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Searching for a Compromise in Multiple Referendum

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Abstract We consider a multiple referendum setting where voters cast approval ballots, in which they either approve or disapprove of each of finitely many dichotomous issues. A program is a set of socially approved issues. Assuming that individual preferences over programs are derived from ballots by means of the Hamming distance criterion, we consider two alternative notions of compromise. The majoritarian compromise is the set of all programs supported by the largest majority of voters at the minimum utility loss. A program is an approval compromise if it is supported by the highest number of voters at a utility loss at most half of the maximal achievable one. We investigate the conditions under which issue-wise majority voting allows for reaching each type of compromise. Finally, we argue that our results hold for many other preferences that are consistent with the observed ballots.

Keywords Approval balloting · Majority rule · Multiple referendum · Voting paradox · Compromise

1 Introduction

An approval ballot is a list of yes/no positions regarding finitely many dichotomous issues. A typical example of voting situations where citizens cast approval ballots

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is multiple referendum, or any situation where voters are interpreted as bargainers designing an acceptable agreement over several issues by selecting Yes or No on each issue (see Brams et al. 2004). Another case where ballots can be equivalently written as approval ballots is multiple elections with two competing parties: several positions (e.g. president, governor, mayor) have to be filled by the candidate of one of the two parties, and each voter supports only one party position-wise. In both cases, approval ballots are aggregated, through some aggregation rule, into a social outcome, or program, which gives the collective decision regarding each of the issues. The most commonly used aggregation rule is the issue-wise majority rule, under which an issue is socially approved if and only if it is at least as approved than disapproved. In this paper, we study the extent to which issue-wise majority voting allows for reaching a program which provides a compromise among voters.

Defining a compromise requires to model how voters compare programs. Equivalently, approval ballots have to be extended into preferences over the set of all programs. We assume that preferences are represented by Hamming utility functions, for which the utility level given to a program is equal to the number of issues minus the number of those which the ballot and the program disagree on.[1]

The issue-wise majority rule (hereafter Maj) shares some nice properties under the assumption of Hamming preferences. It is strategy-proof and produces utilitarian programs, i.e. which maximize the sum of voters' utilities.[2] However, it is also well-known that it may fail at choosing a Condorcet winner. The Ostrogorski paradox precisely relates to such a possibility (see Rae and Daudt 1976; Bezembinder and Van Acker 1985; Deb and Kelsey 1987; Kelly 1989; Nurmi 1999; Laffond and Lainé 2006, 2009, 2010). Furthermore, the outcome of the majority rule may also be such that a majority of the voters disagree on a majority of candidates (known as the Anscombe paradox, see Anscombe 1976; Wagner 1983, 1984 and Nurmi 1999).

We focus on two specific notions of compromise. The first is the majoritarian compromise, which is closely related to the Bucklin Vote, or Grand Junction Vote.[3]

1 Assuming Hamming preferences should not be regarded as a very restrictive assumption. Indeed, both symmetry among issues and separability are properties that promote issue-wise majority voting as a compromising rule. Furthermore, we argue at the end of the paper that our results still hold for a larger domain of preferences that are consistent with ballots.

2 However, as shown in Cuhadaroglu and Lainé (2009), Maj may produce an outcome that is almost Pareto-dominated. They also characterize, under the assumption that voters rank first the program given in their ballot, the largest domain of separable preferences for which the majority rule is Pareto-efficient. The reader may refer to Brams et al. (2007) for further results. The reader may also refer to Özkal-Sanver and Sanver (2006) for a more general study of the Pareto-efficiency of candidate-wise choice rules.

3 The difference between the majoritarian compromise and the Bucklin vote is that, in the former, voters rank all candidates. The Bucklin vote is itself a special case of Fallback Voting (Brams and Sanver 2009). Furthermore, the majoritarian compromise is also a particular case of a q-Fallback Bargaining rule (Brams and Kilgour 2001), and is a refinement of the Median Voting Rule (Bassett and Persky 1999), in the sense that the set of outcomes selected by majoritarian compromise is a subset of the set of outcomes of the Median Voting Rule. The majoritarian compromise is also closely related in spirit to the axiom of efficient compromise introduced in Özkal-Sanver and Sanver (2004). An alternative is an efficient compromise if there exists no other alternative that is supported by at least as many voters at a lower utility loss, and a voting rule satisfies the efficient compromise axiom if and only if it always picks efficient compromises. Özkal-Sanver and Sanver (2004) show that all Condorcet-consistent voting rules, as well as the Borda count, violate this axiom.

A program is a majoritarian compromise if it is supported by at least half of the voters at the minimum utility loss. Few results are already known for the standard single-winner setting where one alternative has to be chosen from finitely many ones, and where voters' preferences over the alternatives are represented by linear orders (see Sertel 1986; Sertel and Yılmaz 1999, and Giritligil and Sertel 2005). First, the maximal utility loss achieved at a majoritarian compromise cannot exceed half of the alternatives; second, a majoritarian compromise may be neither a Borda winner nor the Condorcet winner; furthermore, it is subgame-perfect implementable. We show below that, in a multiple referendum setting, a majoritarian compromise may also be neither a Borda winner nor a Condorcet winner.

The second notion, called the approval $\alpha-$compromise, follows a dual approach. A program is an $\alpha-$compromise if it maximizes the number of voters whose utility level is decreased by at most $\alpha\%$ of its maximal value.[4] It is easily seen that this number of voters will be at least one half of the electorate when $\alpha = \frac{1}{2}$, and this latter case provides an interesting benchmark, called approval compromise. Nonetheless, we show that the majoritarian compromise may be disjoint from the approval compromise.

Below are studied two new compound majority voting paradoxes: the MC paradox describes situations where Maj fails at reaching the majoritarian compromise, whereas the C paradox refers to cases where it fails at reaching the approval compromise.

We show that both paradoxes may prevail, and are logically independent. Moreover, we show that they are related to either the Ostrogorski paradox or the Anscombe paradox. Furthermore, we prove that conditions that allow for avoiding the latter two paradoxes are no longer sufficient to avoid the compromise ones. In particular, we prove that (1) both the MC and the C paradoxes may hold even when voters are almost unanimous issue-wise, (2) however, a strict version of the MC paradox (which states that the majority outcome is not a majoritarian compromise, but also its opposite is a majoritarian compromise) cannot hold if there is enough consensus issue-wise, (3) under a mild richness property, all (resp. almost all) sets of ballots face the C paradox (resp. the MC paradox) for some distribution of the votes among ballots.

The rest of the paper is organized as follows: the model of multiple referendum voting is presented in Part 2, together with the alternative notion of compromise. Part 3 is devoted to the formal definition of both the MC paradox and the C paradox. In Part 4, we establish their relationship with the Ostrogorski and the Anscombe paradoxes. We also prove that both can hold even under quasi-unanimous issue-wise opinions. Moreover, we investigate the properties of sets of ballots which preclude the compromise paradoxes for any distribution of the voters among ballots. The paper ends up with additional comments about how our results can be extended to a larger class of preferences over programs. Finally, all proofs are postponed to an Appendix.

[4] The notion of approval $\alpha-$compromise is equivalent to the threshold procedure introduced in Fishburn and Pekeč (2004). See also Kilgour (2010).

2 Compromises in Multiple Referendum

2.1 Notations and Definitions

We denote by $\mathbb{N}$ the set of non-negative integers. A set $\mathcal{N} = \{1, \ldots, n, \ldots, N\}$ of voters faces a set $\mathcal{Q} = \{1, \ldots, q, \ldots, Q\}$ of yes-no issues, where $N, Q \in \mathbb{N}$ are both variable. Each voter n indicates by means of an *approval ballot* x_n which issues she approves of. Formally, $x_n = (x_n^q)_{q=1,\ldots,Q} \in \{0,1\}^Q$, where $x_n^q = 1 \Leftrightarrow n$ accepts q, $x^q = 0 \Leftrightarrow n$ rejects q. A *program* is a subset of collectively accepted issues. Formally, a program is an element $x = (x^q)_{q=1,\ldots,Q}$ of $\{0,1\}^Q$, where $x^q = 1 \Leftrightarrow q$ is accepted, $x^q = 0 \Leftrightarrow q$ is rejected. The program opposite to x is defined by $(-x) = (-x^q)_{q=1,\ldots,Q}$, where $-x^q = 1 \Leftrightarrow x^q = 0$.

A *voting set* is an element $V = (x_1, \ldots, x_H) \in \{0,1\}^{QH}$, where $H \in \mathbb{N}$, and $x_h \neq x_{h'}$ for all $h \neq h' \in \{1, \ldots, H\}$. Given a voting set V, a *voting profile* is a pair $P = (V, \pi)$, with $\pi = (\pi_1, \ldots, \pi_H) \in \mathbb{N}^H$, where π_h is the number of voters having cast the ballot x_h. We denote by Π_Q the set of all possible profiles $P = (V, \pi)$ with Q issues, and we define $\Pi = \cup_{Q\geq 1} \Pi_Q$.

