Symmetric Coalitional Binomial Semivalues

Francesc Carreras · María Albina Puente

Abstract   We introduce here a family of mixed coalitional values. They extend the binomial semivalues to games endowed with a coalition structure, satisfy the property of symmetry in the quotient game and the quotient game property, generalize the symmetric coalitional Banzhaf value introduced by Alonso and Fiestras and link and merge the Shapley value and the binomial semivalues. A computational procedure in terms of the multilinear extension of the original game is also provided and an application to political science is sketched.

Keywords   Cooperative game · Binomial semivalue · Coalition structure · Multilinear extension

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1 Introduction

The parallel axiomatic characterization stated by Feltkamp (1995) shows that the only difference between the Shapley value (1953) and the Banzhaf value (Owen 1975), as

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allocation rules for all cooperative games, is that the former satisfies efficiency whereas the latter satisfies the total power property. Differences of this kind become important when one is facing a group decision and negotiation problem and wishes to choose a suitable allocation rule for solving the problem. The properties of a value should always be a main argument for either selecting it or rejecting it in each particular case.

In the framework of cooperative games with a coalition structure, other essential differences also arise between the Owen value (1977) and the modified Banzhaf value or Owen–Banzhaf value (1982). The Owen–Banzhaf value fails to satisfy the property of symmetry in the quotient game and the quotient game property, which are satisfied by the Owen value.

Alonso and Fiestras (2002) suggested a modification of the Owen–Banzhaf value that satisfies these two properties and can therefore be compared with the Owen value in terms analogous to Feltkamp (1995). Our aim here is to introduce the notion of coalitional binomial semivalue as a wide generalization of the Alonso–Fiestras value (essentially: $p \in [0, 1]$ instead of $p = 1/2$) in order to get the family of *symmetric coalitional binomial semivalues* that still satisfy the property of symmetry in the quotient game and the quotient game property, so that they differ from the Owen value just in satisfying a total power property instead of efficiency. These new values are especially suited for the study of cooperative games where the players show some (common) tendency as to the size of the coalitions they would agree to form. This tendency is defined by parameter p. Let us include a motivating example.

Example 1.1 In Sect. 5 we apply this family of values to the analysis of an interesting political problem: the Catalonia Parliament during Legislature 2003–2007, prematurely finished in 2006. Curiously, the analysis remains still valid for Legislature 2006–2010 since, in spite of the modification of the seat distribution issued from the elections held in November 1, 2006, the strategic possibilities are exactly the same.

In Catalonia, politics is based on two main axes: the classical left-to-right axis and a crossed one going from Spanish centralism to Catalanism (Catalan nationalism) (see Fig. 1). Thus, in 2003 Esquerra Republicana de Catalunya (ERC), a radical nationalist and left-wing party, was faced to the dilemma of choosing among either a Catalanist majority coalition with Convergència i Unió (CiU) or a left-wing majority coalition with the Partit dels Socialistes de Catalunya (PSC) and Iniciativa per Catalunya-Verds (ICV), which was finally formed in 2003 and has been repeated in 2006.

A classical approach to study the problem would consist in using either (a) the Shapley value and the Owen value or, alternatively, (b) the Banzhaf value and the Owen–Banzhaf value, in order to evaluate the strategic possibilities of each party in both setups. The results are given in Table 1, where (NO) means no coalition formation, (C) means that CiU + ERC forms, and (LW) means that PSC + ERC + ICV forms.

According to the Shapley and Owen values used in (a), ERC would strictly prefer joining CiU instead of PSC and ICV. The same conclusion is obtained according to the Banzhaf and Owen–Banzhaf values used in (b). In both cases, the results fail to provide a mathematical explanation of ERC's actual decision (to join PSC and ICV).

Instead, by using binomial semivalues, and symmetric coalitional binomial semivalues whenever a coalition structure exists, the conclusion of the theoretical analysis
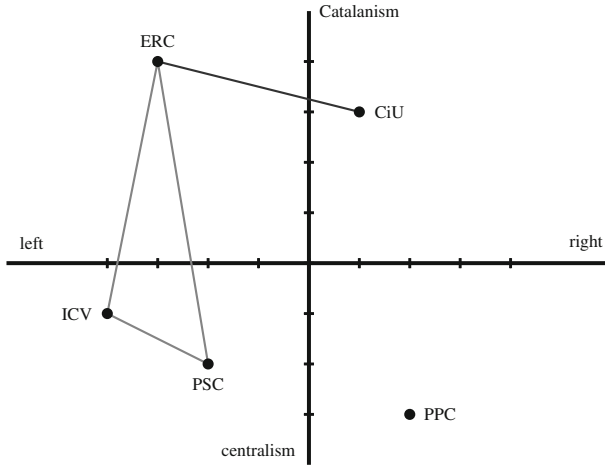
Fig. 1 Political positions in the Catalonia Parliament 2003–2007

Table 1 Classical measures of power in the Catalonia Parliament 2003–2007

	(a)			(b)		
	(NO)	(C)	(LW)	(NO)	(C)	(LW)
CiU	0.4000	0.5000	0.0000	0.6250	0.5000	0.0000
PSC	0.2333	0.0000	0.3889	0.3750	0.0000	0.3750
ERC	0.2333	0.5000	0.3889	0.3750	0.5000	0.3750
PPC	0.0667	0.0000	0.0000	0.1250	0.0000	0.0000
ICV	0.0667	0.0000	0.2222	0.1250	0.0000	0.1250

is that ERC, the crucial agent in this scenario, was not necessarily forced to participate in the left-wing tripartite government but would have got more political power in joining CiU depending on the tendency of the parties. The reader is referred to Example 5.3 for a detailed analysis.

The organization of the paper is as follows. In Sect. 2, a minimum of preliminaries is provided. Section 3 is devoted to define and study the symmetric coalitional binomial semivalue, and it includes an axiomatic characterization that parallels (Owen 1977) for the Owen value. In Sect. 4 we present a computation procedure for the symmetric coalitional binomial semivalue. Section 5 contains a remark on simple games and two detailed examples and, finally, we have included in Sect. 6 a historical note.

2 Preliminaries

2.1 Games and Semivalues

Let N be a finite set of *players* and 2^N be the set of its *coalitions* (subsets of N). A *cooperative game* on N is a function $v : 2^N \to \mathbb{R}$, that assigns a real number $v(S)$

to each coalition $S \subseteq N$, with $v(\emptyset) = 0$. A game v is *monotonic* if $v(S) \leq v(T)$ whenever $S \subseteq T \subseteq N$. A player $i \in N$ is a *dummy* in v if $v(S \cup \{i\}) = v(S) + v(\{i\})$ for all $S \subseteq N\backslash\{i\}$, and *null* in v if, moreover, $v(\{i\}) = 0$. Two players $i, j \in N$ are *symmetric* in v if $v(S \cup \{i\}) = v(S \cup \{j\})$ for all $S \subseteq N\backslash\{i, j\}$.

Endowed with the natural operations for real-valued functions, *i.e.* $v + v'$ and λv for all $\lambda \in \mathbb{R}$, the set of all cooperative games on N is a vector space $\mathcal{G}_N$. For every nonempty coalition $T \subseteq N$, the *unanimity game* u_T is defined by $u_T(S) = 1$ if $T \subseteq S$ and $u_T(S) = 0$ otherwise, and it is easily checked that the set of all unanimity games is a basis for $\mathcal{G}_N$. Finally, every permutation θ of N induces a linear automorphism of $\mathcal{G}_N$ given by $(\theta v)(S) = v(\theta^{-1}S)$ for all $S \subseteq N$ and all v.

By a *value* on $\mathcal{G}_N$ we will mean a map $f : \mathcal{G}_N \rightarrow \mathbb{R}^N$, that assigns to every game v a vector $f[v]$ with components $f_i[v]$ for all $i \in N$.

Following Weber's (1988) axiomatic description, $\Psi : \mathcal{G}_N \rightarrow \mathbb{R}^N$ is a *semivalue* iff it satisfies the following properties:

(i) *linearity*: $\Psi[v + v'] = \Psi[v] + \Psi[v']$ *(additivity)* and $\Psi[\lambda v] = \lambda \Psi[v]$ for all $v, v' \in \mathcal{G}_N$ and $\lambda \in \mathbb{R}$;
(ii) *anonymity*: $\Psi_{\theta i}[\theta v] = \Psi_i[v]$ for all θ on N, $i \in N$, and $v \in \mathcal{G}_N$;
(iii) *positivity*: if v is monotonic, then $\Psi[v] \geq 0$;
(iv) *dummy player property*: if $i \in N$ is a dummy in game v, then $\Psi_i[v] = v(\{i\})$.

