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RESEARCH ARTICLE

Solutions for the null-surface formulation in 2 + 1 dimensions leading to spacetimes of Petrov types I, II, and D

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Abstract

The only nontrivial exact solutions reported to-date for the field equations of the null-surface formulation (NSF) of general relativity are for the $(2+1)$-dimensional version of the theory, where three such solutions are known. This work presents a new family of NSF solutions. The corresponding general relativistic spacetimes are shown to span three different Petrov types, depending upon the choices that are made for various parameters. All of the scalar invariants for the spacetimes are constant, as are all of the eigenvalues of the Cotton-York tensor. The physical nature of a possible source term is discussed in detail, and two of the previously known NSF solutions are presented as special cases. The new family of solutions was derived by assuming additive separability—meaning that the dependent variable in the field equations is represented as a sum. This effectively turns the main NSF field equation (which is a partial differential equation) into an ordinary differential equation that is exactly solvable. The possibility of adapting this approach to the $(3+1)$-dimensional version of the NSF is discussed.

Keywords Null-surface formulation · NSF · Metricity condition · Low dimensional gravity

Contents

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1 Introduction

For each choice of u, the equation $u = Z(x^a)$ represents a surface in spacetime. The null-surface formulation (NSF) of general relativity employs *families* of surfaces that are labelled by one or more angular variables [1–5]. The requirement that the surfaces be null with regard to a spacetime metric $g_{bc}(x^a)$, together with the requirement that the Einstein equations hold, then leads to the field equations for the NSF. These equations are more complicated than the usual Einstein equations. No nontrivial solutions have been found for the original $(3 + 1)$-dimensional NSF, which was first proposed by Kozameh and Newman [1], and by Frittelli et al. [2–5]. In higher dimensions, no nontrivial solution has been found for the $(n + 1)$-dimensional NSF ($n > 3$), due to Gallo [6]. Being a theory of null surfaces, the NSF does not distinguish between spacetimes that are related by a conformal transformation. Saying that an NSF solution is nontrivial means that the solution corresponds to a spacetime that is *not* conformally flat. The NSF does not exist in $1+1$ dimensions because all $(1+1)$-dimensional spacetimes are conformally flat. However, $(2 + 1)$-dimensional NSF, which was introduced by Forni et al. [7, 8], Tanimoto [9], and Silva-Ortigoza [10], has been widely studied.

The interest in the NSF in $2 + 1$ dimensions stems partly from the correspondence with the early work of Cartan on classifying third-order ordinary differential equations [11–14]. Although simpler than in higher dimensions, the $(2 + 1)$-dimensional field equations have nonetheless proved difficult to solve. Only three nontrivial solutions have been found to date [15–18]. The purpose of the present paper is to introduce a new family of $(2 + 1)$-dimensional solutions. It encompasses different Petrov types and includes two of the previously known solutions as special cases. Although the results are only valid in $2 + 1$ dimensions, the approach could be applicable to higher dimensions.

A brief review of the NSF in $2 + 1$ dimensions is given in the next section. The subsequent sections present the new family of solutions and investigate their properties.

2 NSF in 2 + 1 dimensions

Consider the NSF in $2 + 1$ dimensions [7–10]. A family of surfaces is described by the equation $u = Z(x^a; \varphi)$, where x^a $(a = 0, 1, 2)$ are spacetime coordinates and $\varphi \in S^1$ is an angular variable whose role is to label the surfaces. For fixed (u, φ), this equation defines a surface $S_{(u,\varphi)}$. The NSF requires $S_{(u,\varphi)}$ be a *null* with respect to some spacetime metric $g_{bc}(x^a)$. Thus for arbitrary values of the parameter φ, the gradient of Z satisfies

$$g^{bc}(x^a)Z_{,b}(x^a;\varphi)\,Z_{,c}(x^a;\varphi)=0, \tag{1}$$

where $Z_{,a}\equiv\partial_a Z\equiv\partial Z/\partial x^a$. Equation (1) and its derivatives with respect to φ lead to *metricity conditions*. These ensure that the requirement of nullness can be satisfied. The first step in deriving the metricity conditions is to introduce coordinates, known as intrinsic coordinates, that are naturally adapted to the surfaces [3, 7–9]:

$$\begin{aligned} u &:= Z(x^a;\varphi),\\ \omega &:= \eth u\equiv \eth Z(x^a;\varphi),\\ \rho &:= \eth\omega\equiv\eth^2 u\equiv\eth^2 Z(x^a;\varphi), \end{aligned}$$

where $\eth:=\partial/\partial\varphi$ denotes the derivative with respect to φ when the x^a are held fixed. In principle, the equations above can be inverted to give

$$x^a=x^a(u,\omega,\rho,\varphi). \tag{2}$$

Instead of using Z as the dependent variable in the NSF, it is simpler to use its third derivative. This is denoted by Λ and, using Eq. (2), is defined by

$$\Lambda(u,\omega,\rho,\varphi):=\eth^3 Z(x^a(u,\omega,\rho,\varphi);\varphi).$$

The field equations of the NSF and the metric g_{ab} are expressed in terms of Λ, rather than in terms of Z [7–10].

The intrinsic coordinates u, ω and ρ are φ-dependent, and it can be shown that the action of the differential operator $\eth$ on a function $f(u,\omega,\rho,\varphi)$ is given by [7–9]

$$\eth=\eth'+\omega\,\partial_u+\rho\partial_\omega+\Lambda\partial_\rho, \tag{3}$$

where $\eth'$ denotes the derivative with respect to φ when u, ω and ρ are held fixed. The (inverse) metric $g^{bc}(x^a)$ does not depend on φ. Thus $\eth g^{bc}=0$. Using the $\eth$ operator of Eq. (3) to repeatedly differentiate Eq. (1) gives the components of the inverse metric g^{ij} with respect to the u,ω,φ coordinates,

$$g^{ij}=g^{ab}\eth^i Z_{,a}\eth^j Z_{,b}, \tag{4}$$

where $\eth^i$ indicates differentiating i times, $i=0,1,2$. In general, g^{ij} will depend on φ. Equations (1) and (4) immediately imply

$$g^{00}\equiv g^{uu}=g^{ab}Z_{,a}Z_{,b}=0,$$

and

$$g^{01}\equiv g^{u\omega}=g^{ab}Z_{,a}\,\eth Z_{,b}=0.$$

An overall multiplicative factor can be extracted by defining

$$\Omega^2 := g^{11} \equiv g^{\omega\omega} = g^{ab}\partial Z_{,a}\partial Z_{,b},$$

