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EDITOR'S CHOICE (RESEARCH ARTICLE)

Helicity and spin conservation in linearized gravity

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Abstract

The duality-symmetric, Maxwell-like, formulation of linearized gravity introduced by Barnett (New J Phys 16, 2014) is used to generalize the conservation laws for helicity, the spin part of angular momentum, and spin-flux, to the case of linearized gravity. These conservation laws have been shown to follow from the conservation property of the helicity array, an analog of Lipkin's zilch tensor. The analog of the helicity array for linearized gravity is constructed and is shown to be conserved.

Keywords Linearized gravity · Conservation laws · Gravitational waves

Contents

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1 Introduction

The recent observation of gravitational waves provides a good motivation to further analyse the wave nature of Einstein theory and to explore its symmetries and conservation laws, as well as its analogies to Maxwell theory. Symmetries and conservation laws are fundamental features of any field theory. In the case of Maxwell equations, in addition to the Lorentz, conformal, and duality symmetries which were found by Lorentz, Heaviside, Larmor and Bateman roughly in the period 1890-1910, further non-trivial symmetries were found by Lipkin [26] in the 1964, and by Fushchich and Nikitin during the 1970's and 80's, cf. [24] and references therein. See Anco and Pohjanpelto [4] for a classification of the local conservation laws for the Maxwell equations as well as its generalization to spin $s \geq \frac{1}{2}$, and especially for our purposes for linearized gravity [5–7]. Anco and The [8] carried out a classification of local conservation laws for a duality-symmetric formulation for Maxwell theory. The non-classical conservation laws discussed there include Lipkin's zilches and the helicity, originally found by Candlin [20]. The zilch tensor and its associated currents arises as Noether currents for a variational symmetry of the duality-symmetric Maxwell Lagrangian, cf. [2].

Remarkably, a new set of conservation laws including intrinsic spin and orbital angular momentum for Maxwell theory were found in the early 1990's by Allen et al. [3] and van Enk and Nijenhuis [29]. The decomposition of total angular momentum into spin and orbital angular momentum parts is well known but these have been viewed as not representing physical observables. The new conservation laws found in the just cited papers, which were excluded, due to locality assumptions, by the analysis of Anco et al., turn out to play an important role in experiments, and their discovery has led to a burst of activity in the optical literature. These new conservation laws, which include intrinsic orbital angular momentum, spin, and spin flux or infra-zilch, were analysed in the work of Barnett et al. [9,11,18] and Bliokh et al. [17]. See also references in these papers for background. A systematic use of a duality-symmetric formulation of Maxwell theory plays a central role in this work. In particular, in the work of Barnett et al., the symmetries of the Maxwell equations giving rise to the new conservation laws via Noether's theorem were discussed. Further, Cameron et al. have introduced an analog of Lipkin's zilch tensor, called the helicity array, cf. [19]. The helicity array is conserved, and this property implies the conservation laws for helicity, spin and infra-zilch which were just mentioned. The analysis of the conservation laws for intrinsic spin and related quantities in the above mentioned papers is carried out for the Maxwell field in transverse gauge. This gauge condition, and hence also the helicity array fails to be Lorentz invariant. In [1] a Lorentz covariant tensor has been introduced that is conserved for the Maxwell field in Lorenz gauge, and which contains the same information as the helicity array.

The analogy between Maxwell theory and gravity is particularly close if we consider the weak field theory. A Maxwell-like and duality-symmetric formulation for linearized gravity on Minkowski space was introduced by Barnett [10], where the analog of helicity for linearized gravity was derived as the Noether current for the action of duality symmetry. It is worth mentioning at this point that helicity and duality symmetry for Maxwell theory and linearized gravity have previously been studied in terms of the standard formulation, and from a Hamiltonian point of view by Deser and Teitelboim [23], and Henneaux and Teitelboim [25] (for additional discussion on this approach and the generalization to higher spins see [21,22]).

