Group Decis Negot (2015) 24:415–428
DOI 10.1007/s10726-014-9396-4

# A Consensus Gap Indicator and Its Application to Group Decision Making

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Published online: 14 June 2014


**Abstract** A prametric is introduced for describing the consensus gap in group decision making problem and a consensus formation procedure is proposed. It is assumed that each individual's ties-permitted ordinal ranking constitutes a partition of the alternative set, and that the alternatives in a tie are ranked together occupying consecutive positions. A preference sequence matrix is thus constructed with entries indicating the alternatives' potential positions by the expert's preferences. To elicit the group ranking, a certain prametric, namely, the consensus gap indicator is defined for measuring the consensus gap between two preference sequences with ties. Some properties are elaborated, among which an inequality is used to get the potential ties-permitted compromise ranking. An illustrative example is also included.

**Keywords** Group decision making · Ordinal ranking · Consensus gap indicator · Prametric

## 1 Introduction

Group decision making (GDM) is nowadays a fascinating research topic in management science or operational research (Arrow 1951; Dyer and Sarin 1979; Keeney 2013; Xu 2013; Zeng 2013; Zhou et al. 2013). For obtaining a group ranking when the individual provided ordinal rankings, a distance metric model was proposed (see Cook and Seiford 1978; Armstrong et al. 1982; Cook et al. 1997; Cook 2006). In a considered GDM, Cook and Seiford (1978) noted a problem of the presence of multiple solutions to an Assignment Model, but did not offer a solution in that paper. In a subsequent study conducted by Armstrong et al. (1982), a strategy for dealing with the multiple

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solution case was proposed by transforming the Assignment Model into a Constrained Transportation Model. Since an assignment problem can be seen as a special case of a transportation problem, thus, the strategy (see Armstrong et al. 1982) by using a more general model to tackle a more general problem is a natural and excellent choice.

This paper attempts to handle the problem from another prospective by relaxing the metric introduced by Cook and Seiford (1978). Let's consider an example of intransitivity in decision practice. There are three persons: Tom, Jack and John; and two drinks: tea and coffee. It is not rare that: Tom prefers {tea}, John prefers {coffee}, and Jack's preference is {tea,coffee} (namely, Jack prefers 'tea or coffee', a tie appears). The consensus intransitivity comes out that, *Tom and Jack have preferences in common, also Jack and John have preferences in common, but, Tom and John do not necessarily have preferences in common*.

If a mathematical tool could be found to deal with the above-mentioned consensus intransitivity (thus the triangle inequality does not need to be verified), it is of significance to the study of GDM with ties-permitted ordinal ranking involved. The *prametric* used in this paper is such a candidate mathematical tool as to tackle this kind of problem as interpreted in more details in Sects. 2 and 4.

## 2 Prametric

The mathematical concept of distance metric has been used in GDM (see Cook and Seiford 1978). For a function $d$ to be a distance metric, it must satisfy the following axioms:

- $d(x, y) \geq 0$.
- $d(x, x) = 0$.
- $d(x, y) \neq 0$ if $x \neq y$.
- $d(x, y) = d(y, x)$.
- $d(x, z) \leq d(x, y) + d(y, z)$ (triangle inequality).

A prametric is an 'almost metric' by relaxing the last three properties (see Gavshin and Kruusmaa 2008; Skala 2008; Inuma et al. 2009; Typke and Walczak-Typke 2010).

**Definition 2.1** A function $d$ is defined as a prametric by verifying:

- $d(x, y) \geq 0$.
- $d(x, x) = 0$.

In this study, a prametric is introduced to deal with the GDM when the individual provides ordinal ranking with ties. It is not only as a result of that "relaxed method corresponds to relaxed input", but also of the fact that consensus intransitivity happens to the preferences in practice.

## 3 Assumption and Preference Sequence Matrix

We consider a GDM with $m$ experts and $n$ alternatives involved. The individuals provide their preferences as ties-permitted ordinal rankings. To conduct our study, we assume that:

- each individual ties-permitted ordinal ranking constitutes a partition of the alternative set.
- the alternatives in a tie are ranked together occupying consecutive positions.

Under the above assumptions, a ties-permitted ordinal ranking corresponds to a *preference sequence vector* whose entries are sets with consecutive natural numbers. An entry of a preference sequence vector corresponds to an alternative's potential ranking position(s). Different entries of a preference sequence vector constitute a partition of the set $I = \{1, 2, \ldots, n\}$, since we assume that a ties-permitted ordinal ranking constitutes a partition of the alternative set.

The following example, which will be carried through our discussion to make some concepts clear, shows three experts expressing their ordinal preferences about four alternatives.

