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RESEARCH ARTICLE

Holographic entanglement entropy in $T\bar{T}$-deformed CFTs

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Abstract
In this paper, we study the holographic entanglement entropy computation of the ultraviolet, integrable deformation of the 2−dimensional conformal field theory ($T\bar{T}$-deformed conformal field theory) that would be dual to some massive deformations of $3D$ gravity in asymptotically AdS_3 spacetimes. We compute the correction due to the deformation up to the leading order of the deformation parameter in higher curvature 3D gravities such as new massive gravity, general minimal massive gravity, and exotic general massive gravity. We also use the evaluation of the symplectic potential to obtain the entanglement entropy for deformed theories. In each case, we find agreement between the results.

Keywords Holographic Entanglement Entropy · $T\bar{T}$-deformed CFTs · Massive gravity

Contents

1 Introduction

Holographic conjecture is one of the powerful tools to study quantum gravity, in which the quantum gravity in the d spacetime dimensions is equivalent to a quantum field the-

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ory on the $d-1$ dimensions boundary. An important example is a holographic duality between conformal field theory in d dimension and $(d+1)$-dimensional AdS gravity [1–3]. Conformal field theory is by definition a UV complete framework, in which the rules of local quantum field theory apply at all energy scales. CFTs, are critical points of RG flows. It is then natural to ask: can holography be extended to effective QFTs so that the UV behavior isn't described by CFTs? within the context of AdS_3/CFT_2, this question has been answered by Zamolodchikov [4] by considering a general class of exactly solvable irrelevant deformations of 2D CFT. Irrelevant deformations, compared to marginal and relevant deformations, are difficult to understand. Turning on an irrelevant operator will turn on many additional operators at high energies, which modifies the theory in the UV and lead to a loss of predictive power. Although, the $T\bar{T}$ deformation is a irrelevant operator, but it does not have these problems [5–9]. The 2d $T\bar{T}$ operator is a operator constructed of the stress tensor $T_{\mu\nu}$ which can be expressed as

$$\det(T_{\mu\nu}) = \Big[tr(T_{\mu\nu})^2 - T^{\mu\nu}T_{\mu\nu}\Big]. \tag{1}$$

Given a seed theory's Lagrangian $L^{(0)}$, the $T\bar{T}$ flow can be defined by the following flow equation

$$\frac{\partial L^{(\mu)}}{\partial \mu} = \det(T^{(\mu)}_{ab}), \tag{2}$$

μ is the parameter of deformation with dimension $(\text{length})^2$. The flow Eq. (2), defines a curve in the space of quantum field theory parameterized by μ with some properties [10]. When a CFT is deformed by $T\bar{T}$ operator, it doesn't mean we add this operator to the original theory, instead, the deformed theory's Lagrangian $L^{(\mu)}$ is required to satisfy the above flow equation. In [11], the author considered $T\bar{T}$ deformations of the $(0+1)$-dimensional dual to 2d JT Gravity[1] and interpret the deformation as a modification of the JT Gravity boundary conditions. In [12, 13] it has been proposed that $T\bar{T}$ deformation can be obtained by coupling the original theory to topological gravity. In [14] proposed to interpret $T\bar{T}$ deformation as a random geometry. Several methods of determining the exact deformed Lagragian through integrating out vielbeins or metrics are also discovered by [15, 16]. In [17] the authors studied the symmetries of $T\bar{T}$, JT_a and $J\bar{T}$ deformed CFTs, in which they showed that each deformed theory possesses an infinite number of conserved charges. The authors of [18] showed that with a mixed boundary condition at spatial infinity and Chern-Simons formalism of AdS_3 constructed the surface charges and associated algebra in $T\bar{T}$ deformed theories. In [19], by applying covariant phase space methods, the Poisson bracket algebra of boundary observables which is a one-parameter nonlinear deformation of the usual Virasoro algebra of asymptotically AdS_3 gravity deduced. This algebra should be obeyed by the stress tensor in any $T\bar{T}$-deformed holographic CFT. In [20] proposed that within the holographic dual, this deformation represents a geometrical cut-off on a

[1] JT Gravity can be viewed as the dimensional reduction of the Chern-Simons description of 3d gravity.

wall at finite radial distance $r = r_c$ within the bulk that removes the asymptotic region of AdS and places the QFT on it. More precisely if a CFT has a gravity dual, then the deformed theory is dual to the original gravitational theory with the new boundary at $r = r_c$. Many interesting physical quantities such as the partition function, the S-matrix, the energy spectrum, and the entanglement entropy have been computed in the deformed theories, see [21] and the references within.

In this paper following the papers [22, 23] we would like to understand the effect of this deformation on entanglement entropy using holography in the framework of higher derivative massive gravity in $2+1$ dimensions. The holographic method we used to obtain entanglement entropy is the Song-Wen-Xu-like method [24]. We used this method because the symplectic potential depends on the theory and it will give us the possibility to calculate entanglement entropy for different theories of gravity including Chern-simons-like theories.