In this paper we shall use the duality-symmetric formulation of linearized gravity introduced by Barnett in the just cited paper to derive generalizations of the helicity, spin, and infra-zilch conservation laws, and a generalization of the helicity array for linearized gravity on Minkowski space. In view of the role of spin and orbital angular momentum in the interaction of light with matter it is interesting to explore the analogous effects in gravity.

Overview of this paper

Section 2 presents the duality-symmetric formulation of linearized gravity on Minkowski space. Section 3 presents the hierarchy of conservation laws for linearized gravity. The conservation of helicity for linearized gravity is presented in Sect. 3.1, energy-momentum in Sect. 3.2, angular momentum in section and the decomposition of angular momentum into its spin and orbital parts is given in Sect. 3.3. The helicity array for linearized gravity is presented in Sect. 3.4. Section 4 contains some concluding remarks.

2 duality-symmetric formulation of linearized gravity

We shall consider fields on Minkowski space with signature $(-,+,+,+)$, using index notation with Greek indices $\alpha, \beta, \cdots$ taking values $0, \cdots, 3$, and lowercase Latin indices $i, j, \cdots$ taking values $1, 2, 3$. Let (x^α) be Cartesian coordinates on Minkowski space with temporal coordinate $x^0 = t$ and spatial coordinates (x^i), so that the Minkowski metric takes the form

$$\eta_{\alpha\beta} dx^\alpha dx^\beta = -dt^2 + \delta_{ij} dx^i dx^j, \tag{2.1}$$

where δ_{ij} is the Kronecker delta.

Truncating the Einstein-Hilbert Lagrangian

$$\mathcal{L}_{\text{EH}} = \frac{\sqrt{-g}}{16\pi G} \mathcal{R}, \tag{2.2}$$

yields a Lagrangian for linearized gravity which, after adding a total derivative and setting $32\pi G = 1$, takes the form

$$\mathcal{L}_{\text{LG}} = \tfrac{1}{2} (\partial_\beta h^\alpha{}_\alpha \, \partial^\beta h^\gamma{}_\gamma - 2\, \partial_\beta h^\alpha{}_\alpha \, \partial^\gamma h^\beta{}_\gamma - \partial_\gamma h_{\alpha\beta} \, \partial^\gamma h^{\alpha\beta} + 2\, \partial_\gamma h_{\alpha\beta} \, \partial^\beta h^{\alpha\gamma}) \tag{2.3}$$

where $h_{\alpha\beta} = h_{(\alpha\beta)}$ is the linearized metric [15]. In the transverse-traceless gauge,

$$h_{0\alpha} = 0\,, \quad h_{ij}{}^{,j} = 0\,, \quad h^i{}_i = 0, \tag{2.4}$$

which may be consistently imposed on Minkowski space, the Lagrangian $\mathcal{L}_{\rm LG}$ takes the form

$$\mathcal{L}_{\rm LG} = \tfrac{1}{2}\,(\dot{h}_{ij}\,\dot{h}^{ij} - h_{ij,k}\,h^{ij,k} + 2\,h_{ij,k}\,h^{ik,j})\,. \tag{2.5}$$

Adding a total derivative term gives the dynamically equivalent Lagrangian

$$\begin{aligned}\mathcal{L}'_{\rm LG} &= \mathcal{L}_{\rm LG} - \tfrac{1}{2}\,(h_{jk}\,h^{ik,j})_{,i}\\ &= \tfrac{1}{2}\,(\dot{h}_{ij}\,\dot{h}^{ij} - h_{ij,k}\,h^{ij,k} + h_{ij,k}\,h^{ik,j})\end{aligned} \tag{2.6}$$

