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The Evolution of Certainty in a Small Decision-Making Group by Consensus

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Abstract We propose a dynamical systems model for approximating the certainty as a function of communication. Uncertainty is measured during a small group decision-making process, in which participants aim to reach consensus. Assuming that the communication is a one-dimensional continuum variable, both first- and second-order differential models of certainty are analyzed, and then, the general model is obtained by superposition. An experiment was organized, and the data have been used to test the model. A detailed discussion on the assumptions of this approach from the decision theory point of view is also included.

Keywords Subjective certainty · Communication · Group decision-making · Dynamical systems · Evolution of certainty

1 Introduction

Uncertainty is a part of all important decisions in day to day life or in "natural decision environments" (Kobus et al. 2001: 377). Therefore, much attention has been given to uncertainty in decision-making, but less so to uncertainty in group decision-making and, to the best of our knowledge, absolutely none for the way uncertainty evolves in group decisions by consensus.

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In social psychology, confidence measures the extent to which a person is certain that the decision is the correct one and it is usually a scale measurement. The terms uncertainty and confidence are used interchangeably. While uncertainty means that the event which is evaluated has a known probability of occurrence, ambiguity is a different case in which the likelihood of an uncertain event is not even known (Kahn and Sarin 1988). Uncertainty is usually an undesirable state which triggers uncertainty reduction acts, such as information search (Robertson 1980), delegation of responsibility, a decrease in motivation, decision blockage or an oscillation between delaying or continuing the decision, should the uncertainty prove to be irreducible (Zamfir 2005). All of these will be amplified if the consequences of the decision are highly important, meaningful or significant for the subject (Zamfir 2005). For a comprehensive review of the most important issues around uncertainty in group decision-making, please see Sniezek (1992). The following paragraphs will present the aspects of the subjective uncertainty in group decisions that are relevant for showing how certainty evolves in group decision-making by consensus and for the methods employed in this study.

Over the years uncertainty has been modeled either in deterministic or probabilistic ways (for example in game theory), as parametrization (Cromley 1982), as error (in measurement studies), as entropy (Webber 1977) or by using fuzzy logic. More than this, individual subjective certainty has received much more attention than group certainty or the certainty for group decision-making (Sniezek 1992). In recent years, attention has been given to the way the certainty of decisions like, for example, advice-taking changes in interaction (Van Swol 2009) or how interaction increases certainty, but not necessarily the decision quality (Punchochar and Fox 2004; Heath and Gonzales 1995).

Group decision-making has many implications for the certainty and the accuracy of the decision. Group discussions were initially thought to increase the decision accuracy, but recent research has shown that group discussions do not always act in this way, but instead, they only increase the confidence in the group decision (Punchochar and Fox 2004). More than this, the initial assumption at the basis of this belief was shown to be incorrect. Scholars believed that group discussions increase knowledge on the decision task and therefore, decision accuracy will also be raised. Despite these predictions, Fiedler and Kareev (2006) have shown that greater amounts of information might increase uncertainty. Moreover, they have found theoretical and empirical evidence that in contingency assessments the decision quality increases as the information sample size decreases, since smaller random samples have higher variance. On top of this inverse relationship between certainty and knowledge, greater individual certainty is likely to lead to an increase in the social influence of the highly certain group member as opposed to a less certain, but more accurate group member (Zarnoth and Sniezek 1997).

Such disparity between certainty and accuracy has led researchers to calibrate the subjective certainty estimates (usually viewed as probabilities; Fiedler and Kareev 2006; Zarnoth and Sniezek 1997; Sniezek 1992). Without calibration, one may find *overconfidence*, defined as a positive difference between the accuracy rate and the confidence expressed as percentage, or *under-confidence*, defined as a negative difference between the accuracy rate and the confidence expressed as percentage

(Unal et al. 2005; Punchochar and Fox 2004; Slevin et al. 1998). Until now, the over- and the under-confidence are viewed as a result of the limited cognitive possibilities of subjects, and thus, efforts have been made to find ways to correct these so-called errors of judgment or to deal with them in other ways.

This is why attempts have been made to avoid the time-consuming face-to-face group interactions and to obtain a better approximation of group decisions for "intellective" ranking tasks (ranking tasks which have a known solution; Zarnoth and Sniezek 1997: 346). This approximation of face-to-face decisions uses an equation dependent upon the initial decision, its certainty, the "remaining certainty" (the ratio between the individual member's certainty estimates and the sum of certainty estimates within the group), as well as the remaining decisions (Slevin et al. 1998: 184). Employing such a technique or others, such as computerized Group Decision Support Systems (Niederman and Bryson 1998; Slevin et al. 1998) only helps in masking the true nature of uncertainty in group decision-making.

Nevertheless, some research has been conducted to understand how groups "aggregate member preferences in a collective decision" (Laughlin 2011: 63). Several types of decision schemes have been proposed, like central tendency, consensus based, faction-attraction decision schemes and many others, as Hinsz (1999) reveals. Despite these attempts to see how the decisions are combined within groups, the way in which certainty evolves during the decision-making process has not been addressed.

This article was set out to fill this literature gap and to propose a model of the evolution of certainty in group decisions. It does so by arguing that, under the conditions of unidimensional communication (for each unit of communication, there is only one unit of information disseminated in the group discussion), uncertainty in group decision-making by consensus oscillates on an exponential trend and is gradually damped to an equilibrium value. Consequently, certainty measurements taken before the oscillation is completely damped will either lead to an overestimation or an underestimation of the equilibrium value, due to the oscillations. Moreover, because of this, for certain intervals of time before consensus is achieved, a linear approximation of the evolution of certainty, as it is sometimes useful or necessary, will not be justified.

The starting point of this article is the existence of oscillation[1] and exponential evolution to equilibrium[2] of the subjective certainty as a function of the objective certainty, in a small group decision-making process by consensus, which have been concluded by Zamfir (2005) by qualitative methods. According to Zamfir (2005) the *subjective certainty* denotes one's perception of the degree of certainty, while the *objective certainty* is given by the ratio between the available and the necessary knowledge needed to solve a decision task. Unfortunately, this approach does not allow for effective measurements of the subjective certainty as a function of the objective certainty because the objective certainty is almost unmeasurable. In order to overcome this difficulty,

[1] We say that a function of time oscillates if, for a significant period of time, it increases and decreases consecutively for a significant number of times.

[2] The constant solution of a linear differential equation with constant coefficients and constant forcing function is called the equilibrium solution; given another solution, we say that it has an evolution to equilibrium if, as time goes, it becomes closer and closer to the equilibrium solution (asymptotically converges to the equilibrium value) (Medio and Lines 2001: 25).

the relation between objective and subjective certainties as an implicit function is considered and a parameterization of it by means of a new variable, communication, is proposed. Therefore, the space of certainties is considered as function of communication, and its dimension and a basis of this space is estimated, as well. Practically, this is done by means of a dynamical systems model of the subjective certainty evolution in terms of unidimensional communication. The modeling of this problem uses the state space approach.[3] Also, the fitness of the model is analyzed by assessing its accuracy when compared with the data collected through an experiment. To do so, a theoretical model and the results of an experiment showing that this model applies only to a subgroup of participants are presented. The characteristics of this subgroup followed the suggestions of Zamfir (2005): high motivation to fulfill the task and non-constrained communication. In addition, the model proved to be applicable only to those participants who employed a certain (unidimensional) decision strategy to reach consensus. We think that different models, where either the communication is multidimensional (*or even time scale*) instead of unidimensional, or the model is nonlinear and hence the existence and the uniqueness of the solution might be only local (for a very limited interval of time), might be appropriate for the rest of them. These models would also help explain the presence of 'irreducible certainty' (Zamfir 2005) or chaotic behavior.

Briefly, in Sect. 2, the premises of a first-order model and a second-order model are presented. There are important differences between the premises of these two models. For example, the fact that in the second-order model the sensitivity to the *rate of change of certainty* is taken into account, not only the *level of certainty* itself. However, a model that should contain both oscillation phenomena and exponential evolution to equilibrium must be of third order. But, for such a model the premises, based on the same chain of ideas as before, look to us unrealistic. At this point, we have used Hoch's (1987, cited in Stanovich and West 1998) principle in estimating the "consensus effect", that is, "people's tendency to project their own opinions when predicting other people's attitudes, opinions or behaviors" (Stanovich and West 1998: 176). Translating this into our dynamical system model, this means that the full model should be a superposition of the first-order model with the second-order model. The experiment along with its methodological issues and the numerical procedure are presented in Sects. 3 and 4; the data, a classification of the results, and the graphs are in Sect. 5. The conclusion emphasizes how appropriate this model is for approximating the individual and group subjective certainty in the analyzed cases.