2.2 Aggregating Approval Ballots

An aggregation rule is a correspondence from Π to $\cup_{Q\geq 1}\{0,1\}^Q$, which, for any given $Q \in \mathbb{N}$, maps each voting profile with Q issues to one or several programs in $\{0,1\}^Q$. While many alternative aggregation rules have been suggested and studied (see Kilgour 2010, for an overview), we focus in this paper on the rule which is the most commonly used in multiple referendum. The candidate-wise majority rule Maj consists in accepting all those issues which are more often approved than disapproved. Formally Maj is defined by: $\forall Q \in \mathbb{N}, \forall P \in \Pi_Q, x \in Maj(P) \Leftrightarrow \forall q = 1, \ldots, Q, \left|\{n \in \mathcal{N} : x_n^q = x^q\}\right| \geq \left|\{n \in \mathcal{N} : x_n^q \neq x^q\}\right|$. Clearly, the outcome of $Maj(P)$ is unique for all profiles involving an odd number N of voters: no issue can receive as many approvals as disapprovals.

2.3 Preferences Over Programs

Defining a compromise requires to specify how voters compare programs given their ballots. We assume that preferences over programs are complete and separable preorders that are constructed from the ballots by means of the *Hamming distance* criterion. The Hamming distance between two programs $x = (x^q)_{1\leq q\leq Q}$ and $y = (y^q)_{1\leq q\leq Q}$ is the number $d(x, y) = \sum_{q=1}^{Q} |y^q - x^q|$. Voters' preferences over programs are then represented by the *Hamming utility function* $U_n(x) = Q - d(x, x_n)$: voters rank programs according to the number of decisions which their ballot disagrees on. Alternatively, individuals vote for their ideal program, and then compare programs according to the Hamming distance to their ideal.

The Hamming preferences have been used in several related studied (see Brams et al. 2004, 2007; Kilgour et al. 2006; Brams 2008; Laffond and Lainé 2006, 2009).

Assuming separable dichotomous preferences over issues[5] naturally calls for choosing programs through Maj. Moreover Maj is strategy-proof under Hamming preferences,[6] and always selects utilitarian programs, i.e. which maximize the sum of voters' utilities.[7]

2.4 Two Notions of Compromise

2.4.1 Approval α-Compromise

The first concept of compromise is called *approval α-compromise*, and works as follows: say that a voter is satisfied with a program x if she disagrees with x on a proportion at most α of the issues. Then a program is an approval α-compromise if it maximizes the size of the group of voters it satisfies.

Definition 1 Given $\alpha \in [0, 1]$ and a vote profile $P \in \Pi_Q$, a program x is α-satisfactory for n if $U_n(x) \geq Q(1-\alpha)$. The α-support of x is the number $S(x, \alpha, P)$ of voters for whom x is α-satisfactory. Furthermore, x is $\alpha-$ satisfactory if $S(x, \alpha, P) \geq \frac{N}{2}$, and x is an approval α-compromise if, for any other program y, $S(x, \alpha, P) \geq S(y, \alpha, P)$.

The set of α-satisfactory programs (resp. α-compromises) for a voting profile P is denoted by $\mathcal{S}^\alpha(P)$ (resp. $\mathcal{C}^\alpha(P)$). Note that $\mathcal{C}^0(P)$ coincides with the set of plurality winning programs. We get closer to the intuitive meaning of a compromise when giving α a strictly positive value: indeed, one then explicitly considers that voters may jointly agree on a program at the cost of some utility loss.

Yet, if α is small enough, the support of an $\alpha-$compromise may be small, hence keeping the outcome rather far away from the idea of a compromise. Consider the voting set $V = (x_1, \ldots, x_Q)$ where each issue is approved by exactly one voter, and each voter approves of exactly one issue. If $\alpha < \frac{2}{Q}$, then $\mathcal{C}^\alpha = V$, and the support of each $x_n \in V$ consists of $\frac{1}{Q}$ of the electorate. However, an interesting benchmark is reached with $\alpha = 0.5$, since any program in $\mathcal{C}^{0.5}(P)$ involves at least a majority of voters. Indeed, suppose that $S(x, 0.5, P) < \frac{N}{2}$ for some $x \in \mathcal{C}^{0.5}(P)$. From Hamming preferences, we get that $S(-x, 0.5, P) > \frac{N}{2} > S(x, 0.5, P)$, contradicting $x \in \mathcal{C}^{0.5}(P)$.

In order to lighten notations in the sequel, we write $\mathcal{C}(P) = \mathcal{C}^{0.5}(P)$, $S(x, 0.5, P) = S(x, P)$, and $\mathcal{S}^{0.5}(P) = \mathcal{S}(P)$. Furthermore, $\mathcal{C}(P)$ will be called the *approval compromise* at profile P, and elements of $\mathcal{S}(P)$ will be called *satisfactory programs* at P.

[5] For the role played by non-separability in multiple referendum, the reader may refer to Lacy and Niou (2000); Ratliff (2003, 2006); Hodge and Schwallier (2006), and Laffond and Lainé (2010).

[6] The proof is obvious in case where N is odd. If ties are possible, defining strategy-proofness requires to extend preferences over programs to preferences over sets of programs. A way to proceed is to adopt the Kelly axiom (Kelly 1977): for any two subsets A and B of $\{0, 1\}^Q$, A is preferred than B if $U(x) > U(y)$ for all $(x, y) \in A \times B$). Proving that Maj is strategy-proof under the Kelly axiom extending Hanning preferences is an easy task left to the reader.

[7] See Brams et al. (2004, 2007) for a formal proof.

The notion of approval $\alpha-$compromise can equivalently be defined as a threshold aggregation procedure (Fishburn and Pekeč 2004), which involves a binary judgment about the representativeness of a program: either a program has a sufficient overlap (in proportion $(1-\alpha)$) with a voter's ballot to represent that voter, or it does not, and an $\alpha-$ compromise is a program that represents the most voters.[8]

2.4.2 Majoritarian Compromise

The concept of *majoritarian compromise* follows a somehow dual route, which involves a fixed targeted level of representativeness, together with a variable threshold of utility loss. A program x is a majoritarian compromise if (1) x represents at least a majority of voters at the maximal utility loss k (that is, the overlap between x and the ballot cast by a voter in this majority contains at least $(Q-k)$ issues) (2) no other program is supported by a majority whose members suffer from a lower utility loss, and (3) no other program represents a larger fraction of the electorate at the maximal loss k. Formally, given $x \in \{0,1\}^Q$, and $P \in \Pi_Q$, let $Supp_k(x,P) = |\{n \in \mathcal{N} : U_n(x) \geq Q-k\}|$ be the $k-$support of x, where $k \in \{0,\ldots,Q\}$, and let $k^*(P) = Min\{k \in \{0,\ldots,Q\} : Supp_k(x,P) \geq \frac{N}{2}$ for some $x \in \{0,1\}^Q\}$ be the critical loss of the profile P (that is, the minimal utility loss to be accepted for a majority to agree on some program).

Definition 2 A program x is a majoritarian compromise at the profile P if, for any other program y, $Supp_{k^*(P)}(x,P) \geq Supp_{k^*(P)}(y,P)$.

The set of all majoritarian compromises (in short the majoritarian compromise) at profile P is denoted by $\mathcal{MC}(P)$.

2.4.3 Relations Between Compromises

Since there are finitely many programs, both $\mathcal{MC}(P)$ and $\mathcal{C}^\alpha(P)$ (for any $\alpha \in [0,1]$) are non-empty at any profile P. Furthermore, the majoritarian compromise may be disjoint from the approval compromise at some profile, as illustrated in

Example 1 Let $V = (x_1, x_2, x_3) \in \{0,1\}^4$ be the voting set defined by $x_1 = (1,1,1,1), x_2 = (1,0,0,0)$, and $x_3 = (0,0,0,0)$. Consider the voting profile $P = (V,\pi)$, where $\pi_1 = \pi_2 = \pi_3 = 1$. Since no ballot is cast by a majority of voters, then $k^*(P) > 0$. Since $U_2(x_3) = U_3(x_2) = 3$ and $U_2(x_2) = U_3(x_3) = 4$, then $k^*(P) = 1$. Moreover, since no program represents all voters at a loss equal to 1, then $x_2, x_3 \in \mathcal{MC}(P)$. Then, using $x_1 = (-x_3)$ together with the fact that, for any $z, d(x_2,z) = 1 \Rightarrow d(x_3,z) > 1$, we obtain $\mathcal{MC}(P) = \{x_2, x_3\}$. In order to compute $\mathcal{C}(P)$, we just remark that $x = (1,1,0,0)$, $y = (1,0,1,0)$, and $z = (1,0,0,1)$, are the only programs at distance 2 to all ballots which represent all voters, therefore $\mathcal{C}(P) = \{x,y,z\}$, and $\mathcal{MC}(P) \cap \mathcal{C}(P) = \varnothing$.

[8] See also Kilgour (2010) for a complete presentation of alternative aggregation rules, including the threshold procedure.

