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RESEARCH ARTICLE

Exact solutions of a causal viscous FRW cosmology within the Israel–Stewart theory through factorization

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Abstract

In this paper we find exact analytic cosmological solutions for a relativistic dissipative fluid, in the framework of the causal Israel–Stewart theory. We use a general expression for the relaxation time, which is related with the bulk viscosity coefficient, the energy density and pressure of the fluid, and non-adiabatic contribution to the speed of sound. Through the factorization method we find some new exact parametric solutions for the special case $s = 1/2$. For each solution the deceleration parameter, the energy density, the dissipative pressure, the entropy, and the ratio between the dissipative pressure and the fluid's pressure are evaluated as a function of the cosmic time. We finally discuss the kinematic behavior of the solutions and their relationship with their thermodynamic behavior.

Keywords Bulk viscous cosmology · Israel–Stewart theory · Exact solution · Factorization method

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Contents

1 Introduction

The unified dark matter (UDM) models have been explored as alternative to ΛCDM model. One class of them consider a single fluid that behaves as DM, but at late times display an exotic EoS with an effective negative pressure, which drive accelerated expansion. Some of them include generalized perfect fluid models [1,2], logotropic dark fluid [3] or Generalized Chaplygin fluids [4–7]. Other kind of these models are those which consider one dissipative matter component, which were explored at first assuming that dissipative processes played a fundamental role in the dynamics of the early universe. The action of the neutrino viscosity and its effects on the observed large scale isotropy was investigated in [8]. Other processes involving viscous effects in the early times are the evolution of cosmic strings, the classical description of the (quantum) particle production, interaction between matter and radiation, quark and gluon plasma, interaction between different components of dark matter, etc [9].

Instead of a cosmological constant or some kind of dark energy fluid, the presence of bulk viscosity in FRW universes can leads to accelerated expansion due to the negativeness of the viscous pressure and, therefore, could describe the transition observed in our universe from an expanding decelerated era to an accelerated one at present times. These types of models were explores for the first time in [10,11] and later several of them were constrained using cosmological data in [12–17]. Nevertheless, in UDM models the viscous pressure acts as a dynamical DE and predict the above transition earlier than the predict by the ΛCDM model [18,19].

Nowadays the standard model presents problems that have been tackled within viscous models, such as H_0 [12] tension and an excess of radiation in reionization epoch, about $z \approx 17$, measured by the Experiment to Detect the Global EoR Signature (EDGES) [20].

The description of viscous fluids in the framework of the Eckart's theory [21] has been investigated at the background level in the late time cosmology [22] and in inflationary scenarios [10]. Nevertheless, a causal description of non-perfect fluids is necessary to avoid non physical effects such as instabilities and superluminal propagation of the viscous effects. The Israel–Stewart (IS) theory [23–26] describes relativistic non-perfect fluids satisfying causality, but the price to pay is to solve nonlinear ordinary differential equations (ODE) associated to the Hubble parameter. The study of the possible solutions to be found presents the challenge to encounter scenarios which consistently describe the kinematics of the universe, and also thermodynamic conditions required by the theoretical framework. For example, the recent acceleration of the universe and the constraining of the model parameters using the Type Ia supernovae data was performed on [27].

The nonlinear ODE for the Hubble rate which raises within the IS theory has been previously addressed by many authors. The corresponding study of the dynamical system associated to the truncated and full version was undertaken in [28,29]. An extension of these studies, including also radiation and interactions with dark energy, was performed in [30]. More recently, a numerical study of the dynamical analysis for pressureless DM was investigated in [31].

In [9,32–38] several exact solutions for the Einstein field equations (FE) in the framework of dissipative cosmologies have been found for the special case where the coefficient of dissipation is proportional to the square root of the energy density. In all these works the analytic solutions found take the adiabatic contribution equal to one, which simplified calculations, but take for granted that the dissipative effect is propagating at the speed of light. As we show below, in this work, we will take a general expression where the propagation velocity appears as a free parameter.

The aim of this paper is to find new exact solutions to the nonlinear ODE which drives the evolution of a homogeneous and isotropic FRW universe, filled with a simple viscous fluid in the framework of the full causal theory of IS, taking into account a general expression which relates the relaxation time, τ, the bulk viscosity coefficient, ξ, and the pressure and energy density of the fluid, found in [26]

$$\frac{\xi}{(\rho + p)\,\tau} = c_b^2, \tag{1}$$

where c_b is the speed of bulk viscous perturbations (non-adiabatic contribution to the speed of sound in a dissipative fluid without heat flux or shear viscosity). Since the speed of sound within the fluid is given by the expression

$$v^2 = c_s^2 + c_b^2 \leq 1, \tag{2}$$

where c_s is the adiabatic contribution given by

$$c_s^2 = \frac{\partial p}{\partial \rho}, \tag{3}$$

then the non adiabatic contribution can be parameterized in the following form $c_b^2 = \epsilon\,(2-\gamma)$ where $0 < \epsilon \leq 1$, which ensures to have a dissipative speed of sound lower or equal to the speed of light. In various exact solutions found the parameter ϵ is taken equal to one, which ensures causality, but simplifies the physics behind a dissipative fluid.

The equation of evolution is a second order ODE in time for the Hubble parameter, and it can be solved in a systematic way by using the factorization method [38–42], which allows to get solutions in an algebraic manner. This method was widely used for linear ODEs in quantum mechanics since Dirac's works to solve the spectral problem for the quantum oscillator, and had a further development due to Schrodinger's works on the factorization of the Sturm-Liouville equation. The basic concept follows the same pattern already used in linear equations and it works efficiently for ODEs with polynomial nonlinearities. It has been shown that the method is well adapted to the Hubble rate ODE, which raises, for instance, in several cosmological models studied in the context of viscous fluids [38,43,44].