(or, alternatively, $\Omega^2 := -g^{02} \equiv -g^{u\rho}$). It is convenient to introduce γ^{ij} by $g^{ij} = \Omega^2\gamma^{ij}$. The final result is [7, 9]

$$[g^{ij}] = \Omega^2[\gamma^{ij}] = \Omega^2 \begin{pmatrix} 0 & 0 & -1 \\ 0 & 1 & \frac{1}{3}\partial_\rho\Lambda \\ -1 & \frac{1}{3}\partial_\rho\Lambda & \frac{1}{3}\eth(\partial_\rho\Lambda) - \frac{1}{9}(\partial_\rho\Lambda)^2 - \partial_\omega\Lambda \end{pmatrix}. \tag{5}$$

In the present paper, the $(-++)$ signature convention is adopted for the metric g^{ab} and consequently, by Eq. (4), for the metric g^{ij}. It is straightforward to show that the components of the metric g_{ij} are given by

$$[g_{ij}] = \Omega^{-2}[\gamma_{ij}] = \Omega^{-2} \begin{pmatrix} -\frac{1}{3}\eth(\partial_\rho\Lambda) + \frac{2}{9}(\partial_\rho\Lambda)^2 + \partial_\omega\Lambda & \frac{1}{3}\partial_\rho\Lambda & -1 \\ \frac{1}{3}\partial_\rho\Lambda & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \tag{6}$$

where γ_{ij} is called the *unphysical* metric or the *conformal* metric [1, p. 2481]. Thus the unphysical line element can be written

$$ds_\gamma^2 = \left(-\frac{1}{3}\eth(\partial_\rho\Lambda) + \frac{2}{9}(\partial_\rho\Lambda)^2 + \partial_\omega\Lambda\right) du^2 + \frac{2}{3}\partial_\rho\Lambda\, du d\omega - 2du d\rho + d\omega^2.$$

Equations (1) and (6) imply the following metricity conditions [7, 9]:

$$2[\eth(\partial_\rho\Lambda) - \partial_\omega\Lambda - \frac{2}{9}(\partial_\rho\Lambda)^2]\partial_\rho\Lambda - \eth^2(\partial_\rho\Lambda) + 3\eth(\partial_\omega\Lambda) - 6\partial_u\Lambda = 0, \tag{7}$$

$$3\eth\Omega = \Omega\partial_\rho\Lambda. \tag{8}$$

Equations (7) and (8) ensure that Λ will determine a null surface with respect to *some* spacetime metric $g_{bc}(x^a)$. Equation (7) is the main metricity condition. Equation (8) is the secondary metricity condition and fixes the φ-dependence of Ω. Despite the conformal invariance of the theory, Ω cannot be chosen arbitrarily since Ω^2 must equal $g^{\omega\omega}$.

If the two metricity conditions are satisfied, then the Einstein equations, $G_{ij} = \kappa T_{ij}$, will be satisfied if the following equation holds [7]:

$$\partial_\rho^2\Omega = \kappa T_{\rho\rho}\Omega. \tag{9}$$

As usual, the gravitational constant will be assumed to be positive: $\kappa > 0$. Saying that Eq. (9) holds means that it is satisfied when using the same $T_{\rho\rho}$ that occurs on the right side of the Einstein equations, $G_{ij} = \kappa T_{ij}$, with the G_{ij} being determined from the g^{ij} and g_{ij} that are found from Eqs. (5) and (6) by a choice of Λ and Ω that satisfies the metricity conditions.

Collectively, Eqs. (7), (8), and (9) constitute the (coupled set of) *field equations* that must be solved in order to find Λ and Ω. To be nontrivial, the solution must *not* correspond to a spacetime that is conformally flat.

3 Solutions of the NSF field equations

Equation (3) displays the differential operator $\eth$ as a sum of separate terms. This suggests expressing Λ in a like manner. Thus solutions will be assumed to take the following *additively separable* form,

$$\Lambda(u,\omega,\rho) = -a\omega - b\rho + h(\rho + au + b\omega) = -a\omega - b\rho + h(x), \tag{10}$$

where a and b are constants, and where x is defined as $x := \rho + au + b\omega$. The $a\omega$ and $b\rho$ terms are motivated by the coefficients in front of the partial derivatives in Eq. (3). Equations (3) and (10) then lead to an expression for $\eth(\partial_\rho \Lambda)$ that is simple and is independent of ω: $\eth(\partial_\rho \Lambda) = h\,\partial_\rho^2 h$. Both γ_{uu} and $\eth\gamma_{uu}$ are also independent of ω. The main metricity condition, Eq. (7), becomes an *ordinary* differential equation with x as the independent variable:

$$\begin{aligned} h^2\frac{d^3h}{dx^3} - h\frac{dh}{dx}\frac{d^2h}{dx^2} + \frac{4}{9}\left(\frac{dh}{dx}\right)^3 + 4a\frac{dh}{dx} \\ - b\,h\frac{d^2h}{dx^2} + \frac{2}{3}b\left(\frac{dh}{dx}\right)^2 - \frac{2}{3}b^2\frac{dh}{dx} - \frac{4}{9}b^3 + 2ab = 0. \end{aligned} \tag{11}$$

The solution of Eq. (11), together with Eq. (10), represents a solution of the main metricity condition. However, Eq. (11) is so complicated that it is difficult to draw conclusions about the nature of the solution and, in particular, about the properties of the spacetime that the solution represents. For this reason, the constant b will henceforth be assumed to be zero, thereby removing the last five terms on the left side of Eq. (11). Further simplification will be achieved by introducing a new dependent variable $y := h^{2/3}$. After replacing h by $y^{3/2}$ and writing $x = \rho + au$, Eq. (10) becomes

$$\Lambda(u,\omega,\rho) = -a\omega + [y(\rho + au)]^{3/2} \equiv -a\omega + [y(x)]^{3/2}, \tag{12}$$

and Eq. (11), which represents the main metricity condition, becomes

$$\frac{d^3y}{dx^3} + 4ay^{-3}\frac{dy}{dx} = 0.$$

This equation is equivalent to

$$\frac{d}{dx}\left(\frac{d^2y}{dx^2} - 2ay^{-2}\right) = 0,$$

and leads to

$$\frac{d^2 y}{dx^2} - 2ay^{-2} = k, \tag{13}$$

where k is a constant. A further integration can be achieved by multiplying Eq. (13) by dy/dx to give

$$\frac{dy}{dx}\frac{d^2 y}{dx^2} - 2ay^{-2}\frac{dy}{dx} - k\frac{dy}{dx} = 0,$$

which can be written

$$\frac{d}{dx}\left\{\frac{1}{2}\left(\frac{dy}{dx}\right)^2 + 2ay^{-1} - ky\right\} = 0.$$