Because of the absence of local degrees of freedom, General Relativity (GR) in three dimensions is an easier theory for studying the different aspects of gravity. New Massive Gravity(NMG) is a three-dimensional theory of gravity with parity-even, higher derivative action which at the linearized level reduces to massive spin-two Fierz-Pauli theory [25, 26]. General Minimal Massive Gravity (GMMG) which was introduced in [27], is an example of the 3D theory of gravity with actions that make use of two auxiliary one-forms, h and f, which at the level of the field equations can be integrated out, leading to the New Massive Gravity field equations supplemented by the Cotton tensor and by a parity even tensors, J_{ab}. This term with respect to the curvature is quadratic, and therefore the field equations for the metric remain of the fourth-order. These effective Einstein equations cannot be obtained only from a variational principle of the metric as a dynamical field, nevertheless, they are on-shell consistent as is the case in the theories introduced in [28–31]. GMMG avoids the bulk-boundary clash and so possesses positive energy excitations about the maximally symmetric AdS_3 vacuum in addition to a positive central charge within the dual CFT. Such a problem within the previously constructed gravity theories with local degrees of freedom in 2+1-dimensions namely Topologically Massive Gravity and therefore the cosmological extension of Massive Gravity is present [25, 32, 33]. Exotic general massive gravity is another 3D theory of gravity with parity−odd action which describes a propagating massive spin−2 field. The field equations of this theory supplement the Einstein equations with a term that contains up to 3rd of the metric and is made with combinations and derivatives of the Cotton tensor [30]. The different aspects of this model have been studied in [34–39].

The paper is organized as follows: In Sect. 2, we obtained the entanglement entropy for NMG with parity even action, directly by using on-shell action and using the RT-method. In Sect. 3, for the other Chern–Simons-like theories of gravity GMMG and EGMG, we obtain the entanglement entropy and repeat the procedure of the previous section for them. We provide some conclusions in Sect. 4.

2 Entanglement entropy for NMG

The new massive gravity is one of the famous three-dimensional theories of gravity among massive gravity models. This model is second-order in time derivatives, its linearizations around a Minkowski metric are equivalent to the second-order Fierz-Pauli action for a massive spin-2 particle. Furthermore, NMG preserves parity symmetry which is not the case for the topological massive gravity. The action of NMG is described as follows [25, 26]

$$S_{NMG} = \frac{1}{8\pi G}\int d^3x\sqrt{-g}\left[R - 2\lambda - \frac{1}{m^2}\left(R^{\mu\nu}R_{\mu\nu} - \frac{3}{8}R^2\right)\right], \tag{3}$$

where λ and m are the cosmological constant and the mass parameter of NMG, respectively. By a variation of the Lagrangian we obtain

$$E_{\mu\nu} = G_{\mu\nu} + \lambda g_{\mu\nu} - \frac{1}{2m^2}N_{\mu\nu}, \tag{4}$$

with

$$\begin{aligned} N_{\mu\nu} = &-\frac{1}{2}\nabla^2 R g_{\mu\nu} - \frac{1}{2}\nabla_\mu\nabla_\nu R + 2\nabla^2 R_{\mu\nu} + 4R_{\mu a\nu b}R^{ab} \\ &- \frac{3}{2}RR_{\mu\nu} - R_{\alpha\beta}R^{\alpha\beta}g_{\mu\nu} + \frac{3}{8}R^2 g_{\mu\nu}, \end{aligned} \tag{5}$$

and $G_{\mu\nu}$ is the Einstein tensor. To obtain the renormalized action we should take into account the generalized Gibbons-Hawking boundary term for NMG [40] as follows

$$S_{GH} = \int d^2x\sqrt{-\gamma}\left(2K + \hat{f}^{ab}K_{ab} - \hat{f}K\right), \tag{6}$$

where

$$\begin{aligned} K_{\mu\nu} &= -\frac{1}{2}(n_{\mu;\nu} + n_{\nu;\mu}),\ \ \gamma^{\mu\nu} = g^{\mu\nu} - n^\mu n^\nu,\ \ \hat{f}^{ab} = f^{\mu\nu}\gamma_\mu{}^a\gamma_\nu{}^b, \\ f_{\mu\nu} &= \frac{2}{m^2}\left(R_{\mu\nu} - \frac{1}{4}Rg_{\mu\nu}\right). \end{aligned} \tag{7}$$

Now, we consider a deformed CFT on manifold $\mathcal{M}$. So, the entanglement entropy is given by

$$S_{EE} = \lim_{n\to 1}\frac{1}{1-n}\log\frac{Z_n}{Z^n}, \tag{8}$$

where Z_n is the partition function on $\mathcal{M}^n$ which is obtained by using the replica method in which one may provide n copies of the manifold glues them together.