Due to the inclusion of viscosity and the nonlinearity of the evolution equation it is possible to find a wide variety of possible cosmological scenarios, that we will discuss in terms of their kinematic properties, i.e., phases of deceleration or acceleration, and also in terms of their thermodynamic behavior: evolution during the cosmic time of the viscous pressure and grow of the entropy.

This work is organized as follows: in Sect. 2 we write the main equations of the model, where a viscous fluid is described in the framework of the causal Israel–Stewart theory. It is presented the general differential equation for the Hubble parameter, H, which is necessary to solve in order to explore the properties of the cosmic evolution. In Sect. 3, the nonlinear differential equation which drives the evolution of the Hubble parameter is addressed by using the factorization method. In Sect. 4, we found exact solutions for the particular case $s = 1/2$. Finally, in Sect. 5 we present our conclusions considering the kinematic and thermodynamic behavior of the solutions found.

2 The model

In what follows we assume a flat FRW universe filled with only one dissipative matter component. The line element is therefore given by

$$ds^2 = -dt^2 + a^2(t)\left(dx^2 + dy^2 + dz^2\right), \tag{4}$$

and the energy-momentum tensor, which contains only a bulk viscous term, takes the form [25]:

$$T_i^k = (\rho + p + \Pi)\, u_i u^k + (p + \Pi)\, \delta_i^k, \tag{5}$$

where ρ is the energy density, p the thermodynamic pressure, Π the bulk viscous pressure and u_i the four-velocity satisfying the condition $u_i u^i = -1$. We use the units $8\pi G = c = 1$. The gravitational field equations together with the continuity equation, $T^k_{i;k} = 0$, are given by the following expressions

$$2\dot{H} + 3H^2 = -p - \Pi, \tag{6}$$

$$3H^2 = \rho, \tag{7}$$

$$\Pi + \tau\dot{\Pi} = -3\xi H - \frac{1}{2}\tau\Pi\left(3H + \frac{\dot{\tau}}{\tau} - \frac{\dot{\xi}}{\xi} - \frac{\dot{T}}{T}\right), \tag{8}$$

$$\dot{\rho} = -3\left(\gamma\rho + \Pi\right)H, \tag{9}$$

where $H = \dot{a}/a$. Note that Eq. (8) is the transport equation for the viscous pressure Π in the IS framework and takes into account a relaxation time, τ, associated to causal dissipative process.

In order to close the system of equations we are assuming the following equations of state [25]

$$p = (\gamma - 1)\,\rho, \quad T = T_0 \rho^r, \tag{10}$$

where T is the temperature, $\gamma \in [1, 2]$ and $r = \left(1 - \frac{1}{\gamma}\right)$,

$$\xi = \xi_0 \rho^s, \tag{11}$$

ξ is the bulk viscosity coefficient and ξ_0 is a positive constant because of the second law of thermodynamics [45]. This dependence for the bulk viscosity in terms of the density has been widely considered as a suitable function. Using the EoS and Eq. (3) for c_b^2 we obtain that the relaxation time can be written as

$$\tau = \frac{\xi_0}{\epsilon\gamma(2-\gamma)}\rho^{s-1} = \alpha\rho^{s-1}. \tag{12}$$

It is straightforward to see that in the case of a stiff matter fluid, with $\gamma = 2$, Eq. (12) indicates that τ goes to infinity, which means that in this kind of fluid it is not possible to have dissipative effects. This fact can be better understood taking into account the condition $\tau H < 1$, in order to have a consistent fluid description. This physical consistency does not appear in the analytic solutions found in [38], where the adiabatic contribution was taken to be equal from the beginning and then the pressure of the fluid must be zero, in order to satisfy the condition given by Eq. (2). In fact, as we shall show below, the stiff matter case, $\gamma = 2$, is singular and must be studied separately.

The growth of entropy has the following behavior

$$\Sigma\,(t) \approx -3k_B^{-1} \int_{t_0}^{t} \Pi H a^3 T^{-1} dt, \tag{13}$$

where k_B is the Boltzmann constant.

The Israel–Stewart–Hiscock theory is derived under the assumption that the thermodynamical state of the fluid is close to equilibrium, i.e., the non-equilibrium bulk viscous pressure should be small when compared to the local equilibrium pressure $|\Pi| << p = (\gamma - 1)\rho$. Then, we can define the $l(t)$ parameter by $l = |\Pi|/p$. If this condition is violated then one is effectively assuming that the linear theory also holds in the nonlinear regime far from equilibrium. For a fluid description of the matter, the condition ought to be satisfied.

To determine if a cosmological model inflates or not it is convenient to introduce the deceleration parameter $q = dH^{-1}/dt - 1$. The positive sign of the deceleration parameter corresponds to standard decelerating models, whereas the negative sign indicates accelerated expansion.

The fundamental dynamical equation for the Hubble rate is given by

$$-2\dot{H} - 3\gamma H^2 = \Pi, \tag{14}$$

therefore the time variation of the viscous pressure can be expressed as

$$\dot{\Pi} = -\left(2\ddot{H} + 6\gamma H\dot{H}\right). \tag{15}$$

Using the Eqs. (10), (11) and (12) we can obtain the sum of terms in the braket of Eq. (8) in terms of H and $\dot{H}$

$$\frac{\dot{\tau}}{\tau} - \frac{\dot{\xi}}{\xi} - \frac{\dot{T}}{T} = -2\left(\frac{2\gamma - 1}{\gamma}\right)\frac{\dot{H}}{H}. \tag{16}$$

Introducing Eq. (16) in Eq. (8) we obtain

$$\Pi + \tau\dot{\Pi} = -3\xi H - \frac{1}{2}\tau\Pi\left(3H - 2\left(\frac{2\gamma - 1}{\gamma}\right)\frac{\dot{H}}{H}\right). \tag{17}$$