Thus

$$\left(\frac{dy}{dx}\right)^2 + 4ay^{-1} - 2ky + A = 0, \tag{14}$$

where A is a constant. It follows from Eq. (14) that

$$\frac{dy}{dx} = \pm\left(2ky - A - 4ay^{-1}\right)^{1/2},$$

and so

$$\begin{aligned} x &= \pm\int \frac{dy}{\sqrt{2ky - A - 4ay^{-1}}}, \\ &= \pm\int \frac{y\,dy}{\sqrt{y\left(2ky^2 - Ay - 4a\right)}}. \end{aligned} \tag{15}$$

Equation (15) expresses y as an *implicit* function of x and, together with Eq. (12), represents an exact solution of the main metricity condition, Eq. (7). Although the right side of Eq. (15) can be evaluated in terms of elliptic integrals of the first and second kinds, the result is too complicated for the equation to be inverted to give y as *explicit* function of x.

Now consider the secondary metricity condition, Eq. (8),

$$3\,\partial\Omega = \Omega\,\partial_\rho\Lambda.$$

As with Λ, assume that Ω depends upon u and ρ through the combination $\rho + au$, but assume that Ω is independent of ω. Thus $\Omega = \Omega(\rho + au) = \Omega(x)$. Using $\Lambda = -a\omega + h = -a\omega + y^{3/2}$, it follows from Eq. (8) that

$$\Omega = h^{1/3} = y^{1/2}. \tag{16}$$

A family of (implicit) solutions of the (2 + 1)-dimensional null-surface formulation has now been found, and is given by Eqs. (15) and (16). Equation (15) gives an exact answer for y, which, using Eq. (12), gives the answer for Λ, the main dependent variable of the NSF. Finding Λ can be considered an end in itself. Nonetheless, it is still of interest to explore the link with general relativity and to require the third NSF field equation to be satisfied. This will ensure that the Einstein equations hold.

The third NSF field equation, Eq. (9),

$$\partial_\rho^2 \Omega = \kappa T_{\rho\rho} \Omega,$$

implies

$$T_{\rho\rho} = \frac{1}{2\kappa\, y^2} \left\{ y \frac{d^2 y}{dx^2} - \frac{1}{2} \left(\frac{dy}{dx} \right)^2 \right\}. \tag{17}$$

Using Eqs. (13) and (14), it follows from Eq. (17) that

$$T_{\rho\rho} = \frac{1}{4\kappa\, y^2} \left(A + 8ay^{-1} \right) = \frac{1}{4\kappa\, y^2} W, \tag{18}$$

where W is defined by

$$W := A + 8ay^{-1}. \tag{19}$$

For any null vector V, the null-energy condition requires $R_{ij} V^i V^j \geq 0$ or, equivalently, $G_{ij} V^i V^j \geq 0$. The vector $V^\rho = \partial_\rho \equiv (0, 0, 1)$ satisfies $g_{ij} V^i V^j = 0$ and is therefore null. Hence $T_{\rho\rho} \geq 0$, or equivalently $W \geq 0$, is a necessary condition for the null-energy condition to hold.

The properties of the general relativistic spacetime derived from the solutions for Λ and Ω, Eqs. (15) and (16), will be explored in the next section, and in the sections that follow.

4 Metric and curvature

Equations (13) and (14) imply following useful equations:

$$\partial_\rho^2 y = 2ay^{-2} + k, \tag{20}$$

and

$$(\partial_\rho y)^2 = 2ky - A - 4ay^{-1}. \tag{21}$$

Equations (6), (12), (16), (20), and (21) then lead to the following expression for the metric g_{ij}:

$$[g_{ij}] = \Omega^{-2}[\gamma_{ij}] = \begin{pmatrix} -\left(\frac{1}{4}A + 3ay^{-1}\right) & \frac{1}{2}y^{-1/2}\,\partial_\rho y & -y^{-1} \\ \frac{1}{2}y^{-1/2}\,\partial_\rho y & y^{-1} & 0 \\ -y^{-1} & 0 & 0 \end{pmatrix}. \quad (22)$$

Thus

$$ds^2 = \frac{1}{y}\left[-\left(\frac{1}{4}Ay + 3a\right)du^2 + y^{1/2}\,\partial_\rho y\, du d\omega - 2du d\rho + d\omega^2\right].$$

Hence $g := \det[g_{ij}] = -y^{-3}$. The inverse metric is

$$[g^{ij}] = \Omega^2[\gamma^{ij}] = \begin{pmatrix} 0 & 0 & -y \\ 0 & y & \frac{1}{2}y^{3/2}\,\partial_\rho y \\ -y & \frac{1}{2}y^{3/2}\,\partial_\rho y & \frac{1}{2}ky^3 + 2ay \end{pmatrix}. \quad (23)$$

The Christoffel symbols, the Ricci tensor R_{ij}, and the Einstein tensor G_{ij} are listed in the "Appendix". In 2 + 1 dimensions, there are three independent curvature scalars: R, $R_{ij}\, R^{ij} \equiv R^i{}_j R^j{}_i$, and $\det[R_{ij}]/\det[g_{ij}]$ [19, p. 145]. All are found to be constant:

$$R = \frac{1}{32}\,A^2 + ka, \quad (24)$$

$$R_{ij}\, R^{ij} = \frac{3}{1024}\,A^4 + \frac{1}{8}\,A^2\,ka + (ka)^2 = \left(\frac{1}{16}A^2 + R\right)R, \quad (25)$$

$$\frac{\det[R_{ij}]}{\det[g_{ij}]} = -\frac{1}{32}\,A^2\left(\frac{1}{32}\,A^2 + ka\right)^2 = -\frac{1}{32}\,A^2 R^2. \quad (26)$$