2 Equilibrium Evolution Models

Let us consider a small group of people in a decision-making process by consensus, on a single issue (problem), in which the communication flows as a continuum variable, mathematically identified with the real positive semi-axis, while the certainty level is a function of communication. In this state space approach, all the participants and

[3] The state space (phase space, configuration space) is the space of the dependent variables (the variables that specify the state of the system); in the context of the proposed model, it is $\mathbb{R}$ (the real line) (Medio and Lines 2001:11).

the subgroups are *providers* (suppliers) of information as well as *receptors* (buyers) of information and the certainty with respect to the decision-making problem is considered a measurable quantity that reflects a compound concept. Without going into a more detailed analysis of the information relevant for the decision-making process, consider the following functions: $R(\theta)$ the *receptor function*, and $P(\theta)$ the *provider function*, both in the variable θ which denotes the *certainty*. Both functions R and P are real-valued one real variable functions in the state space describing the dynamical system of the decision-making process.

This particular type of modeling involves a holistic perspective upon the individual, more precisely, an individual's particularities are seen as given by both her/his psychological determinants as well as her/his social determinants. Since each participant is regarded in the specific social context she/he is in, her/his *individual characteristics* are defined as the union of her/his social and psychological characteristics. Because a complete list of these social and psychological characteristics cannot be employed, only some of them will be mentioned further on.

Furthermore, each member of the decision group acts simultaneously as a provider and as a receptor of information during the exchange of knowledge, opinions, and arguments. Of course, there are many other group interactions during the decision-making process. In this model, all of them are condensed into these two main functions, of provider and of receptor. These functions are constructed by considering our objectives (to point out the oscillatory and evolution to equilibrium behavior) and the need to keep the model to a reasonable level of simplicity (by considering only the direction and the sense of communication). Thus, from the point of view of a group member, the receptor activity has to be viewed as referring to everything having an external source which may change ones opinion. At the same time, the provider activity has to be viewed as referring to everything that one does (in the broader sense of communication) which may change the opinions of the other members of the group.

The main idea of this approach is to consider the decision-making group as a dynamical system *off the equilibrium of the certainty*: if the group has already achieved a consensus there is no interest in continuing the process. Thus, without loss of generality, the receptor and the provider functions R and P are not assumed to be equal and, consequently, the activity of the group continues, that means, the members of that group *communicate*. Furthermore, t (this is just the usual mathematical notation for "time") is the *variable of communication*, considered as a one-dimensional continuum variable. The reason why we can model the communication as the variable time comes from the fact that in the definition of certainty we take into consideration its rate of change and thus, efficient communication will be communication with a higher rate of change in the objective certainty. Consequently, we emphasize that, in a real decision-making situation, what matters is not reaching the correct solution, because this is seldom known, but rather how certain the actors are that the reached solution is the correct one. The idea is that communication itself leads to an increase in certainty and not necessarily the exchange of sufficient and necessary information about the problem (Oskamp 1982).

Note also that communication in this approach may seem to be used with two separate meanings: some exchange of an amount of information and a continuous flow of information during time. The fact is that these are not separate meanings, these are characteristics of communication. Communication is an exchange of information in time. And the model proposed embeds both characteristics. This assumption is justified since specialists in the field of communication regard it as the means of stimulating cognitive representations within the mind of another person (Nøretranders 2009/1991; Prutianu 2004; Pedler 2001). Cognitive representations may be expressed in terms of amounts of information, as is the case in Computational-representational understanding of mind (CRUM) used in the cognitive sciences (Thagard 2005). This means that in group decision-making, the exchange of some amounts of information takes place for the duration of the decision-making process. Furthermore, it is acknowledged in the literature that communication is a continuous, irreversible process (Prutianu 2004), thus giving it the attributes which allow a comparison with time. The limitation of this perspective on communication is that it does not reflect the possibility that cognitive representations appear simultaneously in the mind, on different channels (Thagard 2005: 134–135). This aspect of communication may have impacted upon 9 of the participants in the experiment (see Sect. 3.3).

Following this preliminary discussion, it is possible to construct two models, corresponding to the first-order approximation and, respectively, the second-order approximation, imposing certain assumptions, and then getting the general model by superposition. The mathematical modeling approach is in line with Medio and Lines (2001).

2.1 First-Order Continuous Dynamical System Model

Recall that $R(\theta)$ is the *receptor function*, and $P(\theta)$ is the *provider function*, both of them real valued functions and depending on the real variable θ which denotes the *certainty*.

The first assumptions on R and P are the following:

(A′) Both functions R and P are linear with respect to the variable θ.

(B′) The function R is decreasing, while the function P is increasing with respect to θ.

A first theoretical justification for these assumptions starts from the following considerations. First, linearity of R and P is just a first-order approximation tool in the analysis of the model and comes mainly from the requirement of keeping the model as simple as possible. The second assumption concerns two common-sense judgments. First, *the higher the certainty, the lower the receptor's interest in getting more information and the higher the provider's interest in offering more information*. Secondly, *the lower the certainty, the higher the receptor's interest in getting more information and the lower the provider's interest in offering more information*.

To further understand these assumptions, recall the supply and demand model in economics. Instead of the market, one has communication, viewed as a dynamic system off the certainty equilibrium. The forces which make this motor running are the supply and the demand of information. Unlike in the classical economic model, in which supply and demand are provided by different actors, in this model, each participant to the group decision is a provider and a receptor of information, at the same time. Suppose we have two such individuals: a and b. Each one of them is both a provider

P and a receptor R of information, so it is possible to denote $P(a)$ the number that represents the amount of information provided by the individual a and $R(a)$, the number that represents the amount of information received by individual a. In the model proposed here, the amount provided by a, $P(a)$, is exchanged, such that b receives the amount $R(b)$, while the amount received by a, namely $R(a)$, is provided by b, namely $P(b)$. Let us consider the interaction between each pair $\{P(b); R(a)\}$ and $\{P(a); R(b)\}$. Suppose individual a is very certain that the information she owns is correct. As a receptor of information with high certainty, she has little interest in getting more information. On the other hand, as a provider of information she is keen on sharing it, also since she does not need to acquire more information. Since b has lower certainty than a, as a receptor, b is very much interested in obtaining the high certainty information which could be supplied by a. Now, suppose that the certainty of a is very low. As a receptor, a tries to obtain more information, such that she can increase her certainty or change her opinion. As a provider of information with low certainty, she might want to communicate less. The intrinsic or social motivations for wanting to communicate less, as a provider of uncertain information may have to do with moral or reputation costs, for example. In this way, it becomes clear that for high levels of certainty, the provider tries to communicate more and the receptor tries to communicate less, while for lower levels of certainty, the provider tries to communicate less and the receptor tries to communicate more. In this line of thought emerge the linear and the increasing (decreasing) behavior of the relationship between the provider (the receptor) and the certainty.

Based on these assumptions, the formal representations of the functions R and P are:

$$R(\theta) = a - b\theta, \quad P(\theta) = c + d\theta, \tag{2.1}$$

where all a, b, c, d are positive real numbers: b and d are positive because of the decreasing/increasing behavior of R and P, while a and c are positive because it is important to grant some nontrivial certainty at the beginning (they can be taken nonnegative as well, but information with null certainty does not seem to be a realistic premise). This restriction is also based on some experimental observations (see Sect. 3) and on the problems imposed by the existence of "irreducible uncertainty" (Zamfir 2005: 64).

Now comes the main assumption:

(C′) As the communication flows, the objective certainty θ considered as a function of communication t changes proportionally with the *excess of certainty of the receptor over the provider*, that is, with $R - P$.

In a mathematical formulation, this assumption can be written as the following equation for the adjustment of the *objective certainty*:

$$\theta(t+h) = \theta(t) + hk[R(\theta(t)) - P(\theta(t))], \tag{2.2}$$

where $h > 0$ is the length of the interval of communication, that is, the interval $(t, t+h)$, and $k > 0$ is a positive number that describes the *speed of change of the* *objective certainty as the response to the excess receptor/provider*. Since k can be absorbed into other coefficients, without loss of generality, from now on, k is assumed to be $k = 1$.