Dividing the above equation by τ, yields

$$\dot{\Pi} + \frac{\Pi}{\tau} + \frac{3\xi H}{\tau} + \frac{1}{2}\Pi\left(3H - 2\left(\frac{2\gamma - 1}{\gamma}\right)\frac{\dot{H}}{H}\right) = 0, \tag{18}$$

where

$$\frac{\Pi}{\tau} = -2\,(3)^{1-s}\,\alpha^{-1}H^{2-2s}\dot{H} - (3)^{2-s}\,\gamma\alpha^{-1}H^{4-2s}, \tag{19}$$

$$\frac{3\xi H}{\tau} = 9\xi_0\alpha^{-1}H^3, \tag{20}$$

$$\frac{3}{2}\Pi H = -3H\dot{H} - \frac{9}{2}\gamma H^3, \tag{21}$$

$$\Pi\left(\frac{2\gamma - 1}{\gamma}\right)\frac{\dot{H}}{H} = 2\left(\frac{2\gamma - 1}{\gamma}\right)H^{-1}\dot{H}^2 + 3\,(2\gamma - 1)\,H\dot{H}. \tag{22}$$

If we substitute the above expressions and Eq. (15) into the Eq. (18), then we obtain the following nonlinear second order ODE for the Hubble rate

$$\ddot{H} - A\frac{\dot{H}^2}{H} + \left(BH + CH^{2(1-s)}\right)\dot{H} + DH^3 + EH^{2(2-s)} = 0, \tag{23}$$

where

$$A = (1 + r) = 2 - \frac{1}{\gamma}, \quad B = 3, \quad C = 3^{1-s}\xi_0^{-1}\epsilon\gamma\,(2 - \gamma),$$

$$D=\frac{9}{4}\gamma\left[1-2\epsilon\left(2-\gamma\right)\right],\quad E=\frac{1}{2}3^{2-s}\xi_0^{-1}\epsilon\gamma^2\left(2-\gamma\right). \tag{24}$$

The general solution to Eq. (23) has two integration constants whose values can be obtained through two initial conditions given as follows

$$H\left(t_0\right)=H_0, \tag{25}$$

$$\dot{H}\left(t_0\right)=-H_0^2\left(q_0+1\right), \tag{26}$$

where H_0 and q_0 are the values of H and q, respectively, at time $t=t_0$.

Furthermore, one is able to obtain analytic parametric solutions to Eq. (23) through the factorization method, as it is shown in the following sections.

3 Solving the Hubble differential equation through factorization.

Let us perform in Eq. (23) the following transformation of the dependent and independent variables

$$H=y^{1/2},\quad d\eta=y^{1/2}dt, \tag{27}$$

then it turns into

$$\frac{d^2y}{d\eta^2}-\frac{A}{2y}\left(\frac{dy}{d\eta}\right)^2+\left(3+Cy^{\frac{1}{2}-s}\right)\frac{dy}{d\eta}+2y(D+Ey^{\frac{1}{2}-s})=0, \tag{28}$$

with the initial conditions

$$y(\eta_0)=H_0^2, \tag{29}$$

$$\frac{dy(\eta_0)}{d\eta}=-2y(\eta_0)(q_0+1), \tag{30}$$

where η_0 is the value of the variable η at time t_0. We consider now the factorization method [39,40] which provides a systematic way to solve nonlinear ODEs. The nonlinear second order equation

$$y''+f\left(y\right)y'^2+g(y)y'+h(y)=0, \tag{31}$$

where $y'=\frac{dy}{d\eta}=D_\eta y$, can be factorized in the form

$$\left[D_\eta-\phi_1(y)y'-\phi_2(y)\right]\left[D_\eta-\phi_3(y)\right]y=0, \tag{32}$$

under the conditions

$$f\left(y\right)=-\phi_1, \tag{33}$$

$$g(y)=\phi_1\phi_3y-\phi_2-\phi_3-\frac{d\phi_3}{dy}y, \tag{34}$$

$$h(y) = \phi_2\phi_3 y. \tag{35}$$

Furthermore, if we assume $\left[D_\eta - \phi_3(y)\right] y = \Omega(y, \eta)$, then the factorized Eq. (32) can be rewritten as follows

$$y' - \phi_3 y = \Omega, \tag{36}$$

$$\Omega' - \left(\phi_1 y' + \phi_2\right) \Omega = 0. \tag{37}$$

The factoring functions ϕ_i are introduced by comparing Eqs. (28) and (31). Then, $\phi_1 = \frac{A}{2y}$, $\phi_2 = a_1^{-1}$ and $\phi_3 = 2a_1(D + Ey^{\frac{1}{2}-s})$, where $a_1(\neq 0)$ is an arbitrary constant, are proposed.

The Eq. (37) is easily solved for the chosen factorizing functions giving as result $\Omega = \kappa_1 e^{\eta/a_1} y^{A/2}$, where κ_1 is an integration constant. Therefore, the Eq. (36) turns into the following equation

$$y' - 2a_1 \left(D + Ey^{\frac{1}{2}-s}\right) y - \kappa_1 e^{\eta/a_1} y^{A/2} = 0, \tag{38}$$

whose solution is also solution of Eq. (28).

Also, the following relationship is obtained from Eq. (34),

$$Aa_1 D - a_1^{-1} - 2a_1 D + a_1 E(A - 3 + 2s)y^{\frac{1}{2}-s} = 3 + Cy^{\frac{1}{2}-s}. \tag{39}$$

This last equation provides the explicit form of a_1 and the relationship among the parameters entering the Eq. (28). Then, the viscous parameter s as a function of the parameters (γ, ϵ) is obtained. By comparing both sides of Eq. (39) leads to obtain

$$Aa_1 D - a_1^{-1} - 2a_1 D = 3,$$
$$a_1 E(A - 3 + 2s) = C,$$

which provide the consistency relationships

$$a_{1\pm} = \frac{3 \pm \sqrt{4D\,(A-2) + 9}}{2D\,(2 - A)} = \frac{2\left(-1 \pm \sqrt{2\epsilon\,(2-\gamma)}\right)}{3\,(2\epsilon\,(\gamma - 2) + 1)}, \tag{40}$$

and

$$s_\pm = \frac{1}{2Ea_{1\pm}}\,(C + Ea_{1\pm}\,(3 - A)) = \frac{1}{2} \mp \frac{\sqrt{\epsilon(1-\gamma/2)}}{\gamma}, \tag{41}$$

Furthermore, $s_+ < 0$ for $1 \le \gamma < -1 + \sqrt{5}$ and $\frac{\gamma^2}{4-2\gamma} < \epsilon \le 1$.