There is an interesting characterization of semivalues, by means of *weighting coefficients*, due to Dubey et al. (1981). Set $n = |N|$. Then: (a) for every *weighting vector* $\{p_k\}_{k=0}^{n-1}$ such that $\sum_{k=0}^{n-1} p_k \binom{n-1}{k} = 1$ and $p_k \geq 0$ for all k, the expression

$$\Psi_i[v] = \sum_{S \subseteq N\backslash\{i\}} p_s[v(S \cup \{i\}) - v(S)] \quad \text{for all } i \in N \text{ and all } v \in \mathcal{G}_N,$$

where $s = |S|$, defines a semivalue Ψ; (b) conversely, every semivalue can be obtained in this way; (c) the correspondence given by $\{p_k\}_{k=0}^{n-1} \mapsto \Psi$ is bijective.

Thus, the payoff that a semivalue allocates to every player in any game is a weighted sum of his marginal contributions in the game. If p_k is interpreted as the probability that a given player i joins a coalition of size k, provided that all the coalitions of a common size have the same probability of being joined, then $\Psi_i[v]$ is the expected marginal contribution of that player to a random coalition he joins.

Well known examples of semivalues are the *Shapley value* φ (Shapley 1953), for which $p_k = 1/n \binom{n-1}{k}$, and the *Banzhaf value* β (Owen 1975), for which $p_k = 2^{1-n}$. The Shapley value φ is the only *efficient* semivalue, in the sense that $\sum_{i \in N} \varphi_i[v] = v(N)$ for every $v \in \mathcal{G}_N$.

Notice that these values are defined for each N. The same happens with the *binomial semivalues*, introduced by Puente (2000) as follows. Let $p \in [0, 1]$ and $p_k = p^k(1-p)^{n-k-1}$ for $k = 0, 1, \ldots, n-1$. Then $\{p_k\}_{k=0}^{n-1}$ is a weighting vector and defines a semivalue that will be denoted as Ψ^p and called the *p-binomial semivalue*. Using the convention that $0^0 = 1$, the definition makes sense also for $p = 0$ and $p = 1$, where

we respectively get the *dictatorial index* Ψ^0 and the *marginal index* Ψ^1, introduced by Owen (1978) and such that $\Psi_i^0[v] = v(\{i\})$ and $\Psi_i^1[v] = v(N) - v(N\setminus\{i\})$ for all $i \in N$ and all $v \in \mathcal{G}_N$. Of course, $p = 1/2$ gives $\Psi^{1/2} = \beta$—the Banzhaf value.

In fact, semivalues are defined on cardinalities rather than on specific player sets: this means that a weighting vector $\{p_k\}_{k=0}^{n-1}$ defines a semivalue Ψ on all N such that $n = |N|$. When necessary, we shall write $\Psi^{(n)}$ for a semivalue on cardinality n and $p_k^{(n)}$ for its weighting coefficients. A semivalue $\Psi^{(n)}$ induces semivalues $\Psi^{(t)}$ for all cardinalities $t < n$, recurrently defined by the Pascal triangle (inverse) formula given by Dragan (1997):

$$p_k^{(t)} = p_k^{(t+1)} + p_{k+1}^{(t+1)} \quad \text{for} \quad 0 \le k < t.$$

A series $\Psi = \{\Psi^{(n)}\}_{n=1}^{\infty}$ of semivalues, one for each cardinality, is a *multisemivalue* if it satisfies Dragan's recurrence formula. Thus, the Shapley and Banzhaf values and all binomial semivalues are multisemivalues.

2.2 Games with Coalition Structure

Let us consider a finite set, say, $N = \{1, 2, \ldots, n\}$. We will denote by $P(N)$ the set of all partitions of N. Each $P \in P(N)$ is called a *coalition structure* or *system of unions* on N. The so-called *trivial coalition structures* are $P^n = \{\{1\}, \{2\}, \ldots, \{n\}\}$ (individual coalitions) and $P^N = \{N\}$ (grand coalition). A *cooperative game with a coalition structure* is a pair $[v; P]$, where $v \in \mathcal{G}_N$ and $P \in P(N)$ for a given N. Each partition P gives a pattern of cooperation among players. We denote by $\mathcal{G}_N^{cs}$ the set of all cooperative games with a coalition structure and player set N.

If $[v; P] \in \mathcal{G}_N^{cs}$ and $P = \{P_1, P_2, \ldots, P_m\}$, the *quotient game* v^P is the cooperative game played by the unions, or, rather, by the set $M = \{1, 2, \ldots, m\}$ of their representatives, as follows:

$$v^P(R) = v\left(\bigcup_{r \in R} P_r\right) \quad \text{for all } R \subseteq M.$$

Unions P_r, P_s are said to be *symmetric* in $[v; P]$ if r, s are symmetric players in v^P.

By a *coalitional value* on $\mathcal{G}_N^{cs}$ we will mean a map $g : \mathcal{G}_N^{cs} \to \mathbb{R}^N$, which assigns to every pair $[v; P]$ a vector $g[v; P]$ with components $g_i[v; P]$ for each $i \in N$.

The *Owen value* (Owen 1977) is the coalitional value Φ defined by

$$\Phi_i[v; P] = \sum_{R \subseteq M\setminus\{k\}} \sum_{T \subseteq P_k\setminus\{i\}} \frac{1}{mp_k} \frac{1}{\binom{m-1}{r}} \frac{1}{\binom{p_k-1}{t}} [v(Q \cup T \cup \{i\}) - v(Q \cup T)]$$

for all $i \in N$ and $[v; P] \in \mathcal{G}_N^{cs}$, where $P_k \in P$ is the union such that $i \in P_k$ and $Q = \bigcup_{r \in R} P_r$. It was axiomatically characterized by Owen (1977) as the only

coalitional value that satisfies the following properties: the natural extensions to this framework of

- *efficiency*
- *additivity*
- *the dummy player property*

and also

- *symmetry within unions*: if $i, j \in P_k$ are symmetric in v then

$$\Phi_i[v; P] = \Phi_j[v; P]$$

- *symmetry in the quotient game*: if $P_r, P_s \in P$ are symmetric in $[v; P]$ then

$$\sum_{i \in P_r} \Phi_i[v; P] = \sum_{j \in P_s} \Phi_j[v; P].$$

The Owen value is a *coalitional value of the Shapley value* φ in the sense that $\Phi[v; P^n] = \varphi[v]$ for all $v \in \mathcal{G}_N$. Besides, $\Phi[v; P^N] = \varphi[v]$. Finally, as Φ is defined for any N, the following property makes sense and is also satisfied:

- *quotient game property*: for all $[v; P] \in \mathcal{G}_N^{cs}$,

$$\sum_{i \in P_k} \Phi_i[v; P] = \Phi_k[v^P; P^m] \quad \text{for all } P_k \in P.$$

The Owen value can be viewed as a two-step allocation rule. First, each union P_k receives its payoff in the quotient game according to the Shapley value; then, each P_k splits this amount among its players by applying the Shapley value to a game played in P_k as follows: the worth of each subcoalition T of P_k is the Shapley value that T would get in a "pseudoquotient game" played by T and the remaining unions on the assumption that $P_k \backslash T$ leaves the game, *i.e.* the quotient game after replacing P_k with T. This is the way to bargain within the union: each subcoalition T claims the payoff it would obtain when dealing with the other unions in absence of its partners in P_k.

The Owen–Banzhaf value B (Owen 1982) follows a similar scheme. The resulting formula parallels that of the Owen value with the sole change of coefficient $1/mp_k \binom{m-1}{r}\binom{p_k-1}{t}$ by $2^{1-m}2^{1-p_k}$. This value, which is a *coalitional value of the Banzhaf value* β, does not satisfy efficiency, but neither symmetry in the quotient game nor the quotient game property. The bargaining interpretation is the same as in the case of the Owen value by replacing everywhere the Shapley value with the Banzhaf value.

Alonso and Fiestras (2002) introduced a modification of the Owen–Banzhaf value. In this case, the coefficient of each marginal contribution is replaced with $2^{1-m}/p_k \binom{p_k-1}{t}$. This *symmetric coalitional Banzhaf value* Π satisfies the same properties as the Owen value, with the sole exception of efficiency—replaced by a total power property—, as well as the quotient game property, and it is a coalitional value of the Banzhaf value.