We start with the deformed CFT defined on the boundary metric in two dimensions with complex coordinates $(x, \bar{x})$ to cover this surface as

$$ds^2 = dx d\bar{x}. \tag{9}$$

By using the following coordinate transformation one can convert the metric to a conformal form as

$$y = \left(\frac{x-a}{x-b}\right)^{\frac{1}{n}} \tag{10}$$

then

$$ds^2 = e^{\phi(y,\bar{y})} dy d\bar{y}, \qquad e^{\frac{\phi(y,\bar{y})}{2}} = nl \frac{|y|^{n-1}}{|y^n - 1|^2} \tag{11}$$

where ϕ is the Liouville field. By using the Fefferman–Graham metric, one can extend the boundary metric (11) to the bulk as [41]

$$ds^2 = \frac{d\rho^2}{4\rho^2} + \frac{1}{\rho} g_{ij}(\rho, X) dX^i dX^j, \quad g_{ij}(\rho, X) = g^{(0)}_{ij}(X) + \rho g^{(1)}_{ij}(X) + \ldots \tag{12}$$

here $X^i = (y, \bar{y})$ and $g^{(0)}_{ij} = e^{\phi} dy d\bar{y}$. So, the bulk metric can be written as follows [42]

$$\begin{aligned} ds^2 = {} & \frac{d\rho^2}{4\rho^2} + \frac{1}{\rho} e^{\phi} dy d\bar{y} + \frac{1}{2} T_\phi dy^2 + \frac{1}{2} \bar{T}_\phi d\bar{y}^2 + \frac{1}{4} R_\phi dy d\bar{y} \\ & + \frac{1}{4} \rho e^{-\phi} \left(T_\phi dy + \frac{1}{4} R_\phi d\bar{y}\right)\left(\bar{T}_\phi d\bar{y} + \frac{1}{4} R_\phi dy\right) \end{aligned} \tag{13}$$

where

$$R_\phi = 4\partial_y \bar{\partial}_y \phi, \quad T_\phi = \partial_y^2 \phi - \frac{1}{2}(\partial_y \phi)^2, \quad \bar{T}_\phi = \bar{\partial}_y^2 \phi - \frac{1}{2}(\bar{\partial}_y \phi)^2. \tag{14}$$

The following coordinate transformations [43]

$$\xi = \sqrt{\frac{e^{\phi}}{\rho}} + \frac{1}{4}\sqrt{\frac{\rho}{e^{\phi}}} |\partial_y \phi|^2, \quad z = y + \frac{1}{2} \frac{\rho e^{\phi} \bar{\partial}_y \phi}{1 + \frac{1}{4}\rho e^{-\phi} |\partial_y \phi|^2} \tag{15}$$

convert the FG coordinate to the Poincare coordinate, which brings us to the following metric

$$ds^2 = \frac{d\xi^2}{\xi^2} + \xi^2 dz d\bar{z}. \tag{16}$$

This metric is a solution for NMG if

$$\lambda = -\left(1 + \frac{1}{4m^2}\right). \tag{17}$$

So, λ and m related by (17) and are not arbitrary. The on-shell action of NMG is given by

$$S_{NMG} = \left(1 + \frac{1}{2m^2}\right)\left[-\frac{1}{\delta^2}\iint e^{\phi} dz d\bar{z} - \frac{1}{2}\iint \psi dz d\bar{z} - \frac{\delta^2}{16}\iint e^{-\phi}\psi^2 dz d\bar{z}\right] \tag{18}$$

here we assumed $\rho = \delta^2$, then the regulator surface is

$$\xi_f = \frac{1}{\delta}e^{\frac{\phi}{2}} + \frac{\delta}{4}e^{-\frac{\phi}{2}}\psi. \tag{19}$$

While on the boundary $n^{\mu} = \xi\delta^{\mu}_{\xi}$, $\gamma = -\xi^4/4$, $g = -\xi^2/4$ and $K = \gamma^{ab}K_{ab} = 2$ one can get the generalized Gibbons-Hawking on-shell action as

$$S_{GH} = \left(1 + \frac{1}{2m^2}\right)\left[\frac{2}{\delta^2}\iint e^{\phi} dz d\bar{z} + \iint \psi dz d\bar{z} + \frac{\delta^2}{8}\iint e^{-\phi}\psi^2 dz d\bar{z}\right]. \tag{20}$$

Here, we select the appropriate counter-term as follows:

$$S_{ct} = \left(1 + \frac{1}{2m^2 l^2}\right)\int d^2x\sqrt{-\gamma}(1 + \delta^2\kappa(z, \bar{z})), \tag{21}$$

The first term in the above counter-term as the usual counterterm of gravitational action removes the divergent term of GH and bulk action. In the second term, we chose κ such that no boundary terms of order δ^2 remain in the boundary action. So, explicitly the action is as follows:

$$\begin{aligned} S_r = -\left(1 + \frac{1}{2m^2}\right)\Bigg[\frac{1}{\delta^2}\iint e^{\phi} dz d\bar{z} + \iint \frac{1}{2}\psi + \kappa(z, \bar{z})e^{\phi(z,\bar{z})} dz d\bar{z} \\ + \frac{\delta^2}{2}\iint\left(\frac{1}{8}e^{-\phi}\psi^2 + \kappa(z, \bar{z})\psi\right) dz d\bar{z}\Bigg]. \end{aligned} \tag{22}$$