However, note that $\mathcal{MC}(P) = \mathcal{C}(P)$ at any profile P such that $k^*(P) = \left\lfloor \frac{Q}{2} \right\rfloor$, that is the largest integer less than or equal to $\frac{Q}{2}$. Furthermore, as pointed out above, the majoritarian compromise may not be unique. Moreover, both $\mathcal{MC}$ and $\mathcal{C}$ always select a (maybe proper) subset of satisfactory programs:

Proposition 1 *For any voting profile P, $\mathcal{MC}(P) \subseteq \mathcal{S}(P)$ and $\mathcal{C}(P) \subseteq \mathcal{S}(P)$. Moreover, there exists P at which $\mathcal{MC}(P) \subset \mathcal{S}(P)$ and $\mathcal{C}(P) \subset \mathcal{S}(P)$.*

3 Voting Paradoxes

A situation where Maj fails at reaching either an approval or a majoritarian compromise pertains to a *compound majority voting paradox*, which points out the possible inconsistency between issue-based and non issue-based aggregation rules.[9] Much attention has been paid in previous studies to two specific paradoxes, namely the *Ostrogorski* and the *Anscombe* paradoxes. Say that a program x is *undefeated* at profile P if there is no other program y such that $|\{n \in \mathcal{N} : U_n(y) \geq U_n(x)\}| > |\{n \in \mathcal{N} : U_n(x) \geq U_n(y)\}|$, that is, if there exists no other program which makes more than half of the voters as least as well off, one voter being better off. The (resp. strict) *Ostrogorski paradox* occurs at profile P if $Maj(P)$ contains a defeated program (resp., and if it is defeated by its opposite). A *Condorcet winner* of a voting profile P is a program that defeats all other programs. The following proposition is proved in Laffond and Lainé (2009):

Proposition 2 *Suppose that N is odd. Then a program x is undefeated at profile P only if x is the outcome of Maj. Moreover, x is undefeated if and only if it is the unique Condorcet winner of P.*

Hence, with an odd number of voters, the Ostrogorski paradox is equivalent to the Condorcet paradox. This justifies even further to pay attention to issue-wise majority voting. Indeed, as long as a Condorcet winner exists, any Condorcet-consistent voting rule will uniquely selects the outcome of Maj.[10]

The *Anscombe paradox* occurs at profile P if $Maj(P) \nsubseteq \mathcal{S}(P)$, that is if an outcome of Maj is not a satisfactory program.

The Ostrogorski and Anscombe paradoxes are not equivalent, though closely related in spirit (Bezembinder and Van Acker 1985; Nurmi 1999). In fact, as pointed out below, the Anscombe paradox is equivalent to the strict Ostrogorski paradox:

Proposition 3 *The strict Ostrogorski paradox occurs at profile P if and only if the Anscombe paradox occurs at P.*

We turn now to the definition of our two 'compromise paradoxes'.

[9] See Nurmi (1999) for a review of compound majority voting paradoxes. Whether using the word 'paradox' is appropriate can be questioned. A possible justification is that some paradoxes relate to the Condorcet paradox.

[10] A voting rule is Condorcet-consistent if it uniquely selects the Condorcet winner whenever it exists.

Definition 3 The Majoritarian Compromise paradox (hereafter MC paradox) occurs at the voting profile P if $x \notin \mathcal{MC}(P)$ for some $x \in Maj(P)$. The MC paradox is strict if, in addition, $-x \in (\mathcal{MC}(P) - Maj(P))$. The (resp., strict) approval compromise paradox (hereafter C paradox) occurs at P if $x \notin \mathcal{C}(P)$ for some $x \in Maj(P)$ (resp., and $-x \in (\mathcal{C}(P) - Maj(P))$).

It is easily checked that both the MC paradox and the C paradox may prevail at some voting profile involving only 2 issues. Indeed, consider $V = (x_1, x_2, x_3, x_4)$, where $x_1 = (1, 1), x_2 = (1, 0), x_3 = -x_1$, and $x_4 = -x_2$. Define the voting profile $P = (V, \pi)$ by: $\pi_1 = 4, \pi_2 = \pi_4 = 1$, and $\pi_3 = 5$. Then, $Maj(P) = \{x_3\}$. Since $S(x_3, P) = \pi_2 + \pi_3 + \pi_4 = 7 < S(x_2, P) = \pi_1 + \pi_2 + \pi_3 = 10$, then the C paradox holds. Moreover, since $k^*(P) = 1$ and $Supp_1(x_h, P) = S(x_h, P)$ for all $h = 1, 2, 3, 4$, then the MC paradox prevails. The following two examples show that both the strict MC paradox and the strict C paradox may also occur.

Example 2 Let $Q = 4$, and consider the voting set $V = (x_1, \ldots, x_5)$, where $x_1 = (1, 1, 1, 0), x_2 = (1, 1, 0, 1), x_3 = (1, 0, 1, 1), x_4 = (0, 1, 1, 1)$, and $x_5 = (0, 0, 0, 0)$. define the voting profile $P = (V, \pi)$, where $\pi_h = 1, 1 \leq h \leq 4$, and $\pi_5 = 3$. We get $Maj(P) = \{x_5\}$, and $\forall h = 1, \ldots, 5, Supp_0(x_h, P) < \frac{N}{2}, Supp_1(-x_5, P) = 4 > \frac{N}{2}$. Hence, $k^*(P) = 1$. Suppose that $-x_5 \notin \mathcal{MC}(P)$. Let $y \in \mathcal{MC}(P)$. Then it must be true that the majority that supports y at a utility loss at most 1 must involve voters for x_5. Hence y must contains at least 3 disapprovals. But this implies that, for all $1 \leq h \leq 4, d(y, x_h) > 1$, so that $\pi_5 > \frac{N}{2}$, a contradiction. Thus, the strict MC paradox holds.

Example 3 Let $Q = 5$, and define $V = V' \cup \{x_1\}$, where $V' = (x_2, \ldots, x_{11})$ contains all ballots with 3 approvals, and $x_1 = (0, 0, 0, 0, 0)$. Consider the profile $P = (V, \pi)$, where $\pi_h = 1, 2 \leq h \leq 11$, and $\pi_1 = 3$. Then $Maj(P) = \{x_1\}$, and $S(x_1, P) = \pi_1 = 3 < S(-x_1, P) = \sum_{h=2}^{11} \pi_h = 10$. Hence, $x_1 \notin \mathcal{C}(P)$. Consider any program y where $t > 0$ issues are accepted, with $t \neq 5$. Suppose $t \in \{3, 4\}$. Since $S(y, P)$ contains no voter having cast a ballot x_1, then $S(y, P) \leq S(-x_1, P)$. Suppose $t = 2$. Then $S(y, P) = 6 < S(-x_1, P)$, since $S(y, P)$ contains exactly 3 voters having cast a ballot in V', together with all 3 voters having cast x_1. Similarly, if $t = 1$, then $S(y, P)$ contains exactly 6 voters having cast a ballot in V', together with all 3 voters having cast x_1. Thus, $S(y, P) = 9 < S(-w, P)$. Hence, $\mathcal{C}(P) = \{-w\}$ and the strict C paradox holds.

As a by-product of these two examples together with Proposition 2, we get that no Condorcet-consistent voting rule always selects either in the majoritarian compromise or in the approval compromise. Furthermore, it is easily shown that, under Hamming preferences, Maj always selects a Borda count winner. Hence, the Borda count may also fail at reaching a compromise.

The next result points out that the two examples above are minimal in terms of number of issues.

Proposition 4 (1) *No voting profile with at most* 3 *issues can face the strict MC paradox*

(2) *No voting profile with at most* 4 *issues can face the strict C paradox*

We address below the following questions:

(1) Are the MC and C paradoxes logically independent? And are they logically related to either the Ostrogorski or the Anscombe paradox?
(2) Which restrictions upon voting profiles allow for avoiding the paradoxes? In particular, it is shown in Wagner (1983, 1984) that the Anscombe paradox never prevails when the issue-wise majority margin is at least 25%. It is easily seen that the same prevails for the Ostrogorski paradox. Furthermore, the domain of voting sets that are immune to the Ostrogorski paradox is characterized in Laffond and Lainé (2006). Can similar results be obtained for both the MC and C paradoxes?

4 Avoiding the MC Paradox and the Approval Paradox

4.1 Relations between Paradoxes

We first establish the relations which prevail between the two compromise paradoxes and both the Ostrogorski and the Anscombe ones. We say that a paradox A is weaker than the paradox B if some profile may exhibit A and not B, while the reverse never holds.

Proposition 5 (1) *The MC paradox is weaker than the Ostrogorki paradox*
(2) *The C paradox is weaker than the Anscombe paradox*
(3) *The Anscombe paradox is weaker than the strict MC paradox*
(4) *The MC paradox and the C paradox are logically independent*

Note that the C paradox is thus weaker than the strict MC paradox. Proposition 5 states that, as long as ballot sets are equally likely, the frequency of the MC paradox (resp. C paradox) is higher than the frequency of the Ostrogorski (resp. Anscombe) paradox. How actually strong are the compromise paradoxes? We address now this question, by searching for two types of sufficient conditions for avoiding them. The first relates to the size of candidate-wise majority margins, while the second looks for the largest class of voting sets which never face the paradox, whatever the number of voters casting each of the ballots.

4.2 Avoiding the Paradoxes: Does the Rule of Three-Fourth Work?

We already mentioned that, if at least 75% of the voters agree issue-wise (the *'three-fourth rule'*), then neither the Anscombe nor the Ostrogorski paradoxes can occur. Our next result shows that, in fact, both the C and the MC paradoxes may hold for any issue-wise majority relative size:

Proposition 6 *For any* $0 < \varepsilon < 100$, *there exists a voting profile* P *such that* $Maj(P)$ *is unique and,* (1) $Maj(P) \notin \mathcal{C}(P)$, *(2)* $(100 - \varepsilon)\%$ *of the voters agree in* P *on each of the issues. Similarly, there exists a voting profile* P' *such that* $Maj(P')$ *is unique and* (1) $Maj(P') \notin \mathcal{MC}(P')$, (2) $(100 - \varepsilon)\%$ *of the voters agree in* P' *on each of the issues.*

Hence, a paradox may hold for any level of issue-wise consensus. More generally, any issue-wise qualified majority rule may fail at ensuring a compromise. We remark that this is no longer true with exactly 3 issues. However, the three-fourth rule no longer works as soon as there are at least 6 issues.

