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RESEARCH ARTICLE

Rastall's theory of gravity: spherically symmetric solutions and the stability problem

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Abstract
We study the stability of static, spherically symmetric solutions of Rastall's theory in the presence of a scalar field with respect to spherically symmetric perturbations. The analysis of perturbations shows that there arises an inconsistency in the sense that time-dependent perturbations do not exist in any order of perturbation theory, and we can conclude that these solutions are pertubatively stable (though nonperturbative time-dependent solutions are not excluded). Possible reasons for this inconsistency are discussed.

Keywords Black holes · Extended theory of gravity · Stability · Exact solutions

1 Introduction

One of the fundamental pillars of the general relativity (GR) theory of gravity is the divergence-free property of the Einstein tensor, leading to the usual conservation laws

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for the matter sources of gravity, expressed in the zero divergence of the energy-momentum tensor.

Rastall's theory [1] is a non-Lagrangian theory of gravity which allows for an energy–momentum tensor of matter with a nonzero divergence. The main original argument for such a radical departure from GR is that, strictly speaking, the conservation laws have only been tested in weak fields, and their generalization to strongly curved space–times may be more problematic than is usually supposed. The departure from the usual GR conservation law is parametrized in the Rastall theory by the dimensionless parameter λ, such that when $\lambda = 1$, GR is recovered.

The Rastall theory has been applied in many different contexts. In cosmology, some studies have been performed for the early [2] and also late-time universe [3]. In some of these applications, a self-interacting scalar field has been considered. A curiosity about the self-interacting scalar field in the Rastall theory is that the corresponding speed of sound c_s can be zero for some value of λ [4], unlike the GR situation where always $c_s = 1$ (in units of the velocity of light). Moreover, it is possible to construct a cosmological model entirely similar to the standard ΛCDM model for what concerns the background and the evolution of linear perturbations, but different for what concerns nonlinear perturbations.

In Ref. [5] black-hole solutions using Rastall's theory in the presence of a scalar field have been obtained. Two classes of exotic black holes have been identified: one where the singularities are located at two spatial infinities separated by horizons; the other where there are two horizons of infinite area connected by a wormhole, and the spatial infinity is essentially replaced by a cosmological singularity. It is very interesting that these structures are geometrically identical to those found using k-essence theories, but with a different behavior of the scalar fields [6]. In Ref. [7] the equivalence (or duality) between these structures was discussed in detail: we have formulated the assumptions under which the two fundamentally different theories lead to space-times with the same metric.

A question that emerges immediately concerns the stability of these exotic black hole-type structures. In Ref. [8], the stability of the corresponding k-essence solutions under spherically symmetric perturbations was analyzed, and their instabilities have been proved. In the present paper, we undertake a similar analysis for the Rastall black hole-type solutions. A surprising aspect of this study is that the linear perturbation analysis turns out to be inconsistent, and the stability or instability of the solutions cannot be determined. More precisely, the linear perturbation equations cannot have time-dependent solutions, and this conclusion can be extended to higher-order perturbations. Such a result looks similar to that of the cosmological perturbation analysis for a self-interacting scalar field, which has also revealed an inconsistency, unless new degrees of freedom of matter are added to the previously existing matter content of the Universe [9]. This may point out at an intrinsic restriction for the Rastall theory and perhaps even for a large class of non-Lagrangian theories of gravity.

2 Basic equations

The field equations of Rastall's theory where the only source of gravity is a scalar field ϕ with an arbitrary self-interaction potential $V(\phi)$ can be written in the form[1]

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = \epsilon\left\{\phi_{;\mu}\phi_{;\nu} - \frac{2-a}{2}g_{\mu\nu}\phi^{;\rho}\phi_{;\rho}\right\} + (3-2a)g_{\mu\nu}V(\phi), \quad (1)$$

$$\Box\phi + (a-1)\frac{\phi^{;\rho}\phi^{;\sigma}\phi_{;\rho;\sigma}}{\phi_{;\alpha}\phi^{;\alpha}} = -\epsilon(3-2a)V_\phi, \quad (2)$$

where a is the free parameter of Rastall's theory, and the equations of GR are restored in the case $a = 1$; $\epsilon = \pm 1$, so that $\epsilon = +1$ corresponds to a canonical scalar field, and $\epsilon = -1$ to a phantom one.