*Example 3.1* Consider three experts are involved for ranking four alternatives. The individual ties-permitted ordinal rankings are:

$$\begin{aligned} &\text{e1: } A_2 \succ A_3 \succ A_1 \succ A_4, \\ &\text{e2: } A_2 \succ A_1 \sim A_3 \succ A_4, \\ &\text{e3: } A_1 \sim A_2 \succ A_3 \sim A_4. \end{aligned}$$

In this example, clearly, the first assumption is satisfied e.g. according to e2's preference the alternative set is partitioned into three subsets $\{A_2\}$, $\{A_1, A_3\}$, and $\{A_4\}$. From the second assumption, the individual ordinal rankings can be changed into partitions of the set $\{1, 2, 3, 4\}$, e.g., e2's preference indicates: $A_2$ is ranked at position $\{1\}$, $A_1$ and $A_3$ in a tie at positions $\{2, 3\}$, and $A_4$ at position $\{4\}$, where $\{1\}$, $\{2, 3\}$ and $\{4\}$ constitute a partition of $\{1, 2, 3, 4\}$. The e2's preference sequence vector is given by

$$(\{2, 3\}, \{1\}, \{2, 3\}, \{4\})^{\mathrm{T}},$$

where the character 'T' is referred to transpose; the first entry, namely $\{2, 3\}$, corresponds to $A_1$'s potential ranking positions, the second to $A_2$'s, and so on.

We denote a preference sequence vector by $(U_i)_{n\times1}$ and denote the cardinal number of a set $U$ by $|U|$. For instance, consider e2's preference sequence vector in Example 3.1, which reads $(U_i)_{4\times1} = (\{2, 3\}, \{1\}, \{2, 3\}, \{4\})^{\mathrm{T}}$. Since the entries of $(U_i)_{4\times1}$ are sets, we get that the corresponding cardinal numbers of its entries are 2, 1, 2, and 1, respectively.

The following proposition characterizes the property of a preference sequence vector.

**Proposition 3.1** *If $(U_i)_{n\times1}$ is a preference sequence vector, then,*

*(1)* $\forall i \in I, U_i \neq \phi$.
*(2)* $\bigcup_{i=1}^{n} U_i = I$.
*(3)* $\forall r, s \in I$, *either* $U_r = U_s$ *or* $U_r \cap U_s = \phi$.
*(4)* $\forall i$, *$U_i$ is a consecutive set over $I$, namely, if $|U_i| > 1$, then the elements of $U_i$ are consecutive natural numbers.*

(5) *If a set e.g. {2,3} is an entry, then, the set {2,3} appears* $|\{2,3\}|$ *times as entries of the preference sequence vector, where* $|\{2,3\}|$ *is the cardinal number of set {2,3}.*

In mathematical language, a partition corresponds to an equivalent relation. Thus the above proposition does hold. The Proposition 3.1 can be explained as follows:

- Item 1 implies that an alternative occupies at least one position. For instance, by e2's preference sequence vector $(U_i)_{4\times 1} = (\{2,3\}, \{1\}, \{2,3\}, \{4\})^{\mathrm{T}}$ in Example 3.1, we know that expert e2 prefers $A_1$ to be ranked at position 2 or 3, $A_2$ at position 1, $A_3$ at position 2 or 3, and, $A_4$ at position 4.
- Item 2 implies that a position is occupied at least by one alternative, e.g. in Example 3.1, e2's preference sequence vector $(U_i)_{4\times 1} = (\{2,3\}, \{1\}, \{2,3\}, \{4\})^{\mathrm{T}}$ represents that position 1 should be occupied by $A_2$, position 2 by $A_1$ or $A_3$, position 3 by $A_1$ or $A_3$, and, position 4 by $A_4$.
- Item 3 implies that for any two alternatives, they will be ranked at different positions except when they are in a tie. As shown by e2's preference sequence vector $(U_i)_{4\times 1} = (\{2,3\}, \{1\}, \{2,3\}, \{4\})^{\mathrm{T}}$ in Example 3.1, $A_2$ occupies position 1 different from $A_4$'s position; however, positions 2 and 3 are for $A_1$ and $A_3$, since they are in a tie.
- Item 4 holds as a result of our second assumption that the alternatives in a tie are ranked together occupying consecutive positions.
- Since an entry of a preference sequence vector corresponds to a partition block with respect to an alternative, item 5 is deduced from item 1, 2, and 3.

For sake of convenience, we also call a vector satisfying all the items in Proposition 3.1 a preference sequence vector.

Further, if we use the preference sequence vectors as columns in a matrix, we get a *preference sequence matrix*, whose $(j,k)$th entry indicates the alternative $j$'s possible position (or positions) based on the expert $k$'s preference. Specifically, if individual preferences are provided as that in Example 3.1, the following preference sequence matrix can be constructed:

$$\begin{array}{c c} & \begin{array}{ccc} e_1 & e_2 & e_3 \end{array} \\ \begin{array}{c} A_1 \\ A_2 \\ A_3 \\ A_4 \end{array} & \begin{pmatrix} \{3\} & \{2,3\} & \{1,2\} \\ \{1\} & \{1\} & \{1,2\} \\ \{2\} & \{2,3\} & \{3,4\} \\ \{4\} & \{4\} & \{3,4\} \end{pmatrix} \end{array}, \tag{3.1}$$

where e.g. the (3, 2)th entry , namely {2, 3}, indicates alternative $A_3$'s possible positions based on expert $e2$'s preference.

