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Logarithm Compatibility of Interval Multiplicative Preference Relations with an Application to Determining the Optimal Weights of Experts in the Group Decision Making

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Abstract We propose the new compatibility of interval multiplicative preference relations (IMPRs) in the group decision making (GDM) and apply it to determine the weights of experts. Firstly, we introduce the operation of interval numbers and define the new conception of logarithm compatibility degree of two interval multiplicative preference relations. Then, we prove the properties of logarithm compatibility of IMPR. It is pointed that if IMPR provided by every expert and its characteristic matrix are of acceptable compatibility, then the synthetic preference relation and the synthetic characteristic matrix are also of acceptable compatibility. Furthermore, we construct a mathematical programming model to determine the optimal weights of experts by minimizing the square logarithm compatibility in the GDM with IMPR and discuss the solution to the model. Finally, a numerical example is illustrated to show that the model is feasible.

Keywords Group decision making · Interval multiplicative preference relation · Logarithm compatibility degree · The weights of experts

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1 Introduction

In the real decision making problems, decision makers usually use various types of preference relations to express their views over alternatives. Xu (2007) presented a survey of preference relations, such as fuzzy preference relation (Xu 2011; Xu and Da 2005; Chiclana et al. 1998), multiplicative preference relation (Saaty 1980) linguistic preference information (Chen et al. 2011), intuitionistic preference relation (Atanassov 1986) etc.

Saaty (1980) introduced multiplicative preference relation which was defined as a reciprocal matrix provided by the decision maker for each pair of alternatives. The eigenvector method was developed to derive the priority vector from a multiplicative preference relation in Saaty (1980). Because of the complexity and uncertainty of problems and the limited amount of information available to decision makers, the input information arguments provided by decision makers are given in the form of interval numerical values rather than the real numerical values. Therefore, Saaty and Vargas (1987) proposed the interval multiplicative preference relation (IMPR), which was expressed by comparison ratios as intervals with lower and upper bounds. The existing investigations on IMPR were mainly focused on obtaining the priority vector. Saaty and Vargas (1987) derived the priority weight intervals from IMPR using a Monte Carlo simulation method. Arbel (1989) built a linear programming model to obtain the priority vector of IMPR. Salo and Hämäläinen (1995) presented a recursive algorithm to get the priority weight intervals for an IMPR. Haines (1998) developed a statistical approach to calculate the priority weight from IMPR. Islam et al. (1997) established a goal programming to derive the weights from inconsistent IMPR. Mikhailov (2002) established a fuzzy programming model to obtain the uncertain weights of partnership selection criteria and uncertain scores of alternative partners. Wang et al. (2005) utilized consistent IMPR to generate consistent interval weights by a linear programming model and derived the interval weights by a nonlinear programming model based on an eigenvector method. Fedrizzi and Brunelli (2009) indicated that the imposed constraint, in which the components of the weight vector sum up to one, does not obtain the proper priority vectors associated with multiplicative preference relations. They proved that this normalization method is incompatible with additive transitivity presented by Tanino (1984).

Group decision making (GDM) is regarded as a process for obtaining a collective preference relation by a number of individual preference relations according to a finite set of alternatives. In the GDM problems, the fundamental and important issue is how to aggregate all individual preference relations into a collective one effectively. The final solution must be derived from the synthesis of individual preferences (Kacprzyk 1986; Kacprzyk et al. 1992).

To achieve a collective preference relation with which the group is satisfied, the individual multiplicative preference relation must be compatible with its characteristic matrix generated by its priority vector. Therefore, Saaty (1994), Saaty and Vargas (2007) introduced the compatibility to judge the difference between two multiplicative preference relations.

Xu (2004) gave the concepts of compatibility degree and compatibility index of two interval fuzzy preference relations based on the distance of two interval numbers and obtained the properties of the synthetic interval fuzzy preference relation under the condition that the interval fuzzy preference relations are of acceptable compatibility, which provided a theoretic basis to apply the interval fuzzy preference relations in group decision making.

Due to time pressure and lack of expertise knowledge, a decision maker usually uses linguistic labels to provide his/her preference information. For example, linguistic labels like fast, very fast, slow can be utilized to evaluate the design of a car. Chen et al. (2011) presented the compatibility for the uncertain additive linguistic preference relations based on lower index of additive linguistic label. They proved the corresponding properties and constructed the optimal model for getting the weights of experts in the group decision making (GDM).

However, no investigation has been devoted to the issue on the compatibility degree of two interval multiplicative preference relations in the existing literatures. Motivated by Xu (2004) and Chen et al. (2011), we propose the new concept of the logarithm compatibility for two interval multiplicative preference relations. We also investigate the properties of the logarithm compatibility and synthetic IMPRs. Its application is to determine the optimal weights of experts by minimizing an incompatibility index in the GDM with IMPRs.

This paper is organized as follows. In Sect. 2, we define the basic concepts of compatibility of IMPRs and give the main results. In Sect. 3, we construct the optimal mathematical model to determine the optimal weights of experts based on logarithm compatibility index of IMPR in the GDM. An illustrative example is discussed in Sect. 4. Finally, we summarize the paper in Sect. 5.

2 Main Results

The input arguments are usually given in the form of interval numerical values rather than non-negative real number values in the process of decision-making. Let's look at operations on interval number as follows.

Let $a = \left[a^L, a^U\right] = \left\{x | a^L \le x \le a^U\right\}$, then a is called an interval number. Especially, a is a real number if $a^L = a^U$. For convenience of expression, let $N = \{1, 2, \ldots, n\}$ and let Θ be the set of all positive interval numbers, i.e., if $a = \left[a^L, a^U\right] \in \Theta$, then $a^U \ge a^L > 0$.