Thus, by choosing $\kappa = \psi/8e^{\phi}$ one can get the renormalized on-shell action is given by

$$S_r = S_{NMG} + S_{GH} + S_{ct} = -\frac{1}{8}\left(1 + \frac{1}{2m^2}\right)\iint\left[\psi + \frac{\delta^2}{2}\psi^2 e^{-\phi}\right] dz d\bar{z}. \tag{23}$$

Therefore, one can rewrite the renormalized action in terms of the Liouville field as

$$S_r = \frac{1}{64\pi G}\left(1+\frac{1}{2m^2}\right)\int dV\left(\partial_i\phi\partial^i\phi+\frac{\delta^2}{2}\left(\partial_i\partial^i e^{\frac{\phi}{2}}\right)^2\right), \tag{24}$$

after integrating by part one can get

$$S_r = \frac{1}{64\pi G}\left(1+\frac{1}{2m^2}\right)\int dS_n\left[\phi\partial_n\phi+\frac{\delta^2}{2}\left(\partial_n e^{-\frac{\phi}{2}}\Box e^{-\frac{\phi}{2}}-e^{-\frac{\phi}{2}}\partial_n\Box e^{-\frac{\phi}{2}}\right)\right]. \tag{25}$$

In order to solve this integral, we adopt

$$z = re^{i\theta}, \quad e^{-\frac{\phi}{2}} = \frac{1}{n\ell}(r^{n+1}+r^{-n+1}-2r\cos(n\theta)), \quad e^{-\phi}\psi^2 = (\partial\bar{\partial}e^{-\frac{\phi}{2}})^2. \tag{26}$$

Then one can get

$$S_{EE} = \frac{1}{4G}\left(1+\frac{1}{2m^2}\right)\left(\frac{1-n^2}{n}\right)\left[\log\left(\frac{\ell}{\delta}\right)+\frac{\delta^2}{n\ell^2}\right], \tag{27}$$

one finally arrives at

$$S_{EE} = \frac{c}{3}\log\left(\sqrt{\frac{24\pi}{\varsigma c}}\ell\right)+\frac{\varsigma c^2}{72\pi\ell^2}, \tag{28}$$

where

$$\varsigma = \frac{8Gm^2\delta^2}{1+2m^2} \tag{29}$$

which is in agreement with the results of CFT side [23, 44]. In the limit $m \to \infty$, $\varsigma = 4\pi G\delta^2$. The entropic $\mathcal{C}-$function in two dimensions for $T\bar{T}$ deformed CFT, is defined as [45]

$$\mathcal{C} = 3\ell\frac{\partial S_{EE}}{\partial\ell} = c-\frac{\varsigma c^2}{12\pi\ell^2}, \qquad c = \frac{3}{4G}\left(1+\frac{1}{2m^2}\right) \tag{30}$$

which depends on the deformation parameter and approaches the central charge of the undeformed CFT (c is the central charge of NMG and it is positive, therefore the dual CFT_2 is unitary.) when $\varsigma = 0$, as expected [46]. It is expected that the holographic entanglement entropy of deformed CFT is obtained by RT-method. By using $z = x + i\tau$, $\xi = 1/\eta$ and going to the polar coordinate in (16), we have

$$ds^2 = \frac{1}{\eta^2}\left[d\eta^2+r^2d\tau^2+n^2dr^2\right]. \tag{31}$$

The entanglement entropy can be calculated using the presymplectic potential by replacing $\delta g_{\mu\nu} = \partial_n g_{\mu\nu}$ as

$$S_{HEE} = \int \Theta\left(\phi_i, \partial_n \phi_i\right)_{n\to 1, r\to 0}, \tag{32}$$

which is integral in direction of bulk after integrating out τ along S^1. For NMG the presymplectic potential is given by [40]

$$\begin{aligned}\Theta^{\mu}_{NMG} &= \theta^{\mu} - \frac{1}{2} f\theta^{\mu} + f^{\rho\sigma} g^{\mu\nu} \bar{\nabla}_{\rho}(\delta g_{\sigma\nu}) - \frac{1}{2} f^{\rho\sigma} \bar{\nabla}^{\mu}(\delta g_{\rho\sigma}) - \frac{1}{2} f^{\mu\nu} g^{\rho\sigma} \bar{\nabla}_{\nu}(\delta g_{\rho\sigma}) \\ &\quad + \frac{1}{2}[\bar{\nabla}^{\mu} f^{\nu\rho} - 2\bar{\nabla} f^{\mu\rho} + g^{\mu\nu}\bar{\nabla}^{\rho} f + g^{\nu\rho}\bar{\nabla}_{\sigma} f^{\sigma\mu} - g^{\nu\rho}\bar{\nabla}^{\mu} f]\delta g_{\nu\rho}, \\ \theta^{\mu} &= g^{\mu\nu}\bar{\nabla}^{\rho}(\delta g_{\nu\rho}) - g^{\rho\sigma}\bar{\nabla}^{\mu}(\delta g_{\rho\sigma}).\end{aligned} \tag{33}$$

So, the presymplectic structure for metric (31) is obtained as

$$\Theta^r = \frac{1}{8\pi G}\left(1 + \frac{1}{2m^2}\right)\frac{\eta^2}{rn^3}. \tag{34}$$