## 4 Consensus Gap Indicator: A Prametric

For eliciting a compromise ranking, we first introduce a prametric called the *consensus gap indicator* for measuring the consensus gap of one preference sequence from another. Subsequently, we examine properties of the consensus gap indicator.

For two preference sequences, having intersection or not is their basic relation. If two preference sequences overlap, we say that the pair of rank positions indicated by these two preference sequences is in a consensus. The consensus gap between two preference sequences is 0 if they overlap (that is, if their intersection is a non empty set). Otherwise, we need to define the consensus gap indicator. It can be done as follows:

**Definition 4.1** Let $S_1$ and $S_2$ be two preference sequences over $I = \{1, 2, \ldots, n\}$. We denote the consensus gap indicator between $S_1$ and $S_2$ as:

$$\Delta(S_1, S_2) = \max\{0, \min S_1 - \max S_2, \min S_2 - \max S_1\}, \tag{4.1}$$

where: $\min S_i$ and $\max S_i$ are the minimum and maximum values of $S_i$, respectively.

The consensus gap indicator of Eq. (4.1) provides a measure for evaluating the consensus gap of two experts' preference sequences: If two experts have common preferred positions with respect to an alternative, then, $\Delta(S_1, S_2) = 0$ (no gap occurs); else, $\Delta(S_1, S_2) > 0$. For illustration, consider the preference sequence matrix (3.1) of Example 3.1. Expert 1 prefers $A_1$ to be ranked at position 3, and expert 2 prefers $A_1$ to be ranked at position 2 or 3. We thus say there is no consensus gap between the preferences of expert 1 and expert 2 with respect to $A_1$. By Eq. (4.1) we have $\Delta(\{3\}, \{2, 3\}) = 0$. However, there is a gap between the preferences of expert 1 and expert 3 with respect to $A_1$, since $\Delta(\{3\}, \{1, 2\}) = 1$.

We have the following results.

**Proposition 4.1** *For two preference sequences $S_1$ and $S_2$ over $I$, $S_1 \cap S_2 \neq \phi$ holds if, and only if, $\Delta(S_1, S_2) = 0$.*

**Proposition 4.2** *For three preference sequences $S_1$, $S_2$ and $S_3$ over $I$, if $S_2 \subseteq S_3$, then, $\Delta(S_1, S_2) \geq \Delta(S_1, S_3)$.*

The definition of consensus gap indicator somehow resembles a widely used distance between two compact sets $K_A$ and $K_B$ as:

$$d(K_A, K_B) = \min\{\|x - y\| : x \in K_A, y \in K_B\}.$$

However, the consensus gap indicator defined by (4.1) is not a distance metric. Although the consensus gap indicator $\Delta$ has the properties:

(1) $\Delta \geq 0$;
(2) $\Delta(\mathbf{\Lambda}, \mathbf{\Lambda}) = 0$;
(3) $\Delta(\mathbf{\Lambda}, \mathbf{\Lambda}') = \Delta(\mathbf{\Lambda}', \mathbf{\Lambda})$;

it does NOT necessarily satisfy:

(4) $\Delta(\mathbf{\Lambda}, \mathbf{\Lambda}') \neq 0$ if $\mathbf{\Lambda} \neq \mathbf{\Lambda}'$.
For instance, we consider the entries in the first row of matrix (3.1). Let $S_1 = \{3\}$ and $S_2 = \{2, 3\}$. We know $S_1 \neq S_2$, but we have $\Delta(S_1, S_2) = 0$.

(5) the triangle inequality $\Delta(\mathbf{\Lambda}, \mathbf{\Lambda}'') \leq \Delta(\mathbf{\Lambda}, \mathbf{\Lambda}') + \Delta(\mathbf{\Lambda}', \mathbf{\Lambda}'')$.
For instance, we again consider the entries in the first row of matrix (3.1). Let $S_1 = \{3\}$, $S_2 = \{2, 3\}$, and $S_3 = \{1, 2\}$. We have $\Delta(S_1, S_2) = 0$, $\Delta(S_2, S_3) = 0$ and $\Delta(S_1, S_3) = 1$. Thus $\Delta(S_1, S_3) > \Delta(S_1, S_2) + \Delta(S_2, S_3)$.

From Definition 4.1, we know that the consensus gap indicator is a kind of prametric. Further, failing to meet Item 4 does not contradict the fact that, with respect to $A_1$, Expert 1 and 2 have preferences in common, namely, no consensus gap happens to Expert 1 and 2's preferences. Furthermore, failing to meet Item 5 does not contradict the fact that *Tom and Jack have consensus preferences, Jack and John have certain consensus preferences, yet Tom and John may still have no consensus preference*. Therefore, the properties of a prametric are sufficient for the purpose of this paper to measure the gap of preferences.