A noteworthy fact, which represents an advantage of the factorization method as opposed to different approaches studied by other authors, is given in Eq. (41), since this equation provides a relationship among parameters in such a way that by fixing s we are able to get a pair of particular values of γ and ϵ; although the case $\gamma = 2$ decouples the parameter ϵ.

The main dynamical variables of the FE are given in parametric form as follows

$$a(\eta) = a_0 \exp(\eta - \eta_0), \tag{42}$$

$$H(\eta) = y^{1/2}(\eta), \tag{43}$$

$$q(\eta) = y^{1/2}(\eta) \frac{d}{d\eta}\left(\frac{1}{H(\eta)}\right) - 1, \tag{44}$$

$$\rho(\eta) = 3y(\eta), \tag{45}$$

$$p(\eta) = 3(\gamma - 1) y(\eta), \tag{46}$$

$$\Pi(\eta) = -\left(3\gamma y(\eta) + \frac{dy}{d\eta}\right), \tag{47}$$

$$l(\eta) = \frac{|\Pi|}{p}, \tag{48}$$

$$\Sigma(\eta) = -3k_B^{-1} \int \Pi(\eta) a^3(\eta) H(\eta) T(\eta)^{-1} y(\eta)^{-1/2} d\eta. \tag{49}$$

The authors have not been able to find the most general solution of Eq. (38). However, this equation can be studied for several particular cases providing general and particular solutions of cosmological interest. In the following section the special case $s = 1/2$ is studied.

4 Exact solutions for $s = 1/2$

4.1 First approach: solutions through Eq. (38) for $s = 1/2$

In what follows we will restrict to study the highly nonlinear differential equation for the Hubble parameter given by Eq. (23), for the case $s = 1/2$. We obtain the following simplified form:

$$\ddot{H} - A\frac{\dot{H}^2}{H} + (B + C) H\dot{H} + (D + E) H^3 = 0, \tag{50}$$

where the constant parameters are given in Eq. (24), and the initial conditions are as given in Eqs. (25) and (26).

Let us consider now the previous factorization procedure since Eq. (50) is a particular case of Eq. (23). Then, according to Eqs. (40) and (41), $a_1 = -2/3$ and $\gamma = 2$, respectively. Note that the method brings us to get only $\gamma = 2$. Furthermore, according to Eq. (38), the Eq. (50) is reduced to the first order differential equation

$$y' + 6y - k_1 e^{-3\eta/2} y^{3/4} = 0, \tag{51}$$

where k_1 is an integration constant, and with general solution given as follows

$$y(\eta) = \frac{1}{256} e^{-6\eta} (k_1\eta + 4k_2)^4, \tag{52}$$

where k_2 is a second integration constant. For the initial conditions given by Eqs. (29) and (30), we get

$$k_1 = 2H_0^{1/2}(2 - q_0), \tag{53}$$

$$k_2 = H_0^{1/2}, \tag{54}$$

and $\eta_0 = 0$ without loss of generality. Also, (from $t(\eta) = \int y^{-1/2} d\eta$)

$$t(\eta) = -\frac{48}{k_1^2}\left(\frac{e^{3\eta}}{3\eta + 12\frac{k_2}{k_1}} + e^{-12k_2/k_1} Ei\left(1, -3\eta - 12\frac{k_2}{k_1}\right)\right), \tag{55}$$

where Ei stands for the exponential integral function.

The Eqs. (52) and (55) represent a parametric general solution of Eq. (50). The FE main quantities are given in the following parametric form

$$a(\eta) = a_0 \exp(\eta - \eta_0), \tag{56}$$

$$H(\eta) = y^{1/2}(\eta), \tag{57}$$

$$q(\eta) = 2\left(1 - \frac{k_1}{k_1\eta + 4k_2}\right), \tag{58}$$

$$\rho(\eta) = 3y(\eta) = p(\eta), \tag{59}$$

$$\Pi(\eta) = -4y(\eta)\left(\frac{k_1}{k_1\eta + 4k_2}\right), \tag{60}$$

$$l(\eta) = \frac{4}{3}\left|\frac{k_1}{k_1\eta + 4k_2}\right|, \tag{61}$$

$$\Sigma(\eta) = \frac{\sqrt{3}}{8}\frac{a_0^3 e^{-3\eta_0}}{k_B T_0}(k_1\eta + 4k_2)^2. \tag{62}$$

However, according to Eq. (12) it is pointed out that this solution is unphysical since the dissipative effects do not vanish as it is expected.

A particular solution can be obtained by setting $q_0 = 2$, which leads to $k_1 = 0$ in Eq. (52) providing the simplified form

$$y(\eta) = k_2^4 e^{-6\eta}, \tag{63}$$

and therefore the time variable

$$t(\eta) = \frac{1}{3k_2^2} e^{3\eta}, \tag{64}$$

so we may recover the scaling solution

$$a = \bar{a}_0 t^{1/3}, \quad \text{where} \quad \bar{a}_0 = a_0 e^{-\eta_0} \sqrt[3]{3k_2^2}, \tag{65}$$

$$H = \frac{1}{3t}, \tag{66}$$

$$q = 2, \tag{67}$$

$$\rho = \frac{1}{3t^2}. \tag{68}$$

It is pointed out that this solution is self-similar and invariant, since it may be obtained as invariant solution from one of the two admitted symmetries ($X_2 = [t, -H]$) of Eq. (50). This suggests us that from the dynamical system point of view, the solution is a singularity and probably a future attractor.

