \le \tilde{y}$ and $k_{x_i} = k_{y_i}$, we have

$$\tilde{\eta}_{ia_j}\left(N, \tilde{m}, \tilde{v}_{\tilde{y}}\right) = \sum_{1\le g\le j} \sum_{\tilde{z}\le\tilde{y}, k_{z_i}=g} h^{\tilde{y}}_{ig}(\tilde{z}_{sub}) \left(\tilde{v}_0\left([\tilde{z}]_{h_l}\right) -_H \tilde{v}_0([\tilde{z}']_{h_l})\right)$$

and

$$\tilde{\eta}_{ia_j}\left(N, \tilde{m}, \tilde{v}_{\tilde{x}}\right) = \sum_{1\le g\le j} \sum_{\tilde{z}\le\tilde{x}, k_{z_i}=g} h^{\tilde{x}}_{ig}(\tilde{z}_{sub}) \left(\tilde{v}_0([\tilde{z}]_{h_l}) -_H \tilde{v}_0([\tilde{z}']_{h_l})\right).$$

From Lemma 3.3, for any given $g \in \{1, 2, \ldots, j\}$, any $l \in \{1, 2, \ldots, q(FM)\}$ and any $\tilde{z} \leq \tilde{x}$, we have

$$\sum_{\tilde{z}\leq\tilde{r}\leq\tilde{y},k_{z_i}=g} h_{ig}^{\tilde{y}}(\tilde{r}_{\text{sub}})(\tilde{v}_0([\tilde{r}]_{h_l}) -_H \tilde{v}_0([\tilde{r}']_{h_l})) -_H \times \sum_{\tilde{z}\leq\tilde{r}\leq\tilde{x},k_{z_i}=g} h_{ig}^{\tilde{x}}(\tilde{r}_{\text{sub}})(\tilde{v}_0([\tilde{r}]_{h_l}) -_H \tilde{v}_0([\tilde{r}']_{h_l})) \geq \left(\sum_{\tilde{z}\leq\tilde{r}\leq\tilde{y},k_{r_i}=g} h_{ig}^{\tilde{y}}(\tilde{r}_{\text{sub}}) - \sum_{\tilde{z}\leq\tilde{r}\leq\tilde{x},k_{r_i}=g} h_{ig}^{\tilde{x}}(\tilde{r}_{\text{sub}})\right)(\tilde{v}_0([\tilde{z}]_{h_l}) -_H \tilde{v}_0([\tilde{z}']_{h_l})).$$

Form the convexity of the associated game $\tilde{v}_0$ and $\sum_{\tilde{z}\leq\tilde{r}\leq\tilde{x},k_{r_i}=g} h_{ig}^{\tilde{x}}(\tilde{r}_{sub}) \leq \sum_{\tilde{z}\leq\tilde{r}\leq\tilde{y},k_{r_i}=g} h_{ig}^{\tilde{y}}(\tilde{r}_{sub})$, we obtain

$$\sum_{\substack{\tilde{z}\leq\tilde{r}\leq\tilde{y},\\ k_{z_i}=g}} h_{ig}^{\tilde{y}}(\tilde{r}_{\text{sub}})\left(\tilde{v}_0\left([\tilde{r}]_{h_l}\right) -_H \tilde{v}_0\left([\tilde{r}']_{h_l}\right)\right) -_H \times \sum_{\substack{\tilde{z}\leq\tilde{r}\leq\tilde{x},\\ k_{z_i}=g}} h_{ig}^{\tilde{x}}(\tilde{r}_{\text{sub}})\left(\tilde{v}_0\left([\tilde{r}]_{h_l}\right) -_H \tilde{v}_0\left([\tilde{r}']_{h_l}\right)\right) \geq 0.$$

Thus, for any $l \in \{1, 2, \ldots, q(FM)\}$, any $i \in \text{Supp}\,\tilde{x}$ and any $0 < j \leq k_{x_i}$, $\tilde{\eta}_{ia_j}(N, \tilde{m}, \tilde{v}_{\tilde{y}}) \geq \tilde{\eta}_{ia_j}(N, \tilde{m}, \tilde{v}_{\tilde{x}})$. □