If we define the consensus gap indicator between two preference sequence vectors as:

**Definition 4.2** Let $\mathbf{\Lambda} = (U_i)_{n\times 1}$ and $\mathbf{\Xi} = (V_i)_{n\times 1}$ be two preference sequence vectors over $I$. We call $\sum_{i=1}^{n} \Delta(U_i, V_i)$ the consensus gap indicator between $\mathbf{\Lambda}$ and $\mathbf{\Xi}$, denoted

$$\Delta(\mathbf{\Lambda}, \mathbf{\Xi}) = \sum_{i=1}^{n} \Delta(U_i, V_i), \tag{4.2}$$

then, we can know that $\Delta(\mathbf{\Lambda}, \mathbf{\Xi}) \geq 0$ and $\Delta(\mathbf{\Lambda}, \mathbf{\Lambda}) = 0$. Thus, the consensus gap indicator between two preference sequence vectors is also a prametric.

The triangle inequality is not necessarily verified by the definition of (4.2). Take the matrix (3.1) of example 3.1 for illustration. As described above, the columns of matrix (3.1) are all preference sequence vectors. By Definition 4.2, we know that the consensus gap indicator between the first and the second column is 0, and so is that between the second and the third column. However, the consensus gap indicator between the first and the third column is 2. Hence, the triangle inequality is not verified by the defined consensus indicator in this example.

We further define the $\subseteq$ relation of two preference sequence vectors as follows.

**Definition 4.3** Let $\mathbf{\Lambda} = (U_i)_{n\times 1}$ and $\mathbf{\Xi} = (V_i)_{n\times 1}$ be two preference sequence vectors over $I$. We call $\mathbf{\Lambda} \subseteq \mathbf{\Xi}$ if, and only if

$$\forall i : U_i \subseteq V_i. \tag{4.3}$$

Take matrix (3.1) of Example 3.1 for illustration. If the preference sequence vector of the first column is denoted by $\mathbf{\Xi}^{(1)}$, and by $\mathbf{\Xi}^{(2)}$ for that of the second column, then, from Definition 4.3 we have $\mathbf{\Xi}^{(1)} \subseteq \mathbf{\Xi}^{(2)}$, since the $\subseteq$ relation holds entry-wise for these two vectors, namely, $\{3\} \subseteq \{2, 3\}$, $\{1\} \subseteq \{1\}$, $\{2\} \subseteq \{2, 3\}$, and $\{4\} \subseteq \{4\}$.

The next result follows directly from Eqs. (4.1)–(4.3) and Proposition 4.2:

**Proposition 4.3** *For three preference sequences vectors* $\mathbf{\Lambda}^{(1)}$, $\mathbf{\Lambda}^{(2)}$ *and* $\mathbf{\Lambda}^{(3)}$ *over* $I$, *if* $\mathbf{\Lambda}^{(2)} \subseteq \mathbf{\Lambda}^{(3)}$, *then,*

$$\Delta(\mathbf{\Lambda}^{(1)}, \mathbf{\Lambda}^{(2)}) \geq \Delta(\mathbf{\Lambda}^{(1)}, \mathbf{\Lambda}^{(3)}). \tag{4.4}$$

*The Proposition 4.3 is meaningful to our approach as shown in Sect. 5.2.*

## 5 Compromise Ranking

### 5.1 An Assignment Model

Since the consensus gap indicator between two preference sequence vectors is also a prametric, an assignment model for eliciting the 'nearest' compromise ranking by minimizing the above defined consensus gap indicator can be constructed as follows. Let $\Pi = (Q_{ij})_{n \times m}$ be the preference sequence matrix corresponding to a problem with $n$ alternatives and $m$ experts involved. The optimization model for obtaining the compromise ranking is given as (M1)

$$\begin{aligned} \min \quad & z = \sum_{i=1}^{n} \sum_{k=1}^{n} y_{ik} D_{ik} \\ s.t. \quad & \sum_{i=1}^{n} y_{ik} = 1, \quad \text{for all } k, \\ & \sum_{k=1}^{n} y_{ik} = 1, \quad \text{for all } i, \\ & y_{ik} = 0, 1, \quad \text{for all } i, k, \end{aligned}$$

where

$$D_{ik} = \sum_{j=1}^{m} \Delta(\{k\}, Q_{ij}) \tag{5.1}$$

means the total consensus gap indicator if alternative $i$ is ranked at position $k$.

### 5.2 A Consensus Formation Procedure

The model M1 can be solved by Hungarian Method (see Kuhn 1955). Sometimes, the above assignment model may present multiple solutions. In this case, a ties-permitted compromise ranking possibly exists which is 'nearer' to the individual preferences than a complete rankings (see Cook and Seiford 1978). Our purpose is to propose a procedure to get the ties-permitted compromise ranking (if exists) based on the multiple solutions.