Note that this solution was found in [46], where it was written in the more general form

$$H(t) = A\,(t_s - t)^{-1}, \tag{69}$$

which allows to explore scenarios with phantom behavior, where a big rip singularity occurs at a finite time t_s in the future. Also the expression

$$H(t) = A\,(t - t_s)^{-1}, \tag{70}$$

is a solution and, depending on the values of the constant A, leads to cosmic evolution with decelerated, linear or accelerated expansions, with constant q parameter [47]. In this sense, the solution found corresponds to a particular case with $q = 2$.

The consistency relationship (41) for the viscous parameter does not allow to study the case $s = 1/2$ for values of γ within the interval $[1, 2)$. However, this constraint is overcome in the following section B.

4.2 Second approach: solutions for $s = 1/2$ and $\gamma \in [1, 2)$

In order to study other cosmological scenarios of interest, we consider for the viscous parameter $s = 1/2$ and $\gamma \in [1, 2)$. The Hubble rate equation (23) reduces to the same form of Eq. (50). However, we perform the more suitable change of variables given in the form

$$H = y^{1/2}, \qquad d\eta = (3 + C)\,H dt. \tag{71}$$

Therefore, Eq. (50) turns into

$$y'' - \frac{A}{2y} y'^2 + y' + 2Ky = 0, \tag{72}$$

where $K = \frac{(D+E)}{(3+C)^2}$.

The Eq. (72) admits the factorization

$$\left[D_\eta - \frac{A}{2y} y' - a_1^{-1}\right]\left[D_\eta - 2a_1 K\right] y = 0, \tag{73}$$

under the consistency condition

$$(A-2)\,a_1 K - a_1^{-1} = 1,$$

which yields the relationship

$$a_{1\pm} = \frac{1 \pm \sqrt{4K\,(A-2)+1}}{2K\,(A-2)}. \tag{74}$$

If we consider $\left[D_\eta - 2a_1 K\right] y = \Omega(y,\eta)$, then the factorized Eq. (73) can be rewritten as follows

$$y' - 2a_1 K y = \Omega, \tag{75}$$

$$\Omega' - \left(\frac{A}{2y} y' - a_1^{-1}\right)\Omega = 0. \tag{76}$$

The Eq. (76) provides

$$\Omega = k_1 e^{\frac{\eta}{a_1}} y^{A/2},$$

where k_1 is an integration constant. So, the Eq. (75) is rewritten in equivalent form

$$y' - 2a_1 K y - k_1 e^{\eta/a_1} y^{A/2} = 0, \tag{77}$$

with general solution

$$y\,(\eta) = 4^{1/(A-2)} e^{2a_1 K\eta} \left(\frac{-k_1 a_1 (A-2)}{\bar{B}} e^{\bar{B}\eta/a_1} + 2k_2\right)^{\frac{2}{2-A}}, \tag{78}$$

where k_2 is an integration constant, $\bar{B} = K a_1^2 (A-2) + 1$, and the parameter a_1 is restricted to values given by Eq. (74). We emphasize the fact that the constant $\bar{B} = 0$ when $\gamma = 2$, and for this reason this particular and special case must be studied separately.

For the initial conditions given in Eqs. (29) and (30), we get

$$k_1 = -\frac{2H_0^{2-A}}{3+C}\left(q_0 + 1 + a_1 K (3+C)\right), \tag{79}$$

$$k_2 = H_0^{2-A}\left(1 + \frac{a_1 (2-A)}{\bar{B}}\left(\frac{q_0+1}{3+C} + a_1 K\right)\right), \tag{80}$$

and $\eta_0 = 0$ without loss of generality.

Also, it is possible to find an explicit parametric equation for the time variable using the Eqs. (71) and (78), as follows

$$t\,(\eta) = \frac{1}{3+C}\int y^{-1/2} d\eta$$

$$= -\frac{k_2^{1/A-2}}{a_1 K (3+C)} {}_2F_1\left(\frac{1}{2-A}, k_3, 1+k_3, \mu(\eta)\right), \tag{81}$$

being ${}_2F_1[., ., ., .]$ the hyper-geometric function with respective arguments, where $k_3 = -\frac{a_1^2 K}{\bar{B}}$ and $\mu(\eta) = \frac{a_1 k_1 (A-2) e^{\bar{B}\eta/a_1}}{2\bar{B}k_2}$.

The FE main dynamical variables are given in the following parametric form

$$a(\eta) = a_0 \exp\left(\frac{\eta - \eta_0}{3+C}\right), \tag{82}$$

$$H(\eta) = y^{1/2}(\eta), \tag{83}$$

$$q(\eta) = -\frac{1}{2}\left(\frac{(3+C)\left(\beta e^{\delta\eta}(2\gamma\delta + \alpha) + 2k_2\alpha\right)}{\beta e^{\delta\eta} + 2k_2}\right) - 1, \tag{84}$$

$$\rho(\eta) = 3y(\eta), \tag{85}$$

$$p(\eta) = 3(\gamma - 1)y(\eta), \tag{86}$$

$$\Pi(\eta) = -y\gamma\left(3 + 2(3+C)\beta\delta\frac{e^{\delta\eta}}{\beta e^{\delta\eta} + 2k_2}\right), \tag{87}$$

$$l(\eta) = \left|\frac{\gamma}{(\gamma - 1)} + \frac{2\gamma(3+C)\beta\delta}{3(\gamma - 1)}\frac{e^{\delta\eta}}{\left(\beta e^{\delta\eta} + 2k_2\right)}\right|, \tag{88}$$