Then, one can achieve the entropy as

$$\begin{aligned}S_{HEE} &= \int_0^{2\pi} d\tau \int \sqrt{-g}\,\Theta^r d\eta = \frac{1}{4Gn^2}\left(1 + \frac{1}{2m^2}\right)\int_{\eta_f}^{\ell} \frac{d\eta}{\eta} \\ &= \frac{1}{4G}\left(1 + \frac{1}{2m^2}\right)\ln\left(\frac{\ell}{\eta_f}\right),\end{aligned} \tag{35}$$

here we have used $\sqrt{-g} = \frac{nr}{\eta^3}$, $\eta_f = 1/\xi_f$ and ℓ is the interval length of subsystem $\mathcal{A}$. Therefore, the explicitly nonperturbative HEE (35) becomes

$$S^{NP}_{HEE} = \frac{1}{4G}\left(1 + \frac{1}{2m^2}\right)\ln\left(\frac{\ell}{\delta}e^{\frac{\phi}{2}} + \frac{\delta}{4}\psi e^{-\frac{\phi}{2}}\right). \tag{36}$$

We have expanded (35) around $\delta \ll 1$, then one can obtain [46]

$$S^{P}_{HEE} = \lim_{\substack{n\to 1\\ r\to 0}} \frac{1}{4G}\left(1 + \frac{1}{2m^2}\right)\left[\ln\left(\frac{\ell e^{\frac{\phi}{2}}}{\delta}\right) + \frac{\psi}{4e^{\phi}}\delta^2\right]. \tag{37}$$

By using (26) one can arrive

$$\lim_{\substack{n\to 1\\ r\to 0}} e^{-\frac{\phi}{2}} = \frac{1}{\ell}, \qquad \lim_{\substack{n\to 1\\ r\to 0}} e^{-\phi}\psi = \frac{4}{\ell^2}. \tag{38}$$

Inserting (38) into the (37) and (36), we have obtained nonperturbative S_{HEE}^{NP} and perturbative S_{HEE}^{P} entropy as follows

$$S_{HEE}^{NP} = \frac{1}{4G}\left(1+\frac{1}{2m^2}\right)\ln\left(\frac{\ell^2}{\delta}+\frac{\delta}{\ell^2}\right), \tag{39}$$

$$S_{HEE}^{P} = \frac{1}{4G}\left(1+\frac{1}{2m^2}\right)\left[\ln\left(\frac{\ell^2}{\delta}\right)+\frac{\delta^2}{\ell^2}\right]. \tag{40}$$

Therefore, the entropic $\mathcal{C}$ function becomes

$$\mathcal{C}^{NP} = c\frac{(\ell^4-\delta^2)}{(\ell^4+\delta^2)}, \tag{41}$$

$$\mathcal{C}^{P} = c-\frac{2\mu c^2}{\ell^4}, \tag{42}$$

in which the nonperturbative $\mathcal{C}^{NP}$ is different from (3.54) of [47], while the perturbative $\mathcal{C}^{P}$ is comparable with perturbative form of (3.54).

3 Entanglement entropy for GMMG

The Lagrangian of general minimal massive gravity theory is a generalization of the Lagrangian of general massive gravity. The Lagrangian of GMMG is given as [27]

$$L_{GMMG} = -\sigma e.R+\frac{\Lambda_0}{6}e.e\times e+h.T(\omega)+\frac{1}{2\mu}\left(\omega.d\omega+\frac{1}{3}\omega.\omega\times\omega\right) -\frac{1}{m^2}\left(f.R+\frac{1}{2}e.f\times f\right)+\frac{\alpha}{2}e.h\times h, \tag{43}$$

where m is the mass parameter of NMG term, Λ_0 is a cosmological constant, μ is a mass parameter of Chern−Simons term, σ is a sign, α is a dimensionless parameter, e is a dreibein, ω is a dualized spin-connection and h and f are auxiliary one-form fields. After integrating out the auxiliary one−form fields f and h, the field equations obtain as

$$\bar{\sigma}G_{\mu\nu}+\bar{\Lambda}_0 g_{\mu\nu}+\frac{1}{\mu}C_{\mu\nu}+\frac{\gamma}{\mu^2}J_{\mu\nu}+\frac{s}{2m^2}K_{\mu\nu}=0\,, \tag{44}$$

where $C_{\mu\nu}$ is the Cotton tensor, $K_{\mu\nu}$ is the Euler−Lagrange derivative of the quadratic part of the NMG Lagrangian with respect to the metric, and $J_{\mu\nu}$ is the quadratic in the curvature tensor introduced in [28]. The parameter s is sign, γ, $\bar{\sigma}$ and $\bar{\Lambda}_0$ are the parameters which defined in terms of other parameters like σ, m and μ.
The metric (16) is a solution for the field Eq. (44) under the condition

$$\lambda = -\frac{4\sigma\mu^2m^2+\gamma m^2-s\mu^2}{4\mu^2m^2}. \tag{45}$$

Therefore, the couplings of the theory related by (45) and are not arbitrary. The Lagrangian (43) can be written as follows

$$L_{GMMG} = -\left(\sigma + \frac{c_f}{m^2}\right) e.R + \frac{1}{2\mu} L_{CS} + c_h e.T + \left[\frac{\Lambda_0}{6} - \frac{c_f^2}{2m^2} + \frac{\alpha c_h^2}{2}\right](e.e \times e), \tag{46}$$

where we have used

$$h = c_h e, \quad f = c_f e. \tag{47}$$

The dreibein components of the metric after Wick rotation can be chosen as

$$e^0 = \frac{d\xi}{\xi}, \qquad e^1 = \frac{\xi}{2}\left(dz - d\bar{z}\right), \qquad e^2 = \frac{\xi}{2}\left(dz + d\bar{z}\right). \tag{48}$$