To propose our strategy, we prescribe:

- The union operation $\bigcup$ of two preference sequence vectors $(U_i)_{n\times 1}$ and $(V_i)_{n\times 1}$ is prescribed as (namely, obtained by entry-wise union)

$$(V_i)_{n\times 1}\bigcup(U_i)_{n\times 1}=(F_i)_{n\times 1}, \tag{5.2}$$

where $F_i = V_i \cup U_i, i = 1, 2, \ldots, n$.
Further, the union of preference sequence vectors $\mathbf{\Lambda}^{(1)}$, $\mathbf{\Lambda}^{(2)}$,..., $\mathbf{\Lambda}^{(m)}$ can be prescribed by using Eq. (5.2) as

$$\bigcup_{j=1}^{m}\mathbf{\Lambda}^{(j)}=\mathbf{\Lambda}^{(1)}\bigcup\mathbf{\Lambda}^{(2)}\bigcup\cdots\bigcup\mathbf{\Lambda}^{(m)}. \tag{5.2$'$}$$

- The cardinal number of a preference sequence vector $\mathbf{\Lambda} = (U_i)_{n\times 1}$ is prescribed as

$$|\mathbf{\Lambda}|=\sum_{i=1}^{n}|U_i|. \tag{5.3}$$

- Let $\mathbf{\Lambda} = (U_i)_{n\times 1}$ be a preference sequence vector over $I$ and $\Pi = (Q_{ij})_{n\times m}$ a preference sequence matrix over $I$. we call $\sum_{i=1}^{n}\sum_{j=1}^{m}\Delta(U_i, Q_{ij})$ the consensus gap indicator of $\mathbf{\Lambda}$ from $\Pi$, denoted

$$\Delta(\mathbf{\Lambda},\Pi)=\sum_{i=1}^{n}\sum_{j=1}^{m}\Delta(U_i, Q_{ij}). \tag{5.4}$$

Our strategy is, based on the multiple solutions of the assignment model, to find out an ordinal ranking which has the nearest consensus gap indicator from the preference sequence matrix. From inequality (4.4), we know that, for two preference sequence vectors $\mathbf{\Xi}^{(1)}$ and $\mathbf{\Xi}^{(2)}$, if $\mathbf{\Xi}^{(1)} \subseteq \mathbf{\Xi}^{(2)}$ then $\mathbf{\Xi}^{(2)}$ may be nearer than $\mathbf{\Xi}^{(1)}$ to a preference sequence matrix. Thus, our method begins by solving the Assignment Model (M1) to obtain all of its solutions, each of which represents a complete ranking of the alternatives. If the model M1 has a unique solution, then, the compromise ranking is obtained. If not, investigate the union vector [defined by Eqs. (5.2) and (5.2$'$)] of all or part of the solutions to search for a preference sequence vector (satisfying the items of Proposition 3.1) as an expected compromise ranking which has the smallest consensus gap from the preference sequence matrix. The procedure is given in details as follows.

I. Suppose the solutions of the assignment model M1 are $\mathbf{\Lambda}^{(1)}, \mathbf{\Lambda}^{(2)}, \ldots, \mathbf{\Lambda}^{(L)}$ constituting a set of $\mathbf{M} = \{\mathbf{\Lambda}^{(1)}, \mathbf{\Lambda}^{(2)}, \ldots, \mathbf{\Lambda}^{(L)}\}$. Initially we set $k := L$.
II. Examine all the $k$-combinations of $\mathbf{M}$. A $k$-combination of a set $S$ is a subset of $k$ distinct elements of $S$. The number of $k$-combinations of $\mathbf{M}$ is $\binom{L}{k}$. For each $k$-combination:
  A. if $k > 1$, inspect the union vector of vectors [defined by Eqs. (5.2) and (5.2$'$)] contained in the considered $k$-combination whether it satisfies all of the items in Proposition 3.1; if Proposition 3.1 is satisfied, the union vector is regarded as a candidate optimal preference sequence vector.
  B. if $k = 1$, the vector contained in the considered $k$-combination is regarded as a candidate optimal preference sequence vector.

III. if every $k$-combination of $\mathbf{M}$ produces a union vector satisfying Proposition 3.1 (Note: the vector contained in a 1-combination is also referred to as a union vector produced by the 1-combination), then, it is not necessary to examine the $\delta$-combination where $\delta = k-1, k-2, \ldots, 1$, thus goto Step IV; else if $k > 1$, then $k := k-1$ and goto Step II.

IV. For all candidate optimal preference sequence vectors, compute their respective consensus gap indicator from the preference sequence matrix, and find out the one with minimal gap as an optimal solution.

In the above procedure, the Step III is suggested based on the inequality (4.4), as stated before, for two preference sequence vectors $\Xi^{(1)}$ and $\Xi^{(2)}$, if $\Xi^{(1)} \subseteq \Xi^{(2)}$ then $\Xi^{(2)}$ may be nearer than $\Xi^{(1)}$ to a preference sequence matrix. For instance (as shown in Example 6.1), if the union vector constructed on the $L$-combination of $\mathbf{M}$ satisfies Proposition 3.1, then, it is not necessary to examine other combination. This is so because all the other combinations are subsets of the $L$-combination, and thus they will not produce a 'nearer' preference sequence vector than that constructed by the $L$-combination. One thing we need to mention here is that, in the worst case, the sub-steps in Step II will be performed for $\binom{L}{L} + \binom{L}{L-1} + \cdots + \binom{L}{1} = 2^L - 1$ times, where $L$ is the number of solutions of the assignment model.