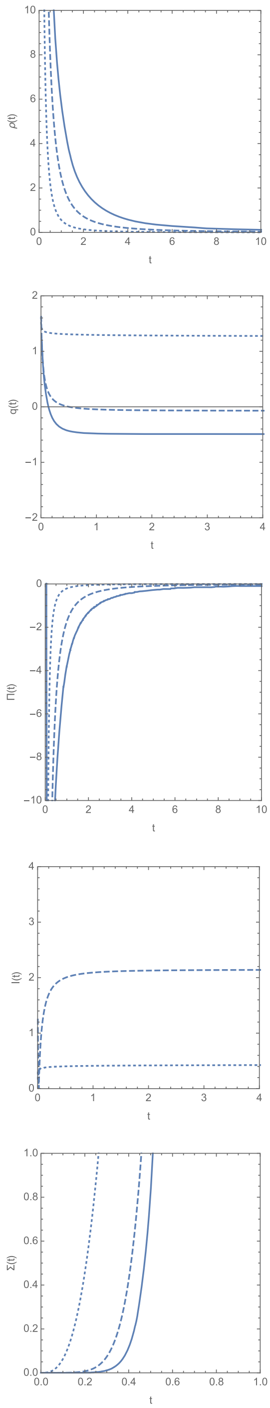
$$\Sigma(\eta) = -\frac{3}{(3+C)k_B}\int \Pi a^3 T^{-1} d\eta, \tag{89}$$

where $\alpha = 2a_1 K$, $\beta = \left(\frac{-k_1 a_1 (A-2)}{2\bar{B}}\right)$ and $\delta = \frac{\bar{B}}{a_1}$.

In order to show the behavior of the solutions we plot the parameters found in terms of the dimensionless variable tH_0, therefore in the following Figures the parameter t corresponds to tH_0. It is straightforward to see, for example, in the solution for the time given by Eq. (91) that introducing k_1 in this expression naturally a factor H_0^{-1} appears in the right hand side, allowing to rewrite expression (91) in terms of tH_0. The constant k_1 is redefined dimensionless to be used in the plots.

In Figure 1, the behavior of the FE main variables is displayed. The first point to highlight in this solution is that the $q(t)$ parameter can present a transition from positive values in the past to negative values in the present era for some values of γ. This means that we have in these cases, solutions representing an expanding decelerated universe in the past, which at some time began to accelerate until to our times. In the ΛCDM model this transition occurred when the energy density associated to the cosmological constant begin to be the dominant component over the dark matter component. In our solution this transition is due mainly to the evolution of the negative pressure due to dissipation. As it is displayed in Figure 1, $|\Pi(t))|$ is a decreasing function of time, which goes to zero as the cosmic time evolves, like the energy density of the matter component does. Nevertheless, in the case $\gamma = 1$, which corresponds to pressureless dark mater, like the matter assumed in the ΛCDM model, the total pressure of the fluid is negative due to $\Pi(t)$ and it is possible to have accelerated expansion at late times.

Fig. 1 Plots of general solution given by Eqs. (78)–(81) for $k_1 = 5$, $k_2 = 1$, $\epsilon = .25$, $\xi_0 = 3$, and $\gamma = 1$ (solid line), $\gamma = 4/3$ (dashed line), $\gamma = 1.9$ (dotted line)



In other cases, for example, when $\gamma = 1.9$, the positive pressure of the fluid is larger than $|\Pi(t))|$ and the total pressure is positive, leading to a decelerated expansion for all cosmic times.

Since all the cases correspond to non-perfect fluids, where a dissipative term is present, the entropy production is always positive.

For the chosen values of the initial conditions through the values of (k_1, k_2) and the parameters ϵ and ξ, the parameter $l = |\Pi|/p$ can be much lower than one only in the case $\gamma = 1.9$, which corresponds to a rather stiff matter fluid. For lower γ this condition can not be fulfilled. In other words, the near equilibrium condition, assumed in the formulation of relativistic non-perfect fluids, requires fluids with an EoS close to $\gamma = 2$ in this solution.

4.2.1 Particular solution 1

It is possible to find a particular solution from Eq. (78) by setting $q_0 = -1 - \frac{3+C}{a_1(2-A)}$, which leads to $k_2 = 0$. For this case, the parametric solution simplifies as follows

$$y(\eta) = e^{2a_1 K\eta}\left(\frac{-k_1 a_1 (A-2)}{2\bar{B}} e^{\bar{B}\eta/a_1}\right)^{2/(2-A)}, \tag{90}$$

and

$$t(\eta) = \frac{2^{1/(2-A)} a_1 (A-2)}{3+C} e^{-a_1 K\eta} \times \left(\frac{-k_1 a_1 (A-2)}{\bar{B}} e^{\bar{B}\eta/a_1}\right)^{1/(A-2)}, \tag{91}$$

where $k_1 = \frac{2\bar{B}}{a_1(2-A)} H_0^{2-A}$.

Then, the FE main quantities are given in the following form

$$a(\eta) = a_0 \exp\left(\frac{\eta-\eta_0}{3+C}\right), \tag{92}$$

$$H(\eta) = y^{1/2}(\eta), \tag{93}$$

$$q(\eta) = -1 - \frac{3+C}{a_1(2-A)}, \tag{94}$$

$$\rho(\eta) = 3y(\eta), \tag{95}$$

$$p(\eta) = 3(\gamma-1)y(\eta), \tag{96}$$

$$\Pi(\eta) = -y(\eta)\left(3\gamma + (3+C)\bar{\alpha}\right), \tag{97}$$

$$l(\eta) = \left|\frac{3\gamma + \bar{\alpha}(3+C)}{3(\gamma-1)}\right|, \tag{98}$$

$$\Sigma(\eta) = 3^{1/\gamma}\frac{(3\gamma + \bar{\alpha}(3+C))}{(3+C)}\frac{a_0^3}{k_B T_0}\int e^{\frac{3(\eta-\eta_0)}{3+C}} y^{2-A} d\eta, \tag{99}$$

where $\bar{\alpha} = 2a_1 K + \frac{2\bar{B}}{a_1(2-A)}$.