Then, the spin connections would be

$$\omega_0^1 = \frac{1}{2}\xi(dz - d\bar{z}), \quad \omega_0^2 = \frac{1}{2}\xi\left(dz + d\bar{z}\right), \tag{49}$$

and therefore the dualized spin-connections are given by

$$\omega^0 = 0, \quad \omega^1 = \frac{\xi}{4}\left(dz + d\bar{z}\right), \quad \omega^2 = \frac{\xi}{4}\left(d\bar{z} - dz\right). \tag{50}$$

The different terms of the action are given by

$$e.R = \frac{\xi}{4} d\xi \wedge dz \wedge d\bar{z}, \qquad e.e \times e = \frac{\xi}{2} d\xi \wedge dz \wedge d\bar{z},$$
$$T = 0, \qquad L_{CS} = 0, \qquad \omega.e = -\frac{\xi^2}{2} dz \wedge d\bar{z}. \tag{51}$$

Then, the on-shell action is given by

$$S_{GMMG} = \frac{1}{4}\left(-\sigma + \frac{\Lambda_0}{3} - \frac{c_f^2}{m^2} + \alpha c_h^2 - \frac{c_f}{m^2}\right) \times \left[\frac{1}{\delta^2}\iint e^{\phi} dz d\bar{z} + \frac{1}{2}\iint \psi dz d\bar{z} + \frac{\delta^2}{16}\iint e^{-\phi}\psi^2 dz d\bar{z}\right]. \tag{52}$$

The boundary actions are given as

$$S_{GH} = \int -\left(\sigma + \frac{c_f}{m^2}\right)\omega.e + \frac{1}{2\mu}\omega.\omega + c_h e.e$$

$$= \frac{1}{2}\left(\sigma + \frac{c_f}{m^2}\right)\left[\frac{1}{\delta^2}\iint e^{\phi} dz d\bar{z} + \frac{1}{2}\iint \psi dz d\bar{z} + \frac{\delta^2}{16}\iint e^{-\phi}\psi^2 dz d\bar{z}\right], \tag{53}$$

$$S_{ct} = -\frac{1}{4}\left(\sigma + \frac{c_f}{m^2}\right)\int e\,(1 + \delta^2\kappa(z,\bar{z})). \tag{54}$$

If $\frac{\Lambda_0}{3} - \frac{c_f^2}{m^2} + \alpha c_h^2 = 0$, then we have

$$S_r = -\frac{1}{8}\left(\sigma + \frac{c_f}{m^2}\right)\left[\iint \psi dz d\bar{z} + \frac{\delta^2}{8}\iint e^{-\phi}\psi^2 dz d\bar{z}\right] \tag{55}$$

by using (26), one can obtain

$$S_{EE} = -\frac{1}{8}\left(\sigma + \frac{c_f}{m^2}\right)\left(\frac{1-n^2}{n}\right)\left[\log\left(\frac{\ell}{\delta}\right) + \frac{\delta^2}{n\ell^2}\right], \tag{56}$$

one finally arrives at

$$S_{EE} = \frac{c'}{3}\log\left(\sqrt{\frac{24\pi}{\varsigma c'}}\ell\right) + \frac{\varsigma c'^2}{72\pi\ell^2}, \quad c' = \frac{c_+ + c_-}{2} \tag{57}$$

where

$$\varsigma = -\frac{64Gm^2\delta^2}{c_f + \sigma m^2}, \quad c_\pm = -\frac{3}{2G}\left(\sigma + \frac{\alpha c_h}{\mu} + \frac{c_f}{m^2} \pm \frac{1}{\mu}\right), \tag{58}$$

where $c_\pm$ are the central charges of GMMG and under the condition $\sigma + \alpha c_h/\mu + c_f/m^2 \pm 1/\mu < 0$ the dual CFT_2 is unitary. The entropic $\mathcal{C}-$function in two dimensions is given as [45]

$$\mathcal{C} = 3l\frac{\partial S_{EE}}{\partial l} = c' - \frac{\varsigma c'^2}{12\pi l^2}, \quad c' = -\frac{3}{2G}\left(\sigma + \frac{c_f}{m^2}\right), \tag{59}$$

which depends on the deformation parameter and approaches c' when $\varsigma = 0$, as expected [46]. The dreibein components of metric (31) can be written as

$$e^0 = \frac{r}{\eta}d\tau, \quad e^1 = \frac{n}{\eta}dr, \quad e^2 = \frac{1}{\eta}d\eta. \tag{60}$$