This first-order dynamical system approach is suitable for *objective certainty*, which is an idealized representation of the certainty. Let us begin with the equation for the adjustment of the objective certainty (2.2) and taking into account that $h > 0$ can be equivalently written as

$$\frac{\theta(t+h) - \theta(t)}{h} = \Big(R(\theta(t)) - P(\theta(t))\Big). \tag{2.3}$$

In order to continue the description of this continuous dynamical system, we have to make our fourth assumption.

(D′) The objective certainty function θ is differentiable with respect to the variable t of communication.

This assumption is a constraint of the continuum-communication model and comes from some general point of view on *real-world phenomena* that originates from Sir Isaac Newton and Gottfried Leibniz foundations of calculus. Thus, this assumption takes into account an ideal situation which clarifies the behavior of the function θ especially for theoretical purposes.

Based on assumption (D′) and taking into account the functional representations of R and P as in (2.1), we can pass to the limit as $h \to 0$ in (2.3) (differentiability of θ implies its continuity and R and P are continuous since they are linear functions) and get

$$\frac{\mathrm{d}\theta}{\mathrm{d}t} = (a - c) - (b + d)\theta, \tag{2.4}$$

where we always have to keep in mind that θ is a function of t, and hence that the ordinary differential Eq. (2.4) is valid for all $t \geq 0$. In addition, this is similar to an initial value problem, which specifies the value of θ at 0, that is $\theta(0) = \theta_0$, a given constant. Due to the fact that (2.4) is a linear first-order differential equation, the general theory (Hirsch and Smale 1974) tells us the general solution is the sum of two particular solutions: the constant solution which is

$$\bar{\theta} = \frac{a - c}{b + d}, \tag{2.5}$$

and the solution of the associated homogeneous differential equation, which is

$$\theta(t) = C\mathrm{e}^{-(b+d)t},$$

where C is a constant that must be determined from the initial condition. Therefore, the general solution to the initial problem (2.4) is

$$\theta(t) = \bar{\theta} + (\theta_0 - \bar{\theta})\mathrm{e}^{-(b+d)t}. \tag{2.6}$$

It is now possible to interpret the graph of the objective certainty as a function of communication, considered as a continuous variable and subject to the differentiability assumption (D′). Since $b, d > 0$ it follows that $-(b+d) < 0$ and hence $t \mapsto \mathrm{e}^{-(b+d)t}$ is a strictly decreasing function. The difference is made by the comparison of the initial value θ_0 and the equilibrium value $\bar{\theta}$.

Thus, if $\theta_0 < \bar{\theta}$, that is, the initial objective certainty is less than the equilibrium objective certainty, the function θ is strictly increasing and stabilizes asymptotically at the equilibrium value $\bar{\theta}$.

If $\theta_0 > \bar{\theta}$, that is, the initial objective certainty is bigger than the equilibrium objective certainty, the function θ is strictly decreasing and stabilizes asymptotically at the equilibrium value $\bar{\theta}$.

Finally, if $\theta_0 = \bar{\theta}$, the function θ is constant because it started exactly at the equilibrium value and this solution is the only one that is stable.

2.2 Second-Order Dynamical System Model

The model described in the previous section assumes that the reactions of the receptors and the providers of information are instantaneous and this was reflected in the assumption in (2.1) that the functions R and P depend only on θ and not on its rate of change θ'. However, a more realistic assumption in the case of the *subjective certainty* is that there is a delay in the reactions of the receptors and the providers of information in the decision-making group, mathematically expressible by the fact that both the receptor and the provider functions vary with θ' as well. Therefore, the representation (2.1) should be modified according to the following assumptions:

(A″) Both functions R and P are linear with respect to the variable θ and its rate of change θ'.

(B″) The function R is separately decreasing, while the function P is separately increasing, with respect to both θ and θ'.

The modification of the assumption (A″), when compared to the assumption (A′), can be justified by the fact that a more realistic model should take into account the higher freedom of the receptors and providers, in the sense of sensitivity to the rate of change of certainty, not only the certainty itself. The assumption (B″) can be justified by analogy with that provided for the assumption (B′).

Based on these assumptions, consider the following formal representations of the functions R and P:

$$R(\theta, \theta') = a - b\theta - \beta\theta', \quad P(\theta, \theta') = c + d\theta + \delta\theta', \tag{2.7}$$

where all $a, b, c, d, \beta, \delta$ are positive real numbers.

The main assumption, in this case, is the following:

(C″) As the communication flows, the *rate of change of certainty* considered as a function of communication t, changes proportionally with the *excess of the receptor over the provider*, that is, with $R - P$.

In a mathematical formulation, this assumption can be written as the following equation for the adjustment of the subjective certainty:

$$\theta(t+h) = \theta(t) + h\theta'(t) + \frac{1}{2}h^2[R(\theta,\theta') - P(\theta,\theta')], \tag{2.8}$$

where $h > 0$ denotes the length of a very small interval of communication, and the factor $1/2$ is inserted, in view of the Taylor formula, for consistency of the constants.

On the other hand, since the interest is in the second-order approximation of θ, assumption (D′) is replaced with the following:

(D″) The subjective certainty function θ is twice differentiable with respect to the variable t of communication.

Mathematically, assumption (D″) is represented by the second order Taylor approximation of θ

$$\theta(t+h) = \theta(t) + h\theta'(t) + h^2\frac{\theta''(t)}{2} + \mathrm{o}(h^3), \tag{2.9}$$

where the notation $\mathrm{o}(h^3)$ refers to other terms of order 3 or higher as functions of h, that will be ignored. Thus, from (2.7), (2.8), and (2.9),

$$\theta(t) + h\theta'(t) + h^2\frac{\theta''(t)}{2} + \mathrm{o}(h^3) = \theta(t) + h\theta'(t) \tag{2.10}$$

$$+ \frac{1}{2}h^2\Big((a-c) - (b+d)\theta(t) - (\beta+\delta)\theta'(t)\Big) + \mathrm{o}(h^3). \tag{2.11}$$

By the uniqueness of the Taylor representations in (2.11) the coefficients corresponding to h^n, for $n = 0, 1, 2, \ldots$, can be identified, accordingly. For $n = 0$ and $n = 1$, nothing is obtained. Identification of coefficients of h^2 in (2.11) gives us the second-order linear differential equation with constant coefficients

$$\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2} + (\beta+\delta)\frac{\mathrm{d}\theta}{\mathrm{d}t} + (b+d)\theta = (a-c), \tag{2.12}$$

in the general non-homogeneous form. This equation is usually accompanied by the initial conditions

$$\theta(0) = \theta_0, \quad \theta'(0) = \theta_0', \tag{2.13}$$

and (2.12) together with (2.13) make the *initial value problem*.

The general theory of solving these types of initial value problems (Hirsch and Smale 1974) makes use of the translation by the equilibrium value

$$\bar{\theta} = \frac{a-c}{b+d}, \tag{2.14}$$

and the associated homogeneous second-order differential equation with constant coefficients

$$\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2} + (\beta+\delta)\frac{\mathrm{d}\theta}{\mathrm{d}t} + (b+d)\theta = 0. \tag{2.15}$$

This latter equation can have basically three types of solutions, based on the solutions of the characteristic polynomial equation a discussion on the solutions of the characteristic polynomial equation

$$\lambda^2 + (\beta+\delta)\lambda + (b+d) = 0. \tag{2.16}$$

To this end, consider its discriminant

$$\Delta = (\beta+\delta)^2 - 4(b+d). \tag{2.17}$$

If $\Delta \geq 0$, that is, the characteristic Eq. (2.16) has real solutions (distinct or not) the function θ has an exponential representation, close to the one obtained by the first-order model.