We shall make use of analogues of vector calculus operations for symmetric 2-tensors here, and now introduce the notation which will be used. Let c_{ij}, d_{ij} be symmetric 2-tensors. We use bold faced letters $\boldsymbol{c}$ for a symmetric, traceless 2-tensor like c_{ij}. Further, we shall use several binary operations. These are

the scalar dot product	$\boldsymbol{c}\cdot\boldsymbol{d} = c_{ij}\,d^{ij}\,,$	(2.7a)
the cross product	$(\boldsymbol{c}\times\boldsymbol{d})_i = \epsilon_i{}^{jk}\,c_{jl}\,d_k{}^l\,,$	(2.7b)
2-tensor dot product	$(\boldsymbol{c}\,:\,\boldsymbol{d})_{ij} = c_{k(i}\,d_{j)}{}^k\,,$	(2.7c)
and the wedge product	$(\boldsymbol{c}\wedge\boldsymbol{d})_{ij} = \epsilon_i{}^{kl}\,\epsilon_j{}^{mn}\,c_{km}\,d_{ln}\,.$	(2.7d)

We also define the divergence and curl of a symmetric 2-tensor e_{ij} as

$$(\nabla\cdot\boldsymbol{e})_i = e_{ij}{}^{,j}\,,\quad (\nabla\times\boldsymbol{e})_{ij} = \epsilon_{(i}{}^{kl}\,e_{j)l,k} \tag{2.8}$$

The symmetric, traceless 2-tensor fields

$$e_{ij} = -\dot{h}_{ij}\,,\quad b_{ij} = \epsilon_i{}^{lm}\,h_{jm,l} \tag{2.9}$$

will play the role of analogues of the electric and magnetic fields in Maxwell theory. Using the notation we have just introduced, we have

$$\begin{aligned}\mathcal{L}'_{\rm LG} &= \tfrac{1}{2}\left[\dot{\boldsymbol{h}}\cdot\dot{\boldsymbol{h}} - (\nabla\times\boldsymbol{h})\cdot(\nabla\times\boldsymbol{h})\right]\\ &= \tfrac{1}{2}\,(\boldsymbol{e}\cdot\boldsymbol{e} - \boldsymbol{b}\cdot\boldsymbol{b})\,,\end{aligned} \tag{2.10}$$

which is closely analogous to the standard Lagrangian for Maxwell theory. The Euler-Lagrange equations which follow from this Lagrangian, when written in our current notation, take the form

$$\nabla\cdot\boldsymbol{e} = 0\,,\qquad \nabla\cdot\boldsymbol{b} = 0\,,\qquad \nabla\times\boldsymbol{e} = -\dot{\boldsymbol{b}}\,,\qquad \nabla\times\boldsymbol{b} = \dot{\boldsymbol{e}}, \tag{2.11}$$

which is close to the free Maxwell equations.

2.1 Duality-symmetric Lagrangian

Barnett [10] exploited the analogy of gravity with Maxwell theory to introduce a duality-symmetric Lagrangian for linearized gravity on Minkowski space, and used this to derive the helicity of the gravitational field. We shall now consider this Lagrangian in the notation introduced above. Let h_{ij}, k_{ij} be a pair of symmetric 2-tensors in transverse-traceless gauge (2.4). Here, k_{ij} is a second (auxiliary) gravitational potential, which is the analogue of the 4-potential C^α in duality-symmetric Maxwell theory. The Lagrangian takes the form

$$\mathcal{L}_{\text{LG-ds}} = \tfrac{1}{4}\left[\dot{\boldsymbol{h}}\cdot\dot{\boldsymbol{h}} - (\nabla\times\boldsymbol{h})\cdot(\nabla\times\boldsymbol{h}) + \dot{\boldsymbol{k}}\cdot\dot{\boldsymbol{k}} - (\nabla\times\boldsymbol{k})\cdot(\nabla\times\boldsymbol{k})\right], \tag{2.12}$$

cf. [16,18] for analogs in the Maxwell case. The Lagrangian in (2.12) is manifestly invariant under the duality reflection

$$h_{ij} \to k_{ij}, \quad k_{ij} \to -h_{ij} \tag{2.13}$$

which generates the continuous $U(1)$ duality rotation

$$h_{ij} \to h_{ij}\,\cos\theta + k_{ij}\,\sin\theta\,, \quad k_{ij} \to k_{ij}\,\cos\theta - h_{ij}\,\sin\theta\,. \tag{2.14}$$