Definition 2.1 Let $a, b \in \Theta,\ a = \left[a^L, a^U\right],\ b = \left[b^L, b^U\right]$ and $p \ge 0$ is a non-negative real number, then
(1) $a = b$, if and only if $a^L = b^L,\ a^U = b^U$;
(2) $a + b = [a^L + b^L, a^U + b^U]$;
(3) $pa = \left[pa^L, pa^U\right]$;
(4) $a \cdot b = \left[a^L b^L, a^U b^U\right]$.

Definition 2.2 Let $c$be a non-negative real number, $a = \left[a^L, a^U\right] \in \Theta$.

(1) If $c \ge 0$, then $a^c = \left[\left(a^L\right)^c, \left(a^U\right)^c\right]$, if $c \le 0$, then $a^c = \left[\left(a^U\right)^c, \left(a^L\right)^c\right]$.
(2) If $c > 1$, then $\log_c a = \left[\log_c a^L, \log_c a^U\right]$. Especially, $c = e$, then $\log a = \left[\log a^L, \log a^U\right]$, and if $0 < c < 1$, then $\log_c a = \left[\log_c a^U, \log_c a^L\right]$.

Let $X = (x_1, x_2, \ldots, x_n)$ be a finite set of alternatives. Saaty (1980) proposed the multiplicative preference relation to describe an expert's preference, which was defined as following:

Definition 2.3 Let $A = (a_{ij})_{n\times n}$ be a matrix. If

$$a_{ij} > 0,\ a_{ij}a_{ji} = 1,\ a_{ii} = 1,\ \forall i, j = 1, 2, \ldots, n, \tag{1}$$

then matrix A is called a multiplicative preference relation on the set X or reciprocal matrix, where a_{ij} denotes the preference degree of the alternative x_i over x_j.

Especially, $a_{ij} = 1$ indicates indifference between x_i and x_j; $a_{ij} > 1$ indicates that x_i is preferred to x_j, $a_{ij} < 1$ indicates that x_j is preferred to x_i.

Saaty (1994), Saaty and Vargas (2007) proposed the notion of compatibility of two multiplicative preference relations and developed a metric for ratio scales of the multiplicative preference relation. For simplicity of calculation and expression, we define the logarithm compatibility degree of multiplicative preference relations as following.

Definition 2.4 Let $A = (a_{ij})_{n\times n}$ and $B = (b_{ij})_{n\times n}$ be two multiplicative preference relations, and let

$$LC(A, B) = \sum_{i=1}^{n}\sum_{j=1}^{n}\left|\log a_{ij} - \log b_{ij}\right|, \tag{2}$$

then $LC(A, B)$ is called the logarithm compatibility degree of A and B.

However, decision makers may have vague knowledge about the preference degrees of one alternative over another, so they cannot estimate their preference with a non-negative real numerical value. Saaty and Vargas (1987) and Xu (2007) introduced the notion of IMPR.

Definition 2.5 An interval multiplicative preference relation (IMPR) on the set X is defined as matrix $\tilde{A} = (\tilde{a}_{ij})_{n\times n}$, $\tilde{a}_{ij} = \left[a_{ij}^L,\ a_{ij}^U\right]$, satisfying

$$a_{ij}^U a_{ji}^L = 1,\ \ a_{ji}^U a_{ij}^L = 1,\ i \neq j,\ a_{ii}^U = a_{ii}^L = 1,\ \forall i, j = 1, 2, \ldots, n, \tag{3}$$

where $\tilde{a}_{ij}$ indicates the interval-valued preference degree of the alternative x_i over x_j, $a_{ij}^U \geq a_{ij}^L \geq 0$, a_{ij}^L and a_{ij}^U are the lower and upper bounds of $\tilde{a}_{ij}$, respectively.

Let M_n be the set of all interval multiplicative preference relations. We define the logarithm compatibility degree of IMPR as following.

Definition 2.6 Let $\tilde{A} = (\tilde{a}_{ij})_{n\times n} \in M_n$, $\tilde{B} = (\tilde{b}_{ij})_{n\times n} \in M_n$, and if

$$ILC(\tilde{A}, \tilde{B}) = \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\left(\left|\log a_{ij}^L - \log b_{ij}^L\right| + \left|\log a_{ij}^U - \log b_{ij}^U\right|\right), \tag{4}$$

then $ILC(\tilde{A}, \tilde{B})$ is called the logarithm compatibility degree of interval multiplicative preference relation $\tilde{A}$ and $\tilde{B}$.

Definition 2.7 Let $\tilde{A} = (\tilde{a}_{ij})_{n\times n} \in M_n,\ \tilde{B} = (\tilde{b}_{ij})_{n\times n} \in M_n$. If $\tilde{a}_{ij} = \tilde{b}_{ij}$, that is, $a_{ij}^U = b_{ij}^U,\ a_{ij}^L = b_{ij}^L,\ i, j = 1, 2, \ldots, n$, then $\tilde{A}$ and $\tilde{B}$ are perfectly compatible.

It is obvious that we can obtain the following Remark 2.1 from Definition 2.6 and Definition 2.7.

Remark 2.1 Let $\tilde{A} = (\tilde{a}_{ij})_{n\times n} \in M_n,\ \tilde{B} = (\tilde{b}_{ij})_{n\times n} \in M_n$, then

(1) $ILC(\tilde{A}, \tilde{B}) \geq 0;\ ILC(\tilde{A}, \tilde{B}) = 0$ if and only if $\tilde{A}$ and $\tilde{B}$ are perfectly compatible.
(2) $ILC(\tilde{A}, \tilde{B}) = ILC(\tilde{B}, \tilde{A})$

Following similar to the proofs from Chen et al. (2011), we can easily get the triangle inequality of logarithm compatibility degree.