## 6 An Illustrative Example and Comparison

### 6.1 Example

To illustrate the application, we consider a simple GDM example.

*Example 6.1* Consider a GDM example of 4 experts evaluating 5 alternatives, $\{A_1, A_2, A_3, A_4, A_5\}$. The ordinal preference is given as

$$\begin{aligned}
&\text{expert1:} \quad A_1 \sim A_2 \succ A_3 \sim A_4 \succ A_5,\\
&\text{expert2:} \quad A_1 \sim A_2 \sim A_3 \succ A_4 \sim A_5,\\
&\text{expert3:} \quad A_1 \sim A_2 \succ A_3 \succ A_4 \sim A_5,\\
&\text{expert4:} \quad A_1 \sim A_2 \succ A_3 \succ A_5 \succ A_4.
\end{aligned}$$

In the following, we use the proposed method to elicit the group ranking through a step-by-step procedure.

(1) The ordinal preference can be transformed directly into preference sequence vector. Thus, the preference sequence matrix is

$$(Q_{ij})_{5\times 4} = \begin{array}{c} \\ A_1 \\ A_2 \\ A_3 \\ A_4 \\ A_5 \end{array} \begin{array}{c} \begin{array}{cccc} \text{expert1} & \text{expert2} & \text{expert3} & \text{expert4} \end{array} \\ \begin{pmatrix} \{1,2\} & \{1,2,3\} & \{1,2\} & \{1,2\} \\ \{1,2\} & \{1,2,3\} & \{1,2\} & \{1,2\} \\ \{3,4\} & \{1,2,3\} & \{3\} & \{3\} \\ \{3,4\} & \{4,5\} & \{4,5\} & \{5\} \\ \{5\} & \{4,5\} & \{4,5\} & \{4\} \end{pmatrix} \end{array}.$$

(2) Using Eq. 5.1 to compute the consensus gap indicator matrix, we have

$$(D_{ik})_{5\times 5} = \begin{pmatrix} 0 & 0 & 3 & 7 & 11 \\ 0 & 0 & 3 & 7 & 11 \\ 6 & 3 & 0 & 3 & 7 \\ 12 & 8 & 4 & 1 & 1 \\ 13 & 9 & 5 & 1 & 1 \end{pmatrix}.$$

We take $D_{23}$ to show the computation in more detail.

$$\begin{aligned} D_{23} &= \sum_{j=1}^{4} \Delta(\{3\}, Q_{2j}) \\ &= \Delta(\{3\}, \{1, 2\}) + \Delta(\{3\}, \{1, 2, 3\}) + \Delta(\{3\}, \{1, 2\}) + \Delta(\{3\}, \{1, 2\}). \end{aligned}$$

By the definition of consensus gap indicator of Eq. (4.1), we have

$$D_{23} = 1 + 0 + 1 + 1 = 3$$

(3) Solving the assignment model M1 based on the consensus gap indicator matrix by the Hungarian Method, we have 4 solutions with a consensus gap of 2. They are:
   - ${\mathbf{\Lambda}^{(1)}}^T = (\{1\}, \{2\}, \{3\}, \{4\}, \{5\})^T$
   - ${\mathbf{\Lambda}^{(2)}}^T = (\{2\}, \{1\}, \{3\}, \{4\}, \{5\})^T$
   - ${\mathbf{\Lambda}^{(3)}}^T = (\{1\}, \{2\}, \{3\}, \{5\}, \{4\})^T$
   - ${\mathbf{\Lambda}^{(4)}}^T = (\{2\}, \{1\}, \{3\}, \{5\}, \{4\})^T$

(4) Performing the steps presented in Sect. 5.2:
   - We have $\mathbf{M} = \{\mathbf{\Lambda}^{(1)}, \mathbf{\Lambda}^{(2)}, \mathbf{\Lambda}^{(3)}, \mathbf{\Lambda}^{(4)}\}$ and $L = 4$. We set $k := 4$.
   - Inspect the union vector constructed from $k$-combinations of $\mathbf{M}$, namely, $(\{1, 2\}, \{1, 2\}, \{3\}, \{4, 5\}, \{4, 5\})^T$. Because the union vector satisfies Proposition 3.1, thus, it is regarded as a candidate optimal preference sequence vector.
   - The number of $L$-combination is $\binom{L}{L} = 1$. And now we know that the union vector constructed from the $L$-combination satisfies Proposition 3.1, we need not examine other combinations, since all the other combinations are subsets of the $L$-combinations of $\mathbf{M}$.
   - Thus, we obtain the optimal preference sequence vector as $(\{1, 2\}, \{1, 2\}, \{3\}, \{4, 5\}, \{4, 5\})^T$, which indicates a ties-permitted compromise ranking as

$$A_1 \sim A_2 \succ A_3 \succ A_4 \sim A_5.$$

Additionally, by Eq. (5.4) the consensus gap of the optimal preference sequence vector from the preference matrix is 0.