In the following, unless otherwise stated, we shall impose the duality constraint

$$\dot{\boldsymbol{h}} = \nabla\times\boldsymbol{k} \tag{2.15a}$$
$$\dot{\boldsymbol{k}} = -\nabla\times\boldsymbol{h} \tag{2.15b}$$

and the transverse traceless gauge condition,

$$h_{ij}{}^{,j} = 0, \quad h^i{}_i = 0 \tag{2.16a}$$
$$k_{ij}{}^{,j} = 0, \quad k^i{}_i = 0. \tag{2.16b}$$

It follows from the Euler-Lagrange equations for $\mathcal{L}_{\text{LG-ds}}$ that the duality constraint holds globally if it holds at $t = 0$. The Maxwellian gravitational fields e_{ij} and b_{ij} in terms of these potentials are

$$\boldsymbol{e} = -\nabla\times\boldsymbol{k} \tag{2.17a}$$
$$\boldsymbol{b} = \nabla\times\boldsymbol{h}\,, \tag{2.17b}$$

which satisfy the Maxwellian equations of motion (2.11), provided that the duality constraint (2.15) is imposed.

2.2 Relation to curvature tensor

The Weyl curvature tensor is the only non-vanishing part of the curvature, and hence governing the propagation of gravitational waves, through free space. The decomposition of the Weyl curvature tensor into 'electric' and 'magnetic' parts E_{ij} and B_{ij} is defined as

$$E_{ij} = C_{i0j0} \quad , \quad B_{ij} = {}^{*}C_{i0j0} \tag{2.18}$$

where $C_{\alpha\beta\gamma\delta}$ is the Weyl curvature tensor and ${}^{*}C_{\alpha\beta\gamma\delta} = \frac{1}{2}\epsilon_{\alpha\beta\mu\nu}\, C^{\mu\nu}{}_{\gamma\delta}$ is its Hodge dual. From Gauss-Codazzi equations for vacuum spacetime, we can find

$$-\,(\nabla \times K)_{ij} = B_{ij} \tag{2.19a}$$

$${}^{3}R_{ij} - K_{im}\, K^{m}{}_{j} + K_{ij}\, tr K = E_{ij} \tag{2.19b}$$

where K_{ij} is the second fundamental form given by $K_{ij} = -\frac{1}{2}\,\dot{h}_{ij} = -\frac{1}{2}\, e_{ij}$. The Ricci tensor of spacial slices is ${}^{3}R_{ij} = -\frac{1}{2}\,\partial_k \partial^k\, h_{ij} = \frac{1}{2}\,(\nabla \times \boldsymbol{b})_{ij}$. Hence from (2.19a)

$$\tfrac{1}{2}\,\nabla \times \boldsymbol{e} = \boldsymbol{B}\ , \tag{2.20}$$

and from (2.19b), using the linearized field equations,

$$\tfrac{1}{2}\,\nabla \times \boldsymbol{b} = \boldsymbol{E}\ . \tag{2.21}$$

These relations allows us to state the generalization of the Biot-Savart operator to the spin-2 case. That is

$$\boldsymbol{e} = 2\,\nabla^{-2}\,\nabla \times \boldsymbol{B} \quad , \quad \boldsymbol{b} = 2\,\nabla^{-2}\,\nabla \times \boldsymbol{E}\ , \tag{2.22}$$

where

$$\nabla^{-2}\,\nabla \times \boldsymbol{B} = \int \frac{\nabla \times \boldsymbol{B}}{|\boldsymbol{x} - \boldsymbol{x}'|}\,\frac{d^3x'}{4\pi}\ . \tag{2.23}$$

One may interpret the fields $\boldsymbol{h}$ and $\boldsymbol{k}$ as constructed from $\boldsymbol{b}$ and $\boldsymbol{e}$, using the above non-local operator. In this way, provided that $\boldsymbol{E}$, $\boldsymbol{B}$ have suitable regularity and fall-off, the fields $\boldsymbol{e}$, $\boldsymbol{b}$, $\boldsymbol{h}$, $\boldsymbol{k}$ are naturally defined in a gauge invariant manner.