In Figure 2, the evolution of the FE main variables of this solution is displayed. Note that this solution represents accelerated or decelerated expansion with constant $q(t)$ parameter. Therefore, independently of the γ value there is no transition from decelerated to accelerated expansion, or vice versa. In this sense, these behaviors are unlike ΛCDM model.

As in the previous solution, $|\Pi(t))|$ is a decreasing function of time, which goes to zero as the cosmic time evolves, like the energy density of the matter component does. The total pressure of the fluid is negative for $\gamma = 1$ and $\gamma = 4/3$ leading to continuous accelerated expansion. In the cases $\gamma = 1.9$, the total pressure is always positive leading to a decelerated expansion for all cosmic times.

As in the previous solution the entropy production is always positive.

The condition $l = |\Pi| / p < 1$ can be fulfilled only in the case $\gamma = 1.9$. As in the previous solution, the near equilibrium requires γ values close to 2.

4.2.2 Particular solution 2

A second particular solution can be obtained by setting $q_0 = -1 - a_1 K(3+C)$, which leads to $k_1 = 0$ in Eq. (78) providing the simplified solution given by

$$y = k_2 e^{2a_1 K\eta}, \tag{100}$$

and

$$t(\eta) = -\frac{1}{(3+C)\sqrt{k_2} a_1 K} e^{-a_1 K\eta}. \tag{101}$$

where $k_2 = H_0^{2-A}$.

The FE main quantities are given in the following form

$$a(\eta) = a_0 \exp\left(\frac{\eta - \eta_0}{3+C}\right), \tag{102}$$

$$H(\eta) = y^{1/2}(\eta), \tag{103}$$

$$q(\eta) = -K a_1(3+C) - 1, \tag{104}$$

$$\rho(\eta) = 3y(\eta) \tag{105}$$

$$p(\eta) = 3(\gamma - 1)y(\eta), \tag{106}$$

$$\Pi(\eta) = -y(\eta)\left(3\gamma + 2K a_1(3+C)\right), \tag{107}$$

$$l(\eta) = \left|\frac{3\gamma + 2K a_1(3+C)}{3(\gamma-1)}\right|, \tag{108}$$

$$\Sigma(\eta) = 3^{1/\gamma} \frac{(3\gamma + 2K a_1(3+C))}{(3+C)} \frac{a_0^3}{k_B T_0} \int e^{\frac{3(\eta-\eta_0)}{3+C}} y^{2-A} d\eta. \tag{109}$$

Plots of the FE main variables are shown in Figure 3.

Furthermore, it is possible to recover the known scaling solution which has been widely studied by several authors [38,43,47]. The Eqs. (100) and (101) together with

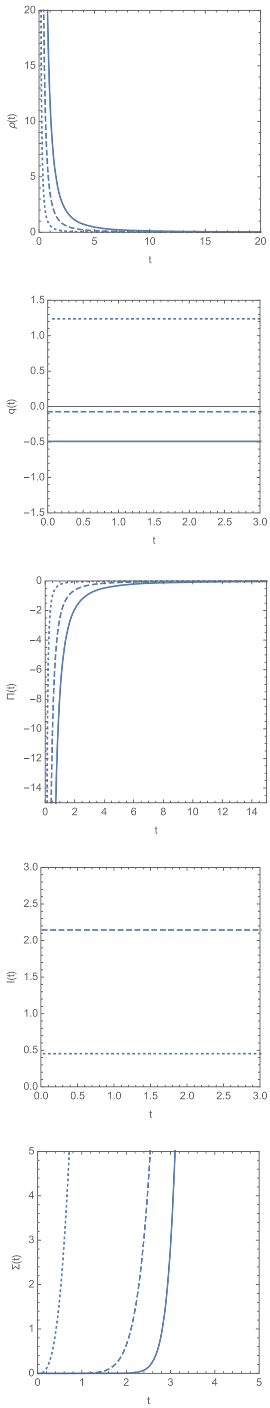


Fig. 2 Plots of particular solution given by Eqs. (90)–(91) for $k_1 = 1$, $\epsilon = .25$, $\xi_0 = 3$, and $\gamma = 1$ (solid line), $\gamma = 4/3$ (dashed line), $\gamma = 1.9$ (dotted line)

the Eqs. (102)–(109) provide the following results

$$a=\bar{a}_0 t^{H_0}, \quad \text{where } \bar{a}_0=a_0 e^{-\frac{\eta_0}{(3+C)}}\left(\frac{H_0}{\sqrt{k_2}}\right)^{-H_0}, \tag{110}$$

$$H(t)=H_0 t^{-1}, \quad \text{where} \quad H_0=-\frac{1}{(3+C)\,a_1 K}, \tag{111}$$

$$q(t)=H_0^{-1}-1, \tag{112}$$

$$\rho(t)=\rho_0 t^{-2}, \tag{113}$$

$$p(t)=3(\gamma-1)\,\rho_0 t^{-2}, \tag{114}$$

$$\Pi(t)=-\Pi_0\rho(t), \quad \text{where } \Pi_0=\left(\gamma+\frac{2}{3}(3+C)\,a_1 K\right), \tag{115}$$

$$l(t)=\frac{\Pi_0}{(\gamma-1)}, \tag{116}$$

$$\Sigma(t)=\frac{3\gamma\rho_0^{2-A}\Pi_0 H_0}{(3H_0\gamma-2)}\frac{\bar{a}_0^3}{k_B T_0}t^{3H_0-\frac{2}{\gamma}}. \tag{117}$$

It is straightforward to see from Figure 3 that these solutions represent accelerated or decelerated expansion with constant $q(t)$ parameter like the above one and, therefore, there is no transition from decelerated to accelerated expansion for all γ values. Note that the only case with accelerated expansion corresponds to $\gamma=1$, nevertheless in the former solution accelerated expansion occurs for $\gamma=1$ and $\gamma=4/3$. This is an indication that in this solution the $|\Pi(t))|$ takes smaller values than the corresponding to the previous case as the cosmic time evolves, which can be seen directly comparing $|\Pi(t))|$ in both Figures, 3 and 2.