Remark 2.2 If $\tilde{A} = (\tilde{a}_{ij})_{n\times n} \in M_n,\ \tilde{B} = (\tilde{b}_{ij})_{n\times n} \in M_n,\ \tilde{C} = (\tilde{c}_{ij})_{n\times n} \in M_n$, then

$$ILC(\tilde{A}, \tilde{C}) \leq ILC(\tilde{A}, \tilde{B}) + ILC(\tilde{B}, \tilde{C}) \tag{5}$$

From Remarks 2.1 and 2.2, we can see that ILC is a kind of distance, which reflects the difference between two interval multiplicative preference relations.

Definition 2.8 Let $\tilde{A} = (\tilde{a}_{ij})_{n\times n} \in M_n,\ \tilde{B} = (\tilde{b}_{ij})_{n\times n} \in M_n$, if

$$ILCI(\tilde{A}, \tilde{B}) = \frac{1}{n(n-1)} ILC(\tilde{A}, \tilde{B}), \tag{6}$$

then $ILCI(\tilde{A}, \tilde{B})$ is called the logarithm compatibility index of IMPR $\tilde{A}$ and $\tilde{B}$.

Definition 2.9 Let $\tilde{A} = (\tilde{a}_{ij})_{n\times n} \in M_n,\ \tilde{B} = (\tilde{b}_{ij})_{n\times n} \in M_n$, If

$$ILCI(\tilde{A}, \tilde{B}) \leq \alpha, \tag{7}$$

then $\tilde{A}$ and $\tilde{B}$ are of acceptable compatibility, where α is the threshold of acceptable compatibility. [in general $\alpha = 0.2$ according to Chen et al. (2011)].

Let $E = (e_1, e_2, \ldots, e_m)$ be a finite set of experts, where m denotes the number of experts in the GDM. Let $\tilde{A}^{(k)} = \left(\tilde{a}_{ij}^{(k)}\right)_{n\times n}$ be the IMPR provided by kth expert, $k = 1, 2, \ldots, m$.

Definition 2.10 Let $\tilde{A}^{(k)} = \left(\tilde{a}_{ij}^{(k)}\right)_{n\times n} \in M_n,\quad k = 1, 2, \ldots, m$, if

$$\bar{\tilde{a}}_{ij} = \prod_{k=1}^{m} \left(\tilde{a}_{ij}^{(k)}\right)^{\omega^{(k)}},\quad \forall\, i, j = 1, 2, \ldots, n, \tag{8}$$

then $\bar{\tilde{A}} = (\bar{\tilde{a}}_{ij})_{n\times n}$ is called the synthetic preference relation of $\tilde{A}^{(k)},\ k = 1, 2, \ldots, m$. $\omega^{(k)}$ is the weight of the kth expert, $\omega^{(k)} \geq 0,\ \sum_{k=1}^{m} \omega^{(k)} = 1$.

Theorem 2.1 *Let* $\tilde{A}^{(k)} = \left(\tilde{a}_{ij}^{(k)}\right)_{n\times n} \in M_n$, $k = 1, 2, \ldots, m$, *then the synthetic preference relation* $\bar{\tilde{A}} = \left(\bar{\tilde{a}}_{\mathbf{ij}}\right)_{n\times n} \in M_n$.

Proof Since $\tilde{A}^{(k)} = \left(\tilde{a}_{ij}^{(k)}\right)_{n\times n} \in M_n$, where $\tilde{a}_{ij}^{(k)} = \left[a_{ij}^{L(k)}, a_{ij}^{U(k)}\right]$, according to Definition 2.5, we have

$$a_{ij}^{U(k)} a_{ji}^{L(k)} = 1,\ a_{ji}^{U(k)} a_{ij}^{L(k)} = 1,\ i \neq j,\ a_{ii}^{U(k)} = a_{ii}^{L(k)} = 1,\ \forall i, j = 1, 2, \ldots, n \tag{9}$$

By the Eq. (8), Definitions 2.1 and 2.2, we get

$$\begin{aligned}\bar{\tilde{a}}_{ij} = \prod_{k=1}^{m} \left(\tilde{a}_{ij}^{(k)}\right)^{\omega^{(k)}} &= \prod_{k=1}^{m} \left(\left[a_{ij}^{L(k)}, a_{ij}^{U(k)}\right]\right)^{\omega^{(k)}} = \prod_{k=1}^{m} \left[\left(a_{ij}^{L(k)}\right)^{\omega^{(k)}}, \left(a_{ij}^{U(k)}\right)^{\omega^{(k)}}\right] \\ &= \left[\prod_{k=1}^{m} \left(a_{ij}^{L(k)}\right)^{\omega^{(k)}} \prod_{k=1}^{m} \left(a_{ij}^{U(k)}\right)^{\omega^{(k)}}\right]\end{aligned}$$

i.e,

$$\bar{a}_{ij}^{L} = \prod_{k=1}^{m} \left(a_{ij}^{L(k)}\right)^{\omega^{(k)}},\ \bar{a}_{ij}^{U} = \prod_{k=1}^{m} \left(a_{ij}^{U(k)}\right)^{\omega^{(k)}} \tag{10}$$

where $\bar{a}_{ij}^{L}$ and $\bar{a}_{ij}^{U}$ are the lower and upper bounds of $\bar{\tilde{a}}_{ij}$, respectively.