Definition 4.2 The core $C_C(N, \tilde{m}, \tilde{v})$ for $\tilde{v} \in CFMC^N$ is denoted by

$$C_C(N, \tilde{m}, \tilde{v}) = \left\{\tilde{w} \in \tilde{\mathbb{R}}^{\sum_{i\in N} m_i} \mid \sum_{i\in N} \tilde{w}_{ia_{m_i}} = \sum_{l=1}^{q(FM)} \tilde{v}_0([\tilde{m}]_{h_l})(h_l - h_{l-1}), \times \sum_{i\in \text{Supp}\,\tilde{x}} \tilde{w}_{ix_i} \geq \sum_{l=1}^{q(\tilde{x})} \tilde{v}_0([\tilde{x}]_{h_l})h_l - h_{l-1}), \forall \tilde{x} \in FM\right\}.$$

Theorem 4.8 *If the associated game $\tilde{v}_0$ of $\tilde{v} \in CFMC^N$ is convex, then*

$$\left(\tilde{\varphi}_{ia_j}^C(N, \tilde{m}, \tilde{v})\right)_{i\in N, j\in\{1,\ldots,m_i\}} \in C_C\,(N, \tilde{m}, \tilde{v}),$$

where $\tilde{\varphi}^C(N, \tilde{m}, \tilde{v})$ as shown in Eq. (15).

Proof The proof Theorem 4.8 is similar to that of Theorem 4.4. □

From Theorems 4.6 and 4.7, we know that $\tilde{\varphi}^C(N, \tilde{m}, \tilde{v})$ degenerates to be an imputation and a FPMAS for fuzzy games with Choquet integral form, when the domain of

$\tilde{v} \in CFMC^N$ is restricted in the framework of it, and its the associated crisp game is convex. Furthermore, by Theorem 4.8, we know that $\tilde{\varphi}^C(N, \tilde{m}, \tilde{v})$ degenerates to be an element in the core for fuzzy games with Choquet integral form when the domain of $\tilde{v} \in CFMC^N$ is restricted in the setting of it, and its associated crisp game is convex.

Corollary 4.2 *If the associated game $\tilde{v}_0$ of $\tilde{v} \in CFMC^N$ is convex, then $C_C(N, \tilde{m}, \tilde{v}) \neq \varnothing$.*

Example 4.2 Similar to Example 4.1, if there are three decision makers, named 1, 2 and 3, cooperate to complete some project, namely, $N = \{1, 2, 3\}$. The decision makers 1, 2 and 3 have three, three and two activity levels, respectively, where $FM_1 = \{a_0 = 0,\ a_1 = 0.2,\ a_2 = 0.4\}$, $FM_2 = \{a_0 = 0,\ a_1 = 0.3,\ a_2 = 0.4\}$ and $FM_3 = \{a_0 = 0,\ a_1 = 0.6\}$. If the coalition fuzzy values of the associated game $\tilde{v}_0$ of $\tilde{v} \in FMC^N$ are given as follows:

$$\begin{aligned}
\tilde{v}_0(1) &= (1, 2, 3, 5),\ \tilde{v}_0(2) = (1, 3, 4, 5),\ \tilde{v}_0(3) = (2, 3, 4, 5),\\
\tilde{v}_0(1, 2) &= (3, 6, 8, 12),\ \tilde{v}_0(1, 3) = (4, 7, 10, 15),\\
\tilde{v}_0(2, 3) &= (5, 9, 15, 17),\ \tilde{v}_0(1, 2, 3) = (8, 15, 31, 35).
\end{aligned}$$

When the fuzzy coalition values of $\tilde{v} \in FMC^N$ can be expressed by Eq. (14), namely, $\tilde{v} \in CFMC^N$. From Eq. (15), we have

$$\begin{aligned}
\tilde{\varphi}^C_{1a_1}(N, \tilde{m}, \tilde{v}) &= (0.367, 0.68, 1.322, 1.815),\\
\tilde{\varphi}^C_{1a_2}(N, \tilde{m}, \tilde{v}) &= (0.833, 1.58, 3.355, 4.378),\\
\tilde{\varphi}^C_{2a_1}(N, \tilde{m}, \tilde{v}) &= (0.617, 1.374, 2.529, 2.775),\\
\tilde{\varphi}^C_{2a_2}(N, \tilde{m}, \tilde{v}) &= (0.9, 1.964, 3.805, 4.082),\\
\tilde{\varphi}^C_{3a_1}(N, \tilde{m}, \tilde{v}) &= (1.867, 3.057, 6.045, 6.575).
\end{aligned}$$