The spacetime can be analysed further by considering a null congruence, generated by a null vector n. Choose $n = (0, 0, \Omega^2) = (0, 0, y)$, which is not only null but is tangent to a null geodesic: $n^i{}_{;\,j}\, n^j = 0$. In 3 + 1 dimensions, three (optical) scalar invariants $\tilde{\omega}$, $\tilde{\sigma}$, and $\tilde{\theta}$ represent, respectively, rotation (twist), shear, and expansion. (The tilde $\tilde{}$ has been inserted to distinguish the optical invariants from the analogous invariants for a fluid with timelike velocity vector U, which will be discussed later.) In 2 + 1 dimensions, $\tilde{\omega}$ and $\tilde{\sigma}$ are always zero. Hence, as pointed out by Chow et al. [20] and emphasized by Podolsky et al. [21], a $(2+1)$-dimensional spacetime must be either Robinson-Trautman ($\tilde{\tau} = \tilde{\sigma} = 0, \tilde{\theta} \neq 0$) or else Kundt ($\tilde{\tau} = \tilde{\sigma} = \tilde{\theta} = 0$). The spacetime of the present paper has nonzero expansion,

$$\tilde{\theta} = -\frac{1}{2}\,n^i{}_{;\,i} = -\frac{1}{4}\,\partial_\rho y = \mp\frac{1}{4}\,\left(2ky - A - 4ay^{-1}\right)^{1/2},$$

and, consequently, the spacetime is Robinson-Trautman (except for the trivial case of flat spacetime: $k = A = a = 0$).

5 Petrov classification

In 2 + 1 dimensions, the Weyl tensor is identically zero. Its role is filled by the Cotton-York tensor, $C^i{}_j$. The Cotton-York tensor is defined by [22]

$$C^i{}_j := \varepsilon^{ikp} \left(R_{pj} - \frac{1}{4} R\, g_{pj} \right)_{;\,k} \tag{27}$$

where $\varepsilon^{ikq} := -(-g)^{-1/2}[ikq]$, with $[ikq]$ denoting the antisymmetric symbol and $[u\omega\rho] = 1$. The matrix whose elements are $C^i{}_j$ will be denoted by $[C^i{}_j]$. The Cotton-York tensor is identically zero if and only if the 2 + 1 spacetime is conformally flat. For the solution considered in the present paper, the components $C^i{}_j$ are listed in the "Appendix". Clearly, $[C^i{}_j] \neq 0$, and so the spacetime is not conformally flat and is hence nontrivial (apart from the exceptional circumstance in which two of the integration constants are zero: $A = 0$ plus either $k = 0$ or $a = 0$, which would imply that the spacetime is flat: $R = 0$).

The Cotton-York tensor is traceless, and so its eigenvalues sum to zero:

$$\lambda_1 + \lambda_2 + \lambda_3 = 0.$$

The eigenvalues provide a Petrov-type classification. According to the classification scheme of Garcia-Diaz [22, page 419],

(1) If there are three different eigenvalues (two independent) then, either all eigenvalues are real and the spacetime is of Petrov-type I, or one eigenvalue is real and two are complex, and the spacetime is of Petrov-type I′.

(2) If there are two different eigenvalues (one independent, so that $\lambda_1 = \lambda_2$ and $\lambda_3 = -2\lambda_1$) then consider the matrix $Q \equiv [Q^i{}_j]$ whose elements $Q^i{}_j$ are defined by

$$Q^i{}_j := \left(C^i{}_k - \lambda_3\, \delta^i{}_k \right) \left(C^k{}_j + \frac{1}{2}\lambda_3\, \delta^k{}_j \right).$$

If $Q = 0$, the spacetime is of Petrov-type D. If $Q \neq 0$, the spacetime is of Petrov-type II.

(3) If all eigenvalues are zero, then, either the matrix of the Cotton-York tensor components is zero, $[C^i{}_k] = 0$, and the spacetime is of Petrov-type O and is conformally flat and so is trivial, or $[C^i{}_k] \neq 0$ and $[C^i{}_j\, C^j{}_k] = 0$ and the spacetime is of Petrov-type N, or else, if neither $[C^i{}_k]$ nor $[C^i{}_j\, C^j{}_k]$ is zero then the spacetime is of Petrov-type III.

The present authors have reported an NSF solution corresponding to a spacetime of Petrov-type N in an earlier work [18]. For the solution in the present paper, given by Eqs. (12) and (15), the Cotton-York tensor is complicated function of $y(x)$. In spite of this, the eigenvalues of $C^i{}_j$ are all constant, and can be simply expressed in terms of A and the scalar curvature R:

$$\lambda_1 = \frac{1}{4} A R, \tag{28}$$

$$\lambda_2 = -\frac{1}{8} A R + 2^{-1/2} R^{1/2} \left(\frac{1}{16} A^2 + R \right), \tag{29}$$

$$\lambda_3 = -\frac{1}{8} A R - 2^{-1/2} R^{1/2} \left(\frac{1}{16} A^2 + R \right). \tag{30}$$

In general, the eigenvalues are all different. If they are all real, the solution represents a spacetime of Petrov-type I. If one eigenvalue is real and two are complex, the solution represents a spacetime of Petrov-type I′. By Eqs. (29) and (30), eigenvalues can only be complex if $R < 0$. By Eq. (24), this would imply $ka < 0$ and $|ka| > A^2/32$.

For the special case $a = 0$, $A \neq 0$, $k \neq 0$, the eigenvalues are:

$$\lambda_1 = \lambda_2 = \frac{1}{128} A^3, \quad \lambda_3 = -\frac{1}{64} A^3, \tag{31}$$

which leads to

$$Q^i{}_j := \left(C^i{}_k + \frac{1}{64} A^3 \, \delta^i{}_k \right) \left(C^k{}_j - \frac{1}{128} A^3 \, \delta^k{}_j \right) = 0.$$

Thus the spacetime is of Petrov-type D. This solution corresponds to a perfect fluid solution that was found earlier by the present authors [15]. A subcase of this solution comes from choosing $a = k = 0$, $A \neq 0$. The eigenvalues remain the same as those given in Eq. (31). The expression for C^i_j still leads to $Q = 0$, which means that the solution is still of Petrov-type D. This solution was also reported earlier by the present authors, but is unphysical because the energy conditions are violated [16].