Example 2.1 ([Alonso and Fiestras 2002](#)) Let us take $n = 5$ and consider the unanimity game u_N and the coalition structure $P = \{P_1, P_2\}$ where $P_1 = \{1, 2, 3\}$ and $P_2 = \{4, 5\}$. Notice that the quotient game is $u_N^P = u_M$, where $M = \{1, 2\}$. It is not difficult to obtain the following values:

$$\beta[u_N] = (1/16, 1/16, 1/16, 1/16, 1/16),$$
$$\beta[u_N^P] = (1/2, 1/2),$$
$$B[u_N; P] = (1/8, 1/8, 1/8, 1/4, 1/4).$$

As P_1 and P_2 are symmetric in $[u_N; P]$, it follows that the Owen–Banzhaf value B fails to satisfy the property of symmetry in the quotient game. Neither the quotient game property is fulfilled by B in this instance. Instead

$$\Pi[u_N; P] = (1/6, 1/6, 1/6, 1/4, 1/4)$$

so that both properties are satisfied by the Alonso–Fiestras value Π (here and elsewhere).

Since the Banzhaf value β is a particular p-binomial semivalue ($p = 1/2$), this example also shows that the *coalitional p-binomial semivalues*, which can be obtained from the work by [Albizuri and Zarzuelo](#) ([2004](#)) or [Amer and Giménez](#) ([2003](#)) by applying Owen's scheme to any p-binomial semivalue, satisfy, in general, none of both properties. That's why we will generalize Alonso and Fiestras' procedure.

3 The Symmetric Coalitional Binomial Semivalue

In this section we define and study a "coalitional version" of each p-binomial semivalue for games with coalition structure. This includes, besides the explicit formula, an axiomatic characterization and an interpretation in terms of a two-step bargaining process, among unions, first, and among players within each union later. We begin by dealing with binomial semivalues. We recall that Ψ^p denotes, for each $p \in [0, 1]$, the p-binomial semivalue acting on a fixed $\mathcal{G}_N$.

Definition 3.1 Let $p \in [0, 1]$. A value f on $\mathcal{G}_N$ satisfies the *p-binomial total power property* if

$$\sum_{i \in N} f_i[v] = \sum_{i \in N} \sum_{S \subseteq N \setminus \{i\}} p^s (1-p)^{n-s-1} [v(S \cup \{i\}) - v(S)] \quad \text{for all } v \in \mathcal{G}_N.$$

Lemma 3.2 *Let $\emptyset \neq S \subseteq N$, $s = |S|$ and $i \in N$. Then $\Psi_i^p[u_S] = p^{s-1}$ if $i \in S$, and $\Psi_i^p[u_S] = 0$ otherwise.*

Proof Let $i \in S$. By the definition of the weighting coefficients of Ψ^p we have

$$\Psi_i^p[u_S] = \binom{n-s}{0} p^{s-1}(1-p)^{n-s} + \binom{n-s}{1} p^s(1-p)^{n-s-1}$$
$$+ \cdots + \binom{n-s}{n-s} p^{n-1}$$
$$= p^{s-1}[(p+(1-p)]^{n-s} = p^{s-1}.$$

If $i \notin S$, the dummy player property yields $\Psi_i^p[u_S] = 0$. □

Proposition 3.3 *The unique semivalue on $\mathcal{G}_N$ that satisfies the p-binomial total power property is the p-binomial semivalue Ψ^p.*

In other words, $\sum_{i \in N} \Psi_i[v] = \sum_{i \in N} \Psi_i^p[v]$ for all $v \in \mathcal{G}_N$ implies $\Psi = \Psi^p$.

Proof (a)   It is obvious that the p-binomial semivalue Ψ^p satisfies the p-binomial total power property.

(b)   Using linearity and the fact that the unanimity games form a basis of $\mathcal{G}_N$, it suffices to prove that any semivalue Ψ satisfying the p-binomial total power property coincides with Ψ^p on each unanimity game. Let u_S be such a game for some $S \subseteq N$. From the dummy player property we get $\Psi_i[u_S] = 0 = \Psi_i^p[u_S]$ for all $i \notin S$. From the p-binomial total power property it follows that $\sum_{i \in S} \Psi_i[u_S] = sp^{s-1}$. Using anonymity, $\Psi_i[u_S] = p^{s-1}$ for each $i \in S$, and this allocation coincides with $\Psi_i^p[u_S]$ according to Lemma 3.2. □

In particular, setting $p = 1/2$ we obtain that the Banzhaf value β is the only semivalue that satisfies the classical *total power property*:

$$\sum_{i \in N} f_i[v] = \frac{1}{2^{n-1}} \sum_{i \in N} \sum_{S \subseteq N \setminus \{i\}} [v(S \cup \{i\}) - v(S)] \quad \text{for all} \quad v \in \mathcal{G}_N.$$

The Owen (resp., Owen–Banzhaf) value is a natural extension of the Shapley (resp., Banzhaf) value to games with a coalition structure. We generalize this idea.

Definition 3.4 Given a value f on $\mathcal{G}_N$, a *coalitional value of f* is a coalitional value g on $\mathcal{G}_N^{cs}$ such that $g[v; P^n] = f[v]$ for all $v \in \mathcal{G}_N$.

Let g be a coalitional value of the p-binomial semivalue Ψ^p defined for all N, and assume that g satisfies the quotient game property. Then, for a given N and any $[v; P] \in \mathcal{G}_N^{cs}$,

$$\sum_{i \in N} g_i[v; P] = \sum_{k \in M} \sum_{i \in P_k} g_i[v; P] = \sum_{k \in M} g_k[v^P; P^m] = \sum_{k \in M} \Psi_k^p[v^P]$$
$$= \sum_{k \in M} \sum_{R \subseteq M \setminus \{k\}} p^r (1-p)^{m-r-1} [v^P(R \cup \{k\}) - v^P(R)].$$

This motivates the next definition, that is an adaptation of the p-binomial total power property to games with a coalition structure.

Definition 3.5 Let $p \in [0, 1]$. A coalitional value g on $\mathcal{G}_N^{cs}$ satisfies the *coalitional p-binomial total power property* if, for all $[v; P] \in \mathcal{G}_N^{cs}$,

$$\sum_{i \in N} g_i[v; P] = \sum_{k \in M} \sum_{R \subseteq M \setminus \{k\}} p^r (1 - p)^{m-r-1} [v^P(R \cup \{k\}) - v^P(R)].$$

The next statement defines and axiomatically characterizes, for each $p \in [0, 1]$, the *symmetric coalitional p-binomial semivalue*, which will be denoted as Ω^p.

Theorem 3.6 *Let $p \in [0, 1]$. For any N there is a unique coalitional value on $\mathcal{G}_N^{cs}$ that satisfies additivity, the dummy player property, symmetry within unions, symmetry in the quotient game, and the coalitional p-binomial total power property. Given $[v; P] \in \mathcal{G}_N^{cs}$, this value allocates to each player $i \in N$ the real number*

$$\Omega_i^p[v; P] = \sum_{R \subseteq M \setminus \{k\}} \sum_{T \subseteq P_k \setminus \{i\}} p^r (1 - p)^{m-r-1} \frac{1}{p_k \binom{p_k - 1}{t}} [v(Q \cup T \cup \{i\}) - v(Q \cup T)],$$

where $P_k \in P$ is the union such that $i \in P_k$ and $Q = \bigcup_{r \in R} P_r$.

Moreover, Ω^p is a coalitional value of the p-binomial semivalue Ψ^p and satisfies the quotient game property.

Proof (a) (Existence) It suffices to show that the coalitional value Ω^p given by the above formula satisfies the five properties enumerated in the statement.

1. Additivity. It merely follows from the expression of $\Omega_i^p[v; P]$.
2. Dummy player property. Let $i \in N$ be a dummy player in game v and P be any coalition structure. Assume $i \in P_k$. Then $v(Q \cup T \cup \{i\}) - v(Q \cup T) = v(\{i\})$ for all R and T. As, moreover,

$$\sum_{R \subseteq M \setminus \{k\}} p^r (1 - p)^{m-r-1} = 1 \quad \text{and} \quad \sum_{T \subseteq P_k \setminus \{i\}} \frac{1}{p_k \binom{p_k - 1}{t}} = 1,$$

   we conclude that $\Omega_i^p[v; P] = v(\{i\})$.
3. Symmetry within unions. Let $i, j \in P_k \in P$ be symmetric players in game v. For each $R \subseteq M \setminus \{k\}$ and $T \subseteq P_k \setminus \{i, j\}$, let $\Delta(R, T, h) = v(Q \cup T \cup \{h\}) - v(Q \cup T)$ for $h = i, j$. Then, by the symmetric position of i, j in v,

$$f(R, T) = \Delta(R, T, i) - \Delta(R, T, j) = 0 \quad \text{and}$$
$$g(R, T) = \Delta(R, T \cup \{j\}, i) - \Delta(R, T \cup \{i\}, j) = 0,$$

so that

$$\Omega_i^p[v;P] - \Omega_j^p[v;P] = \sum_{R \subseteq M\setminus\{k\}} p^r(1-p)^{m-r-1}$$

$$\sum_{T \subseteq P_k\setminus\{i,j\}} \left[\frac{f(R,T)}{p_k \binom{p_k-1}{t}} + \frac{g(R,T)}{p_k \binom{p_k-1}{t+1}} \right] = 0.$$