In similar way, $\bar{\tilde{a}}_{ji} = \left[\prod_{k=1}^{m} \left(a_{ji}^{L(k)}\right)^{\omega^{(k)}},\ \prod_{k=1}^{m} \left(a_{ji}^{U(k)}\right)^{\omega^{(k)}}\right]$, i.e,

$$\bar{a}_{ji}^{L} = \prod_{k=1}^{m} \left(a_{ji}^{L(k)}\right)^{\omega^{(k)}},\ \bar{a}_{ji}^{U} = \prod_{k=1}^{m} \left(a_{ji}^{U(k)}\right)^{\omega^{(k)}} \tag{11}$$

Therefore, according to the Eqs. (9)–(11), we have

$$\bar{a}_{ij}^{U} \cdot \bar{a}_{ji}^{L} = \prod_{k=1}^{m} \left(a_{ij}^{U(k)}\right)^{\omega^{(k)}} \cdot \prod_{k=1}^{m} \left(a_{ji}^{L(k)}\right)^{\omega^{(k)}} = \prod_{k=1}^{m} \left(a_{ij}^{U(k)} \cdot a_{ji}^{L(k)}\right)^{\omega^{(k)}} = \prod_{k=1}^{m} 1^{\omega^{(k)}} = 1 \tag{12}$$

$$\bar{a}_{ij}^{L} \cdot \bar{a}_{ji}^{U} = \prod_{k=1}^{m} \left(a_{ij}^{L(k)}\right)^{\omega^{(k)}} \cdot \prod_{k=1}^{m} \left(a_{ji}^{U(k)}\right)^{\omega^{(k)}} = \prod_{k=1}^{m} \left(a_{ij}^{L(k)} \cdot a_{ji}^{U(k)}\right)^{\omega^{(k)}} = \prod_{k=1}^{m} 1^{\omega^{(k)}} = 1 \tag{13}$$

$$\bar{a}_{ii}^{L} = \prod_{k=1}^{m} \left(a_{ii}^{L(k)}\right)^{\omega^{(k)}} = \prod_{k=1}^{m} 1^{\omega^{(k)}} = 1,\ \bar{a}_{ii}^{U} = \prod_{k=1}^{m} \left(a_{ii}^{U(k)}\right)^{\omega^{(k)}} = \prod_{k=1}^{m} 1^{\omega^{(k)}} = 1, \tag{14}$$

From the Eqs. (12)–(14) and Definition 2.5, it follows that the synthetic preference relation $\bar{\tilde{A}} = \left(\bar{\tilde{a}}_{\mathbf{ij}}\right)_{n\times n}$ is a IMPR, i.e, $\bar{\tilde{A}} = \left(\bar{\tilde{a}}_{\mathbf{ij}}\right)_{n\times n} \in M_n$, which completes the proof. □

Wang et al. (2005) proposed a two-stage logarithmic goal programming method for generating weights from IMPR. They constructed the model to minimize the inconsistency of IMPR in the first stage and developed to generate priorities under the condition of minimal inconsistency in the second stage. The weights are assumed to be multiplicative rather than additive. A nonlinear programming method was used to aggregate local interval weights into global interval weights.

Suppose that $\tilde{w}^{(k)} = \left(\tilde{w}_1^{(k)}, \tilde{w}_2^{(k)}, \ldots, \tilde{w}_n^{(k)}\right)$ is global interval weight vector generated by two-stage logarithmic goal programming method weights from IMPR $\tilde{A}^{(k)} = \left(\tilde{a}_{ij}^{(k)}\right)_{n\times n}$ provided by kth expert in Saaty and Vargas (2007), where $\tilde{w}_i^{(k)} = \left[w_i^{L(k)}, w_i^{U(k)}\right], i = 1, 2, \ldots, n,\ k = 1, 2, \ldots, m.$

Definition 2.11 Let

$$\tilde{w}_{ij}^{(k)} = \left[w_{ij}^{L(k)}, w_{ij}^{U(k)}\right] = \frac{\tilde{w}_i^{(k)}}{\tilde{w}_j^{(k)}} = \left[\frac{w_i^{L(k)}}{w_j^{U(k)}}, \frac{w_i^{U(k)}}{w_j^{L(k)}}\right] \tag{15}$$

Then $\tilde{W}^{(k)} = \left(\tilde{w}_{ij}^{(k)}\right)_{n\times n}$ is called the characteristic matrix corresponding to IMPR $\tilde{A}^{(k)} = \left(\tilde{a}_{ij}^{(k)}\right)_{n\times n}, k = 1, 2, \ldots, m.$

Definition 2.12 Let $\tilde{W}^{(k)} = \left(\tilde{w}_{ij}^{(k)}\right)_{n\times n}$ be the characteristic matrix corresponding to IMPR $\tilde{A}^{(k)} = \left(\tilde{a}_{ij}^{(k)}\right)_{n\times n}, k = 1, 2, \ldots, m$, if

$$\overline{\tilde{w}}_{ij} = \left[\bar{w}_{ij}^L, \bar{w}_{ij}^U\right] = \prod_{k=1}^m \left(\tilde{w}_{ij}^{(k)}\right)^{\omega^{(k)}} = \left[\prod_{k=1}^m \left(w_{ij}^{L(k)}\right)^{\omega^{(k)}}, \quad \prod_{k=1}^m \left(w_{ij}^{U(k)}\right)^{\omega^{(k)}}\right],$$
$$\forall\, i, j = 1, 2, \ldots, n, \tag{16}$$

then $\overline{\tilde{W}} = \left(\overline{\tilde{w}}_{ij}\right)_{n\times n}$ is called the synthetic preference relation of $\tilde{W}^{(k)}$, $k = 1, 2, \ldots, m$. $\omega^{(k)}$ is the weight of the kth expert, $\omega^{(k)} \geq 0$, $\sum_{k=1}^m \omega^{(k)} = 1$.

According to Theorem 2.1, we know that $\overline{\tilde{W}} = \left(\overline{\tilde{w}}_{ij}\right)_{n\times n} \in M_n$.