For the special case $k = 0$, $a \neq 0$, $A \neq 0$, the eigenvalues are again the same as those given in Eq. (31), but the Cotton-York tensor $C^i{}_j$ leads to the following nonzero answer for Q:

$$[Q^i{}_j] = \frac{3}{(32)^2} A^4 a y^{-2} \begin{pmatrix} -a & 0 & -1 \\ \frac{1}{2} y^{1/2} \partial_\rho y & 0 & \frac{1}{2} y^{1/2} \partial_\rho y \\ a^2 & 0 & a \end{pmatrix}.$$

Hence this solution is of Petrov-type II.

Another special case in which two eigenvalues are the same comes from making the specific choice

$$ka = \frac{3}{32} A^2.$$

It follows that

$$\lambda_1 = \lambda_2 = \frac{1}{32} A^3, \quad \lambda_3 = -\frac{1}{16} A^3,$$

and that $Q = 0$. Hence the corresponding spacetime is of Petrov-type D.

The special case $A = 0, k \neq 0, a \neq 0$ was previously reported by the present authors and corresponds to a spacetime with a non-minimally coupled scalar field source [17]. From Eqs. (28), (29), and (30), it follows that the eigenvalues of the corresponding Cotton-York tensor are:

$$\lambda_1 = 0, \quad \lambda_2 = 2^{-1/2}(ka)^{3/2}, \quad \lambda_3 = -2^{-1/2}(ka)^{3/2}.$$

These three eigenvalues are all different from each other, and so the spacetime is either of Petrov-type I (all eigenvalues real), or of Petrov-type I′ (one eigenvalue real, two eigenvalues complex). Complex eigenvalues would arise if $(ka)^{3/2}$ were imaginary, which would imply $ka < 0$ and would result from k and a taking opposite signs. However, Eq. (15), (with $A = 0$), then implies $k > 0$ and $a < 0$ (for $y > 0$), or else $k < 0$ and $a > 0$ (for $y < 0$). Both of these alternatives, when combined with Eqs. (18) and (19), (for $A = 0$), lead to a violation of the null energy condition. Thus, if the spacetime is required to satisfy the null-energy condition, there will be no complex eigenvalues and the spacetime will be of Petrov-type I.

6 Source

As noted above, the family of NSF solutions introduced in the present paper has a special case, $a = 0, A \neq 0, k \neq 0$, where the corresponding spacetime is of Petrov-type D and is known to be consistent with a perfect fluid source [15]. The family of solutions also has a special case, $A = 0, k \neq 0, a \neq 0$, where the corresponding spacetime is of Petrov-type I and is known to be consistent with a minimally coupled scalar field source [17]. It is well known that spacetimes consistent with a minimally coupled scalar field source are also consistent with a perfect fluid source [23–25]. Thus it is reasonable to explore the possibility that the family of solutions given in the present paper will be consistent with a (possibly imperfect) fluid source. The form of the Ricci tensor, R_{ij}, given in the “Appendix” then suggests the following convenient choice for the fluid velocity vector U,

$$\begin{aligned}\left(U^u,\, U^\omega,\, U^\rho\right) &:= a^{-1/2} y^{1/2} \left(1,\, 0,\, -\frac{1}{8} W y\right),\\ \left(U_u,\, U_\omega,\, U_\rho\right) &= a^{-1/2} y^{-1/2} \left(-\frac{1}{8} W y - a,\, \frac{1}{2} y^{1/2}\, \partial_\rho y,\, -1\right).\end{aligned}$$

The covariant derivative $U_{i\,;\,j}$ can be written as a sum [26, p. 85], [27, p. 5293]

$$U_{i\,;\,j} = \omega_{ij} + \sigma_{ij} + \frac{1}{2}\,\theta\, h_{ij} - \dot{U}_i\, U_j,$$

where the projection tensor h_{ij}, the acceleration vector $\dot{U}_i$, the vorticity tensor ω_{ij}, the expansion θ, and the shear tensor σ_{ij} are defined by

$$\begin{aligned} h_{ij} &:= g_{ij} + U_i U_j, \\ \dot{U}_i &:= U_{i\,;\,j}\, U^j, \\ \omega_{ij} &:= \frac{1}{2}\,(U_{i\,;\,k}\, h^k{}_j - U_{j\,;\,k}\, h^k{}_i), \\ \theta &:= U^i{}_{;\,i}, \\ \sigma_{ij} &:= \frac{1}{2}\,(U_{i\,;\,k}\, h^k{}_j + U_{j\,;\,k}\, h^k{}_i) - \frac{1}{2}\,\theta\, h_{ij}. \end{aligned}$$

The coefficient of θ, both in the expression for $U_{i\,;\,j}$ and in the expression for σ_{ij}, is $\frac{1}{2}$ (instead of $\frac{1}{3}$, as would be the case in $3+1$ dimensions). This follows because the trace $h^i{}_i$ of the projection tensor is 2 (instead of 3). The components of $U_{i\,;\,j}$ and other kinematic quantities are listed in the "Appendix". Despite the complexity of the solution Λ (or, equivalently, g_{ab}), the scalar expansion θ, scalar vorticity ω, and scalar shear σ are simply expressed:

$$\begin{aligned} \theta &:= U^i{}_{;\,i} = 0, \\ \omega^2 &:= \frac{1}{2}\omega^{ij}\,\omega_{ij} = \frac{1}{4}R^2 a^{-2} y^2, \\ \sigma^2 &:= \frac{1}{2}\sigma^{ij}\,\sigma_{ij} = \frac{1}{128}A^2 R a^{-2} y^2. \end{aligned}$$

In analysing the Einstein field equations, $G_{ij} = R_{ij} - \frac{1}{2}R\, g_{ij}$, the velocity U^i and the projection tensor h_{ij} can be used to write the stress energy tensor as a sum of separate terms [26, p. 91], [27, p. 5294]

$$T_{ij} = \mu\, U_i U_j + p\, h_{ij} + q_i U_j + U_i q_j + \pi_{ij},$$

where $q_i U^i = 0$, $\pi_{ij} U^j = 0$, and $\pi^i{}_i = 0$. The function μ denotes the total mass-energy density, p the isotropic pressure, q^i the heat-flux vector, and π_{ij} the trace-free anisotropic pressure. It follows that

$$T^i_i = 2p - \mu,$$

and

$$\kappa\,(2p - \mu) = -\frac{1}{2}\,R.$$

Hence the Einstein field equations can be rewritten

$$R_{ij} = \kappa[T_{ij} - (2p - \mu) g_{ij}],$$

and then decomposed using U^i and $h^j_{\ k}$ to produce the following equivalent set of equations:

$$R_{ij}\, U^i U^j = 2\kappa\, p,$$
$$R_{ij}\, U^i h^j_{\ m} = -\kappa\, q_m,$$
$$R_{ij}\, h^i_{\ m}\, h^j_{\ n} = \kappa\, (\mu - p)\, h_{mn} + \kappa\, \pi_{mn}.$$