3 Conservation laws of linearized gravity

3.1 Helicity

Invariance of the Lagrangian density (2.12) under the duality transformation (2.14) leads to the conservation of the current

$$J^{\alpha} = \frac{\partial \mathcal{L}_{\text{LG-ds}}}{\partial(h_{ij,\alpha})}\, k_{ij} - \frac{\partial \mathcal{L}_{\text{LG-ds}}}{\partial(k_{ij,\alpha})}\, h_{ij}\ , \quad \partial_{\alpha}\, J^{\alpha} = 0\,, \tag{3.1}$$

whose components in transverse-traceless gauge are

$$J^0 = \tfrac{1}{2}\,(\dot{h}_{ij}k^{ij} - \dot{k}_{ij}h^{ij}) = \tfrac{1}{2}\,(\boldsymbol{h}\cdot\boldsymbol{b} - \boldsymbol{k}\cdot\boldsymbol{e}) \tag{3.2a}$$

$$J^i = \tfrac{1}{2}\,\epsilon^{ijk}(e_{lj}\,h^l{}_k + b_{lj}\,k^l{}_k) = \tfrac{1}{2}\,(\boldsymbol{e}\times\boldsymbol{h} + \boldsymbol{b}\times\boldsymbol{k})_k\,. \tag{3.2b}$$

This current is analogous to the electromagnetic helicity current (cf. [19,28]), but it will be shown later that the flux J^k is one half of the spin density vector S^k, which is the consequence of describing the gravitational field by a symmetric 2-tensor. Thus we define the helicity $\mathcal{H}$ and spin $\boldsymbol{S}$ of the gravitational field to be[1]

$$\mathcal{H} \equiv 2\,J^0 = \boldsymbol{h}\cdot\boldsymbol{b} - \boldsymbol{k}\cdot\boldsymbol{e}\,, \tag{3.3}$$

$$\boldsymbol{S} \equiv 2\,\boldsymbol{J} = \boldsymbol{e}\times\boldsymbol{h} + \boldsymbol{b}\times\boldsymbol{k}\,. \tag{3.4}$$

The helicity conservation law is

$$\dot{\mathcal{H}} + \nabla\cdot\boldsymbol{S} = 0\,. \tag{3.5}$$

3.2 Energy-momentum tensor

The canonical energy-momentum tensor for the duality-symmetric Lagrangian (2.12) is

$$T_\alpha{}^\beta = \delta_\alpha{}^\beta\,\mathcal{L}_{\text{LG-ds}} - \frac{\partial\mathcal{L}_{\text{LG-ds}}}{\partial(h_{ij,\beta})}\,h_{ij,\alpha} - \frac{\partial\mathcal{L}_{\text{LG-ds}}}{\partial(k_{ij,\beta})}\,k_{ij,\alpha}\,, \quad \partial_\beta\,T_\alpha{}^\beta = 0\,, \tag{3.6}$$

whose components in transverse-traceless gauge are

$$-T_0{}^0 \equiv \mathcal{E} = \tfrac{1}{2}\,(\dot{h}_{ij}\dot{h}^{ij} + \dot{k}_{ij}\dot{k}^{ij}) = \tfrac{1}{2}\,(\boldsymbol{e}\cdot\boldsymbol{e} + \boldsymbol{b}\cdot\boldsymbol{b}) \tag{3.7a}$$