Theorem 2.2 *If* $ILCI\left(\tilde{A}^{(k)}, \tilde{W}^{(k)}\right) \leq \alpha,\ \forall k = 1, 2, \ldots, m$. *Then* $ILCI(\overline{\tilde{A}}, \overline{\tilde{W}}) \leq \alpha$

Proof Since $ILCI\left(\tilde{A}^{(k)}, \tilde{W}^{(k)}\right) \leq \alpha$. By the Eqs. (4) and (6), we have

$$\frac{1}{n(n-1)} \left[\frac{1}{2}\sum_{i=1}^n \sum_{j=1}^n \left(\left|\log a_{ij}^{L(k)} - \log w_{ij}^{L(k)}\right| + \left|\log a_{ij}^{U(k)} - \log w_{ij}^{U(k)}\right|\right)\right] \leq \alpha,$$
$$\forall k = 1, 2, \ldots, m. \tag{17}$$

Therefore, according to Definitions 6, 8, 10, 12 and Eq. (17), we get

$$
\begin{aligned}
ILCI(\overline{\tilde{A}}, \overline{\tilde{W}}) &= \frac{1}{n(n-1)} ILC(\overline{\tilde{A}}, \overline{\tilde{W}}) \\
&= \frac{1}{n(n-1)} \left[\frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \left(\left| \log \bar{a}_{ij}^{L} - \log \bar{w}_{ij}^{L} \right| + \left| \log \bar{a}_{ij}^{U} - \log \bar{w}_{ij}^{U} \right| \right) \right] \\
&= \frac{1}{n(n-1)} \left[\frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \left(\left| \log \prod_{k=1}^{m} \left(a_{ij}^{L(k)} \right)^{\omega^{(k)}} - \log \prod_{k=1}^{m} \left(w_{ij}^{L(k)} \right)^{\omega^{(k)}} \right| \right.\right. \\
&\quad \left.\left. + \left| \log \prod_{k=1}^{m} \left(a_{ij}^{U(k)} \right)^{\omega^{(k)}} - \log \prod_{k=1}^{m} \left(w_{ij}^{U(k)} \right)^{\omega^{(k)}} \right| \right) \right] \\
&= \frac{1}{n(n-1)} \left[\frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} (\left| \sum_{k=1}^{m} \omega^{(k)} \left(\log a_{ij}^{L(k)} - \log w_{ij}^{L(k)} \right) \right| \right. \\
&\quad \left. + \left| \sum_{k=1}^{m} \omega^{(k)} \left(\log a_{ij}^{U(k)} - \log w_{ij}^{U(k)} \right) \right|) \right] \\
&\le \frac{1}{n(n-1)} \left[\frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \left(\sum_{k=1}^{m} \omega^{(k)} \left| \log a_{ij}^{L(k)} - \log w_{ij}^{L(k)} \right| + \sum_{k=1}^{m} \omega^{(k)} \left| \log a_{ij}^{U(k)} - \log w_{ij}^{U(k)} \right| \right) \right] \\
&= \sum_{k=1}^{m} \omega^{(k)} \left\{ \frac{1}{n(n-1)} \left[\frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \left(\left| \log a_{ij}^{L(k)} - \log w_{ij}^{L(k)} \right| + \left| \log a_{ij}^{U(k)} - \log w_{ij}^{U(k)} \right| \right) \right] \right\}
\end{aligned}
$$

$\le \sum_{k=1}^{m} \omega^{(k)} \alpha = \alpha \sum_{k=1}^{m} \omega^{(k)} = \alpha \cdot 1 = \alpha$. i.e, $ILCI(\overline{\tilde{A}}, \overline{\tilde{W}}) \le \alpha$, which completes the proof. □

Theorem 2.2 shows that if $\tilde{A}^{(k)}$ and $\tilde{W}^{(k)}$are of acceptable compatibility, $k = 1, 2, \ldots, m$, then the synthetic preference relation $\overline{\tilde{A}}$ and $\overline{\tilde{W}}$ is also of acceptable compatibility.

3 Determining the Optimal Weights of Experts Based on Square Logarithm Compatibility Index of Interval Multiplicative Preference Relations in the GDM

From Remark 2.1 and 2.2, we know that logarithm compatibility index reflects the difference between two interval multiplicative preference relations. We assume that the IMPR provided by kth expert is worse than its corresponding characteristic matrix in the consistency. Therefore, we can use the logarithm compatibility index between the IMPR and its corresponding characteristic to measure the reliability of information given by kth expert. As a result, the criterion of determining the optimal weights of experts is to minimize the difference between the synthetic preference relation $\overline{\tilde{A}}$ and its corresponding characteristic matrix $\overline{\tilde{W}}$ in the group decision making problems, which leads to aggregate individual information with IMPRs effectively.

For the convenience of calculation, we define the new square logarithm compatibility index of the synthetic preference relation $\overline{\tilde{A}}$ and $\overline{\tilde{W}}$ as following.