If $\Delta < 0$, that is,

$$\beta + \delta < 2\sqrt{b+d}, \tag{2.18}$$

then the Eq. (2.16) has *two complex conjugate solutions* λ and $\bar{\lambda}$, where

$$\lambda = \frac{-(\beta+\delta) + \mathrm{i}\sqrt{(\beta+\delta)^2 - 4(b+d)}}{2}, \tag{2.19}$$

and the general solution to the initial value problem (2.12)–(2.13) is

$$\theta(t) = \left((\theta_0 - \bar{\theta})\cos\omega t + \frac{\theta_0' + \alpha(\theta_0 - \bar{\theta})}{\omega}\sin\omega t\right)\mathrm{e}^{-\alpha t} + \bar{\theta}, \tag{2.20}$$

where we have denoted

$$\alpha = \frac{\beta+\delta}{2}, \quad \omega = \frac{\sqrt{(\beta+\delta)^2 - 4(b+d)}}{2}. \tag{2.21}$$

In this case, we have the *oscillatory solution* θ as in (2.20) which is stable (that is, $\alpha > 0$), and the oscillatory solution asymptotically tends to the equilibrium value $\bar{\theta}$.

2.3 The General Model

From the earlier sections, it appears that the first- and second-order models capture either the exponential or the oscillatory characteristic that we strive to capture. Therefore, a model that encompasses both oscillation and exponential evolution to equilibrium must be of third order, but generalizing in the same way as before has no empirical or even theoretical justification. To obtain the third order model in a legitimate way, the principle proposed by Hoch (1987, cited in Stanovich and West 1998) for estimating the "consensus effect" was used. Hoch observed the tendency to use ones own opinion as a reference point when projecting the opinion of others (Stanovich and West 1998). This means that a superposition between the first- and second-order models might be more useful for our purposes. Consequently, the general model denotes the mathematical model which bares both the particular characteristics of the first- and second-order models presented earlier. Furthermore, the main characteristics of the general model are: the fact that it is the superposition of the solutions to the first- and second-order dynamical systems and the fact that it is framed by the interactionist perspective. This section will first explain the rationale for using superposition, the methodological limits of these main characteristics and then present the model itself.

The rationale behind the use of the superposition emerged during the pretest experiments along with the observation that the participants were not able to make the difference in their appreciations between *the certainty of the new decision with respect to the initial individual decision* (CII) and *the certainty of the new decision itself* (CIF). Before making this distinction, the certainty evaluations used different reference points: the second estimate referred to the initial one and the others to their previous ones. By forcing a difference and then considering the difference between these, one obtains the evolution of the subjective certainty during the group discussion only with respect to the first solution. Let us call this the *absolute subjective certainty* (CI).

There are two methodological shortcomings of this approach. The first one appears when thinking that imposing a difference is like asking a directive question. In an ideal world, the participants should not be influenced by this, but in the real world, the respondents might feel obliged to indicate a difference even if there really is not any. A different corollary of this problem could manifest in that the participants might want to seem consistent with their first estimate of their certainty.

The second shortcoming appears when thinking that the subtraction is done between subjective measurements. The main problem around using subjective scale measurements for mathematical operations is that they do not have a meaning outside a particular framework or context: each individual's scale is that context. In this case, we can only make sure that subtractions are done within one person's personal scale and not among individuals, thus keeping in mind the fact that their interpretation should only be contextual (Zamfir 1980). Also, the evolutions of certainty for individuals with similar characteristics are computed. These characteristics are given by: the initial knowledge on the decision task, the homogeneity of the group, the motivation and the presence of non-constrained communication. Nevertheless, these should only be interpreted as giving a general tendency of particular groups of individuals.

In addressing a similar problem, Krueger and Zeiger (1993, cited in Stanovich and West 1998) employed Hoch's principle (1987, cited in Stanovich and West 1998) in estimating the "consensus effect" by using a "Projection Index" computed as the difference between the predicted consensus and the real consensus in the group. In this way, they managed to avoid potential biases created by the fact that respondents could not differentiate between their own opinion and their prediction of the consensus given their opinion. Our model works in a similar way, except here, the participants project their certainty with respect to some intermediate decisions. These decisions should (for methodological reasons) have the same reference point, but in reality, they proved not to. Consequently, the difference between two subjective certainties (see Sect. 3.2) was employed.

In this respect, another remark should be made. Since the general model uses the superposition between one model representing an objective certainty and a model representing a subjective certainty, in our data analysis we use the difference between two collected—and thus subjective—certainties (CII and CIF, see Sect. 3.2) to obtain the absolute subjective certainty, one could erroneously conclude that one of these collected certainties must be an objective certainty. Since our model is a linear dynamical system, the solution space is a linear space and what we are actually looking for is a basis in the subspace of solutions. Therefore, the basis we find by measurement might not necessarily be the canonical basis.

The second characteristic of the model is its affiliation to the interactionist perspective. Since the model is constructed as the superposition of the solutions to a first- and a second-order dynamical system, this is supported by the idea that the dynamics of subjective uncertainty emerges in the interaction (Zamfir 1980, 2005; Hirokawa and Poole 1996) between the receptor and the provider.

Initially, this model had been built to account for subjective certainties irrespective of the group or individual variables that might have influenced its evolution. As this proved to be untrue, firstly because communication could not always be unidimensional, and secondly because of other unpredictable influences, we decided to see in which cases this model did apply. Limited resources have allowed us to check only a few of the variables that characterize the individuals and their group contexts which abide to the model, thus leaving aside other variables, like gender, social abilities or roles in social networks and so on that could have influenced it.

Having these issues in mind, one can now proceed to combining the two models obtained before, that is Eqs. (2.6) and (2.20) using the superposition principle. The general equilibrium evolution model is obtained, with the certainty θ as a function of the continuum variable t, as a linear combination of the representations (2.6) and (2.20),

$$\theta_{\mathbf{c}}(t) = c_1 + \big(c_2 \cos(c_3 t) + c_4 \sin(c_3 t)\big) \cdot \exp(c_5 t) + c_6 \exp(c_7 t). \qquad (2.22)$$

Here, the vector $\mathbf{c} = (c_1, c_2, c_3, c_4, c_5, c_6, c_7)$ is the coefficient vector of the general solution and it has to be determined (approximated) from the data.

Moreover, this general model has seven parameters $c_1, c_2, \ldots, c_7$ but only three of them are significant for the problem considered in this article, namely: the parameter c_3 shows the oscillatory behavior of the certainty function, while the parameters c_5 and c_7 show the underlying damping process. There is an important difference between the parameters c_3 on one side and the parameters c_5 and c_7 on the other side. Because c_3 represents a frequency of oscillation, we have to consider its equivalence class $[c_3]$ modulo 2π, while c_5 and c_7 represent the damping speed, which are simply real numbers. Thus, as long as $[c_3]$ stays away from $[0]$ it means that the oscillatory process happens. Furthermore, $c_5 \ll 0$ ($a \ll b$ means a is "significantly" smaller than b)

means a damping of the oscillation process, and $c_5 \ll 0$ and $c_7 \ll 0$ means that the process evolves toward equilibrium. Consequently, a statistical analysis that is meant to support our claim that both oscillatory and evolution to equilibrium occur, based on the experimental data, should refer only to these three parameters.

3 The Experiment

Whether the method employed to collect the data for this study is an experiment or not is not immediately obvious and not trivial to argue. This happens because it is at the boundary of the experimental and the observational. The proposed data collection method is closest to the ABA single-case design (Chadwick et al. 1984) because it measures the dependent variable (subjective certainty) before and after the introduction of the independent variable (group decision-making aimed at reaching consensus). However, it is not an experimental design because on top of the two classical measurements a series of other measurements of the dependent variable were introduced. The additional measurements give a reflection of the dependent variable during the time the independent variable takes place. This methodological choice is strictly connected to the research question of this study: how does the subjective certainty *evolve* during group decision-making aimed at reaching consensus.

On the other hand, the definitions of what an experiment is also bring the chosen data collection method closer to an experiment than to any other method. Webster (2005: 423), in the *Encyclopaedia of Social Measurement* offers one definition of what an experiment is: “a simplified constructed reality in which an investigator controls the level of an independent variable(s) before measuring a dependent variable(s)”. From this point of view, our data collection method is an experiment. On the other hand, from the point of view of the interdependence between the research goal and the method applied to study it, Dean and Voss (1999: 1) point out that experiments may be constructed: “(i) to determine the principal causes of variation in a measured response, (ii) to find the conditions that give rise to a maximum or minimum response, (iii) to compare the responses achieved at different settings of controllable variables, and (iv) to obtain a mathematical model in order to predict future responses.” The choice of the data collection method proposed here is guided by the fourth goal suggested by Dean and Voss. In light of these arguments, the data collection method employed in this study is an experiment, although it is indeed, not a standard experiment.