Proposition 7 *Under the three-fourth rule, no profile involving at most* 3 *issues can face the MC paradox. But there exists a profile with 6 issues where the MC paradox holds under the three-fourth rule.*

The intermediate 4-issue and 5-issue cases remain to be studied. Furthermore, Proposition 5(3) implies that the strict MC paradox never occurs under the three-fourth rule.

4.3 Avoiding the Paradoxes: Paradox-Free Voting Sets

Given a number $Q \geq 3$ of issues, a subset $\mathcal{E}$ of $\{0, 1\}^Q$ is called a MC-free (resp. C-free) domain if, for any voting set $V \subseteq \mathcal{E}$, and for any voting profile $P = (V, \pi)$, the MC paradox (resp. C paradox) does not prevail at P. Moreover, a voting set V is *rich* if $\forall x \in V$, one has $-x \in V$, and a domain $\mathcal{E}$ is said to be rich if it only contains rich voting sets. Richness means that, whenever a type of ballot can be cast by some voter, then its opposite can also be cast by another voter. We denote by $\mathcal{E}_Q^{MC}$ (resp. $\mathcal{E}_Q^{C}$) the set of rich MC-free (resp. C-free) domains with Q issues, and we define $\mathcal{E}^{MC} = \cup_{Q \geq 3} \mathcal{E}_Q^{MC}$ (resp. $\mathcal{E}^{C} = \cup_{Q \geq 3} \mathcal{E}_Q^{C}$).

It follows from richness that each voting set contains an even number H of ballot types. Hence, a rich voting set $V = (x_1, \ldots, x_Z)$ can be rewritten as pairs of opposite ballots, that is $V = (\widetilde{x_1}, \ldots, \widetilde{x_H})$, where $H = \frac{Z}{2}$ and, for all $h = 1, \ldots, H$, $\widetilde{x_h} = \{x_h, -x_h\}$ and $x_h^1 = 1$.

We first recall a characterization the largest rich Ostrogorski-free domain of voting sets.

Proposition 8 (Laffond and Lainé 2006) *For any $Q \geq 3$, any single-switch voting set with Q issues never faces the Ostrogorski paradox. Moreover, the domain of single-switch voting sets is the largest* (*for inclusion*) *rich Ostrogorski-free domain.*

The single-switchness property essentially says that issues can be ordered in such a way that each of the ballots presents at most one switch between approvals and disapprovals. Several preliminary definitions are required for a formal definition. Consider a rich voting set $V = (\widetilde{x_1}, \ldots, \widetilde{x_H})$ with Q issues, together with a subset $B \subseteq \{1, \ldots, Q\}$ of issues. The $B-$*relabelling* of V is the voting set $V^B = (\widetilde{y_1}, \ldots, \widetilde{y_H})$, where, for all $h = 1, \ldots, H$, $\widetilde{y_h} = \{y_h, -y_h\}$ is defined by (1) $\forall q \in B, y_h^q = 1 \Leftrightarrow x_h^q = 0$, and (2) $\forall q \notin B, y_z^q = 1 \Leftrightarrow x_z^q = 1$. The relabelling of a voting set thus consists in reversing in each ballot approvals and disapprovals regarding some subset of issues.

Given a permutation σ of $\{1, \ldots, Q\}$, the $\sigma-$ *permutation* of V is the voting set $V_\sigma = (\widetilde{x_{1\sigma}}, \ldots, \widetilde{x_{H\sigma}})$, where, for all $h = 1, \ldots, H$, $\widetilde{x_{h\sigma}} = \{x_{h\sigma}, -x_{h\sigma}\}$ is defined by:

$\forall q \in \mathcal{Q}, x_{h\sigma}^{\sigma(q)} = 1 \Leftrightarrow x_h^q = 1$. A permutation of a voting set consists in reshuffling the set of issues without modifying the voters' positions regarding each of the issues.

Furthermore, two voting sets V and V' are said to be *equivalent* if there exist $B \subseteq \{1, \ldots, Q\}$ and a permutation σ of $\{1, \ldots, Q\}$ such that $V' = V_\sigma^B$. Then V has a *single-switch representation* if $\forall h = 1, \ldots, H$, there exists at most one $q(h) \in \{1, \ldots, Q-1\}$ such that $x_h^{q(h)} \neq x_h^{q(h)+1}$. Moreover, V is said to be *single-switch* it it is equivalent to a voting set having a single-switch representation. Let $\mathcal{E}_Q^{SS}$ stand for the set of single-switch rich voting sets involving Q issues, and let $\mathcal{E}^{SS} = \cup_{Q\geq 3}\mathcal{E}_Q^{SS}$.

Using Propositions 5 and 8, we get that $\mathcal{E}^{MC} \subseteq \mathcal{E}^{SS}$. Our next result characterizes $\mathcal{E}^{MC}$, which appears to be a very small subset of $\mathcal{E}^{SS}$.

Proposition 9 *$\mathcal{E}^{MC}$ is the set of all single-switch voting sets which contain at most two pairs of opposite ballots $\{x, -x\}$ and $\{y, -y\}$ such that $d(x, y) \leq 1$.*

This obviously shows that the MC paradox is almost unavailable. An even more negative result holds about the C paradox: no rich set of voting sets secures that the majority rule outcome is an approval compromise.

Proposition 10 $\mathcal{E}^C = \varnothing$

5 Discussion

This paper shows that, in multiple referendum, when preferences over outcomes are built from approval ballots by using the Hamming distance criterion, issue-wise majority voting may produce neither a majoritarian nor an approval compromise. Moreover, we show that these two 'compound majority voting paradoxes' may hold when strong restrictions are made upon the issue-wise majority margins. We also show that, for (resp. almost) any rich voting set, to be interpreted as sets of potentially cast ballots, the approval (resp. majoritarian) compromise paradox may happen at some distribution of the votes among ballots.

Assuming Hamming preferences over outcomes should not be seen as a limitation of the analysis. Indeed, the Hamming preferences strongly favor candidate-wise majority voting as a way to reach a compromise. Beyond separability, the Hamming preference also equally weights all the issues. Dropping this last property leads to even more negative results. To see why, say that a utility function U_x over the set of programs is consistent with the ballot x if, for any two programs y and z, the set $A(y) = \{q : x^q = y^q\}$ of issues which y and x agree on, contains $A(z)$, then $U_x(y) > U_x(z)$. This consistency property, which induces a partial ordering over programs, is introduced in Özkal-Sanver and Sanver (2006),[11] and has a very intuitive meaning: voters cast their most preferred program as ballot, and always prefer a program that is unambiguously closer to their ballot. An immediate corollary of our results is the following:

[11] Özkal-Sanver and Sanver (2006) show that, when we allow for any separable preference profile consistent with a voting profile P, then Maj may produce a Pareto-dominated program.

Proposition 11 *Suppose that, for any ballot x, any preference over programs that is consistent with x is allowed. Then*

(1) *There exists a voting profile P where the MC paradox (resp. the C paradox) holds*
(2) *Both the MC paradox and the C paradox may hold even when voters are almost issue-wise unanimous*
(3) *If $Q \geq 3$, there is no rich MC-free (resp. C-free) voting set*

The first two assertions trivially come from the fact that the Hamming utility function represents preferences that are consistent with ballots. This also implies from Proposition 10 that there is no rich C-free voting set. Consider the possibility for a rich MC-free voting set V. From Proposition 9, we know that $V = (x, -x, y, -y) \in \mathcal{E}^{SS}$, where $d(x, y) \leq 1$. Through a relevant relabelling of ballots x and y, one can define $V = (x_1, -x_1, x_2, -x_2)$, where (1) $x_1^q = 1 \Leftrightarrow 1 \leq q \leq q^*$, and (2) $x_2^q = 1 \Leftrightarrow 1 \leq q \leq (q^* - 1)$.

Then consider the voting profile P where each ballot x_1, $-x_1$ and $-x_2$ is cast by one voter, while two voters cast the ballot x_2. Moreover, suppose that (1) all voters but the two voting x_2 have Hamming preferences over programs, and (2) the two voters with ballot x_2 have the same utility function U such that $Q = U(x_2) > U(w) = Q - 1 > U(z)$, for any $z \neq x_2, w$, where w is defined by $w^q = 1 \Leftrightarrow q \in \{1, \ldots, q^* + 1\}$. It is easily checked that U is consistent with x_2. One get that $Supp_0(x, P) < \frac{N}{2}$ for any x, and that $Supp_1(w, P) = 3 > \frac{N}{2} > Supp_1(x, P)$ for any $x \neq w$. Thus $\mathcal{MC}(X) = \{w\}$. Finally, $Maj(P) = \{x_2\}$, so that the MC paradox holds.

Several routes to further research may be worth being followed. First, the characterization of voting sets that are free from the strict MC paradox remains to be done. Second, we only address here the case where there is no restriction upon the number of issues which can be socially accepted.[12] How do our results evolve under such a restriction? Third, is the majoritarian compromise subgame-perfect implementable in the present setting?