4. Coalitional p-binomial total power property. Let $[v;P] \in \mathcal{G}_N^{cs}$. Fixing $k \in M$, for every $R \subseteq M\setminus\{k\}$ we consider the game $v_R \in \mathcal{G}_{P_k}$ defined by

$$v_R(T) = v(Q \cup T) - v(Q) \quad \text{for all } T \subseteq P_k.$$

The Shapley value gives, for each $i \in P_k$,

$$\varphi_i[v_R] = \sum_{T \subseteq P_k\setminus\{i\}} \frac{1}{p_k \binom{p_k-1}{t}} [v(Q \cup T \cup \{i\}) - v(Q \cup T)].$$

Using the efficiency of φ, we get

$$\sum_{i \in P_k} \varphi_i[v_R] = v_R(P_k) = v(Q \cup P_k) - v(Q) = v^P(R \cup \{k\}) - v^P(R).$$

Hence

$$\sum_{i \in P_k} \Omega_i^p[v;P] = \sum_{R \subseteq M\setminus\{k\}} p^r(1-p)^{m-r-1}[v^P(R \cup \{k\}) - v^P(R)]$$

$$= (\Psi^P)_k^{(m)}[v^P]$$

and, finally,

$$\sum_{i \in N} \Omega_i^p[v;P] = \sum_{k \in M} \sum_{R \subseteq M\setminus\{k\}} p^r(1-p)^{m-r-1}[v^P(R \cup \{k\}) - v^P(R)].$$

5. Symmetry in the quotient game. It readily follows from the relationship

$$\sum_{i \in P_k} \Omega_i^p[v;P] = (\Psi^P)_k^{(m)}[v^P],$$

stated in the previous point, and the anonymity (whence symmetry) of the p-binomial semivalue Ψ^P.

(b) (Uniqueness) Let g be a coalitional value on $\mathcal{G}_N^{cs}$ satisfying the five properties. Using additivity and the fact that the unanimity games form a basis of $\mathcal{G}_N$, it suffices to show that g is completely determined by its action on any pair of the form $[\lambda u_S; P]$, where $\lambda \in \mathbb{R}$, $\emptyset \neq S \subseteq N$ and $P \in P(N)$.

By the dummy player property, $g_i[\lambda u_S; P] = 0$ if $i \notin S$. This leaves us with players $i \in S$.

Let $S' = \{k \in M : S \cap P_k \neq \emptyset\}$ and, for every $k \in S'$, $S_k' = S \cap P_k$. It is easy to see that $(\lambda u_S)^P = \lambda u_{S'}$. From the coalitional p-binomial total power property, and applying Lemma 3.2, we have

$$\sum_{i \in N} g_i[\lambda u_S; P] = \sum_{k \in M} (\Psi^P)_k^{(m)}[\lambda u_{S'}] = \sum_{k \in S'} (\Psi^P)_k^{(m)}[\lambda u_{S'}] = \lambda s' p^{s'-1}.$$

Now, from symmetry in the quotient game, if $k \in S'$ then

$$\sum_{i \in S_k'} g_i[\lambda u_S; P] = \sum_{i \in P_k} g_i[\lambda u_S; P] = \lambda p^{s'-1}$$

and, finally, using symmetry within unions,

$$g_i[\lambda u_S; P] = \frac{\lambda p^{s'-1}}{s_k'} \quad \text{for any } i \in S_k'.$$

As $S = \bigcup_{k \in S'} S_k'$, this concludes the proof that g is univocally determined.

(c) Ω^P is a coalitional value of the p-binomial semivalue Ψ^P. Indeed, for $P = P^n$ the explicit formula of Ω^P reduces to

$$\Omega_i^P[v; P^n] = \sum_{R \subseteq N \setminus \{i\}} p^r (1-p)^{m-r-1} [v(R \cup \{i\}) - v(R)] = \Psi_i^P[v].$$

Finally, the quotient game property: as we have seen when showing the symmetry in the quotient game in part (a) of this proof, and using the preceding property for $\mathcal{G}_M^{cs}$,

$$\sum_{i \in P_k} \Omega_i^P[v; P] = (\Psi^P)_k^{(m)}[v^P] = \Omega_k^P[v^P; P^m]. \qquad \square$$

Remark 3.7 (a) The symmetric coalitional p-binomial semivalue is a natural (and wide) generalization of Alonso and Fiestras' symmetric coalitional Banzhaf value, since $\Omega^{1/2} = \Pi$.

(b) Ω^P relates not only to the p-binomial semivalue Ψ^P (of which it is a coalitional value) but also to the Shapley value φ, as

$$\Omega^P[v; P^N] = \varphi[v] \quad \text{for any } v \in \mathcal{G}_N.$$

Thus, in some manner, Ω^P establishes a "coalitional path" that links φ and Ψ^P.

(c) From Theorem 3.6 it follows that the only axiomatic difference between the Owen value Φ and the symmetric coalitional p-binomial semivalue Ω^p is that the former satisfies efficiency whereas the latter satisfies the coalitional p-binomial total power property, in a way that parallels the distinction between the Shapley value φ and the p-binomial semivalue Ψ^p.

(d) It is worth mentioning that in the parallel axiomatizations of the Owen value and the symmetric coalitional p-binomial semivalue, additivity might be replaced with linearity, and the dummy player property with the null player property. We have chosen the first possibility in each case (additivity and dummy player property).

(e) The symmetric coalitional p-binomial semivalue also merges the Shapley value and the p-binomial semivalue. It is the result of a two-step bargaining procedure similar to that of the Owen value. In our case, the unions play the quotient game among themselves and each one receives the payoff given by the p-binomial semivalue Ψ^p, and then this payoff is efficiently shared within the union according to the Shapley value φ.

4 A Computation Procedure

The *multilinear extension* (Owen 1972) of a game $v \in \mathcal{G}_N$ is the real-valued function defined on $\mathbb{R}^N$ by

$$f(x_1, x_2, \ldots, x_n) = \sum_{S \subseteq N} \prod_{i \in S} x_i \prod_{j \notin S} (1 - x_j) v(S).$$

As is well known, both the Shapley and Banzhaf values of any game v can be easily obtained from its multilinear extension. Indeed, $\varphi[v]$ can be calculated by integrating the partial derivatives of the multilinear extension of the game along the main diagonal $x_1 = x_2 = \cdots = x_n$ of the cube $[0, 1]^N$ (Owen 1972), while the partial derivatives of that multilinear extension evaluated at point $(1/2, 1/2, \ldots, 1/2)$ give $\beta[v]$ (Owen 1975). This latter procedure extends well to any p-binomial semivalue (see Puente 2000; Freixas and Puente 2002 or Amer and Giménez 2003) by evaluating the derivatives at point $(p, p, \ldots, p)$.

In the context of games with a coalition structure, the multilinear extension technique has been also applied to computing the Owen value Φ (Owen and Winter 1992), as well as the Owen–Banzhaf value B (Carreras and Magaña 1994) and the symmetric coalitional Banzhaf value Π (Alonso et al. 2005). In this section we present a method to compute the symmetric coalitional p-binomial semivalue Ω^p by means of the multilinear extension of the game.

Theorem 4.1 *Let $p \in [0, 1]$ and $[v; P] \in \mathcal{G}_N^{cs}$ be a cooperative game with a coalition structure. Then the following steps lead to the symmetric coalitional p-binomial semivalue of any player $i \in P_k$ in $[v; P]$.*

1. *Obtain the multilinear extension* $f(x_1, x_2, \ldots, x_n)$ *of game* v.
2. *For every* $r \neq k$ *and all* $h \in P_r$, *replace the variable* x_h *with* y_r. *This yields a new function of* x_j *for* $j \in P_k$ *and* y_r *for* $r \in M \setminus \{k\}$.
3. *In this function, reduce to 1 all higher exponents, i.e. replace with* y_r *each* y_r^q *such that* $q > 1$. *This gives a new multilinear function that we denote as* $g((x_j)_{j \in P_k}, (y_r)_{r \in M \setminus \{k\}})$.
4. *In the function obtained in step 3, substitute each* y_r *by* p. *This provides a new function* $\alpha_k((x_j)_{j \in P_k})$ *defined by*

$$\alpha_k((x_j)_{j \in P_k}) = g((x_j)_{j \in P_k}, (p)_{r \in M \setminus \{k\}}).$$

5. *Finally, the symmetric coalitional* p-*binomial semivalue of player* $i \in P_k$ *in* $[v; P]$ *is given by*

$$\Omega_i^p[v; P] = \int_0^1 \frac{\partial \alpha_k}{\partial x_i}(z, z, \ldots, z) dz.$$

Proof Steps 1–3 have already been used by Owen and Winter (1992), Carreras and Magaña (1994) and Alonso et al. (2005) in dealing with the Owen value, the Owen–Banzhaf value and the symmetric coalitional Banzhaf value, respectively. It will be useful to recall their common argument here.