$$-T_0{}^i \equiv P^i = -\tfrac{1}{2}\,(\epsilon^{ijk}\,b_{nj}\,\dot{h}^n{}_k - \epsilon^{ijk}\,e_{nj}\dot{k}^n{}_k) = (\boldsymbol{e}\times\boldsymbol{b})_i \tag{3.7b}$$

$$T_i{}^0 \equiv P^o_i = -\tfrac{1}{2}\,(\dot{h}^{jk}\,h_{jk,i} + \dot{k}^{jk}\,k_{jk,i}) = \tfrac{1}{2}\,[\boldsymbol{e}\cdot(\nabla)\boldsymbol{h} + \boldsymbol{b}\cdot(\nabla)\boldsymbol{k}]_i \tag{3.7c}$$

$$T_i{}^j \equiv \sigma_i{}^j = \tfrac{1}{2}\,(-\epsilon^{jkl}\,b^n{}_k\,h_{nl,i} + \epsilon^{jkl}\,e^n{}_k\,k_{nl,i}) \tag{3.7d}$$

where we have used the notation $[\boldsymbol{e}\cdot(\nabla)\boldsymbol{h}]_i = e^{jk}\,h_{jk,i}$. Here, $\mathcal{E}$ is the energy density, P^i is the energy flux density, P^o_i is the orbital momentum density and $\sigma_i{}^j$ is the spacial stress tensor, cf. [16].

3.3 Angular momentum

Like in Maxwell theory [27], the canonical angular momentum current in linearized gravity can be separated as

$$M^{\alpha\beta\gamma} = \tilde{L}^{\alpha\beta\gamma} + \tilde{S}^{\alpha\beta\gamma}\,, \quad \partial_\alpha\,M^{\alpha\beta\gamma} = 0 \tag{3.8}$$

[1] This definition of gravitational helicity differs from the one defined in [10] by a factor of 2.

where the nonconserved orbital and spin angular momenta, $\tilde{L}^{\alpha\beta\gamma}$ and $\tilde{S}^{\alpha\beta\gamma}$, are

$$\tilde{L}^{\alpha\beta\gamma} = r^{\alpha} T^{\beta\gamma} - r^{\beta} T^{\alpha\gamma} \tag{3.9a}$$

$$\tilde{L}^{ij0} = e_{kl}\, h^{kl,[j}\, r^{i]} + b_{kl}\, k^{kl,[j}\, r^{i]} \tag{3.9b}$$

$$\tilde{L}^{ijk} = \epsilon_{lmn}\, b_p^{\ m}\, h^{pn,[i}\, r^{j]} + \epsilon_{lmn}\, e_p^{\ m}\, k^{pn,[i}\, r^{j]} \tag{3.9c}$$

and with $(\mathcal{M}_{ij})_{ml} = 2\delta_{m[i}\delta_{j]l}$,

$$\tilde{S}_{ij}{}^{\gamma} = \frac{\partial \mathcal{L}_{\text{LG-ds}}}{\partial(\partial_{\gamma} h_{mn})} \left[(\mathcal{M}_{ij})_{ml}\, h_{ln} + (\mathcal{M}_{ij})_{nl}\, h_{ml}\right] + \frac{\partial \mathcal{L}_{\text{LG-ds}}}{\partial(\partial_{\gamma} k_{mn})} \left[(\mathcal{M}_{ij})_{ml}\, k_{ln} + (\mathcal{M}_{ij})_{nl}\, k_{ml}\right] \tag{3.10a}$$

$$\tilde{S}_{ij}{}^{0} = \tfrac{1}{2}\,(e_{n[i}\, h_{j]n} + b_{n[i}\, k_{j]n}) \tag{3.10b}$$