3.1 Methodology

Using the notations introduced in Sect. 2.3, the absolute subjective certainty, denoted by CI, was obtained by subtracting the certainty of the new decision with respect to the initial individual decision (CII) from the certainty of the new decision itself (CIF). In order to observe a tendency for individuals with similar characteristics, group graphical representations of the absolute subjective certainty were also computed. The following variables were used to group participants by: *initial knowledge on the decision task, homogeneity of the decision group with respect to the initial knowledge on the decision task, the motivation to fulfill the task and the perception of* *non-constrained communication*. These variables, the methodological implications of using them and of constructing the individual subjective certainty graphs are discussed below.

While consensus is aimed at, but not necessarily achieved, *high motivation* and *non-constrained communication* were considered to be necessary conditions. The evolution of the absolute subjective certainty is given by the evolution of the difference between how certain one participant was of the new decision and how certain she/he was of the new decision with respect to the old decision. Non-constrained communication and exchange of information in this case means group communication that is not externally constrained. Based on the fact that decisions by consensus among peers have two major consequences that are of interest to us: non-constrained communication (Zamfir 1980, 2005; Moscovici and Doise 1994) and high satisfaction with the way the decision was made (Moscovici and Doise 1994), we told participants that the aim was to reach consensus and we measured the existence of non-constrained communication and the existence of high satisfaction with the way in which the decision was taken.

On the other hand, efforts were made to ensure that there were no decision groups in "irreducible uncertainty" (Zamfir 2005: 64)—a situation described by the impossibility of reducing the uncertainty by knowledge accumulation, either because with the current knowledge it is not possible to assimilate new information in due time or because there is no distinct way of reducing the uncertainty except knowledge accumulation, which would take too long to take place—by providing them with some initial basic information that would help solve the problem, but would not make the decision trivial. This choice was also made after the pilot studies revealed that participants in irreducible uncertainty could not accumulate new information in due time.

In order to obtain a tendency of the individual certainty, two steps were followed. First, individuals were grouped by some variables (their initial knowledge about the task they had to decide upon and the homogeneity of their decision group in terms of knowledge about the decision task). Secondly, the *group certainty* was computed as the average between equally time-framed strings of collected certainty estimates, interpolated with data from the theoretical model, where needed. The homogeneity of the group was established by the standard deviation of all respondents'—in a decision group—initial opinion about the problem's solution. The difference between the initial individual solution to the task and the correct one was computed in order to have an estimate of the respondent's initial knowledge about the decision task. By combining the initial conditions—the existence of non-constrained communication and high motivation—with the homogeneity of the group in terms of knowledge and the initial level of knowledge about the decision task described above, nine groups (out of which one was void) of participants distributed over three degrees of group homogeneity and three degrees of initial knowledge over the problem were obtained. From this point on, we shall call these simply *groups* to distinguish them from the decision groups. The homogeneity of the group has natural values between 1 and 3, 1 denoting groups with higher homogeneity and 3 denoting groups with lower homogeneity. The initial knowledge also takes values between 1 and 3, where 1 means an initial solution that was closer to the correct one and 3 means an initial solution that was further away from the correct one. These figures appear in the group numbers in Table 3 from Sect. 5

In approximating the tendency of the absolute subjective certainty, two different computation methods were available. While comparing the two methods, CId denoted the group absolute subjective certainty computed as a difference between averaged CII and CIF and CIm denoted the group absolute subjective certainty computed as the average of the differences between individual CII and CIF estimates. This distinction yielded different results only for the largest group and suggested that in case of a larger scale experiment one should also check whether this difference is significant. Interpolation was used for generating the visual representations which needed equally sized strings of data. Nevertheless, the measures that evaluated the model are not influenced by this.

3.2 Experimental Design

The experiment was built around measuring the dependent variable (subjective certainty) before, during and after introducing the independent variable (communication in a decision group). Each group (4–6 people) was shown a problem to be solved by consensus [NASA moon survival task (Teleometrics International 2007)]:

You are a member of a crew scheduled to meet the mother-ship on the bright face of the Moon. Due to some malfunction your ship was forced to land 322 kilometers away from the meeting point. During the landing procedure, most of the equipment on board was destroyed. Since your survival depends upon reaching the mother-ship, you have to choose the most important items available in order to walk the distance to the mother-ship. There are 15 items left intact after the impact. Your mission is to order them according to their importance such that your crew will reach the meeting point. Number with 1 the most important item and with 10 the least important item.

- *Match box*
- *Condensed food*
- *15 meters of nylon thread*
- *Parachute silk*
- *Portable heating unit*
- *Two 45 caliber pistols*
- *1 box of condensed milk*
- *2 oxygen tanks of 45 kg each*
- *Stellar map (around the moon)*
- *Automatic inflation rescue vest*
- *Magnetic compass*
- *19 liters of water*
- *Signaling missiles*
- *First aid kit including needles and a syringe*
- *Radio emitter and transmitter with solar batteries*

Before interacting with the other members of the group, each person was asked to write her/his own solution for the problem and how certain she/he was about it on a seven step scale (where 7 is "very certain" and 1 is "not certain"). During the discussion each participant was asked to write the new solutions, the time at which it occurred and how certain she/he was about them as well as how certain she/he was of the initial solution, as soon as she/he realized a change in her/his old opinion. The aim of the discussion was to reach a consensus. When consensus was reached or when the group decided that it is impossible to reach, each participant was asked to:

– individually write the solution,
– write how certain she/he was about it and about the initial solution,
– write how certain she/he was about the group decision,
– write how satisfied they were with the way the decision was made as a measure of having aimed to reach consensus and implicitly of having experienced non-constrained communication.
– and how interesting they had found the decision problem, as a measure of the motivation to participate in the experiment.

3.3 Sample

The experiment was applied to 97 students of sociology or social work (who did not attend decision theory classes), taken in groups of 4–6 participants, selected by purposive sampling. Six of the participants were excluded because they had not filled all items in the form. As discussed previously, to ensure that consensus was aimed at, only participants with high motivation to participate and high satisfaction with the way the decision was made were selected for further analysis. This left for analysis only 48 people.

One unexpected outcome in the experiment compelled the use of another selection criterion. When part of the participants chose a decision strategy which was not linear, meaning that they changed their mind several times about the same item of the sequence they had to order, we had to put aside 9 more subjects, thus having a final sample of 39 people. Participants were basically asked to record the time when they changed their mind during the group discussion and the new decision according to this change. When some participants changed their mind about two or three items at the same time, and consequently changed their mind several times about the same items, it meant that, at the same time, for each unit of communication, there was more than one unit of information disseminated in the group discussion. Since, in our model, communication is an exchange of information, to be able to express the certainty in terms of communication, a linear relationship should take place between time and communication or, in other words, between time and the exchange of knowledge. Because of this, we named the two ways of dealing with the decision task personality[4] type B[5] (nine participants—approximately 19 %) and personality type A (39 participants—approximately 81 %). This behavior is in line with the theory of the adaptive decision-maker (Payne et al. 1993), but may also be explained by the fact that several cognitive representations were processed at the same time (Thagard 2005). Consequently, because the decision strategy chosen by type B does not allow for a one-dimensional approximation of communication, in this paper, only type A was analyzed.

[4] In the context described at the beginning of Sect. 2 regarding the view of individual characteristics as comprising of both social and individual characteristics.

[5] We thank Cătălin Zamfir for bringing this issue to our attention.

Nevertheless, the sample size is not small since the analysis is performed on time series. For time series, the sample size is usually between 20 and 60 (Beck and Katz 1995, 2007). Since there is no information about the population, the sample size cannot be computed by using the usual methods of Bayesian statistics because the errors for time series are not independent. Therefore, the question that one should ask is how would the coefficients behave if the sample size would have been smaller. To answer this question, a bootstrapping or re-sampling method (Hastie et al. 2009) has been undertaken for smaller samples, of, for example, 15 participants. Basically, bootstrapping is a general tool for assessing statistical accuracy by re-sampling the original data (Hastie et al. 2009). The conditions required for coefficients $[c_3]$, c_5 and c_7 will then be analyzed for all sub-samples. In our case, 1000 sub-samples of size 15 have been drawn from the pool of participants and for each sub-sample the mean of the three coefficients of interest has been computed. The resuls revealed the fact that the distribution of sub-sample means for $[c3]$ is entirely placed above 0 ($Mean = 0.272, SEM = 0.002, SD = 0.069$) and that the distribution of sub-sample means for c_5 ($Mean = -0.08, SEM = 0.001, SD = 0.05$) and c_7 ($Mean = -0.045, SEM = 0.001, SD = 0.042$) are entirely placed below 0, thus fulfilling the condition imposed for these coefficients. In conclusion, had the sample been smaller, the coefficients would still fulfill the conditions needed for oscillation and evolution to equilibrium.