By the second and third steps, we get a multilinear function where all terms corresponding to coalitions S such that $S \cap P_r \neq \emptyset$ and $(N \setminus S) \cap P_r \neq \emptyset$ for some $r \in M \setminus \{k\}$ vanish. Indeed, in step 2, the terms corresponding to these coalitions include expressions like $c y_r^{q_1} (1 - y_r)^{q_2}$, with $q_1, q_2 \in \mathbb{N}$, and in step 3 these terms turn on $c(y_r - y_r)$ thus getting zero.

Hence, the only coalitions S for which the corresponding term of the (initial) multilinear extension may not vanish after steps 2 and 3 are those of the form $S = Q \cup T$, where $T \subseteq P_k$ and $Q = \bigcup_{r \in R} P_r$ for some $R \subseteq M \setminus \{k\}$. The function arising from step 3 is therefore

$$g((x_j)_{j \in P_k}, (y_r)_{r \in M \setminus \{k\}})$$
$$= \sum_{T \subseteq P_k} \sum_{R \subseteq M \setminus \{k\}} \prod_{j \in T} x_j \prod_{j \in P_k \setminus T} (1 - x_j) \prod_{r \in R} y_r$$
$$\prod_{r \notin R \cup \{k\}} (1 - y_r) v(Q \cup T).$$

Substituting each y_r by p (step 4) gives

$$\alpha_k((x_j)_{j \in P_k}) = \sum_{T \subseteq P_k} \sum_{R \subseteq M \setminus \{k\}} \prod_{j \in T} x_j \prod_{j \in P_k \setminus T} (1 - x_j) p^r (1 - p)^{m - r - 1} v(Q \cup T).$$

By differentiating function $\alpha_k((x_j)_{j \in P_k})$ with respect to x_i

$$\frac{\partial \alpha_k}{\partial x_i}((x_j)_{j \in P_k}) = \sum_{T \subseteq P_k \setminus \{i\}} \sum_{R \subseteq M \setminus \{k\}} \prod_{j \in T} x_j \prod_{j \in P_k \setminus (T \cup \{i\})} (1 - x_j) p^r (1 - p)^{m-r-1}$$
$$[v(Q \cup T \cup \{i\}) - v(Q \cup T)].$$

Finally, by step 5,

$$\int_0^1 \frac{\partial \alpha_k}{\partial x_i}(z, z, \ldots, z) dz = \sum_{T \subseteq P_k \setminus \{i\}} \sum_{R \subseteq M \setminus \{k\}} p^r (1 - p)^{m-r-1}$$
$$[v(Q \cup T \cup \{i\}) - v(Q \cup T)]$$

$$\int_0^1 z^t (1 - z)^{p_k - t - 1} dz = \sum_{T \subseteq P_k \setminus \{i\}} \sum_{R \subseteq M \setminus \{k\}} p^r (1 - p)^{m-r-1} \frac{t!(p_k - t - 1)!}{p_k!}$$
$$\times [v(Q \cup T \cup \{i\}) - v(Q \cup T)]$$
$$= \Omega_i^p[v; P]. \qquad \square$$

5 A Remark and Two Examples

Simple games form an especially interesting class of cooperative games. Not only as a test bed for many cooperative concepts, but also for the variety of their interpretations, often far from game theory. In particular, they are frequently applied to describe and analyze collective decision-making mechanisms—weighted majority games play a crucial role here—, and the notion of voting power is closely attached to them.

Shapley and Shubik (1954) were the first to adapt a cooperative tool—the Shapley value—to simple games, using it as a measure of power. Shapley (1962) stated a series of arguments inviting to a self-contained treatment of this class of games, and Dubey (1975) initiated this line when providing an axiomatic characterization of the Shapley–Shubik power index as a solution concept on the class of simple games, for which he introduced the *transfer property* in order to replace the useless additivity property.

A cooperative game v on N is *simple* if it is monotonic and $v(S) = 0$ or 1 for every $S \subseteq N$. A coalition $S \subseteq N$ is *winning in v* if $v(S) = 1$ (otherwise it is called *losing*), and $W(v)$ denotes the set of winning coalitions in v. Due to monotonicity, the set $W^m(v)$ of all *minimal* winning coalitions determines $W(v)$ and hence the game. A simple game v is a *weighted majority game* if there are nonnegative *weights* $w_1, w_2, \ldots, w_n$ allocated to the players and a positive *quota q* such that

$$v(S) = 1 \quad \text{iff} \quad \sum_{i \in S} w_i \geq q.$$

We then write $v = [q; w_1, w_2, \ldots, w_n]$. (For additional details on simple games, we refer the interested reader to e.g. Carreras and Freixas 1996, Taylor and Zwicker 1999 or Carreras 2001.)

Let $\mathcal{SG}_N$ denote the set of all simple games on a given player set N. A *power index* on $\mathcal{SG}_N$ is a function $f : \mathcal{SG}_N \to \mathbb{R}^N$. All properties stated for values in this paper—with

the sole exception of additivity and linearity—make sense for power indices. As $\mathcal{SG}_N$ is a lattice under the standard composition laws given by $(v \vee v')(S) = \max\{v(S), v'(S)\}$ and $(v \wedge v')(S) = \min\{v(S), v'(S)\}$, we will say that a power index f satisfies the *transfer property* if

$$f[v \vee v'] = f[v] + f[v'] - f[v \wedge v'] \quad \text{for all } v, v' \in \mathcal{SG}_N.$$

Carreras et al. (2003) gave an axiomatic characterization and a combinatorial description in terms of weighting coefficients for (the restrictions of) semivalues as power indices, which parallel the corresponding ones for semivalues on general cooperative games.

Let $\mathcal{SG}_N^{cs}$ be the set of all simple games with a coalition structure on N. A *coalitional power index* on $\mathcal{SG}_N^{cs}$ is a function $g : \mathcal{SG}_N^{cs} \to \mathbb{R}^N$. All properties stated for coalitional values in this paper—excluding again additivity and linearity—, as well as the natural extension of the transfer property, make sense for coalitional power indices. Vázquez et al. (1997) carried out an axiomatic characterization of the (restricted) Owen value as a coalitional power index by means of efficiency, the transfer property, the dummy player property, symmetry within unions and symmetry in the quotient game.

In a similar way, we have found a "parallel" axiomatic characterization of the symmetric coalitional binomial semivalues as power indices (that is, restricted to $\mathcal{SG}_N^{cs}$) that we state without proof because it is very similar to that of Theorem 3.6.

Corollary 5.1 *Let $p \in [0, 1]$. For any N there is a unique coalitional power index on $\mathcal{SG}_N^{cs}$ that satisfies the coalitional p-binomial total power property, the transfer property, the dummy player property, symmetry within unions and symmetry in the quotient game. It is the restriction of the symmetric coalitional p-binomial semivalue Ω^P to $\mathcal{SG}_N^{cs}$.*

Besides, this index satisfies the quotient game property and reduces to the (restricted) p-binomial semivalue Ψ^p when $P = P^n$ and to the Shapley–Shubik power index φ when $P = P^N$. □

Example 5.2 Let us consider the 5-person weighted majority game

$$v = [68; 46, 42, 23, 15, 9]$$

and the coalition structure $P = \{\{1, 4\}, \{2, 5\}, \{3\}\}$. We will compute $\Psi^P[v]$ and $\Omega^p[v; P]$ for any $p \in [0, 1]$.