Appendix

Proof of Proposition 1 It follows from Hamming preferences that, for any Q, for any $P \in \Pi_Q$ and any $x \in \{0, 1\}^Q$, $S(x, P) < \frac{N}{2} \Rightarrow S(-x, P) > \frac{N}{2}$. Consider $x \in \mathcal{C}(P)$; then, $S(x, P) \geq S(-x, P)$ implies that $S(x, P) \geq \frac{N}{2}$, hence $x \in \mathcal{S}(P)$. Consider $x \in (\mathcal{MC}(P) - \mathcal{S}(P))$. Since $S(x, P) < \frac{N}{2}$, then $k^*(P) > \frac{Q}{2}$. But $S(-x, P) > \frac{N}{2}$ $\Leftrightarrow \left|\left\{n \in \mathcal{N} : U_n(-x) \geq \frac{Q}{2}\right\}\right| > \frac{N}{2}$ implies that $k^*(P) \leq \frac{Q}{2}$, a contradiction. This proves the first two assertions. In order to prove the last two ones, it suffices to consider the profile in Example 1, where $\mathcal{S}(P) = \{0, 1\}^4 - \{x_1, -x_2\}$. □

Proof of Proposition 3 Consider a voting profile P where $x \in Maj(P)$, and suppose that $-x$ defeats x at P. It follows that $\left|\left\{n : U_n(-x) \geq \frac{Q}{2}\right\}\right| > \left|\left\{n : U_n(x) \geq \frac{Q}{2}\right\}\right|$.

[12] An alternative interpretation is that issues are candidates, and that a committee with a given fixed size has to be chosen from the set of candidates. Approval ballots indicate the names of the approved candidates.

We deduce that $\left|\left\{n : U_n(x) \geq \frac{Q}{2}\right\}\right| < \frac{N}{2}$, so that $x \notin \mathcal{S}(P)$. Conversely, suppose that $x \notin \mathcal{S}(P)$. Then $-x \in \mathcal{S}(P)$. Thus, $\left|\left\{n : U_n(-x) \geq \frac{Q}{2}\right\}\right| \geq \frac{N}{2}$ and $\left|\left\{n : U_n(x) \geq \frac{Q}{2}\right\}\right| < \frac{N}{2}$. Since $\left\{n : U_n(-x) \geq \frac{Q}{2}\right\} = \{n : U_n(-x) \geq U_n(x)\}$ and $\{n : U_n(x) \geq \frac{Q}{2}\} = \{n : U_n(x) \geq U_n(-x)\}$, then $|\{n : U_n(-x) \geq U_n(x)\}| > |\{n : U_n(x) \geq U_n(-x)\}|$, which ensures that $-x$ defeats x. □

Proof of Proposition 4 Given a ballot x_h, we denote by π_h^- the number of ballots $-x_h$. Both assertions are obvious if $Q = 1$. Let $Q = 2$, $x_1 = (1, 1)$ and $x_2 = (1, 0)$. Moreover, let $a_h = \pi_h - \pi_h^-$, $h = 1, 2$. Suppose w.l.g. that $-x_1 \in Maj(P)$ and $x_1 \notin Maj(P)$ at some profile P. Thus, the next two inequalities must hold: (1) $\pi_1^- + \pi_2^- \geq \pi_1 + \pi_2 \Leftrightarrow a_1 + a_2 \leq 0$, and (2) $\pi_1^- + \pi_2 \geq \pi_1 + \pi_2^- \Leftrightarrow a_1 \leq a_2$. Hence, $a_1 \leq 0$. Now, $S(-x_1, P) = \pi_1^- + \pi_2 + \pi_2^-$, while $S(x_1, P) = \pi_1 + \pi_2 + \pi_2^-$. Thus, $S(x_1, P) > S(-x_1, P) \Leftrightarrow a_1 > 0$, a contradiction. It follows that $-x_1 \in \mathcal{C}(P)$, therefore the C paradox does not hold. Furthermore, by Proposition 1, $\mathcal{MC}(P) \subseteq \mathcal{S}(P)$, so that $k^*(P) \leq 1$. If $k^*(P) = 0$, then obviously $-x_1 \in \mathcal{MC}(P)$. If $k^*(P) = 1$, then $Supp_1(-x_1, P) = S(-x_1, P)$, so that $x_1 \notin \mathcal{MC}(P)$.

Now, suppose that $Q = 3$. Let $x_1 = (1, 1, 0)$, $x_2 = (1, 0, 1)$, $x_3 = (0, 1, 1)$, $x_4 = (1, 1, 1)$. Let $P = (V, \pi) \in \Pi_3$ be such that, w.l.g., $-x_4 \in Maj(P)$ and $x_4 \notin Maj(P)$. It follows that the next three inequalities hold (with at least one being strict): (1) $a_1 + a_2 + a_4 \leq a_3$, (2) $a_1 + a_3 + a_4 \leq a_2$, and (3) $a_2 + a_3 + a_4 \leq a_1$. Summing them leads to: $a_1 + a_2 + a_3 + 3.a_4 < 0$ (*). Furthermore, since $\mathcal{MC}(P) \subseteq \mathcal{S}(P)$, then $k^*(P) \leq 1$. If $k^*(P) = 0$, then $\pi_4^- > \frac{N}{2}$ ensures that $-x_4 \in \mathcal{MC}(P)$, and thus $x_4 \notin \mathcal{MC}(P)$. Suppose that $k^*(P) = 1$. Then, $x_4 \in \mathcal{MC}(P) \Rightarrow [Supp_1(x_4, P) \geq Max\{Supp_1(-x_h, P), h = 1, 2, 3\}]$. This implies the following inequalities:

- $\pi_1 + \pi_2 + \pi_3 + \pi_4 \geq \pi_1 + \pi_2 + \pi_3^- + \pi_4^- \Leftrightarrow a_3 + a_4 \geq 0$
- $\pi_1 + \pi_2 + \pi_3 + \pi_4 \geq \pi_1 + \pi_2^- + \pi_3 + \pi_4^- \Leftrightarrow a_2 + a_4 \geq 0$
- $\pi_1 + \pi_2 + \pi_3 + \pi_4 \geq \pi_1^- + \pi_3 + \pi_3 + \pi_4^- \Leftrightarrow a_1 + a_4 \geq 0$

Hence, one get by summation that $a_1 + a_2 + a_3 + 3.a_4 \geq 0$, in contradiction with (*). Thus, the strict MC paradox cannot prevail.

Since $S(x, P) = Supp_1(x, P)$ for all $x \in V$, the same contradiction holds if the strict C paradox holds. This proves assertion (1), as well as assertion (2) for $Q = 3$.

Finally, suppose that $Q = 4$. The proof is similar to the one above. Let $x_1 = (1, 1, 1, 1)$, $x_2 = (1, 1, 1, 0)$, $x_3 = (1, 1, 0, 1)$, $x_4 = (1, 0, 1, 1)$, $x_5 = (0, 1, 1, 1)$, $x_6 = (1, 1, 0, 0)$, $x_7 = (1, 0, 1, 0)$, and $x_8 = (1, 0, 0, 1)$. We get that, for any profile P where w.l.g. $-x_1 \in Maj(P)$ and $x_1 \notin Maj(P)$: $a_5 \geq \sum_{h\neq 5} a_h$, $\sum_{h=4,7,8} a_h \geq \sum_{h\neq 4,7,8} a_h$, $\sum_{h=3,6,8} a_h \geq \sum_{h\neq 3,6,8} a_h$ and $\sum_{h=2,6,7} a_h \geq \sum_{h\neq 2,6,7} a_h$., with at least one strict inequality. From summation, we get: $4.a_1 + 2.\sum_{h=2,3,4,5} a_h < 0$ (*)

Moreover: $S(x_1, P) \geq S(-x_5, P) \Leftrightarrow a_1 + a_5 \geq 0$, $S(x_1, P) \geq S(-x_4, P) \Leftrightarrow a_1 + a_4 \geq 0$, $S(x_1, P) \geq S(-x_3, P) \Leftrightarrow a_1 + a_3 \geq 0$, and $S(x_1, P) \geq S(-x_2, P) \Leftrightarrow a_1 + a_2 \geq 0$. Summing these inequalities leads to: $4.a_1 + \sum_{h=2,3,4,5} a_h \geq 0$, in contradiction with (*). Thus, $x_1 \notin \mathcal{C}(P)$, and the strict C paradox does not hold. □

Proof of Proposition 5 *Proof of assertion (1):* Suppose first that the strict Ostrogorski paradox holds at some voting profile P. If $w \in Maj(P)$, we get: $|\{n \in \mathcal{N} : U_n(-w) \geq U_n(w)\}| > |\{n \in \mathcal{N} : U_n(w) \geq U_n(-w)\}|$.

Thus, $\left|\left\{n \in \mathcal{N} : U_n(-w) \geq \frac{Q}{2}\right\}\right| > \left|\{n \in \mathcal{N} : U_n(w) \geq \frac{Q}{2}\right|$. Hence, since $S(w, P) < \frac{N}{2}$, then $w \notin \mathcal{S}(P)$, and we get from proposition 1 that $w \notin \mathcal{MC}(P)$. Next, suppose that w is defeated by $z \neq -w$. Let $A = \{1 \leq q \leq Q : w^q \neq z^q\}$. Denoting by P/A the restriction of P to A and x/A the restriction of $x \in \{0, 1\}^Q$ to A, we have that $w/A \notin \mathcal{MC}(P/A)$. Let $\widetilde{y} \in \mathcal{MC}(P/A)$. It follows from construction that $k^*(P/A) \leq k^*(P)$. Suppose first that $k^*(P/A) < k^*(P)$. Define the program y by: $y^q = (\widetilde{y}/A)^q, q \in A$, and $y^q = w^q, q \in A$. It follows from the separability of preferences that $|\{n \in \mathcal{N} : U_n(y) \geq Q - k^*(P)\}| > |\{n \in \mathcal{N} : U_n(w) \geq Q - k^*(P)|$. Thus $w \notin \mathcal{MC}(P)$. Finally, if $k^*(P/A) = k^*(P)$, then $Supp_{k^*(P)}(y/A, P/A) > Supp_{k^*(P)}(w/A, p/A)$. Since both y and w coincide on all $q \notin A$, it directly follows that $Supp_{k^*(P)}(y, P) > Supp_{k^*(P)}(w, P)$, which proves that any profile facing the Ostrogorski paradox also faces the MC paradox.