3.4 Hypothesis

Having in mind all these methodological aspects, the main hypothesis of this study is:

Hypothesis If participants in a group where decision-making is by consensus employ a one-dimensional communication strategy, the certainty during the decision-making process oscillates and has an evolution to equilibrium.

In order to evaluate this hypothesis, based on the model presented in Sect. 2.3, the coefficients $[c_3]$, c_5 and c_7 shall be evaluated against the requirements for oscillation and evolution to equilibrium presented in the same section. This means that $[c_3]$ is expected to be significantly ($p < 0.05$) greater than [0] and that c_5 and c_7 are expected to be significantly ($p < 0.05$) lower than 0. These coefficients were obtained for each participant by using the collected individual absolute subjective certainty. The numerical procedure employed to obtain the coefficients from the collected data will be described in Sect. 4.

4 Numerical Procedure

Following the experiment, the collected data have been represented by two sequences

$$\text{xdata} = (x_1, x_2, x_3, \ldots, x_N), \quad \text{ydata} = (y_1, y_2, y_3, \ldots, y_N), \tag{4.1}$$

where N is the number of samples, x_j is the time sequence, expressed in seconds, and y_j are the collected data for the certainty, corresponding to the time x_j. In our case,

N has values between 13 and 20 for individuals and, respectively, between 40 and 80 for groups, the data x_j are between 0 and 120, while the data y_j have values between -6 and $+7$.

After collecting the experimental data, it is necessary to find the best approximation, within our abstract model, for which the *least-square approximation method*[6] was used, basically by minimizing the following function

$$F(f) = \sum_{j=1}^{N} \Big(f(x_j) - y_j \Big)^2, \tag{4.2}$$

which is obtained as the Euclidean distance in $\mathbb{R}^N$ between the vectors $[f(x_1), f(x_2), \dots, f(x_N)]$ and $(y_1, y_2, \dots, y_N)$. In order to use the available approximation functions, for example, lsqcurvefit in MATLAB, the class of functions $\mathcal{F}$ within the optimal solution needs to be specified. This is the pragmatic motivation for developing the abstract mathematical model in Sect. 2. Thus, by the general formula obtained in (2.20), one obtains the class of functions

$$\mathcal{F} = \{ f_{\mathbf{c}} \mid \mathbf{c} \in \mathbb{R}^7 \}, \tag{4.3}$$

where for each vector $\mathbf{c} = (c_1, c_2, c_3, c_4, c_5, c_6, c_7)$

$$f_{\mathbf{c}}(x) = c_1 + \big(c_2 \cos(c_3 x) + c_4 \sin(c_3 x) \big) \cdot \exp(c_5 x) + c_6 \exp(c_7 x). \tag{4.4}$$

Finally, these yield the problem of minimization of the function

$$G(\mathbf{c}) = \sum_{j=1}^{N} \Big[c_1 + \big(c_2 \cos(c_3 x_j) + c_4 \sin(c_3 x_j) \big) \cdot \exp(c_5 x_j) + c_6 \exp(c_7 x) - y_j \Big]^2, \tag{4.5}$$

where each component c_j of the vector $\mathbf{c} = (c_1, c_2, c_3, c_4, c_5, c_6, c_7)$ is sought in a "confidence interval" $[c_j^{\min}, c_j^{\max}]$.

The application requires a shooting data (starting point) of the iterations

$$\mathbf{c}(0) = \big(c_1(0), c_2(0), c_3(0), c_4(0), c_5(0).c_6(0), c_7(0) \big),$$

within the confidence interval. The initial values of $\mathbf{c} = (c_1, c_2, c_3, c_4, c_5, c_6, c_7)$ for the model in (2.22) were generated with an improvement of 70 %.

[6] Beck and Katz (2007) show that least-squares approximations for time series in cross-section data are appropriate only if the errors are spherical. This means that the variance of the distribution of variances of residuals is relatively small ($VVR = 1.649589$) and that the correlation matrix of the residuals is diagonal (has close to 0 correlations outside of the diagonal). The results of this analysis showed that the use of the ordinary least-squares approximation method employed is justified and appropriate.

5 Results

In order to check the oscillatory character and the evolution to equilibrium of the absolute subjective certainty, we must check that $[c_3]$ (see Sect. 2.3), from Eq. 2.22, is significantly different from [0] and that c_5 and c_7, from the same equation are significantly < 0. All values of these three coefficients are given in the Appendix, and it becomes apparent that only one of the values of $[c_3]$ is 0 and that both c_5 and c_7 are negative. From Table 1 and the boxplots in Figs. 1, 2, 3, we may see that the distributions of c_5 and c_7 are skewed to the left around an expected value of $-.07578$ (SEM $=.0216$) and $-.04489$ (SEM $=.0256$), respectively. At the same time, the distribution of $[c_3]$ is skewed to the right (skewness $=4.892$; SE $=.378$). The subjective certainty evolution with the average coefficients is given in Fig. 4. Note that although the boxplots look scattered, they are within the required ranges, given in Sects. 2.3 and 3.1. The extreme values in the distribution of c_3 are different than 0,

Table 1 Statistics for the coefficients of the absolute subjective certainty

Coefficients for the absolute certainty	[C3]	C5	C7
N Valid	39	39	39
N Missing	0	0	0
Mean	.37889	−.07578	−.04489
SEM	.10888	.02167	.02567
Median	.20290	−.02350	−.00350
SD	.67994	.20318	.16026
Variance	.462	.041	.026
Skewness	4.892	−3.512	−5.865
SE of Skewness .378	.378	.378	.378
Minimum	.0000	−1.0000	−1.0000
Maximum	4.1495	.2243	−.0010

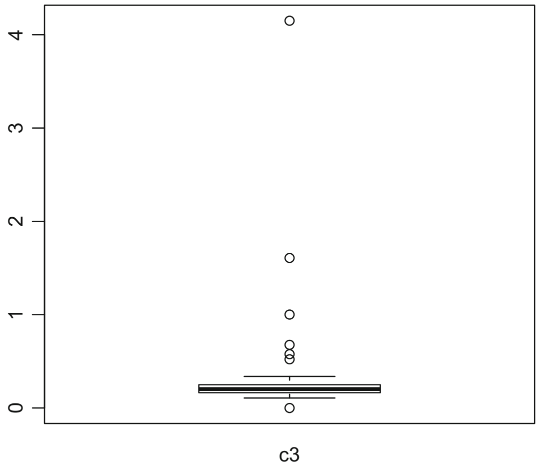

Fig. 1 Boxplot for coefficient c_3 from Eq. (2.22)

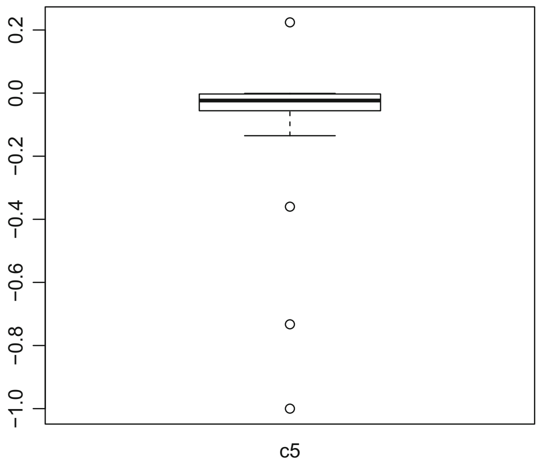

Fig. 2 Boxplot for coefficient c_5 from Eq. (2.22)

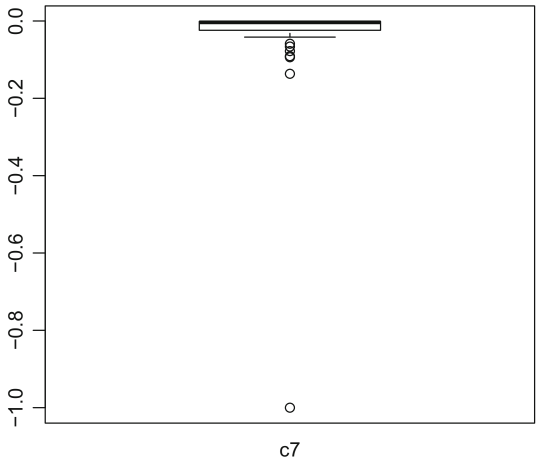

Fig. 3 Boxplot for coefficient c_5 from Eq. (2.22)