The set of minimal winning coalitions of the game is

$$W^m(v) = \{\{1, 2\}, \{1, 3\}, \{1, 4, 5\}, \{2, 3, 4\}, \{2, 3, 5\}\}$$

and the multilinear extension of v is

$$
\begin{aligned}
f(x_1, x_2, x_3, x_4, x_5) &= x_1 x_2 + x_1 x_3 - x_1 x_2 x_3 + x_1 x_4 x_5 + x_2 x_3 x_4 + x_2 x_3 x_5 \\
&\quad - x_1 x_2 x_3 x_4 - x_1 x_2 x_3 x_5 - x_1 x_2 x_4 x_5 \\
&\quad - x_1 x_3 x_4 x_5 - x_2 x_3 x_4 x_5 + 2 x_1 x_2 x_3 x_4 x_5.
\end{aligned}
$$

Taking into account that players 2 and 3, on one hand, and players 4 and 5 on the other, are symmetric in v, the computation method for binomial semivalues stated by Puente (2000) (see also Amer and Giménez 2003) gives

$$\Psi_1^p[v] = \frac{\partial f}{\partial x_1}(p, p, p, p, p) = p(1 - p)(2 + 2p - 2p^2),$$

$$\Psi_3^p[v] = \Psi_2^p[v] = \frac{\partial f}{\partial x_2}(p, p, p, p, p) = p(1 - p)(1 + 2p - 2p^2),$$

$$\Psi_5^p[v] = \Psi_4^p[v] = \frac{\partial f}{\partial x_4}(p, p, p, p, p) = p(1 - p)(2p - 2p^2).$$

In order to compute $\Omega^p[v]$ we use Theorem 4.1. As $M = \{1, 2, 3\}$, steps 1–3 give

$$g_1(x_1, x_4, y_2, y_3) = y_2 y_3 + x_1(y_2 + y_3 - 2y_2 y_3),$$
$$g_2(y_1, x_2, x_5, y_3) = y_1 y_3 + x_2(y_1 - y_1 y_3) + x_5(y_1 - y_1 y_3) + x_2 x_5(y_3 - y_1),$$
$$g_3(y_1, y_2, x_3) = y_1 y_2 + x_3(y_1 + y_2 - 2y_1 y_2),$$

and step 4 leads to

$$\alpha_1(x_1, x_4) = g_1(x_1, x_4, p, p) = p^2 + x_1(2p - 2p^2),$$
$$\alpha_2(x_2, x_5) = g_2(p, x_2, x_5, p) = p^2 + x_2(p - p^2) + x_5(p - p^2),$$
$$\alpha_3(x_3) = g_3(p, p, x_3) = p^2 + x_3(2p - 2p^2).$$

Step 5 concludes the procedure and gives

$$\Omega_1^p[v; P] = \int_0^1 \frac{\partial \alpha_1}{\partial x_1}(z, z)dz = 2p - 2p^2, \ \Omega_4^p[v; P] = \int_0^1 \frac{\partial \alpha_1}{\partial x_4}(z, z)dz = 0,$$

$$\Omega_2^p[v; P] = \int_0^1 \frac{\partial \alpha_2}{\partial x_2}(z, z)dz = p - p^2, \quad \Omega_5^p[v; P] = \int_0^1 \frac{\partial \alpha_2}{\partial x_5}(z, z)dz = p - p^2,$$

$$\Omega_3^p[v; P] = \int_0^1 \frac{\partial \alpha_3}{\partial x_3}(z)dz = 2p - 2p^2.$$

Thus

$$\Omega^p[v; P] = (2p - 2p^2, p - p^2, 2p - 2p^2, 0, p - p^2).$$

Table 2 Weighting coefficients of some p-binomial semivalues Ψ^p for $n = 5$

	p=0.1	p=0.4	$p = 0.5$ (Banzhaf)	p=0.8
$p_0 = (1-p)^4$	0.6561	0.1296	0.0625	0.0016
$p_1 = p(1-p)^3$	0.0729	0.0864	0.0625	0.0064
$p_2 = p^2(1-p)^2$	0.0081	0.0576	0.0625	0.0256
$p_3 = p^3(1-p)$	0.0009	0.0384	0.0625	0.1024
$p_4 = p^4$	0.0001	0.0256	0.0625	0.4096

In the following example, we shall apply some values and coalitional values (mainly Ψ^p and Ω^p) to the analysis of an interesting political structure: the Catalonia Parliament 2003–2007. All values have been computed using the multilinear extension technique, as illustrated in the preceding example.

In the papers by Straffin (1988), Laruelle (1999) and Laruelle and Valenciano (2001), the Banzhaf value β is suggested as a power measure more suitable than the Shapley value. The natural generalization to semivalues has been argued by Laruelle and Valenciano (2001), Carreras and Freixas (2002), and Carreras et al. (2003). By considering here binomial semivalues, we look at the Banzhaf value in perspective, as will be shown by the results of our analysis.

Therefore, our study of alliances will be based on the bargaining process corresponding to the symmetric coalitional binomial semivalues Ω^p: first, a power notion is shared among unions in the quotient game by means of the Banzhaf value or a binomial semivalue; then, the power so got by each union is shared among its members by using the Shapley value. This will reflect that both bargaining steps are of different nature. Indeed, notice that, once an alliance is formed—and, especially, if it supports a coalition government—, cabinet ministries, parliamentary and institutional positions, budgets, and other political responsibilities have to be distributed *efficiently* among the parties of the coalition, hence in a way as closely as possible to the one suggested by the Shapley value. At this point, the quotient game property and symmetry in the quotient game become very relevant properties. In fact, they are connected because if a coalitional value satisfies the quotient game property (as is the case for all Ω^p) and it is a coalitional value of the Banzhaf value (or a p-binomial semivalue) then symmetry in the quotient game follows from the anonymity of β (or of Ψ^p).

Which is the reason for letting p range from 0 to 1? Notice that a reasonable regularity assumption on players' behavior is that the probability to form coalitions follows a monotonic (increasing or decreasing) behavior. Then, it is not difficult to see that the only semivalues such that $p_{k+1} = \lambda p_k$ for all k (maybe the simplest form of monotonicity) are precisely the p-binomial semivalues, in which case $\lambda = \frac{p}{1-p}$ for any $p \in [0, 1]$. For example, $p = 0.1$ means that the players are very reticent to form coalitions, whereas $p = 0.8$ means that great coalitions are likelier. The neutral case $p = 0.5$ corresponds to the Banzhaf value. Table 2 shows, for $n = 5$, the weighting coefficients of Ψ^p for several values of p.

As we will see, almost all allocations $\Psi^p_i[v]$ and coalitional allocations $\Omega^p_i[v; P]$ will show factors $p(1-p)$; the sole exceptions are the cases where i is a dictator or

Table 3 Initial power distribution in the Catalonia Parliament 2003–2007

	$\Psi_i^p[v]$	$p = 0.1$	$p = 0.4$	$p = 0.5$	$p = 0.8$
1. CiU	$p(1-p)(2+2p-2p^2)$	0.1962	0.5952	0.6250	0.3712
2. PSC	$p(1-p)(1+2p-2p^2)$	0.1062	0.3552	0.3750	0.2112
3. ERC	$p(1-p)(1+2p-2p^2)$	0.1062	0.3552	0.3750	0.2112
4. PPC	$p(1-p)(2p-2p^2)$	0.0162	0.1152	0.1250	0.0512
5. ICV	$p(1-p)(2p-2p^2)$	0.0162	0.1152	0.1250	0.0512
$\tau^p[v]$	$p(1-p)(4+10p-10p^2)$	0.4410	1.5360	1.6250	0.8960

veto player (if we were dealing with improper games, we should add winner players to this short list). Furthermore, the maximum or the minimum of all these allocations for each player will be attained in case $p = 0.5$, that respectively correspond to the Banzhaf value $\beta = \Psi^{1/2}$ or to the Alonso–Fiestras coalitional value $\Pi = \Omega^{1/2}$. These properties would not have been discovered if only the case $p = 0.5$ were considered.

Example 5.3 (*The Catalonia Parliament, Legislature 2003–2007*) Five parties elected members to the Catalonia Parliament (135 seats) in the elections held on November 16, 2003, giving rise to a seat distribution that can be represented by the weighted majority game

$$v = [68; 46, 42, 23, 15, 9].$$

Let us briefly describe ideologically the agents in this game:

1: CiU (Convergència i Unió), Catalan nationalist middle-of-the-road coalition of two federated parties.
2: PSC (Partit dels Socialistes de Catalunya), moderate left-wing socialist party, federated to the Partido Socialista Obrero Español.
3: ERC (Esquerra Republicana de Catalunya), radical Catalan nationalist left-wing party.
4: PPC (Partit Popular de Catalunya), conservative party, Catalan delegation of the Partido Popular.
5: ICV (Iniciativa per Catalunya-Verds), coalition of Catalan eurocommunist parties, federated to Izquierda Unida, and ecologist groups ("Verds").

Notice that, as pointed out in Example 5.2,

$$W^m(v) = \{\{1, 2\}, \{1, 3\}, \{1, 4, 5\}, \{2, 3, 4\}, \{2, 3, 5\}\},$$

so that players 2 and 3 on one hand, and 4 and 5 on the other, are symmetric in v.