The choice of velocity vector U^i leads to relatively simple expressions for the components q_i of the heat-flux vector (which are given in the "Appendix"), and for the pressure p and the mass-energy density μ:

$$p = \frac{1}{16\kappa} a^{-1} y A R,$$
$$\mu = \frac{1}{8\kappa} a^{-1} y A R + \frac{1}{2\kappa} R.$$

Heat flow is related to temperature and acceleration by the temperature gradient law [26, p. 96],

$$q^i = -\lambda\, h^{ij} \left(T_{,\,j} + T\, \dot{U}_j\right),$$

where λ denotes the coefficient of thermal conductivity. It can be checked that, for the spacetime currently being considered, this law is satisfied identically. However, the usual phenomenological equation of state relating anisotropic pressure to the shear is *not* satisfied: $\pi_{ij} \neq -2\eta\sigma_{ij}$ (η being the coefficient of shear viscosity). Situations where this equation of state may not hold have been discussed by MacCallum et al. [28].

7 Results and future prospects

A family of $(2+1)$-dimensional null-surface formulation solutions has been presented and involves three parameters, A, a, and k. This family of NSF solutions is then used to deduce the metric for the corresponding general relativistic spacetimes. The scalar invariants for the spacetimes are listed in Eqs. (24), (25), and (26), and are all constant. The eigenvalues of the Cotton-York tensor are also constant. They are listed in Eqs. (28), (29), and (30), and provide a Petrov-type classification. An imperfect fluid was introduced as a possible source, and simple formulas were derived for the mass-energy density, the pressure, the scalar vorticity, the scalar shear, and for the scalar expansion (which was found to be zero). The temperature gradient law was satisfied. The usual phenomenological equation of state relating anisotropic pressure to the shear did not hold.

The idea of solving the NSF field equations and using the resulting NSF solution to determine a general relativistic spacetime metric raises the question of whether or not new or difficult-to-find general relativistic solutions could be found by this method—namely by first solving the NSF equations, and then using the NSF solution to determine a spacetime metric that solves the Einstein equations. Present and past experience suggests that this is true. For the family of solutions given in the present paper, the special choice $\{k \neq 0, a \neq 0, A = 0\}$ corresponds to a solution that was reported earlier by the present authors. This solution belongs to a general class of $2+1$ solutions with minimally coupled scalar field sources, but the solution had not been explicitly constructed prior to the NSF approach [17, 29]. A second example is provided by an NSF solution, found by the present authors but not related to the present paper, where the corresponding spacetime metric (of Petrov type N and with nonconstant curvature scalars) did not correspond to any previously known solution of the Einstein equations [18].

Since the solutions that make up the family all satisfy the NSF main metricity condition, they automatically satisfy Cartan's metricity condition [8, 11–14, 30, 31]. Cartan was concerned with classifying third-order ordinary differential equations and the link between (2 + 1)-dimensional NSF and Cartan's analysis can be seen in the definition of Λ given in Sect. 2,

$$\Lambda(u, \omega, \rho, \varphi) := \partial^3 Z(x^a(u, \omega, \rho, \varphi); \varphi),$$

with $u = Z(x^a; \varphi)$ denoting the family of null surfaces, i.e. the null foliation of the spacetime. The differential operator ∂ indicates differentiation with respect to the angle φ and can be conveniently indicated by a prime$'$. Thus [8, p. 1584]

$$Z''' = \Lambda(Z, Z', Z'', \varphi), \tag{32}$$

where, following the definitions in Sect. 1, $u := Z$, $\omega := Z'$, and $\rho := Z''$. It follows that, for the family of solutions for Λ given by Eqs. (12) and (15), the third-order equation given in Eq. (32) can be written:

$$Z''' = -aZ' + [y(aZ + Z'')]^{3/2}. \tag{33}$$

Equation (33) would need to be solved in order to find the null surfaces, Z. This has already been achieved and reported for the solutions of the present paper in the special case where $a = 0, A \neq 0, k \neq 0$ [15]. An explicit answer for Z can also be found for another special case of the solution of the present paper, namely when $a = k = 0, A \neq 0$. In this instance, Λ is simply [16]:

$$\Lambda = \rho^{3/2},$$

which corresponds to choosing $A = -1$ in Eq. (14). In spite of its simplicity, this solution is nontrivial, i.e. the resulting spacetime is not conformally flat. The fact that $\Lambda = Z''' = \rho'$ gives

$$\rho' = \rho^{3/2},$$

which can be integrated to give

$$\rho = \frac{4}{(\varphi + x)^2},$$

where x is a constant of integration. By writing $\rho = \omega'$, the equation can be integrated with respect to φ a second time to give

$$\omega = -\frac{4}{(\varphi + x)} + y,$$

where y is a constant of integration. After noting that $\omega = u' \equiv Z'$, a further integration yields the answer for Z:

$$Z = -4 \ln(\varphi + x) + y\varphi + z, \tag{34}$$

where z is a constant of integration. (The integration constants are denoted by x, y, and z because they can be regarded as spacetime coordinates.) Although special choices of the parameters a, A, and k lead to simple answers for Z, such as that given in Eq. (34), it is not possible to find an exact solution of Eq. (33) for general values of the parameters—particularly since $y(aZ + Z'')$ is a complicated *implicit* function of $u \equiv Z$ and $\rho \equiv Z''$.

The search for NSF solutions in 2 + 1 dimensions is motivated by the wish to discover clues for finding NSF solutions in 3 + 1 dimensions (where no nontrivial solutions have been found to-date). It should also be noted that the relationship between the (2 + 1)-dimensional NSF and Cartan's theory of ordinary differential equations generalizes to higher dimensions, where (3 + 1)-dimensional NSF is related to the theory of coupled partial differential equations [30–32]. However, the NSF in 3 + 1 dimensions is complicated by the fact that Λ is complex and is a quantity of spin-weight 2 [33, 34]. Surfaces are labelled by *two* angular variables, which are usually chosen to be the complex stereographic coordinate ζ and its complex conjugate $\bar{\zeta}$, with $\zeta = e^{i\varphi} \cot(\theta/2)$. Thus, for each constant choice of u, the equation $u = Z(x^a; \zeta, \bar{\zeta})$ represents a family of surfaces that is parametrized by ζ and $\bar{\zeta}$ [1–5].