$$\tilde{S}_{ij}{}^{k} = \tfrac{1}{4}\,(\epsilon_{kln}\, h_{l[i}\, b_{j]n} - \epsilon_{kl[i}\, h_{j]n}\, b_{nl} - \epsilon_{kln}\, k_{l[i}\, e_{j]n} + \epsilon_{kl[i}\, k_{j]n}\, e_{nl}) \tag{3.10c}$$

and the spin density 3-vector is obtained as

$$S_i = \tfrac{1}{2}\,\epsilon_{ijk}\, \tilde{S}^{jk0} = \epsilon_{ijk}\,(e^{mj}\, h_m{}^{k} + b^{mj}\, k_m{}^{k}) = (\boldsymbol{e}\times\boldsymbol{h} + \boldsymbol{b}\times\boldsymbol{k})_i. \tag{3.11}$$

Similarly to the Maxwell case, the conservation of spin and orbital angular momenta of the gravitational field cannot be found directly from $\tilde{S}^{\alpha\beta\gamma}$, $\tilde{L}^{\alpha\beta\gamma}$ which are not separately conserved. We have

$$\partial_{\gamma}\, \tilde{S}^{\alpha\beta\gamma} = -\partial_{\gamma}\, \tilde{L}^{\alpha\beta\gamma} = T^{\alpha\beta} - T^{\beta\alpha} \neq 0. \tag{3.12}$$

To find the proper conservation laws, we need to modify the spin and orbital angular momentum fluxes in a way that total angular momentum conservation remains unchanged [16]. The false spin flux obtained from nonconserved spin current $\tilde{S}^{\alpha\beta\gamma}$ in (3.10c) is

$$\tilde{\Sigma}_{ij} = \tfrac{1}{2}\,\epsilon_{ikl}\, \tilde{S}^{kl}{}_{j} = \delta_{ij}\,\mathcal{H} - b_{ki}\, h^{k}{}_{j} - \tfrac{1}{2}\, b_{kj}\, h^{k}{}_{i} + e_{ki}\, k^{k}{}_{j} + \tfrac{1}{2}\, e_{kj}\, k^{k}{}_{i} \tag{3.13}$$

Working in the transverse-traceless gauge, we define $\Delta^{\alpha\beta\gamma}$ by

$$\Delta^{\alpha\beta 0} = \Delta^{00\gamma} = 0 \tag{3.14a}$$

$$\Delta^{i0j} = -\Delta^{0ij} = \tfrac{1}{2}\,(h_{ki}\, e^{k}{}_{j} + k_{ki}\, b^{k}{}_{j}) \tag{3.14b}$$

$$\Delta^{ijk} = \tfrac{1}{2}\,\epsilon^{ijl}\,(h_{ml}\, b^{m}{}_{k} - k_{ml}\, e^{m}{}_{k}) \tag{3.14c}$$

The first of these three equations is the condition for not altering the spin and orbital angular momentum densities. The second one modifies the boost angular momentum flux and the third one modifies the spin and orbital angular momentum fluxes. The latter modification results in (symmetric) spin flux

$$\Sigma_{ij} = \tfrac{1}{2}\,\epsilon_{ikl}\,(\tilde{S}^{kl}{}_j - \Delta^{kl}{}_j) = \delta_{ij}\,\mathcal{H} + 2\,e_{k(i}\,k_{j)}{}^k - 2\,b_{k(i}\,h_{j)}{}^k\,, \tag{3.15}$$

which together with the spin density (3.11) satisfies the continuity relation

$$\dot{S}_i + \partial_j\,\Sigma_i{}^j = 0\,. \tag{3.16}$$

3.4 Helicity array for linearized gravity

Similar to the Maxwell case, the spin flux of the linearized gravitational field is conserved,

$$\dot{\Sigma}_{ij} + \partial_k\,N_{ij}{}^k = 0 \tag{3.17}$$

where the flux N_{ijk} can be obtained easily by computing the time derivative of Σ_{ij} and observe that it is in fact a total derivative. This results in

$$N_{ijk} = \delta_{ij}\,S_k - 2\,h_{n(i}\,k_{j)n,k} + 2\,k_{n(i}\,h_{j)n,k} \tag{3.18}$$

