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Uncertain Power Average Operators for Aggregating Interval Fuzzy Preference Relations

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Abstract In this paper, we investigate group decision making problems based on interval fuzzy preference relations. We define an uncertain power weighted average (UPWA) operator and an uncertain power ordered weighted average (UPOWA) operator, on the basis of the power average operator of Yager (IEEE Trans Syst Man Cybern A 31:724–731, 1988) and the uncertain geometric mean. In the situations where the weights of experts are known, we develop a method based on the UPWA operator for group decision making with interval fuzzy preference relations; and in the situations where the weights of experts are unknown, we develop a method based on the UPOWA operator for group decision making with interval fuzzy preference relations.

Keywords Group decision making · Power average operator · Uncertain power weighted average operator · Uncertain power ordered weighted average operator · Interval fuzzy preference relation

1 Introduction

In Xu (2001), Xu defined the concept of interval fuzzy preference relation, whose elements are interval numbers and each of them is provided by a decision maker to express his/her preference degree range of one object over another. Since then, the similarity measures and priority methods of interval fuzzy preference relations have

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been investigated extensively (Xu 2004; Jiang 2007; Lan and Liu 2007; Xu and Chen 2008). The interval fuzzy preference relation is very useful for group decision making under uncertainty, and most of the related investigations have been focused on the combinations of interval fuzzy preference relations with other types of preference information such as numerical preference relations, linguistic preference relations, interval utility values; and interval multiplicative preference relations, etc. (Herrera et al. 2005; Xu 2007; Xu and Chen 2008). But there is little investigation on the group decision making based on interval fuzzy preference relations (Xu 2004). An important issue of this research topic is how to aggregate all individual interval fuzzy preference relations provided by decision makers into group interval fuzzy preference relation. Xu (2004) utilized the well known additive weighted averaging (AWA) operator to fuse all individual opinions into a group opinion. Nevertheless, in the process of group decision making, some individuals may provide unduly high (or low) preferences to their preferred (or repugnant) objects. The AWA operator can only consider the weights of decision makers, but does not take into account the information about the relationship between the values being fused, and thus cannot eliminate the influence of unfair arguments on the decision result. In Yager (2001), Yager developed a power average (PA) and a power ordered weighted (POWA) operator to provide aggregation tools which allow exact argument values to support each other in the aggregation process, i.e., the weighting vectors of these two operators depend upon the input arguments and allow values being aggregated to support and reinforce each other. Based on the PA and POWA operators, in this paper we first develop an uncertain power weighted average (UPWA) operator and an uncertain power ordered weighted average (UPOWA) operator. Then, in the cases where the weights of decision makers are known, we employ the UPWA operator to develop a method for group decision making based on interval fuzzy preference relations, and in the cases where the information about the weights of decision makers is unknown, we employ the UPOWA operator to develop a method for group decision making based on interval fuzzy preference relations. An illustrative example is also given to demonstrate our proposed approaches.

2 Power Average Operators

Let $a_i\,(i=1,2,\ldots,n)$ be a collection of exact arguments, and $w=(w_1,w_2,\ldots,w_n)^T$ be the weight vector of $a_i\,(i=1,2,\ldots,n)$, where $w_i \geq 0,\ i=1,2,\ldots,n$ and $\sum_{i=1}^{n} w_i = 1$. Yager (2001) defined a power weighted average (PWA) operator, as shown below:

$$\mathrm{PWA}\,(a_1,a_2,\ldots,a_n)=\frac{\sum_{i=1}^{n} w_i\,(1+T(a_i))\,a_i}{\sum_{i=1}^{n} w_i\,(1+T(a_i))} \tag{1}$$

where

$$T(a_i)=\sum_{\substack{j=1\\ j\neq i}}^{n} w_j\,Sup\left(a_i,a_j\right) \tag{2}$$

and $Sup(a_i, a_j)$ is the support for a_i from a_j, which satisfies the following three properties:

(1) $Sup(a_i, a_j) \in [0, 1]$;
(2) $Sup(a_i, a_j) = Sup(a_j, a_i)$;
(3) $Sup(a_i, a_j) \geq Sup(a_s, a_t)$, if $|a_i - a_j| < |a_s - a_t|$.

Especially, if $w = (1/n, 1/n, \ldots, 1/n)^T$, i.e., all of the objects being aggregated are of equal importance, then the PWA operator (1) reduces to the power average (PA) operator:

$$\mathrm{PA}(a_1, a_2, \ldots, a_n) = \frac{\sum_{i=1}^{n} (1 + T'(a_i))a_i}{\sum_{i=1}^{n} (1 + T'(a_i))} \tag{3}$$

where

$$T'(a_i) = \frac{1}{n} \sum_{\substack{j=1 \\ j \neq i}}^{n} Sup(a_i, a_j) \tag{4}$$

Moreover, based on the ordered weighted averaging (OWA) operator (Yager 1988) and the PA operator (3), Yager (2001) defined a power ordered weighted average (POWA) operator as follows:

$$\mathrm{POWA}(a_1, a_2, \ldots, a_n) = \sum_{i=1}^{n} u_i a_{index(i)} \tag{5}$$

where

$$u_i = g\left(\frac{R_i}{TV}\right) - g\left(\frac{R_{i-1}}{TV}\right), R_i = \sum_{j=1}^{i} V_{index(j)},$$
$$TV = \sum_{i=1}^{n} V_{index(i)}, \quad V_{index(j)} = 1 + T''\left(a_{index(i)}\right) \tag{6}$$

and

$$T''\left(a_{index(i)}\right) = \frac{1}{n} \sum_{\substack{j=1 \\ j \neq i}}^{n} Sup\left(a_{index(i)}, a_{index(j)}\right) \tag{7}$$

which denotes the support of the ith largest argument by all the other arguments, where $Sup\left(a_{index(i)}, a_{index(j)}\right)$ indicates the support of jth largest argument for the ith largest argument, $index$ is an indexing function such that $index(i)$ is the index of the ith largest of the arguments $a_j (j = 1, 2, \ldots, n)$, and $g: [0, 1] \to [0, 1]$ is a basic unit-interval monotonic (BUM) function, having the properties: (1) $g(0) = 0$; (2) $g(1) = 1$; and (3) $g(x) \geq g(y)$, if $x > y$.