Since the associated game $\tilde{v}_0$ is convex, we have $(\tilde{\varphi}^C_{ia_j}(N, \tilde{m}, \tilde{v}))_{i\in\{1,2,3\}, j\in\{1,\ldots,m_i\}} \in C_C(N, \tilde{m}, \tilde{v})$, and $(\tilde{\varphi}^C_{ia_j}(N, \tilde{m}, \tilde{v}))_{i\in\{1,2,3\}, j\in\{1,\ldots,m_i\}}$ is a FMPMAS for $\tilde{v} \in CFMC^N$ in this example.
If we restrict Eq. (4) in the setting of $CFMC^N$, then

$$\begin{aligned}
\tilde{\psi}_{1a_1}(N, \tilde{m}, \tilde{v}) &= \tilde{\varphi}^C_{1a_1}(N, \tilde{m}, \tilde{v}),\\
\tilde{\psi}_{1a_2}(N, \tilde{m}, \tilde{v}) &= (0.467, 0.9, 2.033, 2.533),\\
\tilde{\psi}_{2a_1}(N, \tilde{m}, \tilde{v}) &= \tilde{\varphi}^C_{2a_1}(N, \tilde{m}, \tilde{v}),\\
\tilde{\psi}_{2a_2}(N, \tilde{m}, \tilde{v}) &= (0.283, 0.59, 1.276, 1.306),\\
\tilde{\psi}_{3a_1}(N, \tilde{m}, \tilde{v}) &= \tilde{\varphi}^C_{3a_1}(N, \tilde{m}, \tilde{v}).
\end{aligned}$$

From Eq. (14), we know that the game $\tilde{v} \in CFMC^N$ given in Example 4.2 is strictly monotone increasing with respect to the players' participations. But we have

$$\tilde{\psi}_{2a_2}(N, \tilde{m}, \tilde{v}) < \tilde{\psi}_{2a_1}(N, \tilde{m}, \tilde{v}) \quad \text{and} \quad \tilde{\psi}_{2a_2}(N, \tilde{m}, \tilde{v}) < \tilde{v}(a_2 e^2),$$

which contradicts with the people's intrusion.

This is also the season that we use Eq. (5) to denote the Shapley value for FMC^N. Furthermore, it is not difficult to know the Hukuhara difference cannot be used in this example.

This section has researched two kinds of FMC^N, which are extensions of fuzzy games introduced by Meng and Zhang (2010) and Tsurumi et al. (2001). Since these classes of FMC^N build the specific relationship with the associated games, the properties for these kinds of FMC^N can be obtained by researching the associated games, which are much simpler.

However, we only study two kinds of FMC^N, and it will be interesting to discuss other kinds of FMC^N, such as fuzzy multichoice games with proportional values and fuzzy characteristic functions and fuzzy multichoice games with weighted functions and fuzzy characteristic functions, which are extensions of fuzzy games given by Butnariu (1980) and Butnariu and Kroupa (2008).

5 Conclusions

Based on the extension Hukuhara difference, the model for fuzzy multichoice games with fuzzy characteristic functions has been introduced. A Shapley value is studied, which is an extension of the Shapley value for fuzzy games presented by Li and Zhang (2009). Moreover, we show that the defined Shapley value is a MFPMAS when the given fuzzy multichoice games are convex. In order to better understand this kind of multichoice games, we pay more attention to research two special kinds of fuzzy multichoice games with fuzzy characteristic functions, which are extensions proposed by Meng and Zhang (2010) and Tsurumi et al. (2001). However, we only study a Shapley value for fuzzy multichoice games with fuzzy characteristic functions, and it will be interesting to discuss other Shapley values.

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