In 3 + 1 dimensions, instead of ∂, the appropriate covariant differential operators are $\eth$ (called *eth*) and $\bar{\eth}$, and were first introduced by Newman and Penrose [33]. When operating on a quantity η of spin weight s, the operators $\eth$ and $\bar{\eth}$ are defined by [33–35]

$$\eth\eta = 2[(1 + \zeta\bar{\zeta})/2]^{1-s}\, \partial \left\{[(1 + \zeta\bar{\zeta})/2]^{s}\eta\right\}/\partial\zeta,$$
$$\bar{\eth}\eta = 2[(1 + \zeta\bar{\zeta})/2]^{1+s}\, \partial \left\{[(1 + \zeta\bar{\zeta})/2]^{-s}\eta\right\}/\partial\bar{\zeta}.$$

The operator $\eth$ causes spin weight to increase by one, and $\bar{\eth}$ causes spin weight to decrease by one. The fact that Λ is of spin-weight 2 is a consequence of its definition, $\Lambda := \eth^2 Z$, and of Z (i.e. u) being of spin-weight zero. The four so-called *intrinsic coordinates* result from operating on Z with $\eth$ and $\bar{\eth}$ [3, p. 4988]. The coordinates u and ρ are real and of spin-weight zero. (In 3 + 1 dimensions, ρ is often denoted by R

or r [1–5, 36].) In moving from $2+1$ to $3+1$, the single coordinate ω is replaced by two complex coordinates, ω and its complex conjugate $\bar{\omega}$, which are of spin weights $+1$ and -1, respectively.

The $(2+1)$-dimensional solution given in the present paper assumes additive separability,

$$\Lambda(u,\omega,\rho) = -a\omega - b\rho + h(\rho + au + b\omega) = -a\omega - b\rho + h(x), \tag{35}$$

(with b possibly chosen as zero), and is based upon the expression for ∂ given in Eq. (3), namely:

$$\partial = \partial' + \omega\,\partial_u + \rho\,\partial_\omega + \Lambda\,\partial_\rho.$$

The corresponding equation for $\eth$ is [3, p. 5000]

$$\eth = \eth' + \omega\,\partial_u + \Lambda\,\partial_\omega + \rho\,\partial_{\bar{\omega}} + K\,\partial_\rho, \tag{36}$$

where

$$K := \bar{\eth}\Lambda - 2\omega = (1 - \partial_\rho\Lambda\,\partial_\rho\bar{\Lambda})^{-1}\,(\bar{J}\,\partial_\rho\Lambda + J),$$

and

$$J := -2\omega + \bar{\omega}\,\partial_u\Lambda + \bar{\eth}'\Lambda + \rho\,\partial_\omega\Lambda + \bar{\Lambda}\,\partial_{\bar{\omega}}\Lambda.$$

Despite the added complication of spin-weight, and of Λ and ω being complex instead of real, the hope would be that Eq. (36) can be a starting point which, combined with an additive separability assumption analogous to that in Eq. (35), could lead to a nontrivial solution in $3+1$ dimensions.

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Data Availability Not applicable to this article as no datasets were generated or analyzed during the current study.

Appendix: Christoffel symbols and curvature tensors

The Christoffel symbols are

$$\begin{aligned}
\Gamma^u{}_{uu} &= -a\Gamma^u{}_{\omega\omega} = -a\Gamma^\omega{}_{\omega\rho} = 2^{-1}ay^{-1}\,\partial_\rho y,\\
\Gamma^u{}_{u\omega} &= 8^{-1}Ay^{-1/2} + ay^{-3/2},\\
\Gamma^u{}_{u\rho} &= \Gamma^u{}_{\omega\rho} = \Gamma^u{}_{\rho\rho} = \Gamma^\omega{}_{\rho\rho} = 0,\\
\Gamma^\omega{}_{uu} &= 2^{-1}Aay^{-1/2} + 3a^2y^{-3/2} - 2^{-1}kay^{1/2},
\end{aligned}$$

$$
\begin{aligned}
\Gamma^{\omega}_{\ u\omega} &= -(16^{-1}A + ay^{-1})\,\partial_\rho y,\\
\Gamma^{\omega}_{\ u\rho} &= 8^{-1}Ay^{-1/2} + ay^{-3/2},\\
\Gamma^{\omega}_{\ \omega\omega} &= -4^{-1}Ay^{-1/2} - ay^{-3/2} + 2^{-1}ky^{1/2},\\
\Gamma^{\rho}_{\ uu} &= -2^{-1}a\,(-4^{-1}A + 3ay^{-1} + 2^{-1}ky)\,\partial_\rho y,\\
\Gamma^{\rho}_{\ u\omega} &= -(16^{-1}Aky^{3/2} + a^2y^{-3/2} + kay^{1/2}),\\
\Gamma^{\rho}_{\ u\rho} &= (16^{-1}A - ay^{-1})\,\partial_\rho y,\\
\Gamma^{\rho}_{\ \omega\omega} &= (2^{-1}\,ay^{-1} + 4^{-1}\,ky)\,\partial_\rho y,\\
\Gamma^{\rho}_{\ \omega\rho} &= 8^{-1}Ay^{-1/2} - 2^{-1}ky^{1/2},\\
\Gamma^{\rho}_{\ \rho\rho} &= -y^{-1}\,\partial_\rho y.
\end{aligned}
$$

Using the abbreviation $W := A + 8a\,y^{-1}$, the Ricci tensor can be written as a matrix, $[R_{ij}]$:

$$
[R_{ij}] = \begin{pmatrix}
\frac{1}{256}\,(A^3 + W^3) + \frac{1}{8}\,Aka & -\frac{1}{64}\,AW\,y^{-1/2}\,\partial_\rho y & \frac{1}{32}\,W^2\,y^{-1}\\
-\frac{1}{64}\,AW\,y^{-1/2}\,\partial_\rho y & -\frac{1}{32}\,A^2\,y^{-1} + \frac{1}{8}\,Ak & -\frac{1}{8}\,A\,y^{-3/2}\,\partial_\rho y\\
\frac{1}{32}\,W^2\,y^{-1} & -\frac{1}{8}\,A\,y^{-3/2}\,\partial_\rho y & \frac{1}{4}\,W\,y^{-2}
\end{pmatrix}.
$$

The scalar curvature, R, is given in Eq. (24).