The dualized spin connections would be

$$\omega^0 = -\frac{n}{2\eta}dr, \quad \omega^1 = \frac{r}{2\eta}d\tau, \quad \omega^2 = \frac{1}{2n}d\tau. \tag{61}$$

The presymplectic form of GMMG is given by [48]

$$\Theta_{GMMG} = -\left(\sigma + \frac{c_f}{m^2}\right)\delta\omega.e + \frac{1}{2\mu}\delta\omega.\omega + c_h\delta e.e. \tag{62}$$

Then, we assume $\delta\omega = \partial\omega/\partial n$, $\delta e = \partial e/\partial n$, one can get

$$\Theta_{GMMG} = -\left(\sigma + \frac{c_f}{m^2}\right)\left[\frac{r}{2\eta^2}dr \wedge d\tau - \frac{1}{2n^2\eta}d\tau \wedge d\eta\right]. \tag{63}$$

So, the entropy for GMMG is as follows

$$\begin{aligned} S &= \int_0^{2\pi} d\tau \int \sqrt{-g}\Theta^r d\eta = \int_0^{2\pi} d\tau \int_{\eta_f}^{\ell} \frac{1}{2n^2}\left(\sigma + \frac{c_f}{m^2}\right)\frac{d\eta}{\eta} \\ &= \frac{1}{4G}\left(\sigma + \frac{c_f}{m^2}\right)\ln\left(\frac{\ell}{\eta_f}\right), \end{aligned} \tag{64}$$

here $\eta_f = 1/\xi_f$. After the series expansion of (64) around $\delta \ll 1$, we have [46]

$$S_{HEE} = \lim_{\substack{n\to 1\\ r\to 0}} \frac{1}{4G}\left(\sigma + \frac{c_f}{m^2}\right)\left[\ln\left(\frac{\ell e^{\frac{\phi}{2}}}{\delta}\right) + \frac{\psi}{4e^{\phi}}\delta^2\right]. \tag{65}$$

Inserting (38) into (64) and (65) one can obtain the results similar to (39)–(41) with the central charges of GMMG.

Entanglement entropy for EGMG

Exotic general massive gravity is a third-way consistency theory in three dimensions with a parity−odd theory describing a propagating massive spin−2 field. A gravity theory that leads to both a unitary theory in the bulk and a positive central charge in the boundary theory when formulated on AdS spaces. The Lagrangian of the theory is given as [30]

$$\begin{aligned} L_{EGMG} = &-\frac{1}{m^2}[f.R(\omega) + \frac{1}{6m^4}f.f \times f \\ &- \frac{1}{2m^2}f.D(\omega)f + \frac{\iota}{2}f.e \times e - m^2 h.T(\omega) \\ &+ \frac{(\iota - m^2)}{2}\left(\omega.d\omega + \frac{1}{3}\omega.\omega \times \omega\right) + \frac{\iota m^4}{3\mu}e.e \times e], \quad \iota = 1 - \frac{m^4}{\mu^2}. \end{aligned} \tag{66}$$

In the metric formalism, the field equation is given as follows

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + \Lambda g_{\mu\nu} + \frac{1}{\mu}C_{\mu\nu} - \frac{1}{m^2}H_{\mu\nu} + \frac{1}{m^4}L_{\mu\nu} = 0, \tag{67}$$

where

$$H_{\mu\nu} = \epsilon_\mu^{\alpha\beta} \nabla_\alpha C_{\nu\beta}, \qquad L_{\mu\nu} = \frac{1}{2}\epsilon_\mu^{\alpha\beta}\epsilon_\nu^{\gamma\sigma} C_{\alpha\gamma} C_{\beta\sigma}, \tag{68}$$

μ and m are mass parameters, $H_{\mu\nu}$ and $L_{\mu\nu}$ are symmetric and traceless tensors. The above Lagrangian can be rewritten by using (47) as

$$L_{EGMG} = -\frac{c_f}{m^2} e.R + \left(c_h + \frac{c_f^2}{2m^4}\right)(e.De) - \frac{1}{m^2}(\iota - m^2)L_{CS} \\ -\frac{1}{m^2}\left[\frac{c_f^3}{6m^4} + \frac{\iota c_f}{2} + \frac{\iota m^4}{3\mu}\right](e.e \times e). \tag{69}$$

By using (51), one can obtain

$$S_{EGMG} = \frac{m^2}{8\mu} \iint \xi_f^2 dz d\bar{z}. \tag{70}$$

The GH term for EGMG is given by

$$L_{GH} = -\frac{c_f}{2m^2} e.\omega + \frac{1}{2}\left(c_h + \frac{c_f^2}{m^4}\right) e.e + \left(1 - \frac{\iota}{m^2}\right)\omega.\omega, \tag{71}$$

then the GH action is given

$$S_{GH} = -\frac{m^2}{4\mu} \iint \xi_f^2 dz d\bar{z}, \tag{72}$$

where we have used

$$c_h = \frac{1}{2}\left(1 - \frac{1}{m^2}\right)\left(1 - \frac{m^4}{\mu^2}\right), \quad c_f = -\frac{m^4}{\mu}. \tag{73}$$