Especially, if $g(x) = x$, then the POWA operator reduces to the PA operator.

3 Uncertain Power Average Operators

Let $\tilde{a}_j = [a_j^L, a_j^U]$ $(j = 1, 2, \ldots, n)$ be a collection of arguments, which take the form of interval numbers, where $0 \leq a_j^L \leq a_j^U,\quad j = 1, 2, \ldots, n$, a_j^L and a_j^U are the lower and upper limits of $\tilde{a}_j$, respectively.

Based on the operational laws of interval numbers (Xu and Zhai 1992) and the PA operator, we define the following uncertain power weighted average (UPWA) operator:

$$\mathrm{UPWA}(\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_n) = \frac{\sum_{i=1}^n w_i(1 + T(\tilde{a}_i))\tilde{a}_i}{\sum_{i=1}^n w_i(1 + T(\tilde{a}_i))} \tag{8}$$

where

$$T(\tilde{a}_i) = \sum_{\substack{j=1 \\ j \neq i}}^{n} w_j Sup(\tilde{a}_i, \tilde{a}_j) \tag{9}$$

with the conditions: $w_i \geq 0,\ i = 1, 2, \ldots, n$ and $\sum_{i=1}^n w_i = 1$. Moreover, $Sup(\tilde{a}_i, \tilde{a}_j)$ is the support for $\tilde{a}_i$ from $\tilde{a}_j$, which satisfies the following three properties:

(1) $Sup(\tilde{a}_i, \tilde{a}_j) \in [0, 1]$;
(2) $Sup(\tilde{a}_i, \tilde{a}_j) = Sup(\tilde{a}_j, \tilde{a}_i)$;
(3) $Sup(\tilde{a}_i, \tilde{a}_j) \geq Sup(\tilde{a}_s, \tilde{a}_t)$, if $d(\tilde{a}_i, \tilde{a}_j) < d(\tilde{a}_s, \tilde{a}_t)$, where d is a distance measure for interval numbers.

Clearly, the closer the two interval numbers $\tilde{a}_i$ and $\tilde{a}_j$, the more similar they are, and the more they support each other. Especially, if $w = (1/n, 1/n, \ldots, 1/n)^T$, then the UPWA operator (8) reduces to the uncertain power average (UPA) operator:

$$\mathrm{UPA}(\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_n) = \frac{\sum_{i=1}^n (1 + T'(\tilde{a}_i))\tilde{a}_i}{\sum_{i=1}^n (1 + T'(\tilde{a}_i))} \tag{10}$$

where

$$T'(\tilde{a}_i) = \frac{1}{n} \sum_{\substack{j=1 \\ j \neq i}}^{n} Sup(\tilde{a}_i, \tilde{a}_j) \tag{11}$$

Now let us look at some properties of the UPWA operator (Note that (12) below indicates that when all the supports are the same, the UPWA operator is simply the uncertain average operator).

Theorem 1 *Let* $Sup(\tilde{a}_i, \tilde{a}_j) = k$*, for all* $i \neq j$*, then*

$$\text{UPWA}(\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_n) == \frac{\sum_{i=1}^{n} w_i \left(1 + k \sum_{\substack{j=1 \\ j\neq i}}^{n} w_j\right)\tilde{a}_i}{\sum_{i=1}^{n} w_i \left(1 + k \sum_{\substack{j=1 \\ j\neq i}}^{n} w_j\right)} \tag{12}$$

Proof Since $Sup(\tilde{a}_i, \tilde{a}_j) = k$ for all $i \neq j$, it follows from (9) that

$$T(\tilde{a}_i) = \sum_{\substack{j=1 \\ j\neq i}}^{n} w_j Sup(\tilde{a}_i, \tilde{a}_j) = k \sum_{\substack{j=1 \\ j\neq i}}^{n} w_j, \quad i = 1, 2, \ldots, n \tag{13}$$

Thus by (8), we have

$$\begin{aligned} \text{UPWA}(\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_n) &= \frac{\sum_{i=1}^{n} w_i (1 + T(\tilde{a}_i))\tilde{a}_i}{\sum_{i=1}^{n} w_i (1 + T(\tilde{a}_i))} \\ &= \frac{\sum_{i=1}^{n} w_i \left(1 + k \sum_{\substack{j=1 \\ j\neq i}}^{n} w_j\right)\tilde{a}_i}{\sum_{i=1}^{n} w_i \left(1 + k \sum_{\substack{j=1 \\ j\neq i}}^{n} w_j\right)} \end{aligned} \tag{14}$$

and hence (12). This completes the proof of Theorem 1.

Theorem 2 (Commutativity). *Let* $(\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_n)$ *be a vector of* n *interval numbers, where* $\tilde{a}_j = [a_j^L, a_j^U]$ $(j = 1, 2, \ldots, n)$*, and* $(\tilde{a}'_1, \tilde{a}'_2, \ldots, \tilde{a}'_n)$ *be any permutation of* $(\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_n)$*, then*

$$\text{UPWA}(\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_n) = \text{UPWA}(\tilde{a}'_1, \tilde{a}'_2, \ldots, \tilde{a}'_n) \tag{15}$$

Proof Since

$$T(\tilde{a}_i) = \sum_{\substack{j=1 \\ j\neq i}}^{n} w_j Sup(\tilde{a}_i, \tilde{a}_j), T(\tilde{a}'_i) = \sum_{\substack{j=1 \\ j\neq i}}^{n} w_j Sup(\tilde{a}'_i, \tilde{a}'_j) \tag{16}$$

where $(\tilde{a}'_1, \tilde{a}'_2, \ldots, \tilde{a}'_n)$ is a permutation of $(\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_n)$, then

$$\frac{\sum_{i=1}^{n} w_i (1 + T(\tilde{a}_i))\tilde{a}_i}{\sum_{i=1}^{n} w_i (1 + T(\tilde{a}_i))} = \frac{\sum_{i=1}^{n} w_i (1 + T(\tilde{a}'_i))\tilde{a}'_i}{\sum_{i=1}^{n} w_i (1 + T(\tilde{a}'_i))} \tag{17}$$

and thus (15) holds, which completes the proof of Theorem 2.