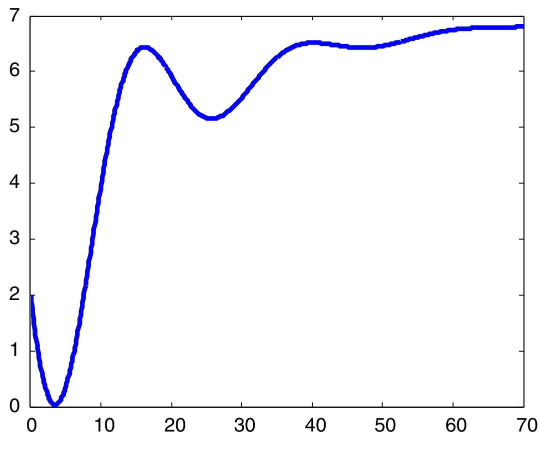

Fig. 4 Expected graphical representation of the individual absolute subjective certainty based on the mean values for c_3, c_5 and c_7 obtained experimentally

Table 2 Central tendencies for the distance to the solution

	Distance to the solution	Relative error
Min	0.0023	0.00016
Max	9.5916	0.8097
Average	3.6791	0.3552
Median	3.3519	0.3433
SD	2.1182	0.2055

Table 3 Results of the evaluation of group graphs by qualitative and quantitative criteria

Gr	Oscillation				Exponential trend				AGDS	AGRE
	CII	CIF	CId	CIm	CII	CIF	CId	CIm		
11	1	1	1	1	0	0	1	1	4.2893	0.3915
13	1	1	1	1	0	1	1	1	1.7474	0.1438
21	1	1	1	1	0	1	1	1	2.1380	0.1535
22	1	1	1	1	0	1	1	1	4.5536	0.2840
23	1	1	1	1	0	1	1	1	1.6838	0.2148
31	1	1	1	1	0	1	1	1	3.1105	0.3215
32	1	1	1	1	0	0	1	1	2.8617	0.1811
33	1	1	1	1	0	1	0	1	2.6077	0.1940
T	8	8	8	8	0	6	7	8	2.874021875	0.23554375

Gr is the group number, *AGDS* is the average group distance to the solution, *AGRE* is the average group relative error, and *T* is for the total

and the extreme values in the distribution of c_5 and c_7 are negative, so they all obey the requirements regarding oscillation and evolution to equilibrium.

Secondly, we evaluated the distance between the shooting data (see Sect. 4) and the output data. Tabl 2 shows the characteristics of the distribution of the distances to the solution which point out how much the shooting point was improved by the curve fitting program. These values range between 0.0023 and 9.5916, with a standard deviation from the mean of approximately 2.12 and a mean of 3.68 points (SE = 0.36). This only speaks of the efficiency of the curve fitting program and, in general, the greater the distance, the better the efficiency.

In Table 3, the evolution of the subjective group certainties (the meaning of each number in the group name is explained in Sect. 3.2) is qualitatively analyzed (1 means that the graph corresponding to the group on the row follows the behavior indicated on the column and 0 means that it does not). All group CIm oscillate on an exponential trend, while 7 of the 8 CId have the same behavior. As these represent the tendency for each group of participants with similar characteristics, the group graphs do not show other behaviors as in the case of the individual ones. They do show that the data for each group tend to approximate the general model without being able to establish significant differences among groups. For example, Figs. 5, 6, 7, 8 and 9 from the Appendix, show a few individual absolute certainty graphs and two group ones.

Table 4 t test for equality of means

	t	df	Sig. (2-tailed)	Mean difference	95 % CI of diff.	
					Lower	Upper
One-sample test						
Test value = 0						
c_3	3.480	38	.001	.37889	.26448	.49330
c_5	−2.329	38	.025	−.07578	−.10997	−.04160
c_7	−1.749	38	.088	−.04489	−.09684	.00706
Test value = 1						
e^{c_5}	−2.545	38	.015	−.05776	−.10370	−.01182
e^{c_7}	−2.112	38	.041	−.03478	−.06811	−.00144

Furthermore, in order to quantitatively establish if $[c_3]$ is significantly different than 0 and that c_5 and c_7 are negative and significantly different than 0 a statistical test should be employed. Despite the fact that $[c_3]$, c_5 and c_7 are not normally distributed (Kolmogorov–Smirnov z test for normality of distribution), based on the Central Limit Theorem for cases in which the distribution of the population is not normally distributed, and the fact that the sample used is conventionally considered a large one ($N > 30$, Mann 2010), it is possible to use the t test of significance to see if there is a significant difference between the means of these coefficients and 0. Some computational problems of approximation arise because the values computed from the data are very close to 0 (see Table 4), and because the t test has difficulties dealing with values close to 0. This is revealed by the non-significant result of the t test when performed upon the computed values of c_7. This is why the value of e^{c_5} and e^{c_7} were computed, and the mean was compared to 1 ($=e^0$). The results of the t test computed upon these values show that $[c_3]$, c_5 and c_7 are significantly different than 0. Moreover, they are so, in the direction indicated by the hypotheses in Sect. 2.3, namely $[c_3]$ is strictly positive and c_5 and c_7 are strictly negative.

From the above analysis, it is possible to conclude that:

– all coefficients involved in the oscillatory and the evolution to equilibrium character are such that the individual subjective certainties evolve on an oscillatory and exponential trend;
– despite the fact that sometimes, evolution to equilibrium may not be completely visible to the eye, as shown by the qualitative analysis, the conditions imposed for oscillation and evolution to equilibrium to occur have been met.

6 Conclusions

To sum up, in this paper, a dynamical system model by the superposition of a first-order model with a second-order model was obtained, which approximates the evolution of the absolute subjective certainty for individuals in externally non-constrained group communication, who had high motivation to participate in the decision process, and applied a decision strategy that allowed for a unidimensional approximation of the communication. The design and methodology of an experiment were presented, together with the data analysis performed by using a numerical approximation procedure within the set of our abstract model. Judging by the number of analyzed cases in which oscillation and evolution to equilibrium have been observed it is possible to say that the model is reasonably confirmed. The evolution to equilibrium was shown to be exponential, ascending and damped. However, for individuals who applied a decision strategy that did not allow for a unidimensional approximation of the communication (approx. 19 % of participants), a generalization of this model based on multidimensional communication should be further tested, in order to complement this paper. Also, this model does not cover the cases of "irreducible uncertainty" (Zamfir 2005).

Some new facts about the evolution of the subjective certainty came to our attention once this mathematical model and the experiment had been done:

1. The subjective certainty generally oscillates, most of the time without reaching its equilibrium during the decision-making process.
2. Under non-constrained group communication, the equilibrium value of the certainty either increases or it remains constant.

The importance of the oscillatory character of the certainty could constitute an argument in support of seeing the certainty in decision-making groups as a dynamical process. This also suggests that local values of the certainty in decision problems should be treated with caution. The second conclusion does not suggest that some (see Sect. 4) individuals in a group decision-making situation tend to converge in their opinions, but rather that, when consensus is aimed at (thus not necessarily achieved), they tend to either be as sure as or more certain than they were before, irrespective of the accuracy of their decision. This finding is consistent with the results obtained by Oskamp (1982) for individual judgments of medical case-studies. As it was shown here, the same happens in group decisions. Still, in generalizing these results, caution is recommended, because this experiment was done on a restricted number of individuals. Concerning the assertion of Zamfir (2005) that subjective certainty as a function of objective certainty has a damped oscillation and evolution to equilibrium, the results obtained here show his claim is plausible, but most likely not the only one.

The linearity of our dynamical system model has the advantage that the solution is global, and hence without imposing limits on the time of running the decision-making process. The cost of this is that, in order to reflect the full complexity of the phenomenon, the degree of the dynamical system should be at least three. However, since applying this model to some groups or individual participants has turned out to be inappropriate in certain cases, nonlinear dynamical systems, with only local valid solutions, might be considered for future investigations. In the latter case, bifurcation and chaos might explain some of the situations when the linear dynamical system model does not reasonably work.