The components of the Einstein tensor, $G_{ij} = R_{ij} - \frac{1}{2}Rg_{ij}$, are

$$
\begin{aligned}
G_{uu} &= \frac{1}{256}(A^3 + W^3) + \frac{1}{8}R(A + 12ay^{-1}) + \frac{1}{8}Aka,\\
G_{u\omega} &= -\frac{1}{64}(AW + 16R)y^{-1/2}\,\partial_\rho y,\\
G_{u\rho} &= \frac{1}{32}y^{-1}(W^2 + 16R),\\
G_{\omega\omega} &= -\frac{1}{32}y^{-1}(A^2 + 16R) + \frac{1}{8}Ak,\\
G_{\omega\rho} &= -\frac{1}{8}Ay^{-3/2}\,\partial_\rho y,\\
G_{\rho\rho} &= \frac{1}{4}Wy^{-2}.
\end{aligned}
$$

The covariant derivatives of the velocity can be written as a matrix, $[U_{i\,;\,j}]$:

$$
\begin{pmatrix}
-\frac{1}{8}a^{1/2}y^{-1/2}\,W\,\partial_\rho y & -\frac{1}{64}a^{-1/2}\,W^2 & -\frac{1}{16}a^{-1/2}y^{-1/2}\,W\,\partial_\rho y\\
a^{-1/2}\left(\frac{1}{2}ka - \frac{1}{64}W^2 + a^2y^{-2}\right) & \frac{1}{16}a^{-1/2}y^{-1/2}\,A\,\partial_\rho y & \frac{1}{2}a^{-1/2}y^{-1}\left(ky - \frac{1}{4}A\right)\\
-a^{1/2}y^{-3/2}\,\partial_\rho y & -\frac{1}{8}a^{-1/2}y^{-1}\,W & -\frac{1}{2}a^{-1/2}y^{-3/2}\,\partial_\rho y
\end{pmatrix}.
$$

The acceleration vector, $\dot{U}_i$, is defined by $\dot{U}_i := U_{i\,;\,j}\, U^j$, and its components are

$$\begin{aligned}
\dot{U}^u &= \left(1 - \frac{1}{16} W a^{-1} y\right)\, \partial_\rho y, \\
\dot{U}^\omega &= \frac{1}{4} W y^{1/2} \left(1 - \frac{1}{16} W a^{-1} y\right) - \frac{1}{2} R a^{-1} y^{3/2}, \\
&= a^{-1} y^{3/2} \left(-R + \frac{1}{2} k a + a^2 y^{-2}\right), \\
\dot{U}^\rho &= -a \left(1 - \frac{1}{16} W a^{-1} y\right)\, \partial_\rho y - \frac{1}{4} R a^{-1} y^2\, \partial_\rho y.
\end{aligned}$$

The components of the heat-flux vector q_i are

$$\begin{aligned}
q_u &= \frac{1}{64\kappa} a^{-3/2} y^{3/2} A W R, \\
q_\omega &= -\frac{1}{16\kappa} a^{-3/2} y A R\, \partial_\rho y, \\
q_\rho &= \frac{1}{8\kappa} a^{-3/2} y^{1/2} A R.
\end{aligned}$$

The only nonzero components of the vorticity tensor, ω_{ij}, are

$$\begin{aligned}
\omega_{u\omega} &= -\omega_{\omega u} = -\frac{1}{16} W R a^{-3/2} y, \\
\omega_{\omega\rho} &= -\omega_{\rho\omega} = \frac{1}{2} R a^{-3/2}.
\end{aligned}$$

The components of the shear tensor, σ_{ij}, are

$$\begin{aligned}
\sigma_{uu} &= -\frac{1}{1024} A W^2 a^{-3/2} y^{3/2}\, \partial_\rho y, \\
\sigma_{u\omega} = \sigma_{\omega u} &= \frac{1}{128} W^2 k a^{-3/2} y^2 - \frac{1}{16} W R a^{-3/2} y, \\
\sigma_{u\rho} = \sigma_{u\rho} &= -\frac{1}{128} A W a^{-3/2} y^{1/2}\, \partial_\rho y, \\
\sigma_{\omega\omega} &= -\frac{1}{32} A k a^{-3/2} y^{3/2}\, \partial_\rho y, \\
\sigma_{\omega\rho} = \sigma_{\rho\omega} &= \frac{1}{16} W k a^{-3/2} y - \frac{1}{2} R a^{-3/2}, \\
\sigma_{\rho\rho} &= -\frac{1}{16} A a^{-3/2} y^{-1/2}\, \partial_\rho y.
\end{aligned}$$

The components of the Cotton-York tensor, ${C^i}_j$, are defined in Eq. (27) and are as follows:

$$
\begin{aligned}
{C^u}_u &= -\frac{1}{4}W\left(R+\frac{1}{32}AW\right),\\
&= -\frac{1}{4}A\left(R+\frac{1}{32}W^2\right)-2Ray^{-1},\\
{C^u}_\omega &= y^{-1/2}\left(\frac{1}{2}R+\frac{1}{32}A^2\right)\partial_\rho y,\\
{C^u}_\rho &= -Ry^{-1}-\frac{1}{16}AWy^{-1},\\
{C^\omega}_u &= ay^{-1/2}\left(\frac{1}{2}R+\frac{1}{32}AW\right)\partial_\rho y,\\
{C^\omega}_\omega &= \frac{1}{128}A^2W+Ray^{-1},\\
{C^\omega}_\rho &= \frac{1}{4}Aay^{-3/2}\,\partial_\rho y,\\
{C^\rho}_u &= \frac{1}{128}AW^2a-2Ra^2y^{-1}+\frac{1}{2}RWa,\\
{C^\rho}_\omega &= -ay^{-1/2}\left(\frac{1}{2}R+\frac{1}{32}A^2\right)\partial_\rho y,\\
{C^\rho}_\rho &= \frac{1}{2}Aa^2y^{-2}-2ka^2y^{-1}+\frac{1}{4}AR+3Ray^{-1},\\
&= \frac{1}{16}AWay^{-1}+\frac{1}{4}AR+Ray^{-1}.
\end{aligned}
$$

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