We show in Table 3 the evaluation of v given by several binomial semivalues Ψ^p. The total power is $\tau^p[v] = \sum_{i \in N} \Psi_i^p[v]$.

It is easy to see that the allocations found for p and $1 - p$ would coincide because the game is *decisive* (proper and strong). Notice that the proportions between the

allocations to the players decrease as p approaches 0.5 from any of the extreme possibilities (0 or 1). Also notice that the maximum allocation (power) for any player and the maximum total power are got for $p = 0.5$ (Banzhaf value).

Next we are interested in the study and comparison of several coalition structures. In each case, we have computed the coalitional value Ω^p for all $p \in [0, 1]$ and also Π (for $p = 1/2$) and the coalitional p-binomial total power $T^p[v; P] = \sum_{i \in N} \Omega_i^p[v; P]$. The cases and results are as follows:

- The left-wing alliance PSC+ICV, as a previous step. The corresponding coalition structure is $P = \{\{2, 5\}, \{1\}, \{3\}, \{4\}\}$, and the coalitional values are:

$$\Pi[v; P] = (1/2, 3/8, 1/2, 0, 1/8),$$
$$\Omega^p[v; P] = (2p - 2p^2, 1.5p - 1.5p^2, 2p - 2p^2, 0, 0.5p - 0.5p^2),$$
$$T^p[v; P] = 6p(1 - p).$$

- The simultaneous alliances CiU+PPC and PSC+ICV, as an alternative previous step. The corresponding coalition structure is $P = \{\{1, 4\}, \{2, 5\}, \{3\}\}$, and the coalitional values are:

$$\Pi[v; P] = (1/2, 1/4, 1/2, 0, 1/4),$$
$$\Omega^p[v; P] = (2p - 2p^2, p - p^2, 2p - 2p^2, 0, p - p^2),$$
$$T^p[v; P] = 6p(1 - p).$$

- The left-wing majority alliance PSC+ERC+ICV. The corresponding coalition structure is $P = \{\{2, 3, 5\}, \{1\}, \{4\}\}$, and the coalitional values are:

$$\Pi[v; P] = (0, 5/12, 5/12, 0, 2/12),$$
$$\Omega^p[v; P] = \left(0, \frac{1 + p - p^2}{3}, \frac{1 + p - p^2}{3}, 0, \frac{1 - 2p + 2p^2}{3}\right),$$
$$T^p[v; P] = 1.$$

Notice that $\Omega_i^p[v; P] > \Psi_i^p[v]$ for all $p \in [0, 1]$ and $i = 2, 3, 5$, and also that $p = 0.5$ gives the maximum of $\Omega^p[v; P]$ for PSC and ERC but, at the same time, the minimum of $\Omega^p[v; P]$ for ICV.
Incidentally, in this case $B[v; P] = (0, 3/8, 3/8, 0, 1/8)$, so that B fails to satisfy the quotient game property and the sharing of the dictatorial power is by no means convincing because of its inefficiency.
- The catalanist majority alliance CiU+ERC. The corresponding coalition structure is $P = \{\{1, 3\}, \{2\}, \{4\}, \{5\}\}$, and the coalitional values are:

$$\Pi[v; P] = (5/8, 0, 3/8, 0, 0),$$
$$\Omega^p[v; P] = \left(\frac{1 + p - p^2}{2}, 0, \frac{1 - p + p^2}{2}, 0, 0\right),$$
$$T^p[v; P] = 1.$$

Table 4 Evaluation according to Ψ^P and Ω^P for $p = 0.5$

Scenario	Value	CiU	PSC	ERC	PPC	ICV
Initial (no alliance)	β	0.6250	0.3750	0.3750	0.1250	0.1250
PSC+ICV	$B = \Pi$	0.5000	0.3750	0.5000	0	0.1250
PSC+ICV and CiU+PPC	$B = \Pi$	0.5000	0.2500	0.5000	0	0.2500
PSC+ERC+ICV	B	0	0.3750	0.3750	0	0.1250
PSC+ERC+ICV	Π	0	0.4167	0.4167	0	0.1667
CiU+ERC	$B = \Pi$	0.6250	0	0.3750	0	0

In this case $\Pi_i[v; P] = \beta_i[v]$ but $\Omega_i^P[v; P] > \Psi_i^P[v]$ for all $p \in [0, 1]$ and $i = 1, 3$ (unless $p = 0.5$, where the equality holds). Here $p = 0.5$ gives the maximum of $\Omega^P[v; P]$ for CiU and the minimum for ERC.

A most convenient way to analyze this set of evaluations of the coalitional behavior will consist in considering different values of p, and we will take 0.1, 0.4, 0.5 (this gives Π) and 0.8. Tables 4, 5, 6, 7 show all these particular allocations but we prefer the following order: $p = 0.5$, $p = 0.4$, $p = 0.8$ and $p = 0.1$.

In Table 4, we find that precoalition PSC+ICV does not increase the power of their members, but it damages the strategic position of CiU and enhances the strategic possibilities of ERC. The alternative (simultaneous precoalitions PSC+ICV and CiU+PPC) does not make better off CiU and PPC but, instead, damages PSC, increases the power of ICV and gives the same position to ERC. Of course, PPC loses its small power even in joining CiU (its only natural partner in this situation) because once PSC+ICV is formed PPC becomes a null player.

An important point arises when considering the majority formation. According to the Owen–Banzhaf value B, forming a winning coalition does not change the power of its members with regard to the initial distribution, although it serves to reduce the outside parties to a null position. Instead, from the viewpoint of the symmetric coalitional Banzhaf value Π, coalition PSC+ERC+ICV clearly increases the power of each one of its members, and hence it suggests to ERC the convenience to choose this coalition (which also satisfies its partners, PSC and ICV) instead of CiU+ERC.

Therefore, we have to point out here that after a short period of negotiations, precisely concerning these two options, alliance PSC+ERC+ICV was actually formed and got the regional government of Catalonia, ending 23 years of CiU governments headed by Jordi Pujol (under absolute majority of this party or with the parliamentary support of PPC). The actual sharing of positions gave the presidency of the government to Pasqual Maragall (PSC) but the presidency of the Parliament and the "Conseller en cap" position (a sort of vice-presidency of the government) to Ernest Benach and Josep Lluis Carod Rovira (both ERC), respectively. The remaining cabinet positions ("conselleries") were distributed in the proportion 8:5:2 among the three parties.

We recall that the allocations on this (decisive) game for a given p are the same as for $1 - p$, so that our comments on Table 5 ($p = 0.4$) are the same as they would be for $p = 0.6$, and the analogue holds for Tables 6 and 7.

Table 5 Evaluation according to Ψ^p and Ω^p for $p = 0.4$

Scenario	CiU	PSC	ERC	PPC	ICV
Initial (no alliance)	0.5952	0.3552	0.3552	0.1152	0.1152
PSC+ICV	0.4800	0.3600	0.4800	0	0.1200
PSC+ICV and CiU+PPC	0.4800	0.2400	0.4800	0	0.2400
PSC+ERC+ICV	0	0.4133	0.4133	0	0.1733
CiU+ERC	0.6200	0	0.3800	0	0

Table 6 Evaluation according to Ψ^p and Ω^p for $p = 0.8$

Scenario	CiU	PSC	ERC	PPC	ICV
Initial (no alliance)	0.3712	0.2112	0.2112	0.0512	0.0512
PSC+ICV	0.3200	0.2400	0.3200	0	0.0800
PSC+ICV and CiU+PPC	0.3200	0.1600	0.3200	0	0.1600
PSC+ERC+ICV	0	0.3867	0.3867	0	0.2267
CiU+ERC	0.5800	0	0.4200	0	0

Table 7 Evaluation according to Ψ^p and Ω^p for $p = 0.1$

Scenario	CiU	PSC	ERC	PPC	ICV
Initial (no alliance)	0.1962	0.1062	0.1062	0.0162	0.0162
PSC+ICV	0.1800	0.1350	0.1800	0	0.0450
PSC+ICV and CiU+PPC	0.1800	0.0900	0.1800	0	0.0900
PSC+ERC+ICV	0	0.3633	0.3633	0	0.2733
CiU+ERC	0.5450	0	0.4550	0	0

By comparing the results given in Table 4 with those of Table 5, where it is assumed that players are not indifferent to join a coalition of any size but, rather, they prefer not too big coalitions (as $p = 0.4$), we notice that forming precoalition PSC+ICV increases a bit the power of its members. Its effects on the outside parties, as well as those of the alternative (simultaneous formation of CiU+PPC) are the same as in Table 4.