Now consider the profile $P = (V, \pi) \in \Pi_3$ where $V = (x_1, \ldots, x_4)$, $x_1 = -x_2 = (0, 1, 1)$, $x_3 = (1, 1, 0)$, $x_4 = (0, 0, 0)$, and $\pi_h = 1$, $h = 1, 2, 3$, $\pi_4 = 2$. We get that $Maj(P) = \{x_4\}$ and $k^*(P) = 1$. Moreover, $Supp_1(x_4, P) = 3$, while $Supp_1(x_2, P) = 4$, so that $x_4 \notin \mathcal{MC}(P)$, and therefore the MC paradox prevails. Finally, using Hamming preferences, it is easily checked that x_4 is a Condorcet winner of P, and hence, the Ostrogorski paradox does not occur. This proves assertion (1). □

Proof of assertion (2): Let P be a voting profile presenting the Anscombe paradox, that is $Maj(P) \nsubseteq \mathcal{S}(P)$. The C paradox follows from $\mathcal{C}(P) \subseteq \mathcal{S}(P)$. Now consider $P = (V, \pi) \in \Pi_3$, where $V = (x_1, \ldots, x_4)$, $x_1 = (1, 1, 0)$, $x_2 = -x_3 = (1, 0, 1)$, $x_4 = (0, 0, 0)$, and $\pi_h = 1$, $h = 1, 2, 3$, $\pi_4 = 2$. One has that $Maj(P) = \{x_4\} \in \mathcal{S}(P)$. Hence, there is no Anscombe paradox. However, $S(x_3, P) = 4 > S(x_4, P) = 3$ implies that $x_4 \notin \mathcal{C}(P)$, therefore the C paradox.

Proof of assertion (3): Suppose that $w \in (Maj(P) - \mathcal{MC}(P))$ and $-w \in (\mathcal{MC}(P) - Maj(P))$. Then $-w \in \mathcal{S}(P)$. We can set w.l.g. that $\forall q$, $w^q = 0$. If Q is odd, then $w \notin \mathcal{S}(P)$, and thus the Anscombe paradox holds. Assume that Q is even and $w \in \mathcal{S}(P)$. Since $-w \in \mathcal{MC}(P)$ and $w \notin \mathcal{MC}(P)$, then $k^*(P) < \frac{Q}{2}$, and $|\{n \in \mathcal{N} : U_n(-w) \geq (q - k^*(P))\}| \geq \frac{N}{2} > |\{n \in \mathcal{N} : U_n(w) \geq (q - k^*(P))\}|$. Equivalently, one faces the situation where (1) at least 50% of the ballots contains at least $k^*(P)$ approvals, (2) at least 50% of the ballots contains $\frac{Q}{2} + k^{**}$ disapprovals, where $1 \leq k^{**} < \left(k^*(P) - \frac{Q}{2}\right)$. Combining (1) and (2) gives a total number of approvals at least $\frac{k^*(P).N}{2} + \frac{N}{2}.(\frac{Q}{2} - k^{**}) > \frac{N.Q}{2}$, which contradicts that $-w \notin Maj(P)$. Thus, $w \notin \mathcal{S}(P)$, and the Anscombe paradox holds.

Now consider the voting profile $P = (V, \pi) \in \Pi_3$, where $V = (x_1, \ldots, x_4)$, $x_1 = (1, 1, 0)$, $x_2 = (1, 0, 1)$, $x_3 = (0, 1, 1)$, $x_4 = (0, 0, 0)$, and $\pi_h = 1$, $h = 1, 2, 3$, $\pi_4 = 2$. Then $Maj(P) = \{x_4\}$. Since $-x_4$ defeats x_4, then the Anscombe paradox occurs. Consider $y = (1, 0, 0)$. Since $Supp_1(y, P) = 4 > \frac{N}{2}$, then $k^*(P) = 1$. Finally, $Supp_1(-x_4, P) = 3 < Supp_1(y, P)$, and therefore, $-x_4 \notin \mathcal{MC}(P)$.

Proof of assertion (4): Consider the profile P defined in Example 1. Since $Maj(P) = \{x_2\} \in \mathcal{MC}(P) - \mathcal{C}(P)$, the C paradox holds, whereas the MC paradox does not. Now define $P = (V, \pi) \in \Pi_6$, where $V = (x_1, \ldots, x_7), x_1 = (1, 1, 1, 0, 0, 0), x_2 = (0, 1, 1, 1, 0, 0), x_3 = (0, 0, 1, 1, 1, 0), x_4 = (0, 0, 0, 1, 1, 1), x_5 = (1, 0, 0, 0, 1, 1), x_6 = (1, 1, 0, 0, 0, 1), x_7 = (0, 0, 0, 0, 0, 0), \pi_h = 1, 1 \leq h \leq 6$, and $\pi_7 = 2$. Then $Maj(P) = \{x_7\}$. Moreover, $S(x_7, P) = N$, so that the C paradox does not hold. Finally, one easily checks that $k^*(P) > 1$. Take $y = (1, 1, 0, 0, 0, 0)$. Then $Supp_2(y, P) = 4 \geq \frac{N}{2}$, while $Supp_2(x_7, P) = 2 < \frac{N}{2}$. Thus, $x_7 \notin \mathcal{MC}(P)$, so that the MC paradox holds.

Proof of Proposition 6 We prove that $\forall \varepsilon > 0$, there exist $Q > 0$ and $P \in \Pi_Q$ such that $Maj(P) = \{w\}, w \notin \mathcal{C}(P)$ and, $\forall q = 1, \ldots, Q, \left|\{n \in \mathcal{N} : x_n^q = w^q\}\right| > N.(1 - \varepsilon)$. Pick up an even integer $Q > 3$, and define P as follows: there exist 3 strictly positive integers α, β, γ such that

- α voters cast the ballot $w^* = (0, \ldots, 0)$
- each ballot containing exactly one approval is cast by α voters
- there are γ ballots x_1 where $x_1^q = 0 \Leftrightarrow q \in \{\frac{Q}{2} + 1, \ldots, Q\}$,
- there are β ballots x_2 where $x_2^q = 0 \Leftrightarrow q \in \{\frac{Q}{2}, \ldots, Q\}$,

It follows that $N = \alpha.(Q+1) + \gamma + \beta$. Noting by 0^q the number of disapprovals given to issue q, we get: $0^q = \alpha.Q, 1 \leq q < \frac{Q}{2}, 0^{Q/2} = \alpha.Q + \beta, 0^q = N - \alpha, q > \frac{Q}{2}$. It follows that, for any $\varepsilon > 0$, there exists Q such that $\left|\{n \in \mathcal{N} : x_n^q = w^{*q}\}\right| > N.(1 - \varepsilon)$ and $Maj(P) = \{w^*\}$. Define the program z by: $z^q = 1 \Leftrightarrow q = 1$. Finally, $S(w^*, P) = \alpha.(Q + 1) + \beta$, and $S(z, P) = N$. Hence, $w^* \notin \mathcal{C}(P)$.[13]

In order to prove the second assertion, let $Q = H+3$, where $H \geq 3$, and let Γ be the subset of ballots defined by: $x \in \Gamma \Leftrightarrow x^1 = x^2 = x^3 = 0$, and $|\{q > 3 : x^q = 1\}| = 3$. Moreover, let $w, y, z \in \{0, 1\}^Q$ be respectively defined by: (1) $w^q = 0, q = 1, \ldots, Q$, (2) $y^q = 1 \Leftrightarrow q = 1, 2$, and (3) $z^q = 1 \Leftrightarrow q = 1, 2, 3$.