Theorem 3 (Idempotency). *Let $\tilde{a}_j (j = 1, 2, \ldots, n)$ be a collection of interval numbers, where $\tilde{a}_j = [a_j^L, a_j^U]$ $(j = 1, 2, \ldots, n)$, if $\tilde{a}_j = \tilde{a}$, for all j, where $\tilde{a} = [a^L, a^U]$, then*

$$\mathrm{UPWA}(\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_n) = \tilde{a} \tag{18}$$

Proof Since

$$\begin{aligned}\mathrm{UPWA}(\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_n) &= \frac{\sum_{i=1}^n w_i(1 + T(\tilde{a}_i))\tilde{a}_i}{\sum_{i=1}^n w_i(1 + T(\tilde{a}_i))} = \frac{\sum_{i=1}^n w_i(1 + T(\tilde{a}_i))\tilde{a}}{\sum_{i=1}^n w_i(1 + T(\tilde{a}_i))} \\ &= \tilde{a}\frac{\sum_{i=1}^n w_i(1 + T(\tilde{a}_i))}{\sum_{i=1}^n w_i(1 + T(\tilde{a}_i))} = \tilde{a}\end{aligned} \tag{19}$$

the proof is completed.

Theorem 4 (Boundedness). *Let $\tilde{a}_j (j = 1, 2, \ldots, n)$ be any collection of interval numbers, where $\tilde{a}_j = [a_j^L, a_j^U]$ $(j = 1, 2, \ldots, n)$, then*

$$\min_i\{\tilde{a}_i\} \leq \mathrm{UPWA}(\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_n) \leq \max_i\{\tilde{a}_i\} \tag{20}$$

Proof In Xu and Da (2002), we introduced a possibility degree formula for the comparison between any two interval numbers $\tilde{a}_i = [a_i^L, a_i^U]$ and $\tilde{a}_j = [a_j^L, a_j^U]$:

$$p(\tilde{a}_i \geq \tilde{a}_j) = \max\left\{1 - \max\left(\frac{a_j^U - a_i^L}{a_j^U - a_j^L + a_i^U - a_i^L},\ 0\right),\ 0\right\} \tag{21}$$

To rank the interval numbers $\tilde{a}_j (j = 1, 2, \ldots, n)$, we compare each pair interval numbers $(\tilde{a}_i, \tilde{a}_j)$, and construct a possibility degree matrix $P = (p_{ij})_{n\times n}$, where $p_{ij} = p(\tilde{a}_i \geq \tilde{a}_j)$, $i, j = 1, 2, \ldots, n$, which satisfy $p_{ij} \geq 0$, $p_{ij} + p_{ji} = 1$, $p_{ii} = 0.5$, $i, j = 1, 2, \ldots, n$. Summing all the elements in each line of the matrix P, we get $p_i = \sum_{j=1}^n p_{ij}$, $i = 1, 2, \ldots, n$, and then we can rank the interval numbers $\tilde{a}_i$ $(i = 1, 2, \ldots, n)$ in descending order in accordance with p_i $(i = 1, 2, \ldots, n)$.

Let $\tilde{\alpha} = \min_i\{\tilde{a}_i\}$ and $\tilde{\beta} = \max_i\{\tilde{a}_i\}$, then

$$\begin{aligned}\mathrm{UPWA}(\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_n) &= \frac{\sum_{i=1}^n w_i(1 + T(\tilde{a}_i))\tilde{a}_i}{\sum_{i=1}^n w_i(1 + T(\tilde{a}_i))} \geq \frac{\sum_{i=1}^n w_i(1 + T(\tilde{a}_i))\tilde{\alpha}}{\sum_{i=1}^n w_i(1 + T(\tilde{a}_i))} \\ &= \tilde{\alpha}\frac{\sum_{i=1}^n w_i(1 + T(\tilde{a}_i))}{\sum_{i=1}^n w_i(1 + T(\tilde{a}_i))} = \tilde{\alpha}\end{aligned} \tag{22}$$

and

$$\begin{aligned}\mathrm{UPWA}(\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_n) &= \frac{\sum_{i=1}^{n} w_i(1+T(\tilde{a}_i))\tilde{a}_i}{\sum_{i=1}^{n} w_i(1+T(\tilde{a}_i))} \leq \frac{\sum_{i=1}^{n} w_i(1+T(\tilde{a}_i))\tilde{\beta}}{\sum_{i=1}^{n} w_i(1+T(\tilde{a}_i))} \\ &= \tilde{\beta}\frac{\sum_{i=1}^{n} w_i(1+T(\tilde{a}_i))}{\sum_{i=1}^{n} w_i(1+T(\tilde{a}_i))} = \tilde{\beta} \end{aligned} \tag{23}$$

thus (20) holds, which completes the proof of Theorem 4.

In what follows, we apply the UPWA operator to group decision making:

Consider a group decision making problem under uncertainty. Let $X = \{x_1, x_2, \ldots, x_n\}$ be a finite set of alternatives, $E = \{e_1, e_2, \ldots, e_m\}$ be a set of decision makers, and $w = (\lambda_1, \lambda_2, \ldots, \lambda_m)^T$ be the weight vector of the decision makers $e_k (k = 1, 2, \ldots, m)$, where $\lambda_k \geq 0, k = 1, 2, \ldots, m$ and $\sum_{k=1}^{m} \lambda_k = 1$. Suppose that the decision maker e_k compares each pair of alternatives (x_i, x_j), and provides his/her preference value range $\tilde{b}_{ij}^{(k)} = [b_{ij}^{L(k)}, b_{ij}^{U(k)}]$, and constructs an interval fuzzy preference relation $\tilde{B}_k = (\tilde{b}_{ij}^{(k)})_{n\times n}$, where

$$\begin{aligned}&b_{ij}^{U(k)} \geq b_{ij}^{L(k)} \geq 0,\ \ b_{ij}^{L(k)} + b_{ji}^{U(k)} = 1,\ \ b_{ji}^{L(k)} + b_{ij}^{U(k)} = 1,\ \ b_{ii}^{L(k)} = b_{ii}^{U(k)} = 0.5, \\ &\quad \text{for all} \quad i, j = 1, 2, \ldots, n \end{aligned} \tag{24}$$

and $\tilde{b}_{ij}^{(k)}$ indicates preference value range of the alternative x_i over x_j provided by the decision maker e_k.

Then based on the UPWA operator, we propose a method for group decision making with interval fuzzy preference relations, which involves the following steps:

3.1 Method I

Step 1. Calculate the supports:

$$Sup\left(\tilde{b}_{ij}^{(k)}, \tilde{b}_{ij}^{(l)}\right) = 1 - d\left(\tilde{b}_{ij}^{(k)}, \tilde{b}_{ij}^{(l)}\right), \quad l = 1, 2, \ldots, m \tag{25}$$

which satisfy the support conditions (1)–(3) in Sect. 3. Without loss of generality, here we let

$$d\left(\tilde{b}_{ij}^{(k)}, \tilde{b}_{ij}^{(l)}\right) = \sqrt{\frac{1}{2}\left(\left(b_{ij}^{L(l)} - b_{ij}^{L(k)}\right)^2 + \left(b_{ij}^{U(l)} - b_{ij}^{U(k)}\right)^2\right)} \tag{26}$$

Step 2. Utilize the weights $\lambda_k (k = 1, 2, \ldots, m)$ of the decision makers $e_k (k = 1, 2, \ldots, m)$ to calculate the weighted support $T(\tilde{b}_{ij}^{(k)})$ of the preference value range $\tilde{b}_{ij}^{(k)}$ by the other preference value ranges $\tilde{b}_{ij}^{(l)} (l = 1, 2, \ldots, m$ and $l \neq k)$:

$$T'(\tilde{b}_{ij}^{(k)}) = \sum_{\substack{l=1 \\ l \neq k}}^{m} \lambda_l Sup\left(\tilde{b}_{ij}^{(k)}, \tilde{b}_{ij}^{(l)}\right) \tag{27}$$

and calculate the weights $v_{ij}^{(k)}\,(k = 1, 2, \ldots, m)$ associated with the preference value ranges $\tilde{b}_{ij}^{(k)}\,(k = 1, 2, \ldots, m)$:

$$v_{ij}^{(k)} = \frac{\lambda_k \left(1 + T\left(\tilde{b}_{ij}^{(k)}\right)\right)}{\sum_{k=1}^{m} \lambda_k \left(1 + T\left(\tilde{b}_{ij}^{(k)}\right)\right)}, \quad k = 1, 2, \ldots, m \tag{28}$$

where $v_{ij}^{(k)} \geq 0, k = 1, 2, \ldots, m$, and $\sum_{k=1}^{m} v_{ij}^{(k)} = 1$.

Step 3. Utilize the UPWA operator (8) to aggregate all the individual interval fuzzy preference relations $\tilde{B}_k = (\tilde{b}_{ij}^{(k)})_{n\times n}\,(k = 1, 2, \ldots, m)$ into the collective preference relation $\tilde{B} = (\tilde{b}_{ij})_{n\times n}$, where

$$\begin{aligned} \tilde{b}_{ij} &= [b_i^L, b_i^U] = \text{UPWA}(\tilde{b}_{ij}^{(1)}, \tilde{b}_{ij}^{(2)}, \ldots, \tilde{b}_{ij}^{(m)}) \\ &= \sum_{k=1}^{m} v_{ij}^{(k)} \tilde{b}_{ij}^{(k)} = \left[\sum_{k=1}^{m} v_{ij}^{(k)} b_{ij}^{L(k)}, \sum_{k=1}^{m} v_{ij}^{(k)} b_{ij}^{U(k)}\right], \quad i, j = 1, 2, \ldots, n \end{aligned} \tag{29}$$

Step 4. Aggregate all the preference value ranges $\tilde{b}_{ij}\,(j = 1, 2, \ldots, n)$ in the ith line of $\tilde{B}$ by using the uncertain average operator:

$$\tilde{b}_i = [b_i^L, b_i^U] = \frac{1}{n}\sum_{j=1}^{n} \tilde{b}_{ij} = \left[\frac{1}{n}\sum_{j=1}^{n} b_{ij}^L,\ \frac{1}{n}\sum_{j=1}^{n} b_{ij}^U\right], \quad i = 1, 2, \ldots, n \tag{30}$$

and get the overall preference value range $\tilde{b}_i$ corresponding to the alternative x_i.

Step 5. Rank all the interval numbers $\tilde{b}_i$ $(i = 1, 2, \ldots, n)$ by using the possibility degree formula (21), and then rank all the alternatives $x_i (i = 1, 2, \ldots, n)$ in accordance with $\tilde{b}_i$ $(i = 1, 2, \ldots, n)$, by which the best alternative can be selected.

By Step 3 in the above method, we have

Theorem 5 *The collective preference relation $\tilde{B} = (\tilde{b}_{ij})_{n\times n}$ aggregated from the individual interval fuzzy preference relations $\tilde{B}_k = (\tilde{b}_{ij}^{(k)})_{n\times n}\,(k = 1, 2, \ldots, m)$ by using the UPWA operator (29) is also an interval fuzzy preference relation.*

Proof Since all $\tilde{B}_k = (\tilde{b}_{ij}^{(k)})_{n\times n}$ $(k = 1, 2, \ldots, m)$ are interval fuzzy preference relations, by (24) and (26), we have

$$
\begin{aligned}
d\left(\tilde{b}_{ij}^{(k)}, \tilde{b}_{ij}^{(l)}\right) &= \sqrt{\frac{1}{2}\left(\left(b_{ij}^{L(l)} - b_{ij}^{L(k)}\right)^2 + \left(b_{ij}^{U(l)} - b_{ij}^{U(k)}\right)^2\right)} \\
&= \sqrt{\frac{1}{2}\left(\left(\left(1 - b_{ij}^{L(l)}\right) - \left(1 - b_{ij}^{L(k)}\right)\right)^2 + \left(\left(1 - b_{ij}^{U(l)}\right) - \left(1 - b_{ij}^{U(k)}\right)\right)^2\right)} \\
&= \sqrt{\frac{1}{2}\left(\left(b_{ji}^{U(l)} - b_{ji}^{U(k)}\right)^2 + \left(b_{ji}^{L(l)} - b_{ji}^{L(k)}\right)^2\right)} \\
&= d\left(\tilde{b}_{ji}^{(k)}, \tilde{b}_{ji}^{(l)}\right)
\end{aligned}
\tag{31}
$$

and then

$$
Sup\left(\tilde{b}_{ij}^{(k)}, \tilde{b}_{ij}^{(l)}\right) = 1 - d\left(\tilde{b}_{ij}^{(k)}, \tilde{b}_{ij}^{(l)}\right) = 1 - d\left(\tilde{b}_{ji}^{(k)}, \tilde{b}_{ji}^{(l)}\right) = Sup\left(\tilde{b}_{ji}^{(k)}, \tilde{b}_{ji}^{(l)}\right)
\tag{32}
$$

Thus, from (27), it follows that

$$
T'\left(\tilde{b}_{ij}^{(k)}\right) = \sum_{\substack{l=1 \\ l\neq k}}^{m} \lambda_l Sup\left(\tilde{b}_{ij}^{(k)}, \tilde{b}_{ij}^{(l)}\right) = \sum_{\substack{l=1 \\ l\neq k}}^{m} \lambda_l Sup\left(\tilde{b}_{ji}^{(k)}, \tilde{b}_{ji}^{(l)}\right) = T'\left(\tilde{b}_{ji}^{(k)}\right)
\tag{33}
$$

and then by (28), we have

$$
v_{ij}^{(k)} = \frac{\lambda_k\left(1 + T\left(\tilde{b}_{ij}^{(k)}\right)\right)}{\sum_{k=1}^{m} \lambda_k\left(1 + T\left(\tilde{b}_{ij}^{(k)}\right)\right)} = \frac{\lambda_k\left(1 + T\left(\tilde{b}_{ji}^{(k)}\right)\right)}{\sum_{k=1}^{m} \lambda_k\left(1 + T\left(\tilde{b}_{ji}^{(k)}\right)\right)} = v_{ji}^{(k)}, \quad k = 1, 2, \ldots, m
\tag{34}
$$

where $v_{ij}^{(k)} \geq 0, k = 1, 2, \ldots, m$, and $\sum_{k=1}^{m} v_{ij}^{(k)} = 1$. Therefore, by (24) and (34), we have

$$
b_{ij}^{U} = \sum_{k=1}^{m} v_{ij}^{(k)} b_{ij}^{U(k)} \geq \sum_{k=1}^{m} v_{ij}^{(k)} b_{ij}^{L(k)} = b_{ij}^{L} \geq 0
\tag{35}
$$

$$
\begin{aligned}
b_{ij}^{L} + b_{ji}^{U} &= \sum_{k=1}^{m} v_{ij}^{(k)} b_{ij}^{L(k)} + \sum_{k=1}^{m} v_{ji}^{(k)} b_{ji}^{U(k)} = \sum_{k=1}^{m} v_{ij}^{(k)} b_{ij}^{L(k)} + \sum_{k=1}^{m} v_{ij}^{(k)} b_{ji}^{U(k)} \\
&= \sum_{k=1}^{m} v_{ij}^{(k)}\left(b_{ij}^{L(k)} + b_{ji}^{U(k)}\right) = \sum_{k=1}^{m} v_{ij}^{(k)} = 1, \quad i, j = 1, 2, \ldots, n
\end{aligned}
\tag{36}
$$

$$
\begin{aligned}
b_{ij}^{U}+b_{ji}^{L} &= \sum_{k=1}^{m} v_{ij}^{(k)} b_{ij}^{U(k)} + \sum_{k=1}^{m} v_{ji}^{(k)} b_{ji}^{L(k)} = \sum_{k=1}^{m} v_{ij}^{(k)} b_{ij}^{U(k)} + \sum_{k=1}^{m} v_{ij}^{(k)} b_{ji}^{L(k)} \\
&= \sum_{k=1}^{m} v_{ij}^{(k)} (b_{ij}^{U(k)} + b_{ji}^{L(k)}) = \sum_{k=1}^{m} v_{ij}^{(k)} = 1, \quad i,j=1,2,\ldots,n \qquad (37)
\end{aligned}
$$

$$
b_{ii}^{L(k)} = \sum_{k=1}^{m} v_{ii}^{(k)} b_{ii}^{L(k)} = \sum_{k=1}^{m} v_{ii}^{(k)} b_{ii}^{U(k)} = 0.5 \sum_{k=1}^{m} v_{ii}^{(k)} = 0.5, \quad i=1,2,\ldots,n \tag{38}
$$

which indicates that $\tilde{B} = (\tilde{b}_{ij})_{n\times n}$ is an interval fuzzy preference relation. This completes the proof of Theorem 5.

In the above method, we have utilized the UPWA operator (whose prominent characteristic is that it allows values being aggregated to support and reinforce each other) to aggregate all individual interval fuzzy preference relations into the collective interval fuzzy preference relation, which can not only consider the importance of each decision maker, but also relieve the influence of the unfair arguments on the decision results by assigning lower weights to those unduly high or unduly low preference value ranges.

4 Uncertain Power Ordered Weighted Average Operators

In this section, we extend the POWA operator to uncertain environments. Let $\tilde{a}_j = [a_j^L, a_j^U]$ $(j=1,2,\ldots,n)$ be a collection of interval numbers. Then, based on the possibility degree formula (21), we define an uncertain power ordered weighted average (UPOWA) operator:

$$
\text{UPOWA}(\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_n) = \sum_{i=1}^{n} u_i \tilde{a}_{index(i)} \tag{39}
$$

where $\tilde{a}_{index(i)}$ is the ith largest interval numbers of $\tilde{a}_j (j=1,2,\ldots,n)$,

$$
\begin{aligned}
u_i &= g\left(\frac{R_i}{TV}\right) - g\left(\frac{R_{i-1}}{TV}\right),\ R_i = \sum_{j=1}^{i} V_{index(j)}, \\
TV &= \sum_{i=1}^{n} V_{index(i)}, \quad V_{index(j)} = 1 + T(\tilde{a}_{index(i)}) \qquad (40)
\end{aligned}
$$

and

$$
T(\tilde{a}_{index(i)}) = \frac{1}{n} \sum_{\substack{j=1 \\ j\neq i}}^{n} Sup(\tilde{a}_{index(i)}, \tilde{a}_{index(j)}) \tag{41}
$$

which denotes the support of the ith largest interval number by all the other interval numbers, i.e., $Sup(\tilde{a}_{index(i)}, \tilde{a}_{index(j)})$ indicates the support of the jth largest interval number for the ith largest interval number.

Especially, if $g(x) = x$, then by (39) and (40), we have

$$\begin{aligned}\text{UPOWA}(\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_n) &= \sum_{i=1}^{n} u_i \tilde{a}_{index(i)} = \sum_{i=1}^{n}\left(g\left(\frac{R_i}{TV}\right) - g\left(\frac{R_{i-1}}{TV}\right)\right)\tilde{a}_{index(i)} \\ &= \sum_{i=1}^{n}\left(\frac{R_i}{TV} - \frac{R_{i-1}}{TV}\right)\tilde{a}_{index(i)} = \frac{\sum\limits_{i=1}^{n} V_{index(i)}\tilde{a}_{index(i)}}{TV} \\ &= \frac{\sum_{i=1}^{n}(1 + T(\tilde{a}_i))\tilde{a}_i}{\sum_{i=1}^{n}(1 + T(\tilde{a}_i))} = \text{UPA}(\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_n) \end{aligned} \tag{42}$$

and thus the UPOWA operator reduces to the UPA operator.

We establish the properties of the UPOWA operator in Theorems 6–9 below:

Theorem 6 *Let $Sup(\tilde{a}_{index(i)}, \tilde{a}_{index(j)}) = k$, for all $i \neq j$, and $g(x) = x$. Then*

$$\text{UPOWA}(\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_n) = \frac{1}{n}\sum_{i=1}^{n}\tilde{a}_i \tag{43}$$

which indicates that when all the supports are the same, and the UPOWA operator is simply the uncertain average operator.

Proof Since $Sup(\tilde{a}_{index(i)}, \tilde{a}_{index(j)}) = k$, for all $i \neq j$, then by (41), we have $T(\tilde{a}_{index(i)}) = T(\tilde{a}_i) = k(n-1), i = 1, 2, \ldots, n$. Thus, from (39), (42) and (10), it follows that

$$\begin{aligned}\text{UPOWA}(\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_n) &= \text{UPA}(\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_n) \\ &= \frac{\sum_{i=1}^{n}(1 + T'(\tilde{a}_i))\tilde{a}_i}{\sum_{i=1}^{n}(1 + T'(\tilde{a}_i))} = \frac{\sum_{i=1}^{n}(1 + k(n-1)/n)\tilde{a}_i}{\sum_{i=1}^{n}(1 + k(n-1)/n)} \\ &= \frac{(1 + k(n-1)/n)}{\sum_{i=1}^{n}(1 + k(n-1)/n)}\sum_{i=1}^{n}\tilde{a}_i = \frac{1}{n}\sum_{i=1}^{n}\tilde{a}_i \end{aligned} \tag{44}$$

which completes the proof of Theorem 6.

Especially, if $Sup(\tilde{a}_i, \tilde{a}_j) = 0$, for all $i \neq j$, i.e., all the supports are zero, then there is not any support in the aggregation process; in this case, the UPOWA operator also reduces to the uncertain average operator.

Similar to Theorems 2–4, we have

Theorem 7 (Commutativity). *Let $(\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_n)$ be a vector of n interval numbers, where $\tilde{a}_j = [a_j^L, a_j^U](j = 1, 2, \ldots, n)$, and $(\tilde{a}'_1, \tilde{a}'_2, \ldots, \tilde{a}'_n)$ be any permutation of $(\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_n)$, then*

$$\text{UPOWA}(\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_n) = \text{UPOWA}(\tilde{a}'_1, \tilde{a}'_2, \ldots, \tilde{a}'_n) \tag{45}$$

Theorem 8 (Idempotency). *Let $\tilde{a}_j = [a_j^L, a_j^U]$ $(j = 1, 2, \ldots, n)$ be a collection of interval numbers, if $\tilde{a}_j = \tilde{a}$, for all j, where $\tilde{a} = [a^L, a^U]$, then*

$$\text{UPOWA}(\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_n) = \tilde{a} \tag{46}$$

Theorem 9 (Boundedness). *Let $\tilde{a}_j = [a_j^L, a_j^U]$ $(j = 1, 2, \ldots, n)$ be a collection of interval numbers, then*

$$\min_i\{\tilde{a}_i\} \leq \text{UPOWA}(\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_n) \leq \max_i\{\tilde{a}_i\} \tag{47}$$

In Sect. 3, we have introduced a method for group decision making where the weights of the decision makers are predefined. If the information about the weights of decision makers is unknown, then we can utilize the UPOWA operator to give a method for group decision making based on interval fuzzy preference relations. This is described below:

4.1 Method II

Step 1. Calculate

$$Sup\left(\tilde{b}_{ij}^{index(k)}, \tilde{b}_{ij}^{index(l)}\right) = 1 - d\left(\tilde{b}_{ij}^{index(k)}, \tilde{b}_{ij}^{index(l)}\right) \tag{48}$$

which indicates the support of the lth largest uncertain preference value $\tilde{b}_{ij}^{index(l)}$ for the kth largest preference value range $\tilde{b}_{ij}^{index(l)}$ of $\tilde{b}_{ij}^{(s)}$ $(s = 1, 2, \ldots, m)$.

Step 2. Calculate the support $T\left(\tilde{b}_{ij}^{index(k)}\right)$ of the kth largest preference value ranges $\tilde{b}_{ij}^{index(k)}$ by the other preference value ranges $\tilde{b}_{ij}^{(l)}$ $(l = 1, 2, \ldots, m$ and $l \neq k)$:

$$T\left(\tilde{b}_{ij}^{index(k)}\right) = \frac{1}{n}\sum_{\substack{l=1\\l\neq k}}^{m} Sup\left(\tilde{b}_{ij}^{index(k)}, \tilde{b}_{ij}^{index(l)}\right) \tag{49}$$

and by (40), calculate the weight $u_{ij}^{(k)}$ associated with the kth largest preference value range $\tilde{a}_{ij}^{index(k)}$, where

$$u_{ij}^{(k)} = g\left(\frac{R_{ij}^{(k)}}{TV_{ij}}\right) - g\left(\frac{R_{ij}^{(k-1)}}{TV_{ij}}\right), \quad R_{ij}^{(k)} = \sum_{l=1}^{k} V_{ij}^{index(l)},$$
$$TV_{ij} = \sum_{l=1}^{m} V_{ij}^{index(l)}, \quad V_{ij}^{index(l)} = 1 + T\left(\tilde{b}_{ij}^{index(l)}\right) \tag{50}$$

$u_{ij}^{(k)} \geq 0, k = 1, 2, \ldots, m$, and $\sum_{k=1}^{m} u_{ij}^{(k)} = 1$, and g is the BUM function described in Sect. 2.

Step 3. Utilize the UPOWA operator (39) to aggregate all the individual interval fuzzy preference relations $\tilde{B}_k = (\tilde{b}_{ij}^{(k)})_{n\times n}(k = 1, 2, \ldots, m)$ into the collective interval fuzzy preference relation $\tilde{B} = (\tilde{b}_{ij})_{n\times n}$, where

$$\tilde{b}_{ij} = [b_{ij}^L, b_{ij}^U] = \text{UPOWA}(\tilde{b}_{ij}^{(1)}, \tilde{b}_{ij}^{(2)}, \ldots, \tilde{b}_{ij}^{(m)}) = \sum_{k=1}^{m} u_{ij}^{(k)} \tilde{b}_{ij}^{index(k)},$$
$$\tilde{b}_{ji} = [b_{ji}^L, \quad b_{ji}^U], \quad b_{ji}^L = 1 - b_{ij}^U, b_{ji}^U = 1 - b_{ij}^L \quad \text{for all } i < j \tag{51}$$

Step 4. See Sect. 3.1.
Step 5. See Sect. 3.1.

By Step 3 of the method above, we have

Theorem 10 *The collective preference relation $\tilde{B} = (\tilde{b}_{ij})_{n\times n}$ aggregated from the individual interval fuzzy preference relations $\tilde{B}_k = (\tilde{b}_{ij}^{(k)})_{n\times n}$ $(k = 1, 2, \ldots, m)$ by using the UPOWA operator* (51) *is also an interval fuzzy preference relation.*

Method II utilizes the UPOWA operator to aggregate all the individual interval fuzzy preference relations into the collective interval fuzzy preference relation, and then employs the uncertain average operator to rank and select the given alternatives. Method II has similar desirable properties similar to those for Method I, and is developed to deal with the situations where the information about the weights of decision makers is unknown. It is a useful complement to Method I.

5 Illustrative Example

Let us consider a group decision making problem that involves the evaluation of four schools $x_j (j = 1, 2, 3, 4)$ of a university (adapted from Xu (2004)). One main criterion used is research. Three decision makers $e_k (k = 1, 2, 3)$ (whose weight vector is $\lambda = (0.5, 0.3, 0.2)^T$) are asked to provide their preferences over the four schools $x_j (j = 1, 2, 3, 4)$ with respect to the criterion research. All the decision makers compare the four alternatives and construct interval fuzzy preference relations, which are listed as follows:

$$\tilde{B}_1 = \begin{pmatrix} [0.5, 0.5] & [0.5, 0.6] & [0.4, 0.7] & [0.4, 0.5] \\ [0.4, 0.5] & [0.5, 0.5] & [0.5, 0.6] & [0.3, 0.4] \\ [0.3, 0.6] & [0.4, 0.5] & [0.5, 0.5] & [0.6, 0.7] \\ [0.5, 0.6] & [0.6, 0.7] & [0.3, 0.4] & [0.5, 0.5] \end{pmatrix}$$

$$\tilde{B}_2 = \begin{pmatrix} [0.5, 0.5] & [0.4, 0.5] & [0.5, 0.8] & [0.3, 0.5] \\ [0.5, 0.6] & [0.5, 0.5] & [0.4, 0.6] & [0.4, 0.5] \\ [0.2, 0.5] & [0.4, 0.6] & [0.5, 0.5] & [0.6, 0.8] \\ [0.5, 0.7] & [0.5, 0.6] & [0.2, 0.4] & [0.5, 0.5] \end{pmatrix}$$

$$\tilde{B}_3 = \begin{pmatrix} [0.5, 0.5] & [0.5, 0.7] & [0.4, 0.7] & [0.3, 0.5] \\ [0.3, 0.5] & [0.5, 0.5] & [0.4, 0.6] & [0.4, 0.5] \\ [0.3, 0.6] & [0.4, 0.6] & [0.5, 0.5] & [0.5, 0.7] \\ [0.5, 0.7] & [0.5, 0.6] & [0.3, 0.5] & [0.5, 0.5] \end{pmatrix}$$

Since the weights of the decision makers are predefined, we utilize Method I to rank and select the given alternatives:

We first utilize (25)–(28) to calculate the weights $v_{ij}^{(k)}(k = 1, 2, 3)$ associated with the preference value ranges $\tilde{b}_{ij}^{(k)}(k = 1, 2, 3)$, which are contained in the matrices $V_k = (v_{ij}^{(k)})_{4\times 4} \quad (k = 1, 2, 3)$ respectively:

$$V_1 = \begin{pmatrix} 0.4630 & 0.4676 & 0.4658 & 0.4621 \\ 0.4676 & 0.4630 & 0.4621 & 0.4618 \\ 0.4658 & 0.4621 & 0.4630 & 0.4657 \\ 0.4621 & 0.4618 & 0.4657 & 0.4630 \end{pmatrix}$$

$$V_2 = \begin{pmatrix} 0.3148 & 0.3118 & 0.3099 & 0.3151 \\ 0.3118 & 0.3148 & 0.3151 & 0.3153 \\ 0.3099 & 0.3160 & 0.3148 & 0.3137 \\ 0.3151 & 0.3153 & 0.3137 & 0.3148 \end{pmatrix}$$

$$V_3 = \begin{pmatrix} 0.2220 & 0.2206 & 0.2243 & 0.2227 \\ 0.2206 & 0.2220 & 0.2227 & 0.2229 \\ 0.2243 & 0.2227 & 0.2220 & 0.2206 \\ 0.2227 & 0.2229 & 0.2206 & 0.2220 \end{pmatrix}$$

Then we utilize the UPWA operator (29) to aggregate all the individual interval fuzzy preference relations $\tilde{B}_k = (\tilde{b}_{ij}^{(k)})_{4\times 4}\,(k = 1, 2, 3)$ into the collective interval fuzzy preference relation:

$$\tilde{B} = \begin{pmatrix} [0.5, 0.5] & [0.4688, 0.5909] & [0.4310, 0.7310] & [0.3462, 0.5000] \\ [0.4091, 0.5312] & [0.5, 0.5] & [0.4462, 0.5999] & [0.3538, 0.4538] \\ [0.2690, 0.5690] & [0.4001, 0.5538] & [0.5, 0.5] & [0.5779, 0.7314] \\ [0.5000, 0.6538] & [0.5462, 0.6462] & [0.2686, 0.4221] & [0.5, 0.5] \end{pmatrix}$$

By using (30), we aggregate all the preference value ranges $\tilde{b}_{ij}(j = 1, 2, 3, 4)$ in the ith line of $\tilde{B}$, and get the overall preference value range $\tilde{b}_i$ corresponding to the alternative x_i:

$$\tilde{b}_1 = [0.4365, 0.5805], \quad \tilde{b}_2 = [0.4273, 0.5212], \quad \tilde{b}_3 = [0.4367, 0.5885],$$
$$\tilde{b}_4 = [0.4537, 0.5555]$$

In order to rank $\tilde{b}_i(i = 1, 2, 3, 4)$, by (21), we construct the possibility degree matrix:

$$P = \begin{pmatrix} 0.5 & 0.6440 & 0.4861 & 0.5159 \\ 0.3560 & 0.5 & 0.3439 & 0.3449 \\ 0.5139 & 0.6561 & 0.5 & 0.5315 \\ 0.4841 & 0.6551 & 0.4685 & 0.5 \end{pmatrix}$$

Summing all the elements in each line of the matrix P, we get

$$p_1 = 2.1460, \quad p_2 = 1.5448, \quad p_3 = 2.2015, \quad p_4 = 2.1077$$

and then we can rank the interval numbers $\tilde{b}_i (i = 1, 2, 3, 4)$ in descending order in accordance with p_i $(i = 1, 2, 3, 4) : \tilde{b}_3 > \tilde{b}_1 > \tilde{b}_4 > \tilde{b}_2$ Therefore, we rank the schools: $x_3 \succ x_1 \succ x_4 \succ x_2$, and thus x_3 is the best school.

If we utilize the traditional uncertain weighted averaging (UWA) operator (Xu 2001):

$$\begin{aligned} \dot{\tilde{b}}_{ij} &= [\dot{b}_i^L, \dot{b}_i^U] = UWA(\tilde{b}_{ij}^{(1)}, \tilde{b}_{ij}^{(2)}, \tilde{b}_{ij}^{(3)}, \tilde{b}_{ij}^{(4)}) = \sum_{k=1}^{4} \lambda_k \tilde{b}_{ij}^{(k)} \\ &= \left[\sum_{k=1}^{4} \lambda_k b_{ij}^{L(k)}, \sum_{k=1}^{4} \lambda_k b_{ij}^{U(k)} \right], \quad i, j = 1, 2, 3, 4 \end{aligned}$$

to aggregate all the individual interval fuzzy preference relations $\tilde{B}_k = (\tilde{b}_{ij}^{(k)})_{4\times 4} (k = 1, 2, 3)$ into the collective interval fuzzy preference relation $\dot{\tilde{B}} = (\dot{\tilde{b}}_{ij})_{4\times 4}$, then we have

$$\dot{\tilde{B}} = \begin{pmatrix} [0.5, 0.5] & [0.43, 0.73] & [0.43, 0.73] & [0.35, 0.50] \\ [0.41, 0.53] & [0.5, 0.5] & [0.45, 0.60] & [0.35, 0.45] \\ [0.27, 0.57] & [0.40, 0.55] & [0.5, 0.5] & [0.58, 0.73] \\ [0.50, 0.65] & [0.55, 0.65] & [0.27, 0.42] & [0.5, 0.5] \end{pmatrix}$$

By using (30), we aggregate all the preference value ranges $\dot{\tilde{b}}_{ij} (j = 1, 2, 3, 4)$ in the ith line of $\dot{\tilde{B}}$, and get the overall preference value range $\dot{\tilde{b}}_i$ corresponding to the alternative x_i:

$$\dot{\tilde{b}}_1 = [0.4375, 0.5800], \quad \dot{\tilde{b}}_2 = [0.4275, 0.5200], \quad \dot{\tilde{b}}_3 = [0.4375, 0.5875],$$
$$\dot{\tilde{b}}_4 = [0.4550, 0.5550]$$

and then by (21), we construct the possibility degree matrix:

$$\dot{P} = \begin{pmatrix} 0.5 & 0.6489 & 0.4872 & 0.5155 \\ 0.3511 & 0.5 & 0.3402 & 0.3377 \\ 0.5128 & 0.6598 & 0.5 & 0.5300 \\ 0.4845 & 0.6623 & 0.4700 & 0.5 \end{pmatrix}$$

Summing all the elements in each line of the matrix $\dot{P}$, we get

$$\dot{p}_1 = 2.1516, \quad \dot{p}_2 = 1.5290, \quad \dot{p}_3 = 2.2026, \quad \dot{p}_4 = 2.1168$$

and thus $\dot{\tilde{b}}_3 > \dot{\tilde{b}}_1 > \dot{\tilde{b}}_4 > \dot{\tilde{b}}_2$, by which we rank the schools: $x_3 \succ x_1 \succ x_4 \succ x_2$, and hence x_3 is also the best school.

In the above numerical results, both the methods have derived the same ranking of the alternatives. Nevertheless, the latter can only consider the weights of the decision makers, but cannot take into account the information about the relationship between the values being fused. Our Method I can, nevertheless, overcome this drawback: it can not only reflect the importance of the decision makers, but also relieve the influence of unduly high or unduly low preference value ranges on the decision results by assigning lower weights to those unfair arguments, and thus can make the decision result more reasonable and reliable.

6 Conclusions

We have extended Yager's power average (PA) operator to uncertain environments and defined an uncertain power weighted average (UPWA) operator and an uncertain power ordered weighted average (UPOWA) operator. We have also established some of their desirable properties. Moreover, we have applied the developed operators to group decision making in different situations. Based on the UPWA operator, we have given a method for group decision making with interval fuzzy preference relations for situations where the weights of decision makers can be predefined. Based on the UPOWA operator, we have given a method for group decision making with interval fuzzy preference relations for situations where the information about the weights of decision makers is unknown. An example is discussed, which shows the effectiveness of our approach.

The proposed operators can be applied to many other fields, such as data mining, information fusion, and pattern recognition, etc. These are interesting topics for further research.

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