## 6.2 Result of the Armstrong–Cook–Seiford Model and Comparison

### *6.2.1 Result of the Armstrong–Cook–Seiford Model*

When the Armstrong–Cook–Seiford approach is applied to a GDM problem similar to Example 6.1, the following notations are used:

- $n$ the number of alternatives,
- $m$ the number of experts,
- $a_i^l$ the rank assigned to alternative $A_i$ by expert $l$,
- $x_{ik/2}$ a 0-1 variable indicating whether or not the alternative $A_i$ is ranked at position $k/2$,
- $d_{ik/2}$ the Cook–Seiford Distance when the alternative $A_i$ is ranked at position $k/2$ by the group.

As regard to Example 6.1, a zero-one programming model can be formulated by using the Armstrong–Cook–Seiford approach (see Armstrong et al. 1982) as:

$$
\begin{aligned}
\min \quad & \sum_{i=1}^{n}\sum_{k=2}^{2n} d_{ik/2}x_{ik/2},\\
s.t. \quad & \sum_{k=2}^{2n} x_{ik/2} = 1 \quad \text{for all } i,\\
& \sum_{i=1}^{n} x_{ik/2} - \sum_{s=1}^{k-1} Y_{k-s,s} = 0, \quad k = 2, 3, \ldots, 2n,\\
& Y_{k-s,s} - Y_{k-s-1,s+1} \geq 0 \quad s \geq k-s,\\
& Y_{r,s} - Y_{s,r} = 0 \quad \text{for all } r, s,\\
& \sum_{r=1}^{n} Y_{r,s} = 1 \quad \text{for all } s,\\
& \sum_{s=1}^{n} Y_{r,s} = 1 \quad \text{for all } r,\\
& x_{ik/2}, Y_{r,s} \in \{0, 1\} \quad \text{for all } i, k, r, s,
\end{aligned}
$$

where $n = 5$ and $d_{ik/2}$, $i = 1, 2, \ldots, n$ and $k = 2, 3, \ldots, 2n$, is defined by (Cook and Seiford 1978; Armstrong et al. 1982)

$$d_{it} = \sum_{l=1}^{m} |a_i^l - t|. \tag{6.1}$$

Based on the individual preference information of Example 6.1 and Eq.(6.1), the $d_{ik/2}$ are calculated as shown in the following (*For sake of convenience, the indices* $k/2, k = 2, 3, \ldots, 2n$, *are replaced by integer numbers, namely, 1 for 2/2, 2 for 3/2, and so on*):

$$
\begin{aligned}
&d_{11} = 2.5, \quad d_{21} = 2.5, \quad d_{31} = 7.5, \quad d_{41} = 13.5, \quad d_{51} = 14.0,\\
&d_{12} = 0.5, \quad d_{22} = 0.5, \quad d_{32} = 5.5, \quad d_{42} = 11.5, \quad d_{52} = 12.0,
\end{aligned}
$$

$$\begin{array}{lllll}
d_{13}=1.5, & d_{23}=1.5, & d_{33}=3.5, & d_{43}=9.5, & d_{53}=10.0,\\
d_{14}=3.5, & d_{24}=3.5, & d_{34}=2.5, & d_{44}=7.5, & d_{54}=8.0,\\
d_{15}=5.5, & d_{25}=5.5, & d_{35}=1.5, & d_{45}=5.5, & d_{55}=6.0,\\
d_{16}=7.5, & d_{26}=7.5, & d_{36}=2.5, & d_{46}=3.5, & d_{56}=4.0,\\
d_{17}=9.5, & d_{27}=9.5, & d_{37}=4.5, & d_{47}=2.5, & d_{57}=2.0,\\
d_{18}=11.5, & d_{28}=11.5, & d_{38}=6.5, & d_{48}=1.5, & d_{58}=1.0,\\
d_{19}=13.5, & d_{29}=13.5, & d_{39}=8.5, & d_{49}=2.5, & d_{59}=2.0.
\end{array}$$

The zero-one programming model is programmed in LINGO and solved on a personal computer. The following results are obtained with an optimal objective function value 5:

$$\begin{array}{lllll}
x_{11}=0, & x_{21}=0, & x_{31}=0, & x_{41}=0, & x_{51}=0,\\
x_{12}=1, & x_{22}=1, & x_{32}=0, & x_{42}=0, & x_{52}=0,\\
x_{13}=0, & x_{23}=0, & x_{33}=0, & x_{43}=0, & x_{53}=0,\\
x_{14}=0, & x_{24}=0, & x_{34}=0, & x_{44}=0, & x_{54}=0,\\
x_{15}=0, & x_{25}=0, & x_{35}=1, & x_{45}=0, & x_{55}=0,\\
x_{16}=0, & x_{26}=0, & x_{36}=0, & x_{46}=0, & x_{56}=0,\\
x_{17}=0, & x_{27}=0, & x_{37}=0, & x_{47}=0, & x_{57}=0,\\
x_{18}=0, & x_{28}=0, & x_{38}=0, & x_{48}=1, & x_{58}=1,\\
x_{19}=0, & x_{29}=0, & x_{39}=0, & x_{49}=0, & x_{59}=0,
\end{array}$$

which indicates a final ranking as:

$$A_1 \sim A_2 \succ A_3 \succ A_4 \sim A_5.$$

### 6.2.2 Comparison and Contrast

From the solving processes of Example 6.1 in Sects. 6.1 and 6.2, we can see that:

(1) The two solution procedures present the same optimal ranking of the alternatives;
(2) The optimal objective function values are different as a result of the definition difference of the consensus gap indicator [used in the approach presented in this paper, as shown by Eqs. (4.1) and (4.2)] and the distance function [used in the Armstrong–Cook–Seiford approach, as shown by Eq. (6.1)];
(3) Even for a GDM problem of small scale as Example 6.1, when the Armstrong–Cook–Seiford model is used, the number of variables and the number of constraints reach up to 70 and 55, respectively. Thus, it is a bit difficult to perform the Armstrong–Cook–Seiford procedure without a computer. By contrast, the procedure presented in this paper is easy to implement by hand when applied to small scale problems. Naturally, these two approaches both need a computer for problems of large scale.

(4) Finally, we should stress that, the Armstrong–Cook–Seiford approach is formalized as a mathematical programming model; by comparison, in addition to the assignment model used in the first part, the approach proposed in this paper is given as a descriptive procedure. Therefore, the Armstrong–Cook–Seiford approach is more convenient to be programmed by computer language than the proposed approach in this paper.

## 7 Conclusion and Prospect

In this paper we show that the prametric approach can be beneficial to GDM by describing preference intransitivity. A prametric, namely, the consensus gap indicator is defined and applied to a kind of group decision making problem. A strategy for tackling the Cook–Seiford multiple solution problem is proposed.

A research prospect is the comparison of the procedure introduced in the present paper with some other classic ones, for example, Kemeny (1959) and Slater (1961). The comparison work may include the theoretical analysis and the computational complexity issues. We should note that, we have made a strict assumption that each individual ties-permitted ordinal ranking constitutes a partition of the alternative set to conduct the present study. Therefore prospects include also how to apply the proposed approach to a problem where the individual preferences take other forms.

**Acknowledgments** The author would like to thank the Editor-in-Chief and the anonymous Referees for their helpful comments and suggestions that have led to an improved version of this paper.

## References

Armstrong RD, Cook WD, Seiford LM (1982) Priority ranking and consensus formation: the case of ties. Manag Sci 28(6):638–645

Arrow KJ (1951) Social choice and individual values. Wiley, New York

Cook WD (2006) Distance-based and ad hoc consensus models in ordinal preference ranking. Eur J Oper Res 172(2):369–385

Cook WD, Kress M, Seiford LM (1997) A general framework for distance-based consensus in ordinal ranking models. Eur J Oper Res 96(2):392–397

Cook WD, Seiford LM (1978) Priority ranking and consensus formation. Manag Sci 24(16):1721–1732

Dyer JS, Sarin RK (1979) Group preference aggregation rules based on strength of preference. Manag Sci 25(9):822–832

Gavshin Y, Kruusmaa M (2008) Comparative experiments on the emergence of safe behaviours. In: TAROS, pp 65–71

Inuma M, Otsuka A, Imai H (2009) Theoretical framework for constructing matching algorithms in biometric authentication systems. In: Procedings of ICB09. Lecture notes in computer science, 5558, Alghero, Italy. Springer, Berlin, pp 806–815

Keeney RL (2013) Foundations for group decision analysis. Decis Anal 10(2):103–120

Kemeny JG (1959) Mathematics without numbers. Daedalus 88(4):577–591

Kuhn HW (1955) The Hungarian method for the assignment problem. Nav Res Logist Q 2:83–97

Skala MA (2008) Aspects of metric spaces in computation. PhD thesis, University of Waterloo, Ontario, p 2

Slater P (1961) Inconsistencies in a schedule of paired comparisons. Biometrica 48:303–312

Typke R, Walczak-Typke A (2010) Indexing techniques for non-metric music dissimilarity measures. In: Rás ZW, Wieczorkowska AA (eds) Advances in music information retrieval. Springer, Berlin, pp 3–17

Xu Z (2013) Compatibility analysis of intuitionistic fuzzy preference relations in group decision making. Group Decis Negot 22:463–482

Zeng S (2013) Some intuitionistic fuzzy weighted distance measures and their application to group decision making. Group Decis Negot 22:281–298

Zhou L, Chen H, Liu J (2013) Continuous ordered weighted distance measure and its application to multiple attribute group decision making. Group Decis Negot 22:739–758