As in the previous solutions, $|\Pi(t))|$ is a decreasing function of time and the entropy production is always positive.

The parameter $l=|\Pi|/p$ can be much lower than one only in the case $\gamma=1.9$, as in the previous solution. For lower γ this condition can not be fulfilled and clearly for $\gamma\approx 1$ the parameter l is too large to be represented in the Figure 3 due to the scale size used.

It is necessary to point out that the three solutions found showed that the condition $l=|\Pi|/p<1$ requires barotropic fluids with γ close to 2. Nevertheless, to be more precise, in these solutions particular values of the initial conditions through the values of (k_1,k_2) and the parameters ϵ and ξ_0 were chosen. Therefore, making numerical calculations with other values of the parameters involved, is possible to find that, for example, $l<1$ for $\gamma\geq 1.5$. A general criteria appears when the type of expansion driven by the solutions is related to the parameter l: if $l<1$, then $q>0$, and the solution represents a decelerated expansion; and when $l>1$, then $q<0$, corresponding to a solution describing accelerated expansion.

Unlike the solutions found in [38], which all of them represents accelerated expansion, these new ones display a wider behavior, as it was discussed above. Nevertheless, like in the former results found in [38], for our solutions accelerated expansion also

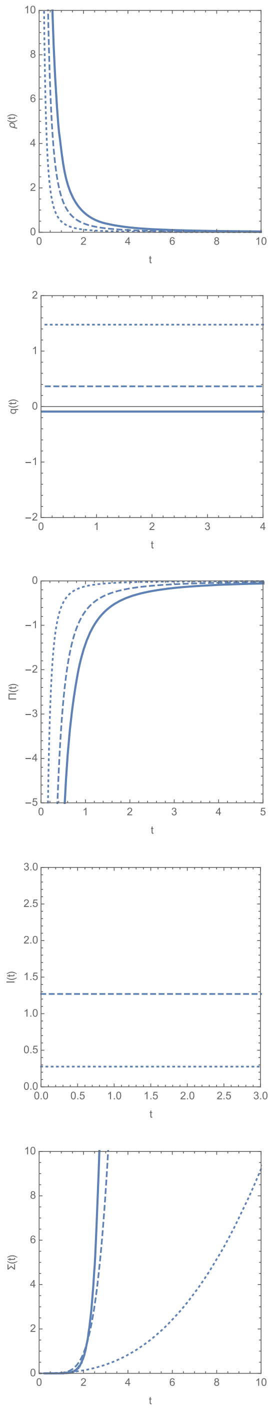


Fig. 3 Plots of particular solution given by Eqs. (100)–(101) for $k_2 = 1$, $\epsilon = .1$, $\xi_0 = 1$, and $\gamma = 1$ (solid line), $\gamma = 4/3$ (dashed line), $\gamma = 1.9$ (dotted line)

occurs with $l >> 1$, indicating the impossibility to satisfy the near equilibrium condition in this case.

5 Conclusion

The nonlinear differential equation for the Hubble parameter, which allows to explore the cosmic evolution, has been solved through the factorization technique for the viscous parameter $s = 1/2$. Some new exact parametric solutions have been found once a transformation of coordinates is performed on the Hubble rate equation. As a first approach, we have obtained the known particular polynomial scaling solution which describes a stiff matter fluid, with $\gamma = 2$, and where the dissipative effects vanish as it is expected. A second approach has been performed to find solutions for other cosmological scenarios of interest for $\gamma \in [1, 2)$. Then, we have found a new general and two particular cosmological solutions for a universe filled with one dissipative matter component, obeying a barotropic EoS. The general solution displays the transition between an expanding decelerated phase and accelerated one at late times, like the ΛCDM model. Although, unlike the ΛCDM model, this transition is due to the negative pressure from dissipation and not to a cosmological constant or even some kind of dark energy.

In the second and third particular solutions, the expansion is accelerated or decelerated with constant $q(t)$ parameter, depending on the γ value and no transition exists.

In all these solutions, $|\Pi(t))|$ is a decreasing function of time like the energy density of the matter component, and the entropy production is always positive.

Also, for the three solutions the condition $l = |\Pi|/p < 1$ is fulfilled only in the case $\gamma = 1.9$. But, as it was pointed out before, in general happens to be that $l < 1$ for $\gamma \geq 1.5$, and for those cases the solutions represent decelerated expansion ($q > 0$). When $l > 1$ the solutions describe an accelerated expansion. As a conclusion, we can say that the near equilibrium condition requires high pressure fluids in the solutions found within the causal framework of relativistic non-perfect fluids.

The general expression for the relaxation time given by Eq. (1), where the adiabatic contribution to the propagation speed is parameterized by ϵ, has shown to lead to a nonlinear differential equation of the causal formalism, whose solutions can describe consistently the effects of viscosity under the condition of near equilibrium. Besides, the general solution displays the transition between decelerated and accelerated expansion, mimicking the behavior of ΛCDM model. As we mentioned before, the solutions found in [38], where the propagation speed was assumed to be one, only exhibit accelerated expansion with $l >> 1$.

The analytical solutions were found for the particular case $s = 1/2$, which introduces important simplifications to the nonlinear differential equation of the formalism. There is no clear physical reason behind this election, on the contrary, negative s values could be more representative of a dissipative dark matter that in the past behaves more like a perfect fluid. We expect to undertake this investigation in a future work.

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