In order to show the existence of oscillatory and evolution to equilibrium nature in a small decision-making group by consensus, our claim is that it is sufficient to consider only linear dynamical system models. Considering a model that reflects at a deeper level the interactive nature of the decision-making group is a much more ambitious task that falls beyond the aims of this research. In addition, it is our belief that a chaotic behavior is very likely to show up in a significant number of real decision-making groups, e.g. in type B participants, which might be reflected by more sophisticated nonlinear dynamical system models. On the other hand, the oscillatory behavior may be a consequence of an inherent chaotic process (even at the mathematical level there is no general agreement of what “chaos” means as a quantitative category) and hence this path was not followed, despite the fact that it looks very interesting and appealing from the social research point of view. However, this may be the subject of a future research, once a nonlinear dynamical system model will be available.

The implications of these results are two fold. On the one hand, the fact that certainty in group decision-making has an oscillatory evolution means that when measuring or using the certainty at any point in time before consensus is achieved may lead to either an overestimation of the certainty or an underestimation of it. This is in line with the literature on overconfidence and under-confidence. The findings of this article are thus likely to give a future explanation for some cases of over- and under-confidence in group decisions.

The second implication of this study is that when using linear approximations of the certainty (as it is done when using initial and final estimates of the certainty, Slevin et al. 1998; Niederman and Bryson 1998), one needs to be very careful when choosing the interval of time in the group discussion. Too short intervals of time before reaching consensus might give too rough and sometimes even inappropriate approximations. At the opposite end, intervals that are either closer to the end of the decision-making process aiming at achieving consensus or intervals that capture as much as possible of the time needed to achieve consensus may be more appropriately approximated in a linear way.

Furthermore, these implications are only valid for the conditions in which the experiment described here was conducted and especially for those participants who employed a unidimensional communication. Further research is needed for the others.

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7 Appendix

See Table 5.

Table 5 Subjective certainty coefficients for all subjects

[C3]	C5	C7
0.1279	−0.0010	−0.0010
0.1859	−0.0330	−0.0909
0.1851	−0.0733	−0.0417
1.0012	−0.0553	−0.0239
0.2206	−0.0565	−0.1365
0.2673	−0.0010	−0.0010
0.1481	−0.0164	−0.0035
0.3385	−0.0235	−0.0033
0.0000	−1.0000	−0.0661
0.2165	−0.0169	−0.0937
0.1432	−0.0195	−0.0010
0.2507	−0.0118	−0.0242
0.5218	−0.3599	−0.0010
0.1976	−0.0010	−0.0078
0.2248	−0.0047	−0.0778
0.1069	−0.0762	−0.0010
0.1484	−0.0246	−0.0010
0.1505	−0.0010	−0.0010
0.2040	−0.0275	−0.0221
0.1801	−0.0247	−0.0010
0.1431	−0.0129	−0.0019
0.1786	−0.0010	−0.0010
0.1852	−0.0010	−0.0136
0.2359	−0.0431	−0.0010
0.6766	−0.7327	−0.0097
0.2409	−0.0305	−1.0000
0.2163	−0.0010	−0.0010
0.1066	−0.0894	−0.0315
0.1604	−0.0010	−0.0010
0.2130	−0.0053	−0.0100
0.1669	−0.0156	−0.0064
0.2029	−0.0307	−0.0049
0.1751	−0.0346	−0.0017
1.6076	−0.0923	−0.0046
0.1920	−0.1174	−0.0010
0.5769	−0.0010	−0.0588
0.2830	−0.1350	−0.0010
0.2471	−0.0076	−0.0010
4.1495	0.2243	−0.0010

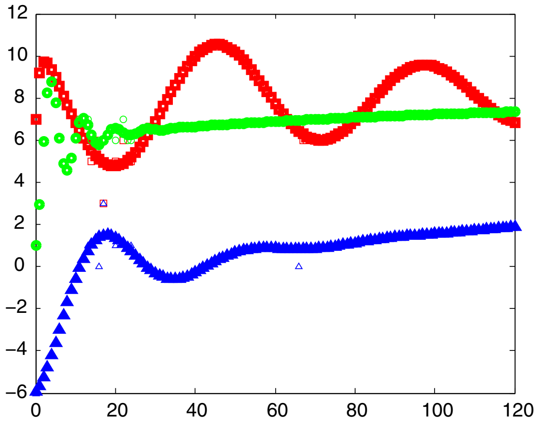

Fig. 5 Participant J from group 21 with the three graphs of CII (*square*), CIF (*circle*), and CId (*triangle*)

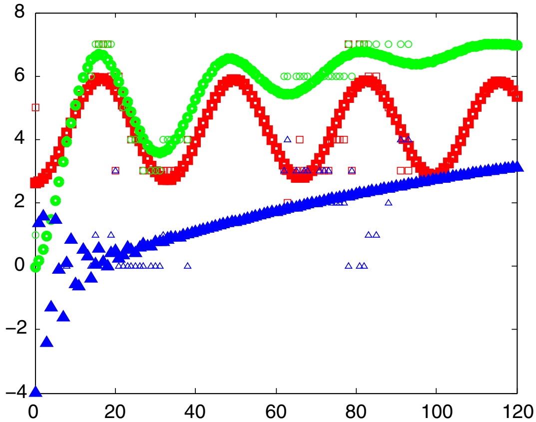

Fig. 6 Participant MY from group 22 with the three graphs of CII (*square*), CIF (*circle*), and CId (*triangle*)

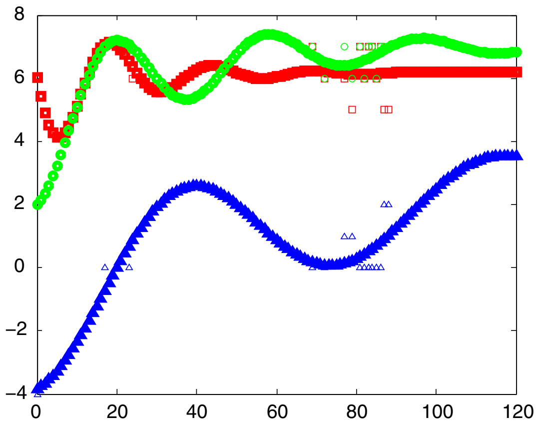

Fig. 7 Participant K from group 23 with the three graphs of CII (*square*), CIF (*circle*), and CId (*triangle*)

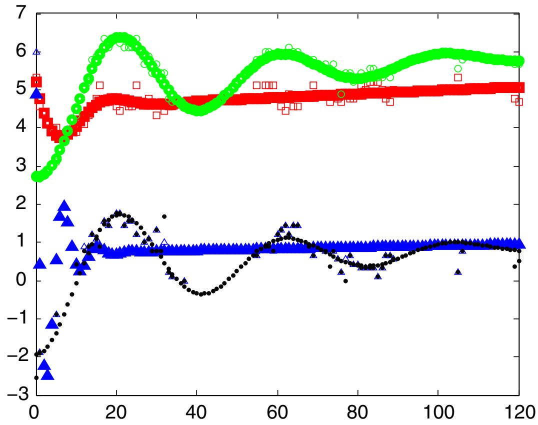

Fig. 8 Group 23 with the four graphs of CII (*square*), CIF (*circle*), and CId (*triangle*) and CIm (*dot*)

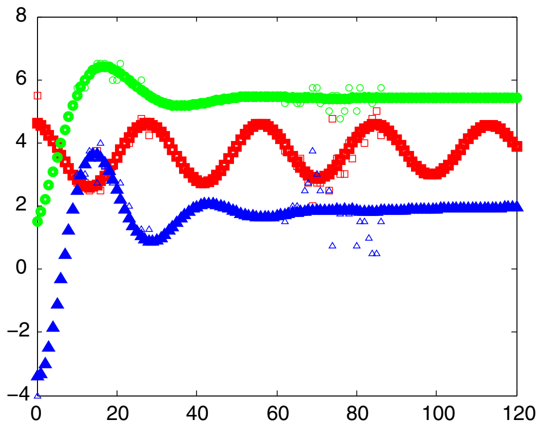

Fig. 9 Group 31 with the four graphs of CII (*square*), CIF (*circle*), and CId (*triangle*) and CIm (*dot*) (last two are overlapping)

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