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The Core of Consistency in AHP-Group Decision Making

J. M. Moreno-Jiménez · J. Aguarón · M. T. Escobar

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Abstract This paper presents a new tool, the Consistency Consensus Matrix, designed to encourage the search for consensus in group decision making when using the Analytic Hierarchy Process (AHP). The procedure exploits one of the characteristics of AHP: the possibility of measuring consistency in judgement elicitation. Using two other tools, Preference Structures and Stability Intervals, we derive the Consistency Consensus Matrix that corresponds to the actor's core of consistency. The performance analysis of the preference structure obtained from this matrix provides us with valuable information in search for knowledge. The new tool is illustrated by means of a case study adapted from a real-life experiment in e-democracy developed for the City Council of Zaragoza (Spain).

Keywords Group decision making · AHP · Consistency Consensus Matrix · Knowledge extraction · E-Democracy

1 Introduction

In order to facilitate the participation of individuals in the e-democracy context, it is necessary to develop new approaches that allow us to capture the perception of reality of all actors involved in the resolution of a problem, and to make the participation of multiple interconnected and spatially distributed actors possible.

To respond to this new perspective in the resolution of highly complex problems, it is necessary to develop open, flexible and adaptive methodologies which, resting on decision systems, provide assistance in the resolution process. With this in mind, we use the Analytic Hierarchy Process (AHP) in a group decision making context and from a cognitive perspective. This is directed towards the extraction of knowledge to support negotiation processes between the actors involved in the resolution of the problem.

J. M. Moreno-Jiménez (✉) · J. Aguarón · M. T. Escobar
Facultad de Económicas, Gran Vía 2, 50005 Zaragoza, Spain
e-mail: moreno@unizar.es

This paper presents a new decision tool, the Consistency Consensus Matrix (CCM), which allows us to identify the core of consistency in AHP group decision making. We assume that there is a finite set of actors that approach the resolution of a group decision making problem with AHP using a common hierarchy. We employ the row geometric mean as the priorisation procedure and the geometric consistency index as the measure of inconsistency. Accordingly, we establish a procedure that allows us to determine an interval judgement matrix (CCM) where each of the entries is the range for the given judgement where all the individuals simultaneously have an acceptable inconsistency with respect to their initial matrices. Some of these entries might be missing.

On the basis of this matrix, we calculate the different preference structures (Moreno-Jiménez and Vargas 1993) associated with the core of consistency of the alternatives being compared, as well as their evolution as function of a fixed inconsistency threshold. This facilitates knowledge extraction derived in two different situations. First, the above information can be considered as the solution to an automatic negotiation process between actors involved in the resolution of the process. Secondly, the above-mentioned information will be used as the starting point of a participatory negotiating process in which actors identify areas of agreement and disagreement in eliciting judgements and establish consensus paths in an interactive manner.

The consensus building procedure we propose does not require any information from the actors beyond that provided in the judgment elicitation phase. However, it can be enriched with the inclusion of additional information, as it is the case of some e-democracy systems such as e-cognocracy (Moreno-Jiménez and Polasek 2003). Together with the consensus building perspective that characterises group decision making situations, there is another important question in the resolution process which is communication between actors. In this regard, we can refer to an Automatic Consensus Building Process, where consensus is reached without direct intervention of actors, and to a Personal Consensus Building Process, in which consensus is reached through participatory dialogue.

In most cases, actors contribute to the resolution process in a personal and active way, communicating their opinions and trying to find a solution as a group. They incorporate the learning and understanding of the problem derived from its resolution into the system. However, there are situations in which personal communication between the actors is not possible. In cases of distributed and asynchronous decision making, we need tools that favour consensus building in a group context. The method we present can be employed as an initial step in any personal consensus building procedure. As an example, this tool is currently employed in a participatory budget project developed for the Zaragoza City Council (see http://www.cmis-app.zaragoza.es/ciudad/presupuestos-participativos/).

The rest of the paper is organised as follows. Section Consensus building in AHP-Group decision making outlines background for the CCM and includes some decisional tools (preference structures, stability intervals and consensus matrices) employed to extract knowledge from the decision making process. Section The Consistency Consensus Matrix (CCM) presents the Consistency Consensus Matrix, the new procedure suggested to deal with Consensus Building in AHP-Group Decision Making. Section Example applies this procedure to a case study adapted from a real experiment in e-democracy developed for the City Council of Zaragoza (Spain). Final Section provides conclusions.

2 Consensus building in AHP-Group decision making

2.1 The analytic hierarchy process (AHP)

AHP (Saaty 1980, 1994) is a multicriteria decision making technique, whose flexibility facilitates solving complex problems in an effective and realistic way. Leaving aside potential limitations of this theory (Barzilai 2001), we will use this approach under a cognitive perspective oriented towards extracting knowledge corresponding to the critical points of the resolution process, and its use in consensus building, although, the underlying ideas of our proposal could be extended to other multicriteria approaches.

AHP combines tangible and intangible aspects to obtain, in a ratio scale, the priorities associated with the alternatives of the problem. The AHP's methodology consists of three steps: modelling, valuation, and priorization and synthesis. Among the different methods used to obtain the local priorities, the eigenvector method proposed by Saaty (1980) and the row geometric mean method (Crawford and Williams 1985) stand out.

One of the advantages of AHP is that it allows us to measure the consistency of the decision maker when eliciting his judgements. If $A = (a_{ij})$ is the pairwise comparison matrix, where a_{ij} is the relative importance of the alternative A_i with respect to the alternative A_j and n is the number of elements, consistency in AHP is defined as the cardinal transitivity between judgements; that is to say, $a_{ij}\, a_{jk} = a_{ik}$ for all $i, j, k = 1,\ldots, n$. Saaty (1980) suggests using the Consistency Ratio (CR) to measure inconsistency when eliciting judgements in the Conventional-AHP, where the Eigenvector Method (EM) is used as the priorization procedure. For the Row Geometric Mean priorization procedure, Aguarón and Moreno-Jiménez (2003) proposed the Geometric Consistency Index (GCI) as an inconsistency measure:

$$GCI = \frac{2}{(n-1)(n-2)} \sum_{i<j} \log^2 e_{ij} \tag{1}$$

where e_{ij} is the error obtained when the ratio ω_i/ω_j is approximated by a_{ij} ($a_{ij} = w_i/w_j\, e_{ij}$). These authors also provide the relationship between GCI and CR, and, for any matrix order, determine the thresholds for GCI that provide an interpretation of the inconsistency level analogous to that suggested by Saaty for the EM. The inconsistency threshold is usually taken as CR = 10% (GCI = 0.3524). Nevertheless, as the proposed procedure seeks the extraction of knowledge, it is possible to use CR values up to 20%.

If the number of judgements elicited by the decision maker when comparing n elements is less than $n(n-1)/2$, it is necessary to use priorization procedures valid for incomplete matrices. Harker (1987) proposed such a procedure is based in the eigenvector method, and Tone (1996) offers a similar one based on the row geometric mean priorization method, which will be used in our approach.

2.2 Group decision making in AHP

AHP has been widely used in group decision making (Saaty 1989; Condon et al. 2003). As the process for eliciting judgements is regarded, there are several ways to obtain a group valuation (Ramanathan and Ganesh 1994; Forman and Peniwati 1998; Escobar and Moreno-Jiménez 2006). Some of these are: (i) consensus between actors; (ii) compromise or voting when consensus cannot be reached; (iii) aggregation of individual judgements; (iv) aggregation of individual priorities; (v) aggretation of individual preference structures and (vi) consideration of interval judgements.

In the case of AHP, the search for consensus can be considered in any of the steps of the AHP methodology. Let us focus on searching for agreement in a final preference structure that reflects the ''core of consistency'' of the process, and that will be accepted by all actors as the common consistency core. The proposed consensus building procedure consists of constructing a group consensus interval matrix for each node of the hierarchy, based on the core of consistency between the actors. From this matrix, where the entries are interval judgements in which the inconsistency of all actors is smaller than a previously fixed value, we derive the preference structure for the group's core of consistency.

2.3 Some decision tools for consensus building

This paper uses two other available decision tools (Moreno-Jiménez et al. 2005a): Preference Structures and Stability Intervals. The *Preference Structure Distribution* of a set of alternatives $A_1,..., A_n$ is defined as the probability distribution of the $n!$ different preference structures, or possible rankings of the n alternatives, obtained when incorporating uncertainty into the model through interval judgements (Moreno-Jiménez and Vargas 1993; Moreno-Jiménez et al. 1999). Using these preference structures, it is possible to design consensus paths between actors, trying to arrive at a common preference structure distribution acceptable to all participants, from which we can determine the most preferred or likely ranking of alternatives.

Stability Intervals (Aguarón and Moreno-Jiménez 2000) are decision tools obtained from an inverse sensitivity analysis of the attribute under study (priority, consistency,...). These intervals indicate the range of values in which a judgement (alternative or criteria) can oscillate while still preserving the previously mentioned property. *Stability Indexes* represent the width of the maximum reciprocal relative stability interval included in the stability interval.

Using the Row Geometric Mean priorization procedure (RGMM), Aguarón and Moreno-Jiménez (2000) obtain the *Priority Stability Intervals* (PSI) for alpha and gamma type problems (α-PSI and γ-PSI). These intervals provide the range of values in which judgements can oscillate without producing a rank-reversal for the best alternative (α-problem), or a rank reversal between any two alternatives (γ-problem). In both cases, the *Priority Stability Indexes* (PSIX) provide the values corresponding to the amplitude of the maximum reciprocal priority relative stability intervals in their respective situations (α and γ).

Aguarón et al. (2003) obtain the *Consistency Stability Interval* (CSI) for a judgement, that is to say, the range of values in which this judgement can oscillate without exceeding a value of the previously fixed inconsistency measure (GCI). The *Consistency Stability Indexes* (CSIX) provide the amplitude of the maximum reciprocal consistency relative stability intervals.

Both decision tools are used in the design of a new one: the Consistency Consensus Matrix, which will be the support for the procedure we propose for building consensus in AHP group decision making.

3 The Consistency Consensus Matrix (CCM)

The Consistency Consensus Matrix reflects the "core of consistency" of the individuals participating in the decision making process. This new procedure rests on the assumptions that: (i) a common hierarchy is accepted by all the participants; (ii) the Row Geometric Mean Method is employed as the priorization procedure; and (iii) the Geometric Consistency Index is used as the inconsistency measure for the RGMM (Aguarón and Moreno-Jiménez 2003).

If a reduced number of actors is considered, the procedure starts by using their initial judgement matrices. If the number of actors is large, as usually occurs in e-democratic processes, we propose as a first step the application of a cluster analysis. In this case, the information employed in order to derive the CCM would be the aggregate judgements of each group or cluster.

Each entry of this group consensus matrix is given by the interval judgement obtained as the intersection between the consistency stability intervals of the different actors (individuals or clusters) for this judgement and the range of the initial judgement values. These intervals correspond to the range for the judgements in which all individuals simultaneously have an acceptable inconsistency. Obviously, fixing in advance a threshold for the GCI, this intersection could be null and, thus, it would be necessary to work with incomplete matrices.

The performance analysis of preference structure distributions, obtained when changing the thresholds for the GCI and the number of consistency stability intervals included in the consensus matrix, provides us with valuable knowledge about patterns, trends, decision opportunities and critical points for the decision making process. A spreadsheet module has been developed for this (Moreno-Jiménez et al. 2005b). The module obtains the CCM matrix and then derives the preference structures associated with it. The theoretical basis for the constuction of the CCM is explained in the following section.

3.1 Theoretical results

We provide now the theoretical results necessary to calculate the probability of the different preference structures associated with the consistency consensus matrix. In this respect, we first derive the priorities and evaluate the inconsistency when using the RGMM with incomplete matrices (Definition 1); secondly, for a fixed value of the GCI, we obtain the consistency stability intervals of the individual judgements (Definition 2 and Theorems 1 and 2) and, finally, we compute the probabilities of the different preference structures (analytically or using simulation).

Let $A = (a_{ij})$ be an incomplete pairwise comparison matrix, that is to say, a reciprocal matrix where the corresponding graph is connected and not complete. For each vertex, i, we define P_i as the set of vertices adjacent to i and N_i as the degree of i. The priorities of a pairwise comparison matrix, $A = (a_{ij})$, calculated by means of the Row Geometric Mean priorization procedure, are obtained (Tone 1996) solving the set of linear equations in ($\log\omega_i$) given by

$$N_i \log \omega_i - \sum_{j \in P_i} \log \omega_i = \sum_{j \in P_i} \log a_{ij} \qquad (i = 1, 2, \ldots, n) \tag{2}$$

A practical rule for constructing the coefficient matrix of the linear equations consists of putting 0 in the missing entries, –1 in the compared ones, and the number of –1s of the corresponding row in the diagonal entries.

Definition 1 Let $A = (a_{ij})$ be an incomplete pairwise comparison matrix, and $C = \{(i, j), i < j, j \in P_i\}$ be the set of non-null entries in the upper triangular block of the matrix. The *inconsistency measure associated with the Row Geometric Mean priorization procedure* (GCI_C) is given by

$$GCI_C = \frac{1}{N - (n-1)} \sum_{i=1}^{n} \sum_{\substack{j \in P_i \\ i<j}} \log^2 e_{ij} \tag{3}$$

where $e_{ij} = a_{ij}\omega_j/\omega_i$ and N represents the total number of comparisons made in the upper triangular block.

Note that $N = \mathrm{card}(C) = \frac{1}{2}\sum_i N_i$.

Definition 2 Given a pairwise comparison matrix, $A = (a_{ij})$, and a variation $\Delta > 0$ for the GCI, we define:

(2a) The *Consistency Relative Stability Interval* (CRSI) for the judgement a_{rs}, is the interval $\left[\underline{\delta}_{rs}(\Delta), \bar{\delta}_{rs}(\Delta)\right]$, in which its relative variations can oscillate without the increment of GCI exceeding Δ.

(2b) The *Reciprocal Consistency Relative Stability Interval* (RCRSI) for the judgement a_{rs} given Δ, is the interval $[\delta_{rs}^{-1}(\Delta), \delta_{rs}(\Delta)]$, in which its relative variations can oscillate without the increment of the GCI exceeding Δ. Its value is given by $\delta_{rs}(\Delta) = \min\{\underline{\delta}_{rs}^{-1}(\Delta), \bar{\delta}_{rs}(\Delta)\}$.

Aguarón et al. (2003) obtain the CRSI for judgement a_{rs} through

$$\underline{\delta}_{rs}(\Delta) = e^{n \log \rho_{min}}$$
$$\bar{\delta}_{rs}(\Delta) = e^{n \log \rho_{max}}$$

when $[\log \rho_{min}, \log \rho_{max}]$ is the interval for $\log \rho_{rs}$, with $\rho_{rs} = \left(\frac{a'_{rs}}{a_{rs}}\right)^{1/n}$, and that:

$$\frac{2n}{(n-1)(n-2)}\left[(n-2)\log^2 \rho_{rs} + 2\log e_{rs} \log \rho_{rs}\right] \leq \Delta \tag{4}$$

Definition 2 can be extended to the case of an alternative or a criterion. When modifying all the judgements of the matrix, the RCRSI are obtained from the next theorem, in which the variables x_{ij} represent the log of the relative change (a'_{ij}/a_{ij}).

Theorem 1 Given a pairwise comparison matrix $A = (a_{ij})$ $i, j = 1,\ldots, n$, the Reciprocal Consistency Relative Stability Interval for the matrix given Δ, $[1/\delta_A(\Delta), \delta_A(\Delta)]$, is determined by $\delta_A(\Delta) = e^{d^*}$, where d^* is the optimal value of the optimization problem:

$$\begin{aligned}
&\text{Min } d = \text{Max } \{|x_{ij}|,\ i,j = 1,\ldots,n\} \\
&s.t. \\
&\tfrac{1}{(n-1)(n-2)}\left[\textstyle\sum_{i,j=1}^{n}\left(\log e_{ij} + x_{ij}\right)^2 - \tfrac{2}{n}\sum_{i=1}^{n}\left(\sum_{j=1}^{n} x_{ij}\right)^2\right] \geq GCI + \Delta \\
&x_{ij} + x_{ji} = 0
\end{aligned} \tag{5}$$

Proof See Appendix.

Corollary 1 Under the previous conditions, when modifying judgements a_{ij} with $(i, j) \in S$, the RCRSI for this set S, $[1/\delta_S(\Delta), \delta_S(\Delta)]$, is determined by $\delta_S(\Delta) = e^{d*}$, where d^* is the optimal value of the optimization problem:

$$\begin{aligned}
&\text{Min } d = \text{Max } \{|x_{ij}|,\ (i,j) \in S\} \\
&s.t. \\
&\tfrac{1}{(n-1)(n-2)}\left[\textstyle\sum_{i,j\in S}\left(\log e_{ij} + x_{ij}\right)^2 - \tfrac{2}{n}\sum_{i=1}^{n}\left(\sum_{j=1}^{n} x_{ij}\right)^2\right] \geq GCI + \Delta \\
&x_{ij} + x_{ji} = 0\ (i,j) \in S \\
&x_{ij} = 0 \text{ for all } (i,j) \notin S
\end{aligned} \tag{6}$$

The iterative procedure we propose to construct the CCM requires that we obtain the RCRSI for a judgement, by allowing the judgements previously included in this matrix to range within their own consistency stability intervals.

Theorem 2 Under the previous conditions, when some judgements of the initial matrix oscillate in interval judgements ($a'_{ij} \in I_{ij}$, $(i, j) \in C$), the RCRSI for the judgement a_{rs}, $[1/\delta_{rs|C}(\Delta), \delta_{rs|C}(\Delta)]$, is determined by $\delta_{rs|C}(\Delta) = e^{d*}$, where d is the optimal value of the optimization problem*:

$$\begin{aligned}
&\text{Min } d = |x_{rs}| \\
&s.t. \\
&\tfrac{1}{(n-1)(n-2)}\left[\textstyle\sum_{(i,j)\in\bar{C}}\left(\log e_{ij} + x_{ij}\right)^2 - \tfrac{2}{n}\sum_{i=1}^{n}\left(\sum_{j=1}^{n} x_{ij}\right)^2\right] \geq GCI + \Delta \\
&a'_{ij} = a_{ij}e^{x_{ij}} \in I_{ij} \text{ for all } (i,j) \in C \\
&x_{ij} + x_{ji} = 0\ (i,j) \in \bar{C} \\
&x_{ij} = 0\ (i,j) \notin \bar{C}
\end{aligned} \tag{7}$$

where $\bar{C} = C \cup (r,s)$, and $C = \{(i, j), i < j, j \in P_i\}$.
Proof Analogous to that of Theorem 1.

This theorem provides the consistency stability intervals for each entry of the CCM. The last step of the procedure consists of obtaining the preference structure distribution derived from the CCM. It should be noted that the CCM is an incomplete interval judgement matrix for which the priorities are obtained using Tone's method (1996) and simulation techniques with reciprocal distributions (Escobar and Moreno-Jiménez 2000). In simple cases, this step can be carried out analytically.

3.2 Algorithm for the CCM

The automatic procedure we propose starts by constructing the CCM for the group in an iterative way. To do this, we first rank the entries from the minimum to the maximum dispersion between the actors' judgements, using, for example, the variance of the log of judgements.

From the list of candidates, we select (Step 1) the one with the smallest dispersion (d_{ij}) that does not form a cycle. Assume it is the (r, s) entry. We then calculate the consistency stability intervals for each actor, $I_{rs}^{[k]}$, $k = 1,..., m$ and their intersection (I_{rs}). We consider that it is appropriate to limit the consensus intervals (I_{rs}^{C}) to their intersection with the initial judgement range ($[\min_k a_{rs}^{[k]}, \max_k a_{rs}^{[k]}]$): in this way we do not consume all the existent inconsistency slack and facilitate the incorporation of more entries in the CCM matrix (Step 2).

As the AHP priorization procedure requires at least $n - 1$ judgements, when the number of non-null entries in the consistency consensus matrix, say N, is smaller than this value ($N < n - 1$), it is necessary to complete the required minimum number of cells ($n - 1$) with some estimations. In our proposal, we associate the geometric mean of the individual judgements to the n-1-N missing entries with smallest dispersion (Step 3).

As regards the calculation of the consistency stability interval for a judgement and each actor which is carried out in Step 2, we follow an iterative method based on the resolution of a mathematical programming model, in which a new constraint is included in each step (Theorem 2). As the procedure incorporates an entry of the matrix in each iteration, corresponding to the consistency stability interval obtained in the previous step, and given that no more than $n(n - 1)/2$ cells can be included, the convergence of the method is guaranteed. Although the procedure may entail some computational complexity, in practice it is usually applied to a reduced number of alternatives and this facilitates its use.

After including the first $n - 1$ consistency stability intervals in the consensus matrix, the procedure can be refined by incorporating additional consistency stability intervals. As mentioned earlier, the automatic procedure followed for the inclusion of judgements in a systematic way does not require the participation of the actors involved. However, this procedure can be enriched with their participation. If there is consensus between the actors with respect to the interval judgement associated with an entry, this interval must be directly incorporated in the consensus matrix. All this redundant information will improve the accuracy of the estimations, as occurs with AHP priorization (Saaty 1980).

3.2.1 *Algorithm*

For each of the actors ($k = 1,..., m$), let $A^{[k]} = (a_{ij}^{[k]})$ be his pairwise comparison matrix of order n, $\mathrm{GCI}^{[k]}$ be the value of its inconsistency measure, and $\Delta^{[k]}$ the variation allowed for this inconsistency measure for which a common fixed threshold (GCI*) is not exceeded. Furthermore, let C be the set of entries already included in the consistency consensus matrix, R be the set of entries which do not satisfy the initial conditions to be incorporated in this matrix (rejected entries), and J the set of entries which have not yet been analysed ($C, R, J \subset I \times I$ with $I = \{1,..., n\}$).

Step 0: Initialisation

Let $C = \emptyset$, $R = \emptyset$, $J = \{(i, j)$, with $i < j\}$ and calculate for all $(i, j) \in J$:

$$y_{ij}^{[k]} = \log a_{ij}^{[k]} \qquad d_{ij} = Var\left(y_{ij}^{[k]}\right) \qquad a_{ij}^{G} = \left(\prod_{k=1}^{m} a_{ij}^{[k]}\right)^{1/m}$$

Step 1: Selection of candidate

Let (r, s) be the entry for which $d_{rs} = \min_{(i,j)\in J} d_{ij}$

$J = J \setminus \{(r,s)\}$. If $C \cup \{(r, s)\}$ has cycles, then $R = R \cup \{(r, s)\}$ and go to Step 3

Step 2: Obtention of a CCM entry

For each $k = 1,\ldots, m$, determine $\delta_{rs}^{[k]}$ by solving the model of Theorem 2. From

$$I_{rs}^{[k]} = \left[\underline{a}_{rs}^{[k]}, \bar{a}_{rs}^{[k]}\right] = \left[a_{rs}^{[k]}/\delta_{rs}^{[k]}, a_{rs}^{[k]}\delta_{rs}^{[k]}\right], \text{we obtain } I_{rs} = \cap_{k=1}^{m} I_{rs}^{[k]}$$

If $I_{rs} = \emptyset$, then $R = R \cup \{(r, s)\}$ and go to Step 3.

Otherwise, $C = C \cup \{(r,s)\}$ and $I_{rs}^{C} = I_{rs} \cap \left[\min_k a_{rs}^{[k]}, \max_k a_{rs}^{[k]}\right]$

If $card(C) = n - 1$, then stop.

Step 3: Completion of the CCM

If $J \neq \emptyset$, then go to Step 1. If not, whilst $card(C) < < n - 1$, then repeat:

Let (r, s) be the entry for which $d_{rs} = \min_{(i,j)\in R} d_{ij}$.

If $C \cup \{(r, s)\}$ has no cycles, then $C = C \cup \{(r, s)\}$ and $I_{rs}^{C} = a_{rs}^{G}$

$R = R \setminus \{(r,s)\}$

4 Example

In order to see how the previously described consensus building process works in practice, we have applied it to a case study adapted from a real-life experiment in e-democracy developed by our research group (http://www.gdmz.unizar.es) for the City Council of Zaragoza.

The case study deals with the budget allocation that the municipal district of El Rabal (Zaragoza) assigns to each of four alternatives (A: Elimination of Architectural Barriers; B: Urban Green Space; C: Youth Activity Areas and D: Street Cleaning) proposed by the Neighbourhood Associations and the Members of the District Council. This participatory budget problem has been modelled with a hierarchy consisting of three criteria, six sub-criteria and the four alternatives (see http://www.cmisapp.zaragoza.es/ciudad/presupuestos-participativos/).

As the number of actors was large (100), we first carried out a cluster analysis with the individual priorities of the alternatives. Among the six identified clusters, we have selected the two that presented the most differentiated behaviour patterns to determine the core of consistency. To illustrate the proposed procedure, we consider two initial matrices corresponding to the aggregate judgements (geometric mean) obtained for these two clusters with respect to the most important sub-criterion (participation). These matrices are:

$$A^{[1]} = \begin{pmatrix} 1,00 & 1,44 & 0,48 & 1,44 \\ 0,69 & 1,00 & 0,48 & 1,00 \\ 2,08 & 2,08 & 1,00 & 2,08 \\ 0,69 & 1,00 & 0,48 & 1,00 \end{pmatrix} \qquad A^{[2]} = \begin{pmatrix} 1,00 & 0,69 & 6,80 & 0,69 \\ 1,44 & 1,00 & 2,92 & 0,58 \\ 0,15 & 0,34 & 1,00 & 0,12 \\ 1,44 & 1,71 & 8,28 & 1,00 \end{pmatrix}$$

The pairwise comparison matrix corresponding to the group obtained through aggregation of individual judgements using the geometric mean method is:

$$A^G = \begin{pmatrix} 1,00 & 1,00 & 1,81 & 1,00 \\ 1,00 & 1,00 & 1,19 & 0,76 \\ 0,55 & 0,84 & 1,00 & 0,50 \\ 1,00 & 1,31 & 1,99 & 1,00 \end{pmatrix}$$

First, applying the row geometric mean method, we have calculated the priority vectors and the geometric consistency indexes for the three matrices (see Table 1). With respect to these results (see Table 2), we can observe that the first cluster prefers alternative C, while the second prefers alternative D. When using the aggregation of individual judgements (a_{ij}^G), the preferred alternative is D.

Let us now apply the algorithm to this example.

Step 0:

We first initialise the values of the sets C, R and J:

$$C = \emptyset, R = \emptyset, J = \{(1,2),(1,3),(1,4),(2,3),(2,4),(3,4)\}.$$

We also calculate the group judgements a_{ij}^G (see the matrix A^G above) and the variances of the log of judgements for the different actors, which are collected in the matrix $D = (d_{ij})$.

Table 1 Individual and group priorities and GCI

Alternative	[1]	[2]	[G]
A	0.235199	0.268823	0.282731
B	0.178712	0.250496	0.237904
C	0.407376	0.055801	0.169528
D	0.178712	0.424880	0.309837
GCI	0.022351	0.169733	0.026948

Table 2 Rankings of alternatives

Rank	[1]	[2]	[G]
1	C	D	D
2	A	A	A
3	B	B	B
4	D	C	C

$$D = \begin{pmatrix} - & 0.134 & 1.756 & 0.134 \\ 0.134 & - & 0.815 & 0.072 \\ 1.756 & 0.815 & - & 2.025 \\ 0.134 & 0.072 & 2.025 & - \end{pmatrix}$$

The entries are ranked from the minimum to the maximum variance of the log of judgements. This provides the order used to select the next entry when constructing the CCM matrix: (2,4), (1,2), (1,4), (2,3), (1,3) and (3,4). As entries (1,2) and (1,4) have the same variance, they are interchangeable in the selection order.

Iteration 1

Step 1
As we have just seen, the entry with the minimum variance is (2,4). This implies that set J is updated: $J = \{(1,2),(1,3),(1,4),(2,3),(3,4)\ \}$. As the set $C \cup \{(2,4)\} = \{(2,4)\}$ has no cycles, the algorithm continues with step 2.
Step 2
The consistency stability intervals for each individual matrix (GCI* = 0.4934) are:

$$I_{24}^{[1]} = [0.246, 4.064], I_{24}^{[2]} = [0.210, 1.682] \text{ and } I_{24}^{[1]} \cap I_{24}^{[2]} \neq \emptyset$$

Then the entry (2,4) of the CCM matrix is given by the intersection of these intervals with the initial range:

$$I_{24}^{C} = I_{24}^{[1]} \cap I_{24}^{[2]} \cap [0.585, 1] = [0.585, 1]$$

In this way, judgement (2,4) from each actor can oscillate within the interval [0.585, 1] without the inconsistency of any of the actors being greater than the threshold fixed in advance.
As the cardinal of the set $C = \{(2,4)\}$ is $1 < n - 1 = 3$, the algorithm continues.
Step 3
The set J is not empty, thus the algorithm leads to the following iteration.
After two similar iterations we reach

Iteration 4

Step 1
The next judgement that should be considered is (2,3), which does not form a cycle with the entries previously considered. Now, J = {(1,3), (3,4)}.
Step 2
Repeating once more the procedure, the consistency stability intervals are obtained for each of the clusters.

Their intersection is non-empty, and then the consistency consensus interval is given by:

$$I_{23}^{C} = I_{23}^{[1]} \cap I_{23}^{[2]} \cap [0.48, 2.92] = [1.63, 1.72]$$

The set C is updated: $C = \{(1,2),(2,3),(2,4)\}$. As $card(C) = 3 = n - 1$ the algorithm ends, although it would be possible to incorporate new judgements in order to enrich the process.

In this example, an inconsistency level of CR = 14% (GCI = 0.4934) was necessary to obtain a CCM with three non-null entries without cycles. This was due to the fact that we selected two of the most differentiated clusters. The resulting CCM for the CR = 14% is given by:

$$A^{C} = \begin{pmatrix} 1 & [\mathbf{0.69}, \mathbf{1.44}] & - & - \\ & 1 & [\mathbf{1.63}, \mathbf{1.72}] & [\mathbf{0.58}, \mathbf{1.00}] \\ & & 1 & - \\ & & & 1 \end{pmatrix}$$

The evolution of the preference structures with respect to the inconsistency level is shown in the following table (Table 3).

As can be observed, from the point of view of the participation criterion, only alternatives A and D appear in the first position for the two most differentiated clusters. The priority of the best alternative (D) obtained from the CCM stabilises around 75%.

5 Conclusions

We have presented a new decision tool for consensus building in AHP-group decision making and which we call the Consistency Consensus Matrix. Assuming that: (i) the individuals agree on a common hierarchy representing the problem; (ii) the Row Geometric Mean is employed as the priorization method and (iii) the Geometric Consistency Index is used to evaluate the inconsistency, then, for a fixed inconsistency threshold, this tool determines the group consistency preference structure distribution which, derived from the core of consistency between the actors, is accepted as the group consensus outcome of the resolution process.

This preference structure has been obtained from the Consistency Consensus Interval Judgment Matrix constructed, in an automatic way, from the initial pairwise comparison matrices provided by the individual actors. The method used to construct this matrix essentially consists of resolving three different mathematical programming problems. First, the selection of a set of $n - 1$ entries that minimise the divergence between the actors' judgements and do not form a cycle. Second, the estimation of the priorities of an incomplete pairwise comparison matrix when using

Table 3 Probabilities of the preference structures for different GCI

	CR GCI	14% 0.4934	15% 0.5286	16% 0.5638	17% 0.5991	18% 0.6343	19% 0.6696	20% 0.7048
$D > B > A > C$		0.5123	0.1921	0.1973	0.192	0.1884	0.1928	0.187
$D > A > B > C$		0.3444	0.5612	0.5588	0.5527	0.5654	0.5637	0.5692
$A > D > B > C$		0.1433	0.2467	0.2439	0.2553	0.2462	0.2424	0.2393
$D > B > C > A$		–	–	–	–	–	0.0011	0.0048

the RGMM as priorization procedure. Third, the computation of the consistency stability interval associated with a cell in an interval judgement matrix.

The automatic consensus building process we propose can also be considered as an initial step in any personal or participatory consensus building algorithm. Moreover, the automatic method can be enriched with personal contributions during the iterative resolution process. This decision tool can be applied in situations with both small and large numbers of individuals. In fact, it is now being employed in a real-life experiment on e-democracy developed for the City Council of Zaragoza, an experiment that involves a large number of actors.

Finally, it should be mentioned that the underlying ideas of the proposed algorithm can be extended to other multicriteria decision making techniques in which the concept of consistency can be defined in a meaningful way. Without entering into a discussion about the validity of the multicriteria decision making approaches and the priorization procedure used, we should place emphasis on the fact that the use of the proposed procedure in extracting the relevant knowledge that assists in the consensus building process is more important than the technique employed in the resolution of group decision problems.

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Appendix

Proof of Theorem 1

Notation Let $A = (a_{ij})$, $i, j = 1,\ldots, n$, be a pairwise comparison matrix, $\omega = (\omega_i)$ be its associated priority vector, e_{ij} the error obtained when estimating the judgement a_{ij} from the priority vector ω ($e_{ij} = a_{ij}\omega_j/\omega_i$), $A' = (a_{ij}')$ the matrix obtained when modifying the original judgements, and t_{ij} the relative variations in the judgements ($t_{ij} = a_{ij}'/a_{ij}$) and $\rho_{ij} = t_{ij}^{1/n}$.

Lemma 1 Given $A = (a_{ij})$, $i, j = 1,\ldots, n$, and its variation $A' = (a_{ij}')$, the new values of the judgement errors when applying the row geometric mean method are:

$$e'_{ij} = e_{ij}\rho_{ij}^{n}\prod_{k=1}^{n}\frac{\rho_{jk}}{\rho_{ik}} \tag{8}$$

Proof. Immediate, on applying the definition of the priorities obtained with the RGMM.

Lemma 2 Given $A = (a_{ij})$, $i, j = 1,\ldots, n$, and its variation $A' = (a'_{ij})$, the new value of the GCI is

$$GCI' = \frac{1}{(n-1)(n-2)}\left[\sum_{i,j=1}^{n}\log^2 e_{ij}t_{ij} - \frac{2}{n}\sum_{i=1}^{n}\left(\sum_{j=1}^{n}\log t_{ij}\right)^2\right] \tag{9}$$

Proof. Applying the expression of the GCI and expression (8) we have:

$$GCI' = \frac{1}{(n-1)(n-2)}\sum_{i,j=1}^{n}\log^2 e'_{ij} = \frac{1}{(n-1)(n-2)}\sum_{i,j=1}^{n}\log^2(e_{ij}\rho_{ij}^{n}\prod_{k=1}^{n}\frac{\rho_{jk}}{\rho_{ik}})$$

where $\rho_{ij} = \left(\frac{a'_{ij}}{a_{ij}}\right)^{1/n}$. Developing the expression of the square of the logarithm:

$$\begin{aligned} GCI' &= \frac{1}{(n-1)(n-2)}\sum_{i,j=1}^{n}\left(\log e_{ij} + \log \rho_{ij}^{n}\prod_{k=1}^{n}\frac{\rho_{jk}}{\rho_{ik}}\right)^2 \\ &= \frac{1}{(n-1)(n-2)} \\ &\times\left(\sum_{i,j=1}^{n}\log^2 e_{ij} + \underbrace{\sum_{i,j=1}^{n}\log^2 \rho_{ij}^{n}\prod_{k=1}^{n}\frac{\rho_{jk}}{\rho_{ik}}}_{=A} + \underbrace{2\sum_{i,j=1}^{n}\log e_{ij}\log \rho_{ij}^{n}\prod_{k=1}^{n}\frac{\rho_{jk}}{\rho_{ik}}}_{=B}\right) \end{aligned} \tag{10}$$

Let us now operate with the term A

$$A = \sum_{i,j=1}^{n}\log^2 \rho_{ij}^{n} + \underbrace{\sum_{i,j=1}^{n}\log^2\prod_{k=1}^{n}\frac{\rho_{jk}}{\rho_{ik}}}_{=A_1} + \underbrace{2\sum_{i,j=1}^{n}\log \rho_{ij}^{n}\log\prod_{k=1}^{n}\frac{\rho_{jk}}{\rho_{ik}}}_{=A_2}$$

For A_1 we have, after simple computations:

$$\begin{aligned} A_1 &= \sum_{i,j=1}^{n}\left(\log\prod_{k=1}^{n}\rho_{jk} - \log\prod_{k=1}^{n}\rho_{ik}\right)^2 \\ &= 2n\sum_{i=1}^{n}\left(\sum_{k=1}^{n}\log\rho_{ik}\right)^2 - 2\sum_{i=1}^{n}\left[\left(\log\prod_{k=1}^{n}\rho_{ik}\right)\left(\log\prod_{j<k=1}^{n}\rho_{jk}\rho_{kj}\right)\right] \end{aligned}$$

As ρ_{jk} and ρ_{kj} are reciprocal, the last product equals 1, the logarithm is null and, therefore, all the second term of A_1 equals 0. With all these, and changing the index k by j, we have:

$$A_1 = 2n\sum_{i=1}^{n}\left(\sum_{j=1}^{n}\log\rho_{ij}\right)^2$$

Let us now consider the value of A_2

$$
\begin{aligned}
A_2 &= 2n\left[\sum_{i,j=1}^{n}\left(\log\rho_{ij}\log\prod_{k=1}^{n}\rho_{jk} - \log\rho_{ij}\log\prod_{k=1}^{n}\rho_{ik}\right)\right]\\
&= 2n\left[\sum_{i,j=1}^{n}\log\rho_{ij}\log\prod_{k=1}^{n}\rho_{jk} + \sum_{i,j=1}^{n}\log\rho_{ji}\log\prod_{k=1}^{n}\rho_{ik}\right] = 4n\sum_{i,j=1}^{n}\left(\log\rho_{ij}\log\prod_{k=1}^{n}\rho_{jk}\right)\\
&= 4n\sum_{j=1}^{n}\left(\log\prod_{k=1}^{n}\rho_{jk}\sum_{i=1}^{n}\log\rho_{ij}\right) = 4n\sum_{j=1}^{n}\left(\sum_{k=1}^{n}\log\rho_{jk}\sum_{i=1}^{n}\log\rho_{ij}\right)\\
&= -4n\sum_{j=1}^{n}\left(\sum_{k=1}^{n}\log\rho_{jk}\sum_{i=1}^{n}\log\rho_{ji}\right) = -4n\sum_{j=1}^{n}\left(\sum_{i=1}^{n}\log\rho_{ji}\right)^2
\end{aligned}
$$

Swapping the indexes i and j:

$$
A_2 = -4n\sum_{i=1}^{n}\left(\sum_{j=1}^{n}\log\rho_{ij}\right)^2
$$

With the values of A_1 and A_2 we have:

$$
A = \sum_{i,j=1}^{n}\log^2\rho_{ij}^{n} - 2n\sum_{i=1}^{n}\left(\sum_{j=1}^{n}\log\rho_{ij}\right)^2
$$

Now, the term B

$$
\begin{aligned}
B &= 2\left(\sum_{i,j=1}^{n}\log e_{ij}\log\rho_{ij}^{n} + \sum_{i,j=1}^{n}\log e_{ij}\log\prod_{k=1}^{n}\frac{\rho_{jk}}{\rho_{ik}}\right)\\
&= 2n\sum_{i,j=1}^{n}\log e_{ij}\log\rho_{ij} + \underbrace{2\sum_{i,j=1}^{n}\log e_{ij}\log\prod_{k=1}^{n}\frac{\rho_{jk}}{\rho_{ik}}}_{B_1}
\end{aligned}
$$

Finally, developing B_1:

$$
\begin{aligned}
B_1 &= 2\sum_{i,j=1}^{n}\log e_{ij}\log\prod_{k=1}^{n}\rho_{jk} - 2\sum_{i,j=1}^{n}\log e_{ij}\log\prod_{k=1}^{n}\rho_{ik}\\
&= 2\sum_{i,j=1}^{n}\log e_{ij}\log\prod_{k=1}^{n}\rho_{jk} + 2\sum_{i,j=1}^{n}\log e_{ji}\log\prod_{k=1}^{n}\rho_{ik}
\end{aligned}
$$

Therefore:

$$
B_1 = 4\sum_{i,j=1}^{n}\left[\log e_{ij}\log\prod_{k=1}^{n}\rho_{jk}\right] = 4\sum_{j=1}^{n}\left[\log\prod_{k=1}^{n}\rho_{jk}\left(\sum_{i=1}^{n}\log e_{ij}\right)\right]
$$

As $\prod_j e_{rj} = 1$ we have $\sum_{i=1}^{n} \log e_{ij} = 0$ and then $B_1 = 0$. The value of B will be:

$$B = 2n \sum_{i,j=1}^{n} \log e_{ij} \log \rho_{ij}$$

Finally, returning to the value of GCI' in expression (10), after simple computations:

$$\begin{aligned}(n-1)(n-2)GCI' &= \sum_{i,j=1}^{n} \log^2 e_{ij} + A + B \\ &= \sum_{i,j=1}^{n} \left(\log e_{ij} + n \log \rho_{ij}\right)^2 - 2n \sum_{i=1}^{n} \left(\sum_{j=1}^{n} \log \rho_{ij}\right)^2\end{aligned}$$

If we now consider $t_{ij} = \frac{a'_{ij}}{a_{ij}}$, $\log t_{ij} = n \log \rho_{ij}$ and then:

$$GCI' = \frac{1}{(n-1)(n-2)} \left[\sum_{i,j=1}^{n} \left(\log e_{ij} + \log t_{ij}\right)^2 - \frac{2}{n} \sum_{i=1}^{n} \left(\sum_{j=1}^{n} \log t_{ij} \right)^2 \right] \qquad \#$$

Proof. If we make the variable change $x_{ij} = \log t_{ij}$, the condition imposed to demand that the increase in inconsistency would be less than Δ is:

$$\frac{1}{(n-1)(n-2)} \left[\sum_{i,j=1}^{n} \left(\log e_{ij} + x_{ij}\right)^2 - \frac{2}{n} \sum_{i=1}^{n} \left(\sum_{j=1}^{n} x_{ij} \right)^2 \right] \leq GCI + \Delta \qquad (11)$$

On determining the maximum value of δ so that if $t_{ij} \in [1/\delta, \delta]$, the inconsistency does not increase its value above Δ, it is equivalent to determining the maximum value of x^* so that if $x \in [-x^*, x^*]$ the inequality above is verified. In other words, we must determine the maximum ball (L_∞ norm) centred in the origin which verifies the inequality (11). The radius of this ball is determined by the closest point in the frontier to the origin in L_∞. In order to obtain the radius we must solve the optimisation problem (5). #

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