$$\begin{aligned}
SILCI(\overline{\tilde{A}},\overline{\tilde{W}}) &= \frac{1}{n(n-1)}\left[\frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\left(\left(\log \bar{a}_{ij}^{L}-\log \bar{w}_{ij}^{L}\right)^2+\left(\log \bar{a}_{ij}^{U}-\log \bar{w}_{ij}^{U}\right)^2\right)\right] \\
&= \frac{1}{2n(n-1)}\sum_{i=1}^{n}\sum_{j=1}^{n}\left[\left(\sum_{k=1}^{m}\omega^{(k)}\left(\log a_{ij}^{L\ (k)}-\log w_{ij}^{L(k)}\right)\right)^2+\left(\sum_{k=1}^{m}\omega^{(k)}\left(\log a_{ij}^{U(k)}-\log w_{ij}^{U(k)}\right)\right)^2\right] \\
&= \frac{1}{2n(n-1)}\sum_{i=1}^{n}\sum_{j=1}^{n}\left(\sum_{k=1}^{m}\sum_{l=1}^{m}\omega^{(k)}\omega^{(l)}\left[\begin{array}{l}\left(\log a_{ij}^{L(k)}-\log w_{ij}^{L(k)}\right)\left(\log a_{ij}^{L(l)}-\log w_{ij}^{L(l)}\right)\\+\left(\log a_{ij}^{U(k)}-\log w_{ij}^{U(k)}\right)\left(\log a_{ij}^{U(l)}-\log w_{ij}^{U(l)}\right)\end{array}\right]\right) \\
&= \sum_{k=1}^{m}\sum_{l=1}^{m}\omega^{(k)}\omega^{(l)}\left(\frac{1}{2n(n-1)}\sum_{i=1}^{n}\sum_{j=1}^{n}\left[\begin{array}{l}\left(\log a_{ij}^{L\ (k)}-\log w_{ij}^{L\ \ (k)}\right)\left(\log a_{ij}^{L\ (l)}-\log w_{ij}^{L\ \ (l)}\right)\\+\left(\log a_{ij}^{U\ (k)}-\log w_{ij}^{U\ \ (k)}\right)\left(\log a_{ij}^{U\ (l)}-\log w_{ij}^{U\ \ (l)}\right)\end{array}\right]\right)
\end{aligned} \tag{18}$$

Let $\Omega=\left(\omega^{(1)},\omega^{(2)},\ldots,\omega^{(m)}\right)^{\mathrm{T}}$, Ω is the weight vector of experts, $D=(d_{kl})_{m\times m}$ is a matrix, where

$$d_{kl}=\frac{1}{2n(n-1)}\sum_{i=1}^{n}\sum_{j=1}^{n}\left[\begin{array}{l}\left(\log a_{ij}^{L(k)}-\log w_{ij}^{L(k)}\right)\left(\log a_{ij}^{L(l)}-\log w_{ij}^{L(l)}\right)\\+\left(\log a_{ij}^{U(k)}-\log w_{ij}^{U(k)}\right)\left(\log a_{ij}^{U(l)}-\log w_{ij}^{U(l)}\right)\end{array}\right].$$

Thus, Eq. (18) is rewritten as $SILCI(\overline{\tilde{A}},\overline{\tilde{W}})=\Omega^{\mathrm{T}}D\Omega$. Therefore, the optimal model of determining weights of experts based on square logarithm compatibility index of IMPR in the GDM is expressed as following:

$$\begin{aligned}
&\text{Min } SILCI(\overline{\tilde{A}},\overline{\tilde{W}})=\Omega^{\mathrm{T}}D\Omega \\
&s.t\begin{cases}\sum_{k=1}^{m}\omega^{(k)}=1,\\ \omega^{(k)}\geq 0, k=1,2,\ldots,m\end{cases}
\end{aligned} \tag{19}$$

Let $R^{\mathrm{T}}=(1,\ \ 1,\ \ \ldots,1)_{1\times m}$, Problem (19) is rewritten as following:

$$\begin{aligned}
&SILCI(\overline{\tilde{A}},\overline{\tilde{W}})=\Omega^{\mathrm{T}}D\Omega \\
&s.t\begin{cases}R^{\mathrm{T}}\Omega=1,\\ \Omega\geq 0\end{cases}
\end{aligned} \tag{20}$$

If we don't consider the constraint that $\Omega\geq 0$ in Problem (20), then

$$\begin{aligned}
&SILCI(\overline{\tilde{A}},\overline{\tilde{W}})=\Omega^{\mathrm{T}}D\Omega \\
&s.t.\ \ R^{\mathrm{T}}\Omega=1,
\end{aligned} \tag{21}$$

Theorem 3.1 *If $\overline{\tilde{A}}$ and $\overline{\tilde{W}}$ are not perfectly compatible, the solution to Problem (21) is*

$$\Omega^{*}=\frac{D^{-1}R}{R^{\mathrm{T}}D^{-1}R} \tag{22}$$

Proof Since $\overline{\tilde{A}}$ and $\overline{\tilde{W}}$ are not perfectly compatible, then $\overline{\tilde{A}} \neq \overline{\tilde{W}}$, i.e., there exists $i_0, j_0 \in \{1, 2, \ldots, n\}, i_0 \neq j_0$, satisfying $\overline{\tilde{a}}_{i_0 j_0} \neq \overline{\tilde{w}}_{i_0 j_0}$, where $\overline{\tilde{a}}_{i_0 j_0} = \left[\bar{a}^L_{i_0 j_0}, \bar{a}^U_{i_0 j_0}\right], \overline{\tilde{w}}_{i_0 j_0} = \left[\bar{w}^L_{i_0 j_0}, \bar{w}^U_{i_0 j_0}\right]$. Therefore, we have $\left(\log \bar{a}^L_{i_0 j_0} - \log \bar{w}^L_{i_0 j_0}\right)^2 + \left(\log \bar{a}^U_{i_0 j_0} - \log \bar{w}^U_{i_0 j_0}\right)^2 > 0$. Thus,

$$SILCI(\overline{\tilde{A}}, \overline{\tilde{W}}) = \frac{1}{n(n-1)} \times \left[\frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\left(\left(\log \bar{a}^L_{ij} - \log \bar{w}^L_{ij}\right)^2 + \left(\log \bar{a}^U_{ij} - \log \bar{w}^U_{ij}\right)^2\right)\right] > 0 \quad (23)$$

Obviously, $D = (d_{kl})_{m\times m}$ is a symmetric matrix. By Eq. (23), we have $SILCI(\bar{A}, \bar{W}) = \Omega^{\mathrm{T}} D\Omega > 0$. Since Ω is the weight vector of experts, $\Omega \geq 0, \Omega \neq 0$. Thus, $D = (d_{kl})_{m\times m}$ is a positive definite matrix, and it is also a nonsingular matrix. We construct the Lagrange function corresponding to the Problem (21),

$$L(\Omega, \lambda) = \Omega^{\mathrm{T}} D\Omega + \lambda(R^{\mathrm{T}}\Omega - 1) \quad (24)$$

where λ is the Lagrange multiplier.