The counter term is given

$$S_{ct} = \frac{m^2}{8\mu} \iint e(1 + \delta^2 \kappa(z, \bar{z})) dz d\bar{z}. \tag{74}$$

The renormalized on-shell action by using a cut-off surface is given as

$$S_r = \frac{m^2}{8\mu}\left[\iint \psi dz d\bar{z} + \frac{\delta^2}{8}\iint e^{-\phi}\psi^2 dz d\bar{z}\right], \tag{75}$$

then, similar to the previous section, using (26) we have

$$S_r = -\frac{m^2}{8\mu}\left(\frac{1-n^2}{n}\right)\left[\log\left(\frac{\ell}{\delta}\right) + \frac{\delta^2}{n\ell^2}\right] \tag{76}$$

one finally arrives at

$$S_r = \frac{c'}{3}\log\left(\sqrt{\frac{24\pi}{\varsigma c'}}\ell\right) + \frac{\varsigma c'^2}{72\pi\ell^2}, \qquad c' = \frac{c_+ + c_-}{2} \tag{77}$$

where

$$\varsigma = -\frac{64G\mu\delta^2}{m^2}, \qquad c_\pm = \frac{3}{2G}\left[-\frac{m^2}{\mu} \pm \left(1 + \frac{m^2}{\mu^2} - \frac{1}{m^2}\right)\right], \tag{78}$$

where $c_\pm$ are the right moving and left moving central charges of the dual CFT$_2$. In the case of $m^2/\mu \mp (1 + m^2/\mu^2 - 1/m^2) < 0$, the dual CFT is unitary. The entropic $\mathcal{C}-$function in two dimensions is defined as [45]

$$\mathcal{C} = 3l\frac{\partial S_{EE}}{\partial l} = c' - \frac{\varsigma c'^2}{12\pi l^2}, \qquad c' = -\frac{3m^2}{2G\mu}, \tag{79}$$

which depends on the deformation parameter and approaches c' when $\varsigma = 0$, as expected [46]. The presymplectic form of EGMG is given [48]

$$\Theta_{EGMG} = \frac{1}{2}\left[\left(c_h + \frac{c_f^2}{m^4}\right)\delta e.e + \left(1 - \frac{\varpi}{m^2}\right)\delta\omega.\omega - \frac{c_f}{m^2}\delta e.\omega\right], \tag{80}$$

this presymplectic using (60) and (61) can be written as

$$\Theta = \frac{c_f}{2m^2}\left[\frac{r}{2\eta^2}dr \wedge d\tau + \frac{1}{2n^2\eta}d\tau \wedge d\eta\right]. \tag{81}$$

Then, the entropy for EGMG is given as follows

$$S = \int_0^{2\pi} d\tau \int \sqrt{-g}\Theta^r d\eta = \int_0^{2\pi} d\tau \int_{\eta_f}^{\ell} \frac{c_f}{4n^2m^2}\frac{d\eta}{\eta} = -\frac{m^2}{16G\mu}\ln\left(\frac{\ell}{\eta_f}\right), \tag{82}$$

here $\eta_f = 1/\xi_f$. In the case of $\delta \ll 1$, one can obtain [46]

$$S_{HEE} = -\lim_{\substack{n\to 1\\ r\to 0}} \frac{m^2}{16G\mu}\left[\ln\left(\frac{\ell e^{\frac{\phi}{2}}}{\delta}\right) + \frac{\psi}{4e^{\phi}}\delta^2\right]. \tag{83}$$

After using (26) and limiting, one can arrive at (77).

4 Conclusion

In this paper, we investigated the holographic entanglement entropy of deformed conformal field theories dual to a cut-off of AdS spacetimes. The holographic entanglement entropy evaluated on a three-dimensional Poincare AdS_3 space with a finite cut-off can be interpreted as the dual field theory deformed by $T\bar{T}$-deformation. We have done these calculations in the framework of higher derivative gravity theories like NMG, GMMG, and EGMG theories. We perform a direct holographic calculation of the entanglement entropy by evaluation of the gravitational action in the bulk spacetime which has been reconstructed from a dual CFT_2 on n-sheeted Riemann surface as a finite cut-off boundary. The correction term corresponds to the deformation which comes from the boundary side affected by the mass parameter of higher derivative theories. For the theories with gravitational anomalies like EGMG and GMMG, the average of the central charges of left and right moving ($c = (c_+ + c_-)/2$) appear in the deformation parameters. By considering the entropic $\mathcal{C}-$functions, the effect of deformation parameters on the central charges of deformed CFTs were studied. We have also obtained the entropy for the theories with a cut-off on AdS_3 spacetimes using the pre-symplectic potential integrated along Euclidean time and along with the depth into the bulk. By expansion around UV cut-off deformation, we find an agreement between the results in the two methods.

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Data Availibility Data sharing not applicable to this article as no datasets were generated or analyzed during the current study.

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