Let $P \in \Pi_Q$ where each ballot in Γ is cast by a unique voter, both y and z are cast by a single voter, and $\alpha = \binom{H}{3} - 1$ voters cast ballot w. Hence, $N = 2.\alpha + 1$. Furthermore, the number of approvals 1^q given to issue q is: $1^1 = 1^2 = 2, 1^3 = 1, 1^q = \binom{H-1}{2} = \beta, q = 4, \ldots, Q$. Hence, for any issue q, $\frac{1^q}{N} \leq \frac{3.\alpha}{H.N}$, so that it can be made as close as wanted to 0 by choosing H large enough. It follows that $Maj(P) = \{w\}$. Denote by π_x the number of ballots $x \in \{0, 1\}^Q$. It is easily checked that:

- $\forall x \in \{0, 1\}^Q, k \leq 1 \Rightarrow Supp_k(x, P) < \frac{N}{2}$
- $Supp_1(w, P) = \pi_w = \alpha - 1, Supp_2(w, P) = \pi_w + \pi_z = \alpha < \frac{N}{2}$
- $Supp_1(y, P) = \pi_y + \pi_z = 2, Supp_2(y, P) = \pi_w + \pi_z + \pi_y = \alpha + 1 > \frac{N}{2}$

Hence, $w \notin \mathcal{MC}(P)$, whereas the issue-wise majority size can be made as close as wanted to 100%. □

Proof of Proposition 7 The first assertion is trivial if $Q < 3$. Let $Q = 3$. We can consider w.l.g. a profile P where $w = (0, 0, 0) \in Maj(P)$ and the three-fourth rule

[13] A by-product of the proof is that the majority rule may fail to lead to the Fallback Bargaining solution (see Brams et al. 2007) even under quasi-unanimous candidate-wise preferences.

applies. It is easy to check that $w \in \mathcal{S}(P)$. Suppose that $w \notin \mathcal{MC}(P)$, and pick up any $y \in \mathcal{MC}(P)$. Since $\mathcal{MC}(P) \subseteq \mathcal{S}(P)$, then $k^*(P) = 1$ and $y \neq (-w)$. Suppose that y contains only one approval, say w.l.g. that $y = (1, 0, 0)$. Denoting by π_x the number of ballots x, and by π_{abc} the number of ballots $x = (a, b, c) \in \{0, 1\}^3$, we get that $Supp_1(y, P) = \pi_w + \pi_y + \pi_{110} + \pi_{101}$, and $Supp_1(w, P) = \pi_w + \pi_y + \pi_{010} + \pi_{001}$. Moreover, the three-fourth rule ensures that:

- $\pi_w + \pi_{001} + \pi_{010} + \pi_{011} \geq 3.(\pi_{111} + \pi_{101} + \pi_{110} + \pi_y)$ (1)
- $\pi_w + \pi_y + \pi_{001} + \pi_{101} \geq 3.(\pi_{111} + \pi_{011} + \pi_{110} + \pi_{010})$ (2)
- $\pi_w + \pi_y + \pi_{010} + \pi_{110} \geq 3.(\pi_{111} + \pi_{101} + \pi_{011} + \pi_{001})$ (3)

Summation leads to $3\pi_w \geq (\pi_{001} + \pi_{010} + \pi_y) + 9.\pi_{111} + 5.(\pi_{101} + \pi_{110} + \pi_{011})$ (4)

Since $w \notin \mathcal{MC}(P)$ implies that $\pi_w < \frac{N}{2}$, then $\pi_{001} + \pi_{010} + \pi_y + \pi_{111} + \pi_{101} + \pi_{110} + \pi_{011} > \frac{N}{2}$ (5). Hence from (4), $\pi_{001} + \pi_{010} + \pi_y + 9.\pi_{111} + 5.(\pi_{101} + \pi_{110} + \pi_{011}) < \frac{3.N}{2}$ (6). Then (6) – (5) leads to: $8.\pi_{111} + 4.(\pi_{101} + \pi_{110} + \pi_{011}) < N$ (7). It follows that $\pi_{111} + \pi_{101} + \pi_{110} + \pi_{011} < \frac{N}{4}$, which implies that $Supp_1(w, P) > \frac{3.N}{4}$. Furthermore, $Supp_1(w, P) < Supp_1(y, P)$ implies that $Supp_1(y, P) = \pi_w + \pi_y + \pi_{110} + \pi_{101} > \frac{3.N}{4}$ (8). From the three-fourth rule, we get: $\pi_{111} + \pi_y + \pi_{101} + \pi_{110} < \frac{N}{4}$ (9). Combining with (8) with (9) gives $\pi_w > \frac{N}{2}$, contradicting $w \notin \mathcal{MC}(P)$.

Hence, y must contain two approvals, say $y = (1, 1, 0)$. This implies that $Supp_1(y, P) = \pi_y + \pi_{010} + \pi_{111} + \pi_{100} < \pi_{010} + \frac{N}{4}$ (from the three-fourth rule). Thus, $Supp_1(w, P) < Supp_1(y, P) \Rightarrow \pi_w + \pi_{100} + \pi_{001} < \frac{N}{4}$, which implies, together with $Supp_1(w, P) \geq \frac{N}{2}$, that $\pi_{010} > \frac{N}{4}$, in contradiction with the three-fourth rule. This proves the first part of the proposition.

In order to prove the second one, define the voting profile $P = (V, \pi)$ by: let $V = \Gamma \cup \{w, y\}$, where $\Gamma = \{x \in \{0, 1\}^6 : x^1 = x^2 = 0, \text{and } |\{q > 2 : x^q = 1\}| = 2\}$, where $\forall q$, $w^q = 0$, and $y^q = 1 \Leftrightarrow q = 1, 2$. Assume that $\pi_x = 1$ for all $x \in \Gamma$, while $\pi_w = \pi_y = 3$. Hence, $N = 12$, and $Maj(P) = \{w\}$. Moreover the majority is exactly 75% for each q. Since $Supp_0(z, P) < 6 < \frac{N}{2}$ for all $z \in V$, then $k^*(P) \geq 1$. Finally, consider z defined by $z^q = 1 \Leftrightarrow q = 1$. Since $Supp_1(z, P) = 6 \geq \frac{N}{2} > Supp_1(w, P) = 3$, then $k^*(P) = 1$ and $w \notin \mathcal{MC}(P)$. □

Proof of Proposition 9 It is obviously seen that any voting set involving exactly one pair of opposite ballots is MC-free. Let V be any rich MC-free voting set that contains at least 2 pairs of opposite ballots. Since $\mathcal{E}^{MC} \subseteq \mathcal{E}^{SS}$, we may assume that V is single-switch, and contains the ballots x_1 and x_2, where (1) $x_1^q = 1 \Leftrightarrow 1 \leq q \leq q_1$, and (2) $x_2^q = 1 \Leftrightarrow 1 \leq q \leq q_2$, with $1 \leq q_1 < q_2 \leq Q$. Let $d = (q_2 - q_1)$ and $d' = (Q - q_2)$.

Suppose first that $\beta = q_1 + d' \geq d > 1$. Let $P = (V, \pi)$ be the voting profile defined by: $\pi_1 = \pi_2 = 2, \pi_1^- = 1$, and $\pi_2^- = 0$. We get $Maj(P) = \{x_2\}$. Then, $Supp_{d-1}(x_2, P) = \pi_2 < \frac{N}{2}$. Define the program y by: $y^q = 1 \Leftrightarrow 1 \leq q \leq q_2 - 1$. Then $Supp_{d-1}(y, P) = \pi_1 + \pi_2 > \frac{N}{2}$, which implies that $x_2 \notin \mathcal{MC}(P)$, a contradiction. Now, suppose that $\beta < d$, and define $P' = (V, \pi)$ by: $\pi_1 = 1, \pi_2 = 2, \pi_1^- = 1 = \pi_2^-$, so that $Maj(P') = \{x_2\}$. If $d' > 0$, pick up the program $z = (1, \ldots, 1)$. Then, $Supp_{\beta-1}(x_2, P') = \pi_2 < \frac{N}{2}$, whereas $Supp_{\beta-1}(z, P') = \pi_1^- + \pi_2 > \frac{N}{2}$.

Therefore, $x_2 \notin \mathcal{MC}(P')$, a contradiction. If $d' = 0$, the same argument applies once z is replaced by z' defined by: $z'^q = 1 \Leftrightarrow$ either $q_1 \leq q \leq q_2$, or $q_1 = 1$.

This proves that any $V \in \mathcal{E}^{MC}$ must contain at most two pairs of opposite single-switch ballots $\{x, -x\}$ and $\{y, -y\}$ such that $d(x, y) \leq 1$.

Now, consider any such V, which contains the ballots x_1 and x_2, where (1) $x_1^q = 1 \Leftrightarrow 1 \leq q \leq q_1 - 1$, and (2) $x_2^q = 1 \Leftrightarrow 1 \leq q \leq q_1$, with $2 \leq q_1 \leq Q$. Consider any profile $P = (V, \pi)$. It is easily checked that $Maj(P) \subseteq \{x_1, x_2, -x_1, -x_2\}$. Suppose w.l.g. that $x_2 \in Maj(P)$. Then $\pi_1 + \pi_2 \geq \frac{N}{2}$ and $\pi_1^- + \pi_2 \geq \frac{N}{2}$. It follows that $k^*(P) \leq 1$. If $k^*(P) = 0$, then, clearly, $x_2 \in \mathcal{MC}(P)$, and all is done. Suppose that $k^*(P) = 1$ and $x_2 \notin \mathcal{MC}(P)$. Thus, there exists $y \in \{0, 1\}^Q$ such that $Supp_1(y, P) > Supp_1(x_2, P) = \pi_1 + \pi_2$. This implies that either all voters for x_2 or all for x_1 are counted in $Supp_1(y, P)$ (note that both categories can be simultaneously counted only if $y = x_1$, and since $Q \geq 3$, then $Supp_1(x_1, P) = \pi_1 + \pi_2$, we get a contradiction). Consider the former case. Let q^* be the unique issue such that $x_2^{q^*} \neq y^{q^*}$. Then either $q^* = q_1$, which implies that $y = x_1$, a contradiction, or $q^* \neq q_1$, which implies that $Supp_1(y, P) = \pi_2$, a contradiction. In the latter case, since $y \neq x_2$, there exists again a unique issue $q^* \neq q_1$ such that $x_1^{q^*} \neq y^{q^*}$. But this implies that $Supp_1(y, P) = \pi_1$, again a contradiction. □