Setting these partial derivatives equal to zero by differentiating Eq. (24) with respect to Ω and λ, we have $\frac{\partial L(\Omega,\lambda)}{\partial \Omega} = 0$, $\frac{\partial L(\Omega,\lambda)}{\partial \lambda} = 0$. i.e.,

$$\begin{cases} 2D\Omega + \lambda R = 0 \\ R^{\mathrm{T}}\Omega - 1 = 0 \end{cases}, \quad (25)$$

We find the optimal solution by solving Eq. (25) $\Omega^* = \frac{D^{-1}R}{R^{\mathrm{T}}D^{-1}R}$. Since $\frac{\partial^2 L(\Omega,\lambda)}{\partial \Omega^2} = D$ is a positive definite matrix, $L(\Omega, \lambda)$ is a strictly convex function. Therefore $\Omega^* = \frac{D^{-1}R}{R^{\mathrm{T}}D^{-1}R}$ is the unique optimal solution to Problem (21), which completes the proof. □

Theorem 3.1 indicates that Problem (24) has the analytical solutions Ω^*. However, Ω^* is not necessarily non-negative. If $\Omega^* \geq 0$, it is also the optimal solution to Problem (23). Otherwise, we can use the software for optimization problems (such as LINGO) to solve the Problem (23).

4 Numerical Example

In this section, we offer a numerical example which is used to determine the experts weights based on square logarithm compatibility index of interval multiplicative preference relations in the GDM. Suppose that $X = (x_1, x_2, x_3, x_4)$ is a finite set of alternatives and $E = (e_1, e_2, e_3)$ is a finite set of experts. Every expert provides his own interval multiplicative preference relation on X in Wang et al. (2005), respectively.

Let

$$A_1 = \begin{bmatrix} [1,1] & [2,5] & [2,4] & [1,3] \\ [1/5,1/2] & [1,1] & [1,3] & [1,2] \\ [1/4,1/2] & [1/3,1] & [1,1] & [1/2,1] \\ [1/3,1] & [1/2,1] & [1,2] & [1,1] \end{bmatrix}$$

$$A_2 = \begin{bmatrix} [1,1] & [1,2] & [1,2] & [2,3] \\ [1/2,1] & [1,1] & [3,5] & [4,5] \\ [1/2,1] & [1/5,1/3] & [1,1] & [6,8] \\ [1/3,1/2] & [1/5,1/4] & [1/8,1/6] & [1,1] \end{bmatrix}$$

$$A_3 = \begin{bmatrix} [1,1] & [2,4] & [3,5] & [3,5] \\ [1/4,1/2] & [1,1] & [1/2,1] & [2,5] \\ [1/5,1/3] & [1,2] & [1,1] & [1/3,1] \\ [1/5,1/3] & [1/5,1/2] & [1,3] & [1,1] \end{bmatrix}$$

Wang et al proposed a two-stage logarithmic goal programming method for generating weights $\tilde{w}^{(1)} = \left(\tilde{w}_1^{(1)}, \tilde{w}_2^{(1)}, \tilde{w}_3^{(1)}, \tilde{w}_4^{(1)}\right)$ from A_1 in the table 3 of Wang et al. (2005), where

$$\tilde{w}_1^{(1)} = \left[w_1^{L(1)}, w_1^{U(1)}\right] = [1.6818, 2.4495], \quad \tilde{w}_2^{(1)} = \left[w_2^{L(1)}, w_2^{U(1)}\right] = [0.7598, 1.1067],$$
$$\tilde{w}_3^{(1)} = \left[w_3^{L(1)}, w_3^{U(1)}\right] = [0.5000, 0.8409], \quad \tilde{w}_4^{(1)} = \left[w_4^{L(1)}, w_4^{U(1)}\right] = [0.6866, 1.0000],$$

Similarly, weights $\tilde{w}^{(2)} = \left(\tilde{w}_1^{(2)}, \tilde{w}_2^{(2)}, \tilde{w}_3^{(2)}, \tilde{w}_4^{(2)}\right)$ is obtained from A_2 in the table 6 of Wang et al. (2005).

$$\tilde{w}_1^{(2)} = \left[w_1^{L(2)}, w_1^{U(2)}\right] = [1.1583, 1.7783], \quad \tilde{w}_2^{(2)} = \left[w_2^{L(2)}, w_2^{U(2)}\right] = [1.4142, 2.2361],$$
$$\tilde{w}_3^{(2)} = \left[w_3^{L(2)}, w_3^{U(2)}\right] = [0.7071, 1.4953], \quad \tilde{w}_4^{(2)} = \left[w_4^{L(2)}, w_4^{U(2)}\right] = [0.2991, 0.4855],$$

weights $\tilde{w}^{(3)} = (\tilde{w}_1^{(3)}, \tilde{w}_2^{(3)}, \tilde{w}_3^{(3)}, \tilde{w}_4^{(3)})$ is generated from A_3 in the table 9 of Saaty and Vargas (2007).

$$\tilde{w}_1^{(3)} = \left[w_1^{L(3)}, w_1^{U(3)}\right] = [2.0598, 3.1623], \quad \tilde{w}_2^{(3)} = \left[w_2^{L(3)}, w_2^{U(3)}\right] = [0.7071, 1.1892],$$
$$\tilde{w}_3^{(3)} = \left[w_3^{L(3)}, w_3^{U(3)}\right] = [0.5623, 0.8633], \quad \tilde{w}_4^{(3)} = \left[w_4^{L(3)}, w_4^{U(3)}\right] = [0.4949, 0.7598],$$