Equation (1) can be rewritten as

$$R_{\mu\nu} = \epsilon\left\{\phi_{;\mu}\phi_{;\nu} + \frac{1-a}{2}g_{\mu\nu}\phi^{;\rho}\phi_{;\rho}\right\} - (3-2a)g_{\mu\nu}V(\phi), \quad (3)$$

We are going to discuss the linear stability of static, spherically symmetric solutions of the theory under spherically symmetric perturbations. Accordingly, we consider the general spherically symmetric metric

$$ds^2 = e^{2\gamma}dt^2 - e^{2\alpha}du^2 - e^{2\beta}d\Omega^2, \quad (4)$$

where α, β, γ are functions of u and t, the radial and time coordinates, respectively, and $d\Omega^2$ is the line element on a unit sphere. We also assume $\phi = \phi(u, t)$.

Before starting the perturbation analysis, let us write out the complete form of Eqs. (2) and (3) for the system under study, without fixing the coordinates u and t:

$$-e^{2(\alpha-\gamma)}\left[\ddot\alpha + 2\ddot\beta + \dot\alpha^2 + 2\dot\beta^2 - \dot\gamma(\dot\alpha + 2\dot\beta)\right] + \gamma'' + \gamma'(\gamma' - \alpha' + 2\beta')$$
$$= \frac{\epsilon}{2}\left[(3-a)e^{2(\alpha-\gamma)}\dot\phi^2 - (1-a)\phi'^2\right] - (3-2a)e^{2\alpha}V, \quad (5)$$

$$-e^{2(\alpha-\gamma)}\left[\ddot\alpha + \dot\alpha(\dot\alpha - \dot\gamma + 2\dot\beta)\right] + \gamma'' + 2\beta'' - \alpha'(\gamma' + 2\beta') + \gamma'^2 + 2\beta'^2$$
$$= \frac{\epsilon}{2}\left[(1-a)e^{2(\alpha-\gamma)}\dot\phi^2 - (3-a)\phi'^2\right] - (3-2a)e^{2\alpha}V, \quad (6)$$

$$-e^{2(\alpha-\gamma)}\left[\ddot\beta + \dot\beta(\dot\alpha - \dot\gamma + 2\dot\beta)\right] + \beta'' + \beta'(\gamma' - \alpha' + 2\beta') - e^{2(\alpha-\beta)}$$
$$= \frac{\epsilon}{2}(1-a)\left[e^{2(\alpha-\gamma)}\dot\phi^2 - \phi'^2\right] - (3-2a)e^{2\alpha}V, \quad (7)$$

$$\dot\beta' + (\beta' - \gamma')\dot\beta - \beta'\dot\alpha = -\frac{\epsilon}{2}\phi'\dot\phi, \quad (8)$$

$$a\phi'' + [\gamma' - a\alpha' + 2\beta']\phi' - e^{2(\alpha-\gamma)}\ddot\phi = \epsilon(3-2a)e^{2\alpha}V_\phi \quad (9)$$

(the overdot stands for $\partial/\partial t$, the prime for $\partial/\partial u$, the subscript ϕ for $d/d\phi$).

[1] The Rastall parameter a is related to the other frequently used parameter λ by $a = \dfrac{3\lambda - 2}{2\lambda - 1}$.

For the static background we have the equations

$$\gamma'' + \gamma'(\gamma' - \alpha' + 2\beta') = -\frac{\epsilon}{2}(1-a)\phi'^2 - e^{2\alpha}W, \tag{10}$$

$$\gamma'' + 2\beta'' - \alpha'(\gamma' + 2\beta') + \gamma'^2 + 2\beta'^2 = -\frac{\epsilon}{2}(3-a)\phi'^2 - e^{2\alpha}W, \tag{11}$$

$$\beta'' + \beta'(\gamma' - \alpha' + 2\beta') - e^{2(\alpha-\beta)} = -\frac{\epsilon}{2}(1-a)\phi'^2 - e^{2\alpha}W, \tag{12}$$

$$a\phi'' + [\gamma' - a\alpha' + 2\beta']\phi' = \epsilon e^{2\alpha}W_\phi, \tag{13}$$

where $W(\phi) = (3-2a)V(\phi)$.

To implement a perturbation analysis, we introduce a fluctuation around the static background functions. This includes the matter function, given by the scalar field, and the metric functions α, β and γ, leading to a set of coupled equations for the perturbations around a static background. Assuming, with some small parameter ε, which allows us to track the perturbative series order by order,

$$\phi(u,t) = \phi(u) + \delta\phi(u,t), \quad \delta\phi \sim \varepsilon \ll 1,$$

and similarly for all other variables, we can write the perturbation equations in the linear order $O(\varepsilon)$ as follows:

$$\begin{aligned} &-e^{2(\alpha-\gamma)}(\delta\ddot{\alpha} + 2\ddot{\delta}\beta) + \delta\gamma'' + