Group Decision and Negotiation (2007) 16:485–506
DOI 10.1007/s10726-006-9068-0

AHP-Group Decision Making: A Bayesian Approach Based on Mixtures for Group Pattern Identification*

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Abstract

This paper proposes a Bayesian estimation procedure to determine the priorities of the Analytic Hierarchy Process (AHP) in group decision making when there are a large number of actors and a prior consensus among them is not required. Using a hierarchical Bayesian approach based on mixtures to describe the prior distribution of the priorities in the multiplicative model traditionally used in the stochastic AHP, this methodology allows us to identify homogeneous groups of actors with different patterns of behaviour for the rankings of priorities. The proposed procedure consists of a two-step estimation algorithm: the first step carries out a global exploration of the model space by using birth and death processes, the second concerns a local exploration by means of Gibbs sampling. The methodology has been illustrated by the analysis of a case study adapted from a real experiment on e-democracy developed for the City Council of Zaragoza (Spain).

Key words: group decision making, AHP, priority, Bayesian analysis, mixtures, MCMC, e-democracy

1. Introduction

The resolution of high complexity problems that have arisen in the Knowledge Society requires the use of approaches which exhibit appropriate behaviour in the multi-actor decision making framework (Saaty 1996; Moreno-Jiménez et al. 1999, 2002). One of these approaches is the Analytic Hierarchy Process (AHP), proposed by Saaty (1977, 1980).

There are two methods usually followed in the literature to obtain the local priorities in AHP-group decision making (Saaty 1989; Ramanathan and Ganesh 1994; Forman and Peniwati 1998): the *Aggregation of Individual Judgements* (AIJ) and the *Aggregation of Individual Priorities* (AIP). Other procedures are based on

* Partially funded under the research project *Electronic Government. Internet-based Complex Decision Making: e-democracy and e-cognocracy* (Ref. PM2004-052) approved by the Regional Government of Aragon (Spain) as part of the multi-disciplinary projects programme.

different orientations. Alho et al. (1996) and Basak (1998, 2001) use hierarchical Bayesian models; Bryson and Joseph (1999) use goal programming; González-Pachón and Romero (1999) and Wei et al. (2000) minimise the maximum discrepancy among the decision makers; Escobar et al. (2000, 2001) consider judgement intervals and reciprocal distributions; Moreno-Jiménez and Escobar (2000) and Escobar and Moreno-Jiménez (2002) minimise the regret of the group; Moreno-Jiménez et al. (2005) search for the core of consistency and, more recently, Escobar and Moreno-Jiménez (2005) combine preference structures with Borda count methods in a new procedure which has been given the name *Aggregation of Individual Preference Structures* (AIPS).

Whilst all these approaches assume, either implicitly or explicitly, the existence of consensus between the individuals involved in the decision-problem resolution, they do not take into account the possibility of the existence of several/various groups of individuals with significant differences with respect to the priorities and the rankings of the alternatives of the problem. For this reason, the synthesis followed in traditional procedures carries with it the risk of providing positions that are not satisfactory for some individuals or groups. This problem could be especially acute if there are a large number of actors involved as, for example, in the e-democracy context (Moreno and Polasek 2003; Ríos Insua et al. 2003).

In this sense, it is necessary to develop methodologies that allow us to identify how many potential groups exist in the population of actors and what are their opinions with regards to the problem alternatives (patterns of behaviour in the judgement elicitation process). This identification and more specifically, the determination of their characteristics provides better knowledge of the decisional process and facilitates the subsequent negotiation processes which eventually lead to a greater degree of consensus in the final decision.

In the case of a large number of actors involved in the decision making process and one of the approaches most widely used in practical situations – the AHP – we advance a methodology to infer the preferences and to identify the patterns of behaviour of the entire population in a local context (a single criterion) from a stochastic perspective (Vargas 1982; Crawford and Williams 1985; Moreno-Jiménez and Vargas 1993; Basak 1998, 2001; Leskinen and Kangas 1998; Escobar and Moreno-Jiménez 2000, Aguarón and Moreno-Jiménez 2003).

The methodology uses a Bayesian hierarchical model based on mixtures of normal distributions to describe the distribution of the individual priorities and for the identification of the existing groups of the decision makers' preferences. Although the literature does not contain many references to Bayesian analysis in AHP, an increasing tendency to use this approach has been observed (Alho et al. 1996; Alho and Kangas 1997; Basak 1998, 2001), especially since the appearance of MCMC methods (Gilks et al. 1996; Robert and Casella 1999). The new method has the advantage of not imposing restrictions on the aggregation procedure for the judgements and priorities used in AHP. Moreover, it can easily be extended to more general situations with several criteria and incomplete and/or vague information about the pairwise comparison matrices.

The paper is organised as follows. Section 2 formulates the problem and describes the procedure proposed to solve it. Section 3 illustrates this methodology by analysing a case study adapted from a real experiment on e-democracy developed for the City Council of Zaragoza (Spain). Section 4 closes the paper with a review of the main conclusions. Finally, we have included two Appendices with the statistical details of the proposed methodology.

2. Methodology

2.1. Notation

Before describing the methodology used in the paper, let us briefly consider the notation and probability distributions employed. Vectors and matrices are denoted in bold: $\mathbf{0}_n$ is the $n \times 1$ null vector; $\mathbf{1}_n$ is the vector of n ones; $\mathbf{I}_n$ is the identity matrix $(n \times n)$ and $\mathbf{e}_{\ell,n} = (\delta_{\ell 1}, \ldots, \delta_{\ell n})'$ with $\delta_{ij} = 1$ if $i = j$ and 0 if $i \neq j$ and $\ell \in \{1, \ldots, n\}$ is the ℓth coordinate vector of $\mathbf{R}^n$. If X, Y are two random variables, $[X|Y]$ denote the conditional distribution X given Y. Furthermore, $N(\mu, \sigma^2)$ denotes the univariate normal distribution with mean μ and variance σ^2 and $N_p(\boldsymbol{\mu}, \Sigma)$ is the multivariate normal distribution with mean vector $\boldsymbol{\mu} \in \mathbf{R}^p$ and variance and covariance matrices $\Sigma_{(p \times p)}$.

2.2. Problem formulation

Let us assume a local context (a single criterion) and let $\mathbf{D} = \{D_1, \ldots, D_r, r \geq 2\}$ be a group of r decision makers that are a sample of a population $\mathbf{P}$.

Let $\{A^{(k)}; k = 1, \ldots, r\}$ be r individual pairwise comparison matrices corresponding to a set of alternatives $\{A_1, \ldots, A_n\}$ where $A^{(k)} = \left(a_{ij}^{(k)}\right)$ is a square positive matrix $(n \times n)$ verifying that $a_{ii}^{(k)} = 1, a_{ji}^{(k)} = \frac{1}{a_{ij}^{(k)}} > 0$ for $i, j = 1, \ldots, n$. The judgement $a_{ij}^{(k)}$ represents the relative importance of the alternative i versus the alternative j for the decision maker D_k, according to the fundamental scale proposed by Saaty (1980).

We consider the multiplicative model with logarithmic-normal errors to estimate the priorities. This model has been commonly used in the literature (Fitchner 1986; Genest and Rivest 1994; Alho and Kangas 1997; Aguarón and Moreno-Jiménez 2000; Laininen and Hämäläinen 2003) and is given by the following expression:

$$a_{ij}^{(k)} = \frac{v_i^{(k)}}{v_j^{(k)}} e_{ij}^{(k)}, \quad i = 1, \ldots, n-1; \;\; j = i+1, \ldots, n; \;\; k = 1, \ldots, r$$

where $\mathbf{v}^{(k)} = \left(v_1^{(k)}, \ldots, v_n^{(k)}\right)$ is the priority vector of the decision maker D_k without normalisation $\left(v_i^{(k)} \geq 0; i = 1, \ldots, n; k = 1, \ldots, r\right)$ and $e_{ij}^{(k)} \sim \mathrm{LN}\left(0, \sigma^{(k)2}\right)$. $\mathrm{LN}(\mu, \sigma^2)$ denotes the log-normal distribution with parameters μ and σ^2 (see, for example, Appendix A of Gelman et al. (2005)).

Taking the logarithms in the above expression we obtain:

$$y_{ij}^{(k)} = \mu_i^{(k)} - \mu_j^{(k)} + \varepsilon_{ij}^{(k)}; \quad i = 1, \ldots, n-1 \tag{1}$$

with $y_{ij}^{(k)} = \log a_{ij}^{(k)}, \mu_i^{(k)} = \log v_i^{(k)}, \varepsilon_{ij}^{(k)} = \log e_{ij}^{(k)} \sim N\left(0, \sigma^{(k)2}\right)$ independent $j = i+1, \ldots, n; k = 1, \ldots, r$. Furthermore, for each decision maker the alternative A_n is fixed as a reference alternative $(\mu_n^{(k)} = 0 \Leftrightarrow v_n^{(k)} = 1 \forall k = 1, \ldots, r)$ with the aim of avoiding identificability problems.

Using a matrix notation, the model (1) can be written as:

$$\mathbf{y}^{(k)} = \mathbf{X}\boldsymbol{\mu}^{(k)} + \boldsymbol{\varepsilon}^{(k)} \text{ with } \boldsymbol{\varepsilon}^{(k)} \sim N_J(\mathbf{0}, \boldsymbol{\sigma}^{(k)2}\mathbf{I}_J) \tag{2}$$

where $J = n(n-1)/2$ is the number of judgements that each decision maker should elicit, $\mathbf{y}^{(k)} = (y_{12}^{(k)}, y_{13}^{(k)}, \ldots, y_{n-1,n}^{(k)})'$, $\boldsymbol{\mu}^{(k)} = (\mu_1^{(k)}, \ldots, \mu_{n-1}^{(k)})'$, $\boldsymbol{\varepsilon}^{(k)} = (\varepsilon_{12}^{(k)}, \varepsilon_{13}^{(k)}, \ldots, \varepsilon_{n-1,n}^{(k)})'$ with $k = 1, \ldots, r$ and $\mathbf{X} = (x_{ij})$ is a $Jx(n-1)$ matrix with $x_{ij} = 1$, $x_{ik} = -1$ and $x_{i\ell} = 0$ if $\ell \neq j, k$ if the ith comparison made for the decision maker that involves alternatives A_j and A_k with $j<k<n$. If this comparison involves alternatives A_j and A_n then $x_{ij} = 1$ and $x_{i\ell} = 0$ if $\ell \neq j$.

The objective of the analysis is to determine the opinion status of the population **P** from which **D** is a sample. With this aim, we want to estimate the distribution $G(\boldsymbol{\mu})$ of the log-priorities vector $\boldsymbol{\mu} = (\mu_1, \ldots, \mu_{n-1})'$ with $\mu_i = \log(v_i); i = 1, \ldots, n-1$ and $\{v_i; i = 1, \ldots, n-1\}$ being the priorities corresponding to a generic individual of **P**. This distribution describes the opinion of the individuals of the population **P** about the alternatives of the problem and constitutes the central objective of the study. From this distribution, it would be possible to ascertain the most probable preference structure of the alternatives, which reflects the different opinion flows of the population. In addition, it allows for the identification of the different opinion groups, which can be extremely useful when establishing subsequent negotiation processes (see Section 3).

To estimate the unknown distribution G, we propose the use of a family of mixtures of normal distributions. Given that any continuous density on the real axis can be accurately approximated by a finite random mixture of normal densities, this family is both wide and flexible enough to model a significant number of situations (Richardson and Green 1997; Stephens 2000). Furthermore, and given that the population could consist of sub-populations that share similar beliefs, this family could be very useful to identify them in a similar way to the traditional Cluster analysis (Hand et al. 2001). In addition, if several opinion groups exist among the actors, with different priorities, it is possible to prove that if model (1) is true, then the MLE distribution of the log-priorities vector $\boldsymbol{\mu}$ will be, (approximately) a mixture of normal densities centred on the MLE priorities of each group.

2.3. Bayesian estimation of priorities

From the prior distribution for the model parameters and the likelihood of the model, the Bayesian approach provides the posterior distribution of the parameters by using Bayes Theorem. Furthermore, using the posterior distribution it will be possible to make inferences about G and to analyse the existence of opinion groups in $\mathbf{P}$. This sub-section describes how to specify these distributions and how to carry out inferences from them by means of Monte Carlo methods.

2.3.1. Prior distribution

As mentioned previously, in what follows we suppose that:

$$\boldsymbol{\mu}^{(k)} \sim G = \sum_{\ell=1}^{m} \pi_\ell N_{n-1}\left(\boldsymbol{\mu}_G^\ell, \Sigma_G^\ell\right) \quad k = 1, \ldots, r \tag{3}$$

Furthermore, it is assumed that:

$$\tau^{(k)} = \frac{1}{\sigma^{(k)2}} \sim \text{Gamma}\left(\frac{n_1}{2}, \frac{d_1}{2}\right); \quad k = 1, \ldots, r \text{ independent} \tag{4}$$

$$\boldsymbol{\pi} = (\pi_1, \ldots, \pi_m)' \sim \text{Dirichlet}(\mathbf{1}_m) \tag{5}$$

$$\boldsymbol{\mu}_G^{(\ell)} \sim N_{n-1}\left(\mathbf{0}_{n-1}, \frac{1}{\tau_{\mu_G}} \mathbf{I}_{n-1}\right); \quad \ell = 1, \ldots, m \text{ independent} \tag{6}$$

$$\Sigma_G^{(\ell)} \sim \text{IW}_{n-1}(n_0, n_0 \mathbf{D}_0); \quad \ell = 1, \ldots, m \text{ independent} \tag{7}$$

$$m \sim \text{ Poisson } (\lambda_0) \text{ truncated in } \{1, 2, \ldots.\} \tag{8}$$

where Gamma (p, a) denotes the gamma distribution with mean p/a and variance p/a^2, Dirichlet$(\alpha_1, \ldots, \alpha_m)$ the Dirichlet distribution, IWishart$_p(n, \mathbf{S})$ the inverse-Wishart distribution of dimension p with n degrees of freedom and scale matrix $\mathbf{S}$, Poisson(λ) denotes the Poisson distribution of mean λ and n_0, D_0, n_1, d_1, $\sigma^2_{\mu_G}$ and λ_0 are known constants. Definitions and characteristics of these distributions can be found in the Appendix A of Gelman et al. (2005).

These prior distributions are conjugated distributions commonly used in Bayesian inference. In particular, expression (3) supposes the existence in $\mathbf{P}$ of m internally homogeneous and externally heterogeneous opinion groups that capture a $\{100\pi_\ell; \ell = 1, \ldots, m\}$ percentage of the population, respectively. The members of each group have similar preferences and in such a way that their individual log-priority distributions are $\left\{N_{n-1}\left(\boldsymbol{\mu}_G^{(\ell)}, \Sigma_G^{(\ell)}\right); \ell = 1, \ldots, m\right\}$. The mean-vectors $\left\{\boldsymbol{\mu}_G^{(\ell)}; \ell = 1, \ldots, m\right\}$ synthesise the individual preferences of the members of these groups and represent their opinions.

We suppose the unknown number of groups oscillates around λ_0 with a distribution given by (8). The size of these groups $\boldsymbol{\pi} = (\pi_1, \ldots, \pi_m)$ is also unknown and we assume, under the insufficiency ratio principle, that it follows a distribution given by (5), where $E[\pi_\ell] = \frac{1}{m}; \ell = 1, \ldots, m$. For the unknown group parameters

$\left\{\left(\boldsymbol{\mu}_G^{(\ell)}, \Sigma_G^{(\ell)}\right); \ell = 1, \ldots, m\right\}$, we consider the prior distributions given by (6) and (7), in which $\left\{E\left[\boldsymbol{\mu}_G^{(\ell)}\right] = \mathbf{0}_{n-1}, E\left[\left(\Sigma_G^{(\ell)}\right)^{-1}\right] = \mathbf{D}_0; \ell = 1, \ldots, m\right\}$ are assumed, that is to say, all the alternatives are equally preferred a priori (principle of insufficient reason). Parameters $\tau_{\mu_G} \geq 0$ and n_0>0 control the influence of (6) and (7) on the estimation of parameters $\left\{\left(\boldsymbol{\mu}_G^{(\ell)}, \Sigma_G^{(\ell)}\right); \ell = 1, \ldots, m\right\}$. The smaller their values, the greater is the diffuseness of these distributions and the smaller their influence. In order to guarantee a proper posterior distribution we impose $n_0 \geq n - 1$. Finally, the parameters $\left\{\sigma^{(k)2}; k = 1, \ldots, r\right\}$ represent the prior unknown inconsistency levels of the decision makers. These values oscillate around $\left\{\left(E\left[\tau^{(k)}\right]\right)^{-1} = \frac{d_1}{n_1}; k = 1, \ldots, r\right\}$ following the distribution (3).

In the following paragraph we propose a procedure to specify the parameters in a non-informative way and according to the expected consistency level of the decision makers.

2.3.2. Specification of the prior distribution for the parameters: n_0, n_1, τ_{μ_G}, D_0, d_1 and λ_0

In order to assume a diffuse prior, we take $n_0 = n - 1$, $n_1 \approx 0$ and $\tau_{\mu_G} \approx 0$ as are usually taken in the Bayesian literature. The values of d_1 and D_0 are chosen, depending on the expected value of the consistency index of the individuals. With this aim, we fix an upper limit, $e_{\max}$>0, for the error committed in the judgement elicitation process in such a way that with a confidence level $1 - \alpha$, $\alpha \in (0, 1)$, it is verified that:

$$P\left[\frac{1}{1 + e_{\max}} \leq e_{ij} = \frac{a_{ij}}{v_i/v_j} \leq 1 + e_{\max}\right] = 1 - \alpha \quad \forall i \neq j \in \{1, \ldots, n\} \tag{9}$$

If model (1) is true, it follows that:

$$P\left[\exp\left[-z_{\alpha/2}\sigma\right] \leq \frac{a_{ij}}{v_i/v_j} \leq \exp\left[z_{\alpha/2}\sigma\right]\right] = 1 - \alpha \tag{10}$$

where z_α is the $(1 - \alpha)$ -quantile of an $N(0, 1)$ distribution. From (9) and (10) we have that $\sigma^2 = \left(\frac{\log(1+e_{\max})}{z_{\alpha/2}}\right)^2 = \sigma_0^2$.

If (4) is the prior distribution of σ^2, we have that $E\left[\frac{1}{\sigma^2}\right] = \frac{n_1}{d_1}$. Equalising this value to σ_0^2, then $d_1 = n_1\left(\frac{\log(1+e_{\max})}{z_{\alpha/2}}\right)^2$ is a possible value of d_1.

In addition, we take $D_0 = \sigma_0^2 \mathbf{I}_{n-1}$ given that this does not postulate any kind of prior relationship among the estimations of the components of $\boldsymbol{\mu}$.

Finally, λ_0 would be an initial estimation of the number of the opinion groups (m) existing in $\mathbf{P}$ ($\lambda_0 = 1$ if it is assumed that there is consensus in the whole population).

2.3.3. Posterior distribution

Let $\theta = \left(\left(\boldsymbol{\tau}^{(k)}\right)_{k=1}^r, \left(\boldsymbol{\mu}^{(k)}\right)_{k=1}^r, m, \left(\boldsymbol{\mu}_G^{(\ell)}, \Sigma_G^{(\ell)}\right)_{\ell=1}^m, \boldsymbol{\pi} = (\pi_\ell)_{\ell=1}^m\right)$ be the parameter vector of the model (2)–(3). Given that the approximation adopted in this paper is a

Bayesian approach, the inferences about $\boldsymbol{\theta}$ are made from the posterior distribution. This distribution is calculated by means of the Bayes theorem and is described in the Appendix A. Due to the fact that it has no tractable analytical form, we use MCMC methods (Robert and Casella 1999) in order to calculate it.

Appendix B describes the algorithm used to draw a sample from this distribution. From this sample it will be possible to make inferences on the different components of $\boldsymbol{\theta}$ and, in particular, on the distribution G. From this information, and if there was consensus, the decision making process would be carried out; otherwise, a negotiation process should begin. In the following section we illustrate the proposed methodology with the study of a real case.

The previous results have been obtained by assuming the decision makers provide all the judgments in the pairwise comparison matrix $(n(n-1)/2$ entries). When a decision maker does not provide all the judgments, equation (2) must be replaced by $y^{(k)} = X^{(k)}\mu^{(k)} + \varepsilon^{(k)}$ with $\varepsilon^{(k)} \sim N_{J_k}\left(0, \sigma^{(k)2} I_{J_k}\right)$, where $X^{(k)}$ is a $J_k x(n-1)$ matrix containing the rows of X corresponding to the J_k judgements elicited by D_k, but the results only remain valid on updating the expressions.

3. Case Study

3.1. Problem formulation

In order to illustrate the proposed methodology, we have applied it to a case study adapted from a real experience of our research group ("Zaragoza Multicriteria Decision Making Group", GDMZ) for the City Council of Zaragoza, (Spain) (http://www.zaragoza.es/presupuestosparticipativos/ElRabal/). This experience consists of an application of the new democratic system, known as e-cognocracy (Moreno-Jiménez and Polasek 2003), to an e-participatory budget allocation problem. It responds to the suggestion made in the framework of e-democracy by Simon French (http://esc.org/TED), as this relates to the incorporation of the different perspectives and the extraction and social diffusion of knowledge (French 2004).

The amount of the budget that the municipal district of El Rabal (Zaragoza) assigns to each one of four alternatives proposed by the Neighbourhood Associations and the Members of the District Council was determined by using AHP as the multicriteria methodological support and the Internet as the communication tool to extract the individual's preferences. The four alternatives, A1: Elimination of Architectural Barriers; A2: Urban Green Space; A3: Youth Activity Areas and A4: Street Cleaning were prioritised by taking into account a total of three criteria and six sub-criteria.

In what follows, we only consider the preferences of the seven Neighbourhood Associations that participated in the project, given by way of their representatives, with respect to one of the most important aspects of the problem – the environmental subcriteria called "Prevention". These Associations, together with the number of their individuals who were considered in this adapted case to elicit their preferences, can be seen in Table 1. For each of these 100 decision makers, a 4×4

Table 1. Associations and votes.

Associations	Votes
Picarral	20
Jota	20
Arrabal	20
Jesús	20
Zalfonada	10
Ríos_de_Aragón	5
Teniente_Polanco	5
Total	100

pairwise comparison matrix (six judgments) was obtained from the initial data provided by the members of the Associations. These matrices reflect the preferences of the actors between the four alternatives with respect of a single criterion (Prevention).

The methodology discussed in Section 2 has been applied by taking $n_0 = 1$, $n_1 = 6$ and $\tau_{\mu_G} = 0.01$ as the values for the parameters of the prior distribution. The values of d_1 and D_0 were assigned using the procedure described in the previous section with $\alpha = 0.01$ and $e_{\max} = 0.85$. These values correspond to the level of inconsistency $(\sigma_0^2 \approx 0.1)$ suggested by Genest and Rivest (1994). Finally, we have taken $\lambda_0 = 1$, i.e., we a priori, expect that G is a normal distribution $(\lambda_0 = E(m) = 1)$.

The algorithm described in Appendix B was run during $\mathrm{IT}_{\max} = 20,000$ with $\mathrm{IT}_{\mathrm{GS}} = 2$ and taking different values for the model parameters. The convergence was quick and determined from visual inspections of the chain and by applying the procedures described in Gelman and Rubin (1992) and Geweke (1992) for a random sample of the parameters $\left\{\boldsymbol{\theta}_1^{(k)} = (\mu_1^{(k)}, \mu_2^{(k)}, \mu_3^{(k)}, \tau^{(k)}); k = 1, \ldots, 100\right\}$. Finally, we discarded the first $\mathrm{it}_0 = 10,000$ iterations and in order to get an approximately random sample, we took $s = 10$, where s is the number of retards to neglect the serial autocorrelation. Thus the size of the used sample is 1000.

3.2. *Priorities distribution*

Posterior distributions of priorities $\{w_i = e^{\mu_i} / \sum_{i=1}^{4} e^{\mu_i};\ i = 1, \ldots, 4\}$, normalised in a distributive mode $(\sum_i w_i = 1)$, are shown in Figure 1. These distributions have been calculated with the help of Kernel estimators applied to a sample of these priorities obtained by Monte Carlo method from the expression (A.1). In addition, point estimations given by the posterior median and 95% Bayesian credibility intervals obtained from the distribution quantities 2.5 and 97.5 were calculated and are showed in Table 2. Several modes in these distributions, representing non-homogeneous decision makers can be observed.

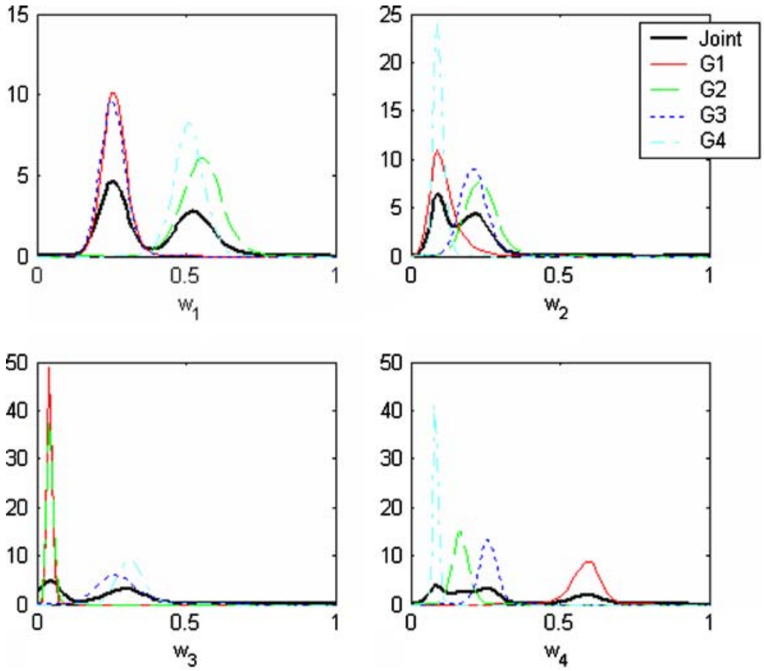

Figure 1. Posterior distribution of the priorities $\{w_{\mathbf{i}}; i = 1, \ldots, 4\}$ joint and for groups.

With the aim of identifying the opinion groups, we first analysed the posterior distribution of the number of components for the mixture (3). In 95.20% of the iterations this number was equal to 4; in 4.60% it was equal to 5 and in the rest, some 0.20%, it was equal to 6. Thus, the number of recommended groups is 4.

In order to identify these groups, we employed a hierarchical cluster algorithm based on the similarity matrix among decision makers $\frac{1}{1000}\sum_{\text{it}=1}^{1000} \mathbf{z}^{(\text{it})} z^{(\text{it})'}$ constructed from the group indicators $\{\mathbf{z}^{(\text{it})}; \text{it} = 1, \ldots, 1000\}$ defined on Appendix A. Figure 2 shows the results obtained with this algorithm when using an average linkage intra-groups (other linkages such as complete or intra-groups provide very similar results). The size of the four groups considered is shown on Table 2. The biggest group is the third one (around 32% of the population), with the size of the remaining groups being similar.

Once the groups were identified, we analysed the opinions shared by their members. We estimated the posterior distribution for the log-priorities in each group by means of the expression:

$$\frac{1}{|A|}\sum_{\text{it}\in A} N_3\left(\boldsymbol{\mu}_G^{(\text{it},\ell)}, \Sigma_G^{(\text{it},\ell)}\right); \quad \ell = 1, \ldots, 4 \tag{11}$$

with $A = \left\{\text{it} \in \{1, \ldots, 1000\} \text{ with } m^{(\text{it})} = 4\right\}$. In (11) we identify the parameters $\{(\boldsymbol{\mu}_G, \Sigma_G); \ell = 1, \ldots, 4\}$ of the mixture (3) according to the increasing values of

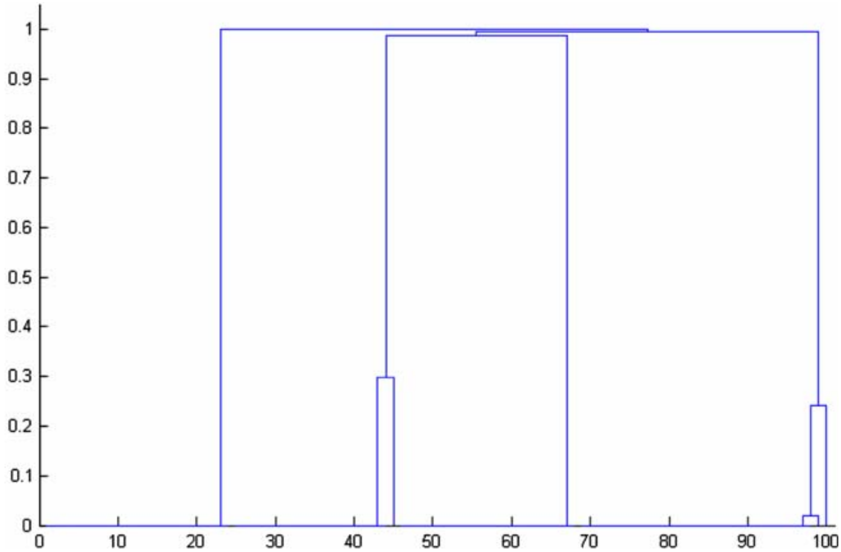

Figure 2. Dendrogram of the hierarchical agglomerative algorithm with average linkage intergroups used to identify the opinion groups.

Table 2. 95% Bayesian credibility intervals of the size of the groups $\{\pi_i; i = 1, \ldots, 4\}$.

Groups	1 (%)	2 (%)	3 (%)	4 (%)
Quantile 2.5	15.60	14.23	22.59	16.42
Mean	23.23	21.08	31.66	24.03
% Decision makers	23.00	21.00	32.00	24.00
Quantile 97.5	32.07	29.11	41.07	31.81

$\{(\mathbf{v}'\boldsymbol{\mu}_G); \ell = 1, \ldots, 4\}$ where $\mathbf{v}$ is the eigenvector of the first discriminate function of a *Linear Discriminate Analysis* of the groups when employing as data the sample of the individual log-priorities $\left\{\left(\boldsymbol{\mu}^{(\mathrm{it},k)}\right)_{k=1}^{100}; \mathrm{it} \in 1, \ldots, 1000 : m^{(\mathrm{it})} = 4\right\}$. From the distributions (11) we calculate, by Monte Carlo methods, point estimations and Bayesian credibility intervals of the normalised priorities of the groups, applying a similar procedure to that described for the distributions corresponding to the joint distribution (A.1). The results are shown in Figure 1 and Table 3.

For each group and the joint distribution (A.1), we also calculated the posterior distributions of the most preferred alternative (Table 4 and Figure 3) and the preference structures (Table 5 and Figure 4). These figures show the perceptual maps obtained when applying a multi-dimensional scaling to these distributions (see Moreno-Jiménez et al. (2005) for more details).

Finally, Figure 5 displays four compositional ternary diagrams (Aitchinson 1986) which represent the MLE estimations of the priorities $\{w_i; i = 1, \ldots, 4\}$ for each decision maker and the sample of normalised priorities $\left\{\left(\mathbf{w}_G^{(\mathrm{it},\ell)} = \right.\right.$

Table 3. 95% Bayesian credibility intervals of the priorities $\{w_i; i = 1, \ldots, 4\}$.

Groups	Priorities	w_1	w_2	w_3	w_4
1	Quantile 2.5	0.1863	0.0448	0.0277	0.4822
	Mean	0.2585	0.1122	0.0428	0.5848
	Quantile 97.5	0.3429	0.2375	0.0616	0.6680
2	Quantile 2.5	0.4153	0.1382	0.0255	0.1151
	Mean	0.5528	0.2364	0.0448	0.1665
	Quantile 97.5	0.6916	0.3504	0.0708	0.2246
3	Quantile 2.5	0.1768	0.1304	0.1547	0.1972
	Mean	0.2527	0.2116	0.2761	0.2580
	Quantile 97.5	0.3393	0.3067	0.4342	0.3180
4	Quantile 2.5	0.4071	0.0626	0.2328	0.0673
	Mean	0.5083	0.0934	0.3143	0.0844
	Quantile 97.5	0.6041	0.1330	0.4076	0.1032
Joint	Quantile 2.5	0.1927	0.0598	0.0294	0.0729
	Mean	0.3129	0.1650	0.1806	0.2741
	Quantile 97.5	0.6349	0.3105	0.3964	0.6404

Table 4. Posterior distributions of the most preferred alternative.

Alternative/group	1 (%)	2 (%)	3 (%)	4 (%)	Joint
A_1	0.03	99.22	24.26	97.70	51.54
A_2	0.09	0.77	7.85	0.00	2.41
A_3	0.00	0.00	46.47	2.30	15.26
A_4	99.88	0.01	21.42	0.00	30.79
Total	100	100	100	100	100

$\left(w_{G1}^{(\text{it},\ell)}, w_{G2}^{(\text{it},\ell)} \cdot w_{G3}^{(\text{it},\ell)} \cdot w_{G4}^{(\text{it},\ell)}\right)\}_{\ell=1}^{m^{(\text{it})}}; \text{it} = 1, \ldots, 1000\}$ where $w_{Gi}^{(\text{it},\ell)} = \frac{\exp\left[\mu_{Gi}^{(\text{it},\ell)}\right]}{\sum_{j=1}^{4} \exp\left[\mu_{Gj}^{(\text{it},\ell)}\right]}; i = 1, .., 4$ with $\boldsymbol{\mu}_G^{(\text{it},\ell)} = \left(\mu_{G1}^{(\text{it},\ell)}, \mu_{G2}^{(\text{it},\ell)} \cdot \mu_{G3}^{(\text{it},\ell)} \cdot \mu_{G4}^{(\text{it},\ell)}\right)$ obtained from the sample (B.3) of the distribution (A.1).

The results obtained show that the opinions of the four groups are clearly different. The decision makers of the groups 1 (about 23% of the population) and 2 (about 21%) are homogeneous in their preferences, and they show (see Table 5 and Figure 4) a marked preference for the rankings $A_4 > A_1 > A_2 > A_3$ (group 1) and $A_1 > A_2 > A_4 > A_3$ (group 2). On the other hand, for the decision makers of the group 4 (24% of the population, approximately) the alternatives A_1 and A_3 are the first and the second most preferred, respectively and they are indifferent between the alternatives A_2 and A_4 (see Table 5). Finally, the third group (about 32% of the population) shows a higher preference for the alternative A_3 (see Table 4) being indifferent in its preferences with respect to the rest of the alternatives (see Table 3 and Figure 5).

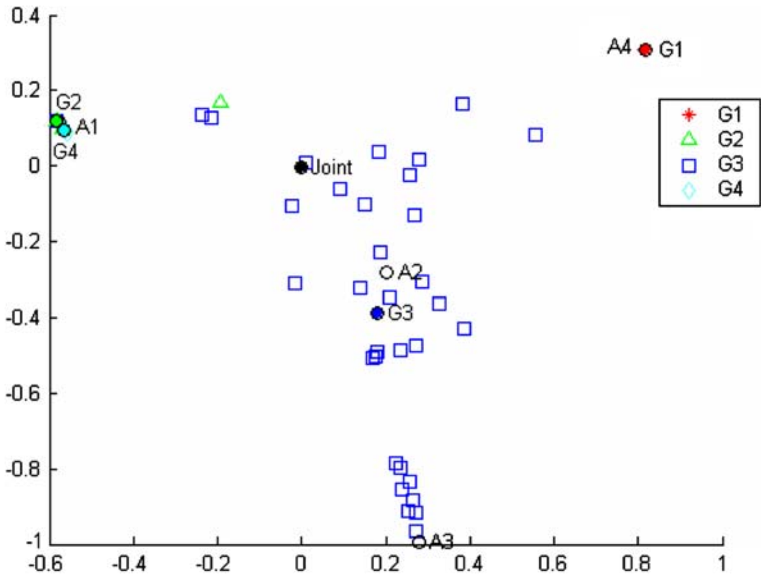

Figure 3. Perceptual map of the individual alfa distributions, distinguishing by groups, of the joint alfa distribution, the alfa distribution for each group $(G_i; i = 1, \ldots, 4)$ and the alfa distributions degenerated in each alternative $\{A_i; i = 1, \ldots, 4\}$.

Table 5. Posterior distributions of preference structures.

Structures/groups	1 (%)	2 (%)	3 (%)	4 (%)	Joint
1243[a]	0.00	91.07	1.89	0.00	20.62
1324	0.00	0.00	0.52	69.20	16.95
1342	0.00	0.00	5.08	29.96	8.27
1423	0.03	8.15	7.50	0.00	3.69
1432	0.00	0.00	9.91	0.00	2.79
3142	0.00	0.00	15.70	0.84	5.15
3412	0.00	0.00	19.26	0.00	6.02
4123	92.73	0.00	5.94	0.00	24.35
4132	3.47	0.00	6.21	0.00	3.02
Other	3.77	0.78	27.99	0.00	9.14
Total	100	100	100	100	100

[a]1243 means $A_1 > A_2 > A_4 > A_3$.

Furthermore, it can be appreciated that the model has captured the preference patterns of the sample in that the majority of this sample $\left\{ w_{Gi}^{(\mathrm{it},\ell)}; \ell = 1, \ldots, m^{(\mathrm{it})}; \mathrm{it} = 1, \ldots, 1000; \right\}$ is concentrated in the zones where the MLE estimations corresponding to each individual are also concentrated (see Figure 5).

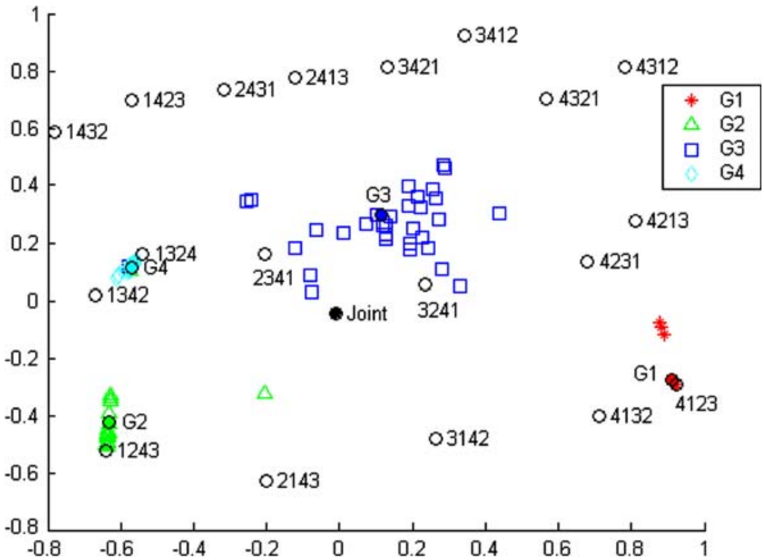

Figure 4. Perceptual map of the individual gamma distributions, distinguishing by groups, of the joint gamma distribution, the gamma distribution for each group $(G_i; i = 1, \ldots, 4)$ and the gamma distributions degenerated in each alternative $\{A_i; i = 1, \ldots, 4\}$.

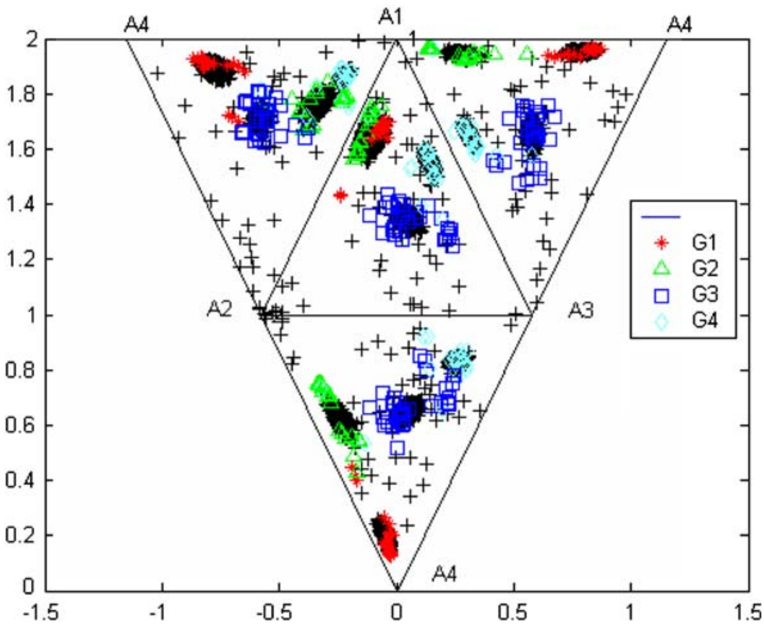

Figure 5. Compositional ternary diagrams of the individual priorities estimated by maximum likelihood (*) distinguishing by groups. In addition, note the superimposed sample of the joint posterior distribution of the priorities $\left\{w_G^{(\ell)}; \ell = 1, \ldots, 4\right\}$ (+).

Table 6. Composition of the groups for associations.

Group/association	Picarral	Jota	Arrabal	Jesús	Zalfonada	Ríos Aragón	Tte. Polanco
1	0	3	10	0	0	5	5
2	0	12	9	0	0	0	0
3	0	3	0	9	20	0	0
4	20	2	1	1	0	0	0
Total	20	20	20	10	20	5	5

Table 7. Sensitivity study of the posterior distribution of the number of groups.

m	$e_{\max} = 0.70$, $\lambda_0 = 1$	$e_{\max} = 0.85$, $\lambda_0 = 1$	$e_{\max} = 1.0$, $\lambda_0 = 1$
3	0.00	0.00	96.75
4	96.45	95.20	3.25
5	3.55	4.80	0.00
6 or more	0.20	0.00	0.00

Table 8. Cross tabulation of groups obtained in the sensitivity study.

		Groups $e_{\max} = 1.0, \alpha = 0.01, \lambda_0 = 1$		
		1	2	3
Groups $e_{\max} = 0.85, \alpha = 0.01, \lambda_0 = 1$	1	23	0	0
	2	1	0	20
	3	0	30	2
	4	0	1	23

Table 6 shows the results obtained when crossing the groups with the Neighbourhood Associations to which each individual of the sample belongs. Here, we can note a clear pattern of voting by associations: group 1 contains the members of *Ríos de Aragón*, *Teniente Polanco* and 50% of *Arrabal*; group 2 those of *La Jota* and 49% of *Arrabal*; group 3 those of *Jesús* and *Zalfonada* and group 4 all the members of *Picarral* and some separate members of other associations.

The detection of the different patterns of behaviour and their social diffusion corresponds to the ideas suggested by Simon French when the author speaks of e-democracy (French 2004).

A sensitivity study was carried out with respect to the parameters of the prior distribution (4)–(8). The most influential parameters were $e_{\max}$, $1 - \alpha$ and λ_0. Table 7 shows the posterior distribution of m for reasonable different values of $e_{\max}$ (results for other values of α and λ_0 were similar and are omitted for the sake of brevity).

It can be noted that the posterior distribution is robust for values of $e_{\max}$ lower than 0.85. However, for values greater than the threshold $e_{\max} = 0.85$, the posterior distribution of m is sensitive and it tends to be concentrated around 3. The increase of the prior inconsistency level (σ_0^2) makes it possible to reduce the number of groups through joining groups 2 and 4 (see Table 8 and Figure 6).