As to the formation of majorities, here, not only in the case of PSC+ERC+ICV but also in the case of CiU+ERC, every party entering such a coalition clearly increases its power. However, from ERC's viewpoint, coalition PSC+ERC+ICV gives again the best fraction of coalitional power.

It is worth mentioning that almost all (initial or coalitional) power allocations given in Table 6 are lower than in the previous cases. The only exceptions are for ICV in PSC+ERC+ICV and ERC in CiU+ERC. Nevertheless, the variations undergone by the initial allocations when the precoalitions form are similar to those found in Table 5. The new feature here is that, in these circumstances ($p = 0.8$), ERC would clearly prefer CiU+ERC instead of PSC+ERC+ICV.

Finally, Table 7 exhibits the same trends as Table 6 but they are even strengthened. Again, ERC would prefer CiU+ERC, and notice that the increase of its power in agreeing to form this coalition would be greater than in the previous case.

It is not difficult to see that ERC would prefer option PSC+ERC+ICV instead of CiU+ERC if, and only if, $p \in (\frac{5-\sqrt{5}}{10}, \frac{5+\sqrt{5}}{10})$, would remain indifferent if $p = \frac{5\pm\sqrt{5}}{10}$ and would prefer CiU+ERC if $p \notin [\frac{5-\sqrt{5}}{10}, \frac{5+\sqrt{5}}{10}]$.

As a conclusion of our analysis, we find that the evaluation of games and games with a coalition structure by means of binomial semivalues and symmetric coalitional binomial semivalues provides a new approach to the study of the coalitional bargaining. Some general properties sketched only on the basis of this instance should deserve further attention. And, finally, the extension of the coalitional theory to probabilistic values might be, in the near future, an interesting research field.

6 A Historical Note

Shapley (1953) (see also Roth 1988; Owen 1995) initiated the value theory for cooperative games. The *Shapley value* applies without restrictions and provides, for every game, a single payoff vector to the players. The restriction of the value to simple games gives rise to the Shapley and Shubik (1954) power index, that was axiomatized by Dubey (1975) introducing the transfer property. As a sort of reaction, Banzhaf (1965) proposed a different power index (see also Coleman 1971, and even Penrose 1946), that Owen (1975) extended to a dummy-independent and somehow "normalized" *Banzhaf value* for all cooperative games. A nice almost common characterization of the Shapley and Banzhaf values would be given by Feltkamp (1995), and a sound interpretative and comparative analysis has been carried out by Laruelle and Valenciano (2001). See also Owen (1978), Dubey and Shapley (1979), Lehrer (1988), Dragan (1996) and Carreras (2005).

Dubey et al. (1981) axiomatically introduced the notion of *semivalue*, that encompasses both the Shapley and Banzhaf values (see also Weber 1979; Einy 1987). Weber (1988) gave an alternative characterization for semivalues and introduced the *probabilistic values* by dropping anonymity.

Many authors have been working on semivalues. We will refer only to the most related to the present paper. Carreras and Freixas (1999) and Carreras and Freixas (2000) introduced *regular* semivalues. Puente (2000) devoted much of her Ph.D. thesis to semivalues and introduced *binomial semivalues* as a natural generalization of the Banzhaf value (see also Freixas and Freixas and Puente (2002), where different ways are provided to evaluate the importance of the components in a given reliability system based on semivalues and probabilistic values). Laruelle and Valenciano (2001) and Carreras et al. (2003) investigated semivalues as power indices, that is, by restricting them to simple games. Finally, Carreras and Freixas (2002) suggested several applications of semivalues based on their versatility.

Games with a *coalition structure* were introduced by Aumann and Drèze (1974), who extended the Shapley value to this new framework in such a manner that the game really splits into subgames played by the unions isolately from each other, and

every player receives the payoff allocated by the restriction of the Shapley value to the subgame he is playing within his union. A second approach was used by Owen (1977), when introducing and axiomatically characterizing his coalitional value (*Owen value*). In this case, the unions play a *quotient game* among themselves, and each one receives a payoff which, in turn, is shared among its players in an internal game. Both payoffs, in the quotient game for unions and within each union for its players, are given by applying the Shapley value. Further axiomatizations of the Owen value have been given by e.g. Hart and Kurz (1983), Peleg (1989), Winter (1992), Amer and Carreras (1995) and Amer and Carreras (2001), Vázquez et al. (1997), Vázquez (1998), Hamiache (1999) and Hamiache (2001), Albizuri and Zarzuelo (2004) and Albizuri (2002).

By applying a similar procedure to the Banzhaf value, Owen (1982) got the *modified Banzhaf value* or *Owen–Banzhaf value* for games with a coalition structure. In this case the payoffs at both levels, that of the unions in the quotient game and that of the players within each union, are given by the Banzhaf value. This modified value was axiomatically characterized only recently, by Albizuri (2001) and Albizuri (2002) and, independently, by Amer et al. (2002). Interesting interpretations of this value as a power index when restricted to simple games can be found in Laruelle and Laruelle and Valenciano (2003).

The natural generalization of semivalues for games with coalition structure has been carried out by Albizuri and Zarzuelo (2004). These authors provide axiomatic characterizations in both cases: the *homogeneous* one, when a common semivalue is used by unions in the quotient game and by players within each one of them (see also Giménez (2001) and Amer and Giménez (2003) for this case), and the *heterogeneous* one, where different semivalues apply in the quotient game and (uniformly) within all unions.

Alonso and Fiestras (2002) realized that the Owen–Banzhaf value fails to satisfy two interesting properties of the Owen value: symmetry in the quotient game and the quotient game property. Then they suggested to modify the two-step allocation scheme and use the Banzhaf value for sharing in the quotient game and the Shapley value within unions. This gave rise to a new, heterogeneous coalitional value that can be compared with the Owen value and satisfies the above properties.

In fact, heterogeneous coalitional values are a particular case of mixed coalitional values. *Mixed* coalitional values were already suggested by Carreras and Magaña (1997) in a more general setup (see also Alonso et al. 2005). The idea is that unions might use any value in the quotient game and, then, the players of each union might use a value different from that of the unions and from those used within other unions. Notice that the unions are, in general, of a different nature from the original, single players, and even from each other, and the quotient game may well possess features not found in the initial game. The question is not, therefore, "why will the unions follow, as entities, a way different from players' one?" but, rather, "why not?" After all, freedom is a human aspiration that we should take into account in our mathematical modeling of the real life behavior, and the contract for forming each union could (in fact, it should) perfectly specify the way to share profits among its members.

Then, a formal notation will help us to better distinguish the several coalitional evaluation criteria that can arise. Let σ be the semivalue used by unions and $\rho_1, \rho_2, \ldots, \rho_m$

be the semivalues respectively used by each union. We denote the compound rule as

$$\sigma \rho_1, \rho_2, \ldots, \rho_m.$$

With this notation, a first level of homogeneity is achieved in case $\sigma \rho, \rho, \ldots, \rho = \sigma \rho^m$ for some common ρ. Thus, Alonso and Fiestras (2002) symmetric coalitional Banzhaf value is $\Pi = \beta \varphi^m$, whereas the symmetric coalitional binomial semivalue introduced in this paper is $\Omega^p = \Psi^p \varphi^m$, where Ψ^p is the binomial semivalue defined by number p. A further homogeneity level is finally found in the case where $\sigma = \rho$, like in Owen's classical extensions: the Owen value is $\Phi = \varphi \varphi^m$ and the Owen–Banzhaf value is $B = \beta \beta^m$.

Great attention has also been paid to the computation of values, usually in terms of the *multilinear extension* (Owen 1972) of the original game. Thus, Owen (1972) refers to the Shapley value φ, Owen (1975) to the Banzhaf value β, and Owen and Winter (1992) to the Owen value Φ. Carreras and Magaña (1994) have applied the same procedure to the Owen–Banzhaf value B, and Carreras and Magaña (1997) have studied the multilinear extension of the quotient game.

More recently, Alonso et al. (2005) apply the multilinear extension method to the Alonso–Fiestras value Π, and also to a "counterpart" value $M = \varphi \beta^m$ introduced by Amer et al. (2002). In Puente (2000) it is shown that the binomial semivalues can be computed in a way very close to that of the Banzhaf value. Giménez (2001) and Amer and Giménez (2003) prove that (a) any other semivalue requires using a geometrical reference system of the semivalue simplex, given by any n different binomial semivalues, and a linear map whose matrix depends on (the partial derivatives of the multilinear extension of) the game and the reference system—it also applies, of course, to the Shapley value, with no integration step—, and (b) the homogeneous coalitional semivalues can be computed by means of a bilinear form whose matrix depends, again, on the game and the reference system. It is worthy mentioning here that in the case of coalitional binomial semivalues (Carreras and Magaña 1994) procedure applies as well.

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