The helicity array for linearized gravity can then can be constructed as

$$\mathcal{N}^{000} \equiv \mathcal{H} = \boldsymbol{h}\cdot\boldsymbol{b} - \boldsymbol{k}\cdot\boldsymbol{e}\,, \tag{3.19a}$$

$$\mathcal{N}^{0i0} = \mathcal{N}^{00i} \equiv S^i = (\boldsymbol{e}\times\boldsymbol{h} + \boldsymbol{b}\times\boldsymbol{k})^i\,, \tag{3.19b}$$

$$\mathcal{N}^{ij0} = \mathcal{N}^{0ij} \equiv \Sigma^{ij} = \delta^{ij}\,\mathcal{H} + 2\,e_k{}^{(i}\,k^{j)k} - 2\,b_k{}^{(i}\,h^{j)k}\,, \tag{3.19c}$$

$$\mathcal{N}^{ijk} \equiv N^{ijk} = \delta^{ij}\,S^k - 2\,h_l{}^{(i}\,k^{j)l,k} + 2\,k_l{}^{(i}\,h^{j)l,k}\,, \tag{3.19d}$$

with the symmetry $\mathcal{N}^{\alpha\beta\gamma} = \mathcal{N}^{(\alpha\beta)\gamma}$. The equation

$$\partial_\gamma\,\mathcal{N}^{\alpha\beta\gamma} = 0 \tag{3.20}$$

contains the linked conservation laws

$$\dot{\mathcal{H}} + \partial_i S^i = 0, \tag{3.21a}$$

$$\dot{S}_i + \partial_j \Sigma_i{}^j = 0, \tag{3.21b}$$

$$\dot{\Sigma}_{ij} + \partial_k N_{ij}{}^k = 0. \tag{3.21c}$$

4 Conclusion

We have presented the duality-symmetric formulation of linearized gravity on Minkowski space, and derived the generalization from Maxwell theory of the conservation laws for helicity, spin, and infra-zilch to the gravitational case.

The fact that the spin and orbital parts of angular momentum are separately conserved and therefore physical observables has had a tremendous impact in optics and in our understanding of the interaction of light and matter. Here we have shown that the spin and orbital parts of angular momentum are separately conserved also in linearized gravity on Minkowski space. It is now interesting to analyse the consequences of this fact for the interaction of gravity with matter as well as with other fields. We remark that Bialynicki-Birula and Bialynicki-Birula [13] have constructed beams of gravitational waves carrying orbital angular momentum. Recently [14], the interaction of such beams with matter has been investigated.

It has been shown by Bialynicki-Birula and Bialynicki-Birula [12] that a gauge-invariant, but non-local, expression for spin and orbital angular momentum of the Maxwell field can be given by using the Biot-Savart law. In effect, this means that from the gauge-invariant fields $\boldsymbol{E}, \boldsymbol{B}$, the potentials $\boldsymbol{A}, \boldsymbol{C}$ are determined by inverting the curl operator $\nabla\times$. The analogous situation holds for linearized gravity. In particular, there is an analog of the Biot-Savart operator for the spin-2 case, which allows one to write the fields $\boldsymbol{e}, \boldsymbol{b}$ in terms of the electric and magnetic parts $\boldsymbol{E}, \boldsymbol{B}$ of the Weyl tensor. Similarly, the fields $\boldsymbol{h}, \boldsymbol{k}$ can be expressed using the Biot-Savart operator in terms of $\boldsymbol{e}, \boldsymbol{b}$. See Sect. 2.2 for details. In future work, we plan to illustrate these conservation laws to some families of solutions of the linearized field equations. We also plan to investigate the generalization of the Lorentz invariant helicity tensor introduced in [1] to the case of gravity.

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