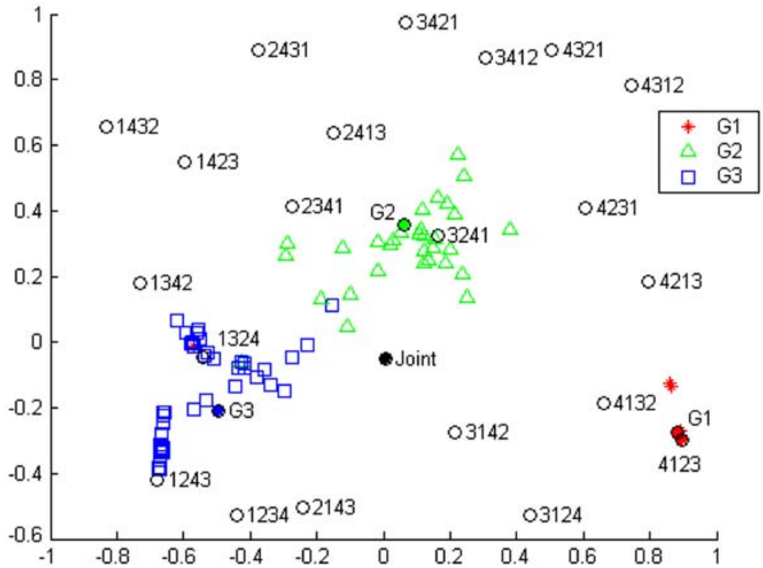

Figure 6. Perceptual map of the individual gamma distributions, distinguishing by groups, of the joint gamma distribution, the gamma distribution for each group $(G_i; i = 1, \ldots, 4)$ and the gamma distributions degenerated in each alternative $\{A_i; i = 1, \ldots, 4\}$ for $e_{\max} = 0.85$, $\lambda_0 = 1$ and $\alpha = 0.01$.

4. Conclusions

This paper proposes a methodology to determine the priorities of the AHP in group decision making with a large number of actors involved in the resolution process, without imposing any prior restrictions about the existence of consensus among them.

Supposing a local context (a single criterion) and a large number of decision makers, the methodology allows us to extract the existing opinion patterns of the decision makers when eliciting their preferences about the alternatives of the problem. Similarly, it makes it possible to estimate the priorities associated with the different identified groups.

We have adopted a Bayesian hierarchical approach, which uses mixtures of normal distributions with an unknown number of components to represent the distribution of the actors' preferences. By using these methods a statistical procedure to identify the existing opinion groups and to estimate their preferences is provided.

The methodology is illustrated by an e-democracy application to a budget allocation problem developed for the City Council of Zaragoza. For one of the most important attributes of the problem, we have identified the different groups for the priorities and rankings of the alternatives.

The proposed approach allows us to consider a wide range of probability distributions to describe the preferences of the actors in a group decision making problem and to work with incomplete and imprecise pairwise comparison matrices. In addition, this methodology can be extended to the case of hierarchies and networks, an area that will be studied in the future.

Acknowledgements

The authors wish to express their thanks to three anonymous referees for their helpful observations and suggestions on two earlier versions of this paper which we believe have served to significantly improve the quality of this revised text. They are also grateful to Stephen Wilkins for his help in preparing the final draft of the manuscript.

Appendix A

Posterior distribution of θ

In order to make the calculation of the posterior distribution easier, the auxiliary vectors $\{\mathbf{z}_k = (z_{k1}, \ldots, z_{km})'; k = 1, \ldots, r\}$ are introduced. These vectors indicate the component of the mixture (3) to which the decision maker D_k belongs, in such a way that $\mathbf{z}_k = \mathbf{e}_{\ell,m}$ with probability $\pi_\ell; \ell = 1, \ldots, m$.

Using the Bayes Theorem the joint posterior distribution of $\boldsymbol{\theta}$ and $\mathbf{z} = (\mathbf{z}_1, \ldots, \mathbf{z}_r)$, $(\boldsymbol{\theta}, \mathbf{z})|\mathbf{y}^{(1)}, \ldots, \mathbf{y}^{(r)}$, will be given by

$$\begin{aligned}
&\left[(\boldsymbol{\theta},\mathbf{z})\middle|\mathbf{y}^{(1)},\ldots,\mathbf{y}^{(r)}\right]\propto\\
&\propto\left[\mathbf{y}^{(1)},\ldots,\mathbf{y}^{(r)}|(\boldsymbol{\theta},\mathbf{z})\right][(\boldsymbol{\theta},\mathbf{z})]=\left\{\prod_{k=1}^{r}\left[\mathbf{y}^{(k)}|(\boldsymbol{\theta},\mathbf{z}_k)\right][\mathbf{z}_k|\boldsymbol{\theta}]\right\}[\boldsymbol{\theta}]\propto\\
&\propto\prod_{k=1}^{r}\left\{\left[\mathbf{y}^{(k)}|\boldsymbol{\mu}^{(k)},\boldsymbol{\tau}^{(k)}\right]\left[\boldsymbol{\tau}^{(k)}\right]\prod_{\ell=1}^{m}\left[\boldsymbol{\mu}^{(k)}\middle|\boldsymbol{\mu}_G^{(\ell)},\Sigma_G^{(\ell)}\right]^{z_{k\ell}}\boldsymbol{\pi}_\ell^{z_{k\ell}}\right\}\prod_{\ell=1}^{m}\left\{\left[\boldsymbol{\mu}_G^{(\ell)}\right]\left[\Sigma_G^{(\ell)}\right]\right\}[\boldsymbol{\pi}|m][m]\\
&\propto\prod_{k=1}^{r}\left\{\left(\tau^{(k)}\right)^{\frac{J}{2}-1}\exp\left[-\boldsymbol{\tau}^{(k)}\frac{\left(\mathbf{y}^{(k)}-\mathbf{X}\boldsymbol{\mu}^{(k)}\right)'\left(\mathbf{y}^{(k)}-\mathbf{X}\boldsymbol{\mu}^{(k)}\right)}{2}\right]I_{(0,\infty)}\left(\boldsymbol{\tau}^{(k)}\right)\right\}\times\\
&\times\prod_{\ell=1}^{m}\left\{\prod_{k=1}^{r}\left\{\left|\Sigma_G^{(\ell)}\right|^{-\frac{z_{k\ell}}{2}}\exp\left[-\frac{z_{k\ell}\left(\boldsymbol{\mu}^{(k)}-\boldsymbol{\mu}_G^{(\ell)}\right)'\left(\Sigma_G^{(\ell)}\right)^{-1}\left(\boldsymbol{\mu}^{(k)}-\boldsymbol{\mu}_G^{(\ell)}\right)}{2}\right]\right\}\right\}\times\\
&\times\prod_{\ell=1}^{m}\left\{\exp\left[-\boldsymbol{\tau}_{\mu_G}\frac{\left(\boldsymbol{\mu}_G^{(\ell)}\right)'\mu_G^{(\ell)}}{2}\right]\left|\Sigma_G^{(\ell)}\right|^{-\frac{n_0+n-1}{2}}\exp\left[-\frac{1}{2}\mathrm{trace}\left(n_0\mathbf{D}_0\left(\Sigma_G^{(\ell)}\right)^{-1}\right)\right]I_S\left(\Sigma_G^{(\ell)}\right)\right\}\times\\
&\times\left\{\prod_{\ell=1}^{m}\pi_\ell^{z_{k\ell}}\right\}I_{P_m}(\boldsymbol{\pi})\frac{\lambda^m}{m!}
\end{aligned}\tag{A.1}$$

where $I_A(x) = 1$ if $x \in A$ and 0 otherwise, $S = \{\Sigma_{(n-1)\times(n-1)}$ half-defined positive and symmetric$\}$ and $\prod_{\mathbf{m}} = \left\{\boldsymbol{\pi} \in \mathbf{R}^m : \sum_{\ell=1}^{m} \pi_\ell = 1 \text{ and } \pi_\ell \geq 0; \ell = 1, \ldots, m\right\}$.

Given that distribution (A.1) has no tractable analytical form, we use MCMC methods in order to draw a sample that allows us to make inferences on the different components of the parameter vector $\boldsymbol{\theta}$. In the following paragraph we describe the algorithm used to draw this sample.