According to the Eq. (15), we have the characteristic matrix $\tilde{W}^{(k)} = \left(\tilde{w}_{ij}^{(k)}\right)_{n\times n}$ corresponding to IMPR $\tilde{A}^{(k)} = \left(\tilde{a}_{ij}^{(k)}\right)_{n\times n}$, $k = 1, 2, 3$, i.e,

$$\tilde{W}^{(1)} = \begin{bmatrix} [1,1] & [1.5197, 3.2239] & [2, 4.8990] & [1.6818, 3.5676] \\ [0.3102, 0.6580] & [1,1] & [0.9036, 2.2134] & [0.7598, 1.6119] \\ [0.2041, 0.5] & [0.45179, 1.1067] & [1,1] & [0.5, 1.2247] \\ [0.2803, 0.5946] & [0.6204, 1.3161] & [0.81651, 2] & [1,1] \end{bmatrix};$$

$$\tilde{W}^{(2)} = \begin{bmatrix} [1,1] & [0.518, 1.2575] & [0.7746, 2.5149] & [2.3858, 5.9455] \\ [0.7953, 1.9305] & [1,1] & [0.9458, 3.1624] & [2.9129, 7.4761] \\ [0.3976, 1.2909] & [0.3162, 1.0573] & [1,1] & [1.4564, 4.9993] \\ [0.1682, 0.4192] & [0.1338, 0.3433] & [0.2000, 0.6866] & [1,1] \end{bmatrix};$$

$$\tilde{W}^{(3)} = \begin{bmatrix} [1,1] & [1.7321, 4.4722] & [2.386, 5.6239] & [2.711, 6.3898] \\ [0.2236, 0.5773] & [1,1] & [0.8191, 2.1149] & [0.9306, 2.4029] \\ [0.1778, 0.4191] & [0.4728, 1.2209] & [1,1] & [0.7400, 1.7444] \\ [0.1565, 0.3689] & [0.4162, 1.0745] & [0.5733, 1.3512] & [1,1] \end{bmatrix};$$

According to the Problem (20), we have

$$D = \begin{bmatrix} 0.0729 & 0.0669 & 0.0172 \\ 0.0669 & 0.4413 & -0.1732 \\ 0.0172 & -0.1732 & 0.2536 \end{bmatrix}$$

By the Eq. (22), we get

$$\Omega^* = \frac{D^{-1}R}{R^{\mathrm{T}}D^{-1}R} = \begin{bmatrix} 0.5545 \\ 0.1562 \\ 0.2893 \end{bmatrix}$$

Since $\Omega^* > 0$, Therefore Ω^* is the optimal solution of the Problem (20). i.e, the experts e_1, e_2, e_3 have their optimal weights

$$\omega^{(1)} = 0.5545, \omega^{(2)} = 0.1562, \omega^{(3)} = 0.2893$$

Furthermore, we calculate the logarithm compatibility index of IMPR $\tilde{A}_k$ and $\tilde{W}^{(k)}$by the Eq. (6), $k = 1, 2, 3$, then

$$ILCI\left(\tilde{A}_1, \tilde{W}^{(1)}\right) = 0.2256$$
$$ILCI\left(\tilde{A}_2, \tilde{W}^{(2)}\right) = 0.5571$$
$$ILCI\left(\tilde{A}_3, \tilde{W}^{(3)}\right) = 0.4203$$

If $\alpha = 0.25$, then

$$ILCI\left(\tilde{A}_1, \tilde{W}^{(1)}\right) < \alpha,\ ILCI\left(\tilde{A}_2, \tilde{W}^{(2)}\right) > \alpha,\ ILCI\left(\tilde{A}_2, \tilde{W}^{(3)}\right) > \alpha$$

i.e.,

IMPR $\tilde{A}_1$ provided by the 1st expert and its characteristic matrix $\tilde{W}^{(1)}$ are of acceptable compatibility. However, $\tilde{A}_2$ provided by the 2nd expert and $\tilde{W}^{(2)}$, $\tilde{A}_3$ provided by the 3rd expert and $\tilde{W}^{(3)}$ are not of acceptable compatibility.

From the computational result, we can see that

$ILCI\left(\tilde{A}_2, \tilde{W}^{(2)}\right) > ILCI\left(\tilde{A}_3, \tilde{W}^{(3)}\right) > ILCI\left(\tilde{A}_1, \tilde{W}^{(1)}\right)$ and $\omega^{(1)} > \omega^{(3)} > \omega^{(2)}$

i.e., The less the logarithm compatibility index of IMPR and its characteristic matrix is, the more weight expert corresponding to the IMPR has.

5 Concluding Remarks

In this paper, we have presented the new logarithm compatibility degree of IMPR in which the input information arguments provided by experts are given in the form of interval numerical values rather than the non-negative real values.

we have been able to obtain the properties of logarithm compatibility degree. Especially, we prove that the synthetic preference relation and the synthetic characteristic matrix are also of acceptable compatibility under the condition that the IMPR provided by every expert and its characteristic matrix are of acceptable compatibility, which provided the scientific base of using the synthetic characteristic matrix for priority instead of the synthetic characteristic matrix in the GDM. Furthermore, we have introduced square logarithm compatibility degree and utilized it for determining the optimal weights of experts. We have also illustrated a numerical example to show the feasibility and effectiveness of the new approach. The result shows that the optimal weights of experts are related to the reliability of information provided by them, which gives the objective approach for weighting the experts by the mathematical model.

In the future, we expect to develop the compatibility of interval linguistic preference information and its properties and applications in the GDM.

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