Proof of Proposition 10 It is sufficient to prove that any rich voting set V containing 2 different pairs of opposite ballots $\{x_1, -x_1\}$ and $\{x_2, -x_2\}$ is exposed to the C paradox. $\mathcal{Q}$ can be partitioned into four sets $B_1, \ldots, B_4$, where $\mid B_h \mid = b_h$, $1 \leq h \leq 4$, and where programs x_1 and x_2 are defined by: (1) $x_1^q = 1 \Leftrightarrow q \in B_1 \cup B_2$, and (2) $x_2^q = 1 \Leftrightarrow q \in B_1 \cup B_3$. Since $x_1 \neq x_2$, then $b_1 + b_4 \neq 0$ and $b_2 + b_3 \neq 0$. Moreover, $Maj(P) \subset \{x_1, x_2, -x_1, -x_2\}$ for any profile $P = (V, \pi)$. Furthermore,

- $d(x_1, x_2) = d(-x_1, -x_2) = b_2 + b_3$
- $d(x_1, -x_2) = d(-x_1, x_2) = b_1 + b_4$

Then, we consider the 2 following possible cases:

Case 1: $b_2 + b_3 \leq b_1 + b_4$

Let $\pi_1 = 5, \pi_1^- = 0, \pi_2 = 3$ and $\pi_2^- = 4$. Then, $Maj(P) = \{x_1\}$. Since $b_2 + b_3 \leq b_1 + b_4$ implies that $b_2 + b_3 \leq \frac{Q}{2}$, then $S(x_1, P) = \pi_1 + \pi_2 = 8$. Suppose that $b_1, b_4 \leq \frac{Q}{2}$. Let y be the program where $y^c = 1 \Leftrightarrow c \in B_1 \cup B_2 \cup B_4$. Then $d(x_1, y) = b_4 \leq \frac{Q}{2}$ and $d(y, -x_2) = b_1 \leq \frac{Q}{2}$. Hence $S(y, P) = \pi_1 + \pi_2^- = 9 > S(x_1, P)$, and the C paradox holds. Suppose that $b_1 > \frac{Q}{2}$. Define the program z by $z^c = 1 \Leftrightarrow c \in B_1' \cup B_2 \cup B_4$, where B_1' is the set of the first $\left\lfloor \frac{Q}{2} \right\rfloor$ issues in B_1. Then $d(x_1, y) = b_1 + b_4 - \left\lfloor \frac{Q}{2} \right\rfloor$ and $d(y, -x_2) = b_1 - \left\lfloor \frac{Q}{2} \right\rfloor$. Since $b_2 + b_3 \neq 0$ implies that $b_1 + b_4 < Q$, then $d(x_1, y) \leq \frac{Q}{2}$ and $d(y, -x_2) \leq \frac{Q}{2}$. Thus, again, $S(z, P) = \pi_1 + \pi_2^- = 9 > S(x_1, P)$, and the C paradox holds. The same argument holds if $b_4 > \frac{Q}{2}$, by defining z such that $z^c = 1 \Leftrightarrow c \in B_1 \cup B_2 \cup B_4'$, where B_4' is the set of the first $\left\lfloor \frac{Q}{2} \right\rfloor$ issues in B_4.

Case 2: $b_2 + b_3 > b_1 + b_4$

Let $\pi_1 = 1, \pi_1^- = 2, \pi_2 = 3$ and $\pi_2^- = 5$. Then, $Maj(P) = \{-x_2\}$. If $b_2, b_3 \leq \frac{Q}{2}$, define w by $w^c = 1 \Leftrightarrow c \in B_4$. Then $d(-x_1, w) = b_3 \leq \frac{Q}{2}$ and $d(w, -x_2) = b_2 \leq \frac{Q}{2}$. Hence, $S(w, P) = \pi_1^- + \pi_2^- = 7 > S(-x_2, P) = \pi_1 + \pi_2^- = 6$, so that the C-paradox holds. If $b_3 > \frac{Q}{2}$, define the program v by $v^c = 1 \Leftrightarrow c \in B_2 \cup B_3' \cup B_4$, where B_3' is the set of the first $\left\lfloor \frac{Q}{2} \right\rfloor$ issues in B_3. Then, $d(-x_1, v) = b_2 + b_3 - \left\lfloor \frac{Q}{2} \right\rfloor \leq \frac{Q}{2}$ (since $b_1 + b_4 > 0 \Rightarrow b_2 + b_3 < Q$) and $d(v, -x_2) = \left\lfloor \frac{Q}{2} \right\rfloor \leq \frac{C}{2}$. Hence, $S(w, P) = \pi_1^- + \pi_2^- = 7 > S(-x_2, P)$, and therefore the C paradox holds. The same argument prevails if $b_2 > \frac{Q}{2}$, once we replace v with the program w, with $w^c = 1 \Leftrightarrow c \in B_2' \cup B_3 \cup B_4$, where B_2' is the set of the first $\left\lfloor \frac{Q}{2} \right\rfloor$ issues in B_2. □

References

Anscombe GEM (1976) On the frustration of the majority by fulfillment of the majority's will. Analysis 36:161,168

Bassett GW, Persky J (1999) Robust voting. Public Choice 99:299–310

Bezembinder T, Van Acker P (1985) The Ostrogorski paradox and its relation to non-transitive choice. Journal of Mathematical Psychology 11:131–158

Brams SJ (2008) Mathematics and Democracy: Designing Better Voting and Fair-Division Procedures. Princeton University Press, Princeton, NJ

Brams SJ, Kilgour DM (2001) Fallback Bargaining. Group Decision and Negotiation 10:287–316

Brams SJ, Kilgour DM, Sanver R (2004) A Minimax procedure for negotiating multilateral treaties. In: Wiberg Matti (ed) Reasoned Choices: Essays in Honor of Hannu Nurmi. Finnish Political Science Association, Turku Finland, pp 108–139

Brams SJ, Kilgour DM, Sanver R (2007) A minimax procedure for electing committees. Public Choice 132:401–420

Brams SJ, Sanver R (2009) Voting systems that combine approval and preference. In: Fishburn Brams S, Gehrlein WV, Roberts FS (eds) The Mathematics of Preference, Choice, and Order: Essays in Honor of Peter C. Springer, Heidelberg, pp 215–237

Cuhadaroglu T, Lainé J (2009) Electing Pareto-optimal committees by approval balloting. Istanbul Bilgi University, Mimeo

Deb R, Kelsey D (1987) On constructing a generalized Ostrogorski paradox: necessary and sufficient conditions. Mathematical Social Sciences 14:161–174

Fishburn PC, Pekeč A (2004) Approval voting for committees. mimeo

Giritligil Kara AE, Sertel MR (2005) Does majoritarian approval matters in selecting a social choice rule? An explanatory panel study. Social Choice and Welfare 25:43–73

Hodge JK, Schwallier P (2006) How separability affects the desirability of referendum elections outcome. Theory and Decision 61:251–276

Kelly JS (1977) Strategy-proof social choice without single-valuedness. Econometrica 45:439–446

Kelly JS (1989) The Ostrogorski paradox. Social Choice and Welfare 6(1):71–76

Kilgour DM (2010) Approval Balloting for Multi-winner elections. In: Laslier JF, Sanver R (eds) Handbook on Approval Voting. Springer, Heidelberg, pp 105–124

Kilgour DM, Brams SJ, Sanver R (2006) How to elect a representative committee using approval balloting. In: Simeone Bruno, Pukelsheim Friedrich (eds) Mathematics and Democracy: Recent Advances in Voting Systems and Collective Choice. Springer, Heidelberg, pp 83–95

Lacy D, Niou EMS (2000) A problem with referendums. Journal of Theoretical Politics 12(1):5–31

Laffond G, Lainé J (2006) Single-switch preferences and the Ostrogorski paradox. Mathematical Social Sciences 52(1):49–66

Laffond G, Lainé J (2009) Condorcet choice and the Ostrogorski paradox. Social Choice and Welfare 32:317–333

Laffond G, Lainé J (2010) Does choosing committees from approval balloting fulfill the electorate's will? In: Laslier JF, Sanver R (eds) Handbook on Approval Voting. Springer, Heidelberg, pp 125–150
Nurmi H (1999) Voting paradoxes and how to deal with them. Springer, Heidelberg
Özkal-Sanver I, Sanver MR (2004) Efficiency in the degree of compromise: A new axiom for social choice. Group Decision and Negotiation 13:375–380
Özkal-Sanver I, Sanver MR (2006) Ensuring Pareto optimality by referendum voting. Social Choice and Welfare 27(1):211–219
Rae D, Daudt H (1976) The Ostrogorski paradox: a peculiarity of compound majority decisions. European Journal of Political Research 4:391–398
Ratliff T.C. (2003) Some startling inconsistencies when electing committees, Social Choice and Welfare, 21, 433–454
Ratliff T.C. (2006) Selecting Committees, Public Choice, 126, 343–355
Sertel MR (1986) Lecture Notes in Microeconomics. unpublished, Boğaziçi University, Istanbul
Sertel MR, Yılmaz B (1999) The Majoritarian Compromise is majoritarian optimal and subgame-perfect implementable. Social Choice and Welfare 16:615–627
Wagner C (1983) The Anscombe paradox and the rule of three-fourth. Theory and Decision 15:303–308
Wagner C (1984) Avoiding the Anscombe paradox. Theory and Decision 16:233–238