Appendix B

Algorithm to draw a sample from the posterior distribution (A.1)

Note that, for each value of m, we have a two-level family of hierarchical models, given by the following equations:

$$\mathbf{EQ}_1 : y_{\mathrm{ij}}^{(\mathrm{k})} = \mu_{\mathrm{i}}^{(k)} - \mu_j^{(\mathrm{k})} + \varepsilon_{\mathrm{ij}}^{(\mathrm{k})}; \quad i = 1, \ldots, n-1; \; j = i+1, \ldots, n; \; k = 1, .., r$$

$$\mathbf{EQ}_{2,m} : \boldsymbol{\mu}^{(k)} \sim G = \sum_{\ell=1}^{m} \pi_\ell N_{n-1}\left(\boldsymbol{\mu}_G^{(\ell)}, \Sigma_G^{(\ell)}\right) \quad k = 1, \ldots, r$$

where the parameter vector $\boldsymbol{\theta}$ is $\boldsymbol{\theta} = (\boldsymbol{\theta}_1, m, \boldsymbol{\theta}_{2,m})$, with $\boldsymbol{\theta}_1 = \left\{\left(\tau^{(k)}, \boldsymbol{\mu}^{(k)}\right),\right.$ $\left. k = 1, \ldots, r\right\}$ and $\boldsymbol{\theta}_{2,m} = \left\{\left(\pi_\ell, \boldsymbol{\mu}_G^{(\ell)}, \Sigma_G^{(\ell)}\right), \; \ell = 1, \ldots, m\right\}$ whose dimensionality changes according to the value of m.

Therefore, there is a space of models underlying the problem, $\{\mathbf{M}_m; m = 1, 2, \ldots\}$, where $\mathbf{M}_m$ is given by equations EQ_1 and $\mathrm{EQ}_{2,m}$. With the aim of exploring this space and obtaining samples of the posterior distribution (A.1), we have followed the methodology based on birth–death Markov processes developed by Stephens (2000).

In order to explain the algorithm, we define the set $\Omega = \bigcup_{m=1}^{\infty} \Omega_m$ where

$$\Omega_m = \left\{\left(\pi_1, \boldsymbol{\mu}_G^{(1)}, \Sigma_G^{(1)}\right), \ldots, \left(\pi_m, \boldsymbol{\mu}_G^{(m)}, \Sigma_G^{(m)}\right) : \boldsymbol{\pi} \in P_m, \boldsymbol{\mu}_G^{(k)} \in R^{n-1}, \Sigma_G^{(k)} \in S\right\}$$

is the parameter space of the distribution G for a fixed number m of mixture components (3). Therefore, Ψ constitutes the model space in which the process of global exploration is developed.

This global exploration process consists of the choice of the movement and its implementation. There are two types of possible movements: incorporation (birth) of a new component of the mixture (3) or removal (death) of one of the components. We define births and deaths on Ψ as follows:

A birth or incorporation of a new component $\left(\pi_{m+1}, \boldsymbol{\mu}_G^{(m+1)}, \Sigma_G^{(m+1)}\right)$ is said to occur when the process jumps from Ω_m to Ω_{m+1} where:

$$\Omega_{m+1}=\left\{\left(\pi_1(1-\pi_{m+1}),\boldsymbol{\mu}_G^{(1)},\Sigma_G^{(1)}\right),\ldots,\left(\pi_m(1-\pi_{m+1}),\boldsymbol{\mu}_G^{(m)},\Sigma_G^{(m)}\right),\left(\pi_{m+1},\boldsymbol{\mu}_G^{(m+1)},\Sigma_G^{(m+1)}\right)\right\}$$

$$\pi_{m+1}\sim\text{Beta}(1,m),\mu_G^{(m+1)}\sim N_{n-1}\left(0_{n-1},\frac{1}{\tau_{\mu_G}}I_{n-1}\right)\text{and}\,\Sigma_G^{(m+1)}\sim\text{IW}(n_0,\mathbf{D}_0) \quad \text{(B.1)}$$

The death or removal of the ℓ th component of Ω_m, $\left(\pi_\ell, \boldsymbol{\mu}_G^{(\ell)}, \Sigma_G^{(\ell)}\right)$ is said to occur when the process jumps from Ω_m to $\Omega_{m-1}^{(-\ell)}$ where: $\Omega_{m-1}^{(-\ell)=}$

$$\left\{\left(\frac{\pi_1}{(1-\pi_\ell)}, \boldsymbol{\mu}_G^{(1)}, \Sigma_G^{(1)}\right), \ldots, \left(\frac{\pi_{\ell-1}}{(1-\pi_\ell)}, \boldsymbol{\mu}_G^{(\ell-1)}, \Sigma_G^{(\ell-1)}\right), \left(\frac{\pi_{\ell+1}}{(1-\pi_\ell)}, \boldsymbol{\mu}_G^{(\ell+1)}, \Sigma_G^{(\ell+1)}\right),\right.$$
$$\left.\ldots, \left(\tfrac{\pi_m}{(1-\pi_\ell)}, \boldsymbol{\mu}_G^{(m)}, \Sigma_G^{(m)}\right)\right\} \quad \text{(B.2)}$$

The algorithm consists of two steps that are alternated in each iteration. In the first step, a global exploration of the model space $\{\mathbf{M}_m, m = 1, 2, \ldots\}$ is carried out by means of birth–death point processes with reversible jumps developed by Stephens (2000). In the second, a local exploration for a model $\mathbf{M}_m$ with m fixed is carried out by applying Gibbs sampling (Robert and Casella 1999) to the model $\mathbf{M}_m$ during a number I of iterations fixed beforehand by the analyst.

The scheme of the algorithm is as follows:

Step 0: Start The maximum number of algorithm iterations, $\text{IT}_{\max}$ and the number of iterations for the Gibbs sampling, IT_{GS}, used for carrying out a Bayesian analysis of each model explored by the algorithm, are fixed. $m^{(0)}$ is extracted from a Poisson(λ_0) and a set of objects

$$\Omega_{m^{(0)}}^{(0)} = \left\{\left(\pi_1^{(0)}, \mu_G^{(0,1)}, \Sigma_G^{(0,1)}\right), \ldots, \left(\pi_{m^{(0)}}^{(0)}, \mu_G^{(0,m^{(0)})}, \Sigma_G^{(m^{(0)})}\right)\right\}$$

is obtained using, for example, the prior distribution (4)–(8). Thereafter, the auxiliary vectors $\mathbf{z}^{(0)} = \left\{\mathbf{z}_1^{(0)}, \ldots, \mathbf{z}_r^{(0)}\right\}$ with $\mathbf{z}_k^{(0)} = \left(z_{k1}^{(0)}, \ldots, z_{km^{(0)}}^{(0)}\right)'$; $k = 1, \ldots, r$ are generated in such a way that $\mathbf{z}_k^{(0)}$ is equal to the ℓ th coordinate vector of $\mathbf{R}^{m^{(0)}}$ with probability $\pi_\ell^{(0)}$; $\ell = 1, \ldots, m^{(0)}$ and, from these $\{\boldsymbol{\mu}^{(0,k)}; k = 1, \ldots, r\}$ are generated by means of the normal distribution $N_{n-1}\left(\sum_{\ell=1}^{m^{(0)}} z_{k\ell}^{(0)} \boldsymbol{\mu}_G^{(0,\ell)}, \sum_{\ell=1}^{m^{(0)}} z_{k\ell}^{(0)} \Sigma_G^{(0,\ell)}\right)$. The iterations counter it is initialised (it $=$ 1) and the following steps are repeated until it>$\text{IT}_{\max}$.

Step 1: Local exploration by means of Gibbs sampling Steps 1(a) to 1(f) are executed during IT_{GS} iterations

Step 1(a): Draw $\left\{\tau^{(\text{it},k)}; k = 1, \ldots, r\right\}$ from

$$\text{Gamma}\left(\frac{J + n_1}{2}, \frac{\left(\mathbf{y}^{(k)} - \mathbf{X}\boldsymbol{\mu}^{(\text{it}-1,k)}\right)'\left(\mathbf{y}^{(k)} - \mathbf{X}\boldsymbol{\mu}^{(\text{it}-1,k)}\right) + d_1}{2}\right)$$

Step 1(b): Set $m^{(\text{it})} = m^{(\text{it}-1)}$ and draw $\{\boldsymbol{\mu}^{(\text{it},k)}; k = 1, \ldots, r\}$ from $N_{n-1}(\mathbf{MED}^{(k)}\mathbf{VAR}^{(k)})$ where

$$\mathbf{MED}^{(k)} = \mathbf{VAR}^{(k)}\left(\tau^{(\text{it},k)}\left(\mathbf{X}'\mathbf{y}^{(k)}\right) + \sum_{\ell=1}^{m^{(\text{it})}} z_{k\ell}^{(\text{it}-1)}\left(\Sigma_G^{(\text{it}-1,\ell)}\right)^{-1}\boldsymbol{\mu}_G^{(\text{it}-1,\ell)}\right)^{-1}$$

$$\mathbf{VAR}^{(k)} = \left(\boldsymbol{\tau}^{(\text{it},k)}(\mathbf{X}'\mathbf{X}) + \sum_{\ell=1}^{m^{(\text{it})}} z_{k\ell}^{(\text{it}-1)}\left(\Sigma_G^{(\text{it}-1,\ell)}\right)^{-1}\right)^{-1}$$

Step 1(c): Draw $\left\{\Sigma_G^{(\text{it},\ell)}; \ell = 1, \ldots, m^{(\text{it})}\right\}$ from $\text{IW}(n_\ell, \mathbf{D}_\ell)$ with $n_\ell = n_0 + 1 + \sum_{k=1}^r z_{k\ell}^{(\text{it}-1)}$

$$\mathbf{D}_\ell = n_0\mathbf{D}_0 + \sum_{k=1} z_{k,\ell}^{(\text{it}-1)}\left(\boldsymbol{\mu}^{(\text{it},k)} - \boldsymbol{\mu}_G^{(\text{it}-1,\ell)}\right)\left(\boldsymbol{\mu}^{(\text{it},k)} - \boldsymbol{\mu}_G^{(\text{it}-1,\ell)}\right)'$$

Step 1(d): Draw $\left\{\boldsymbol{\mu}_G^{(\text{it},\ell)}; \ell = 1, \ldots, m^{(\text{it1})}\right\}$ from $N_{m^{(\text{it}-1)}}\left(\mathbf{MED}_G^{(\ell)}, \mathbf{VAR}_G^{(\ell)}\right)$ where

$$\mathbf{MED}_G^{(\ell)} = \mathbf{VAR}_G^{(\ell)}\left(\sum_{k=1}^{r} z_{k\ell}^{(\text{it}-1)}\left(\Sigma_G^{(\text{it},\ell)}\right)^{-1}\boldsymbol{\mu}^{(\text{it},k)}\right)$$

$$\mathbf{VAR}_G^{(\ell)} = \left(\sum_{k=1}^{r} z_{k\ell}^{(\text{it}-1)}\left(\Sigma_G^{(\text{it},\ell)}\right)^{-1} + \tau_{\mu_G}\mathbf{I}_{n-1}\right)^{-1}$$

Step 1(e): Draw $\left\{\mathbf{z}_k^{(\text{it})}; k = 1, \ldots, r\right\}$ from $\text{Mul}(1, p_1^{(k)}, \ldots, p_{m^{(\text{it})}}^{(k)})$ where

$$p_\ell^{(k)} \propto \boldsymbol{\pi}_\ell^{(\text{it}-1)}\left|\Sigma_G^{(\text{it},\ell)}\right|^{-1/2}\exp\left[-\frac{1}{2}\left(\boldsymbol{\mu}^{(\text{it},k)} - \boldsymbol{\mu}_G^{(\text{it},\ell)}\right)'\left(\Sigma_G^{(\text{it},\ell)}\right)^{-1}\left(\boldsymbol{\mu}^{(\text{it},k)} - \boldsymbol{\mu}_G^{(\text{it},\ell)}\right)\right]; \ell = 1, \ldots, m^{(\text{it})}$$

Step 1(f): Draw $\boldsymbol{\pi}^{(it)} = \left(\boldsymbol{\pi}_1^{(\text{it})}, \ldots, \boldsymbol{\pi}_{m^{(\text{it})}}^{(\text{it})}\right)'$ from Dirichlet $\left(1 + \sum_{k=1}^r z_{k1}^{(\text{it})}, \ldots, 1 + \sum_{k=1}^r z_{km^{(\text{it}-1)}}^{(\text{it})}\right)$. Set it = it + 1.

Step 2: Global exploration of the model space.

Step 2(a): Set $\theta^{(\text{it})} = \theta^{(\text{it}-1)}$, $\mathbf{z}^{(\text{it})} = \mathbf{z}^{(\text{it}-1)}$, $\Omega_{m^{(\text{it})}}^{(\text{it})} = \left\{\left(\pi_\ell^{(\text{it})}, \boldsymbol{\mu}_G^{(\text{it},\ell)}, \Sigma_G^{(\text{it},\ell)}\right); \ell = 1, \ldots, m^{(\text{it})}\right\}$ and t = 0.

Step 2(b): If $m^{(\text{it})} = 1$, go to Step 2(d). Otherwise, calculate the death rate for each component of the mixture

$$\delta_\ell = \frac{L\left(\Omega_{m^{(\text{it})}}^{(\text{it})(-\ell)}\right)}{L\left(\Omega_{m^{(\text{it})}}^{(\text{it})}\right)}; \quad \ell = 1, \ldots, m^{(\text{it})}$$

where $L(\Omega_m) = \prod_{k=1}^{r}\left(\sum_{\ell=1}^m \pi_\ell\varphi\left(\boldsymbol{\mu}^{(k)}; \boldsymbol{\mu}_G^{(\ell)}, \Sigma_G^{(\ell)}\right)\right)$ is the likelihood function for the set of points Ω_m, where $(\mathbf{x}; \boldsymbol{\mu}, \Sigma)$ is the density function of a $N_{n-1}(\boldsymbol{\mu}, \Sigma)$ evaluated in $\mathbf{x} \in \mathbf{R}^{n-1}$.

Step 2(c): Simulate the time of the next jump $t = t + v$, where v is sampled from an exponential distribution with mean $\frac{1}{\lambda_0+\delta}$ where $\delta = \sum_{\ell=1}^{m^{(\text{it})}} \delta_\ell$ is the total death rate of the process. If t>it, return to Step 1 of the algorithm. Otherwise, go to Step 2(d).

Step 2(d): Simulate the type of jump. Here, we distinguish two cases:
Step 2(d1): If $m^{(\text{it})} = 1$, a birth happens following the procedure (B.1).
Step 2(d2): If $m^{(\text{it})}>1$, a birth happens following the procedure (B.1) with probability $\frac{\lambda_0}{\lambda_0+\delta}$, and the death of the components ℓ th following the procedure (B.2) happens with probability $\frac{\delta_\ell}{\lambda_0+\delta}$. In this case, the values of $\left\{\mathbf{z}_k^{(\text{it})} : \mathbf{z}_k^{(\text{it}-1)} = \mathbf{e}_\ell\right\}$ are again reassigned by applying Step 1(e).

In both cases, if the jump carried out is a birth, set $m^{(\text{it})} = m^{(\text{it})} + 1$, and if it is a death, set $m^{(\text{it})} = m^{(\text{it})} - 1$. Return to Step 2(b).

As a consequence of the algorithm a sample of the distribution (A.1) is obtained:

$$\left\{\left(\boldsymbol{\theta}^{(\text{it})}, \mathbf{z}^{(\text{it})}\right) = \left(\left(\left(\tau^{(\text{it},k)}, \boldsymbol{\mu}^{(\text{it},k)}\right)_{k=1}^{r}, m^{(\text{it})}, \left(\pi_\ell^{(\text{it})}, \boldsymbol{\mu}_G^{(\text{it},\ell)}, \Sigma_G^{(\text{it},\ell)}\right)_{\ell=1}^{m^{(\text{it})}}\right), \mathbf{z}^{(it)}\right);\right.$$

$$\left.\text{it} = \text{it}_0, \text{it}_0 + s \ldots, \text{IT}_{\max}\right\} \tag{B.3}$$

where it_0 is the estimated number of iterations necessary for the process convergence to the stationary distribution (A.1) and s is the number of lags to neglect the serial autocorrelation. This value can be estimated following the usual procedures in the literature (Robert and Casella 1999, Cap. 8).

This sample can be used to draw inferences on the different components of $\boldsymbol{\theta}$. In particular, an estimation of distribution G is given by $E[G(\boldsymbol{\mu})|\mathbf{y}]$. This expectation can be calculated using the Blackwell-Rao estimator (Casella and Robert (1996)) given by

$$\frac{1}{(\text{IT}_{\max} - \text{it}_0 + 1)} \sum_{\text{it}=\text{it}_0}^{\text{IT}_{\max}} \sum_{\ell=1}^{m^{(\text{it})}} \pi_\ell^{(\text{it})} N_{n-1}\left(\boldsymbol{\mu}_G^{(\text{it},\ell)}, \Sigma_G^{(\text{it},\ell)}\right) \tag{B.4}$$

From (B.4) it is possible to calculate estimations of the distributions of the priorities $\{w_i; i = 1, \ldots, n\}$ (with or without normalisation), as well as the posterior distribution of the preference structures by means of Monte Carlo methods. From these preference structures it can be inferred if there is consensus between the decision makers, or if there are several modes which make clear the existence of several opinion groups. In this latter case, the groups would be located by using the samples of the individual priorities $\{\boldsymbol{\mu}^{(\text{it},k)}; \text{it} = \text{it}_0, \ldots, \text{IT}_{\max}, k = 1, \ldots, r\}$ and/or the indicators $\{\mathbf{z}^{(\text{it})}; \text{it} = \text{it}_0, \ldots, \text{IT}_{\max}\}$. This could be carried out using classification algorithms. Alternatively, we could use perceptual maps which reflect the individual preferences for each alternative (see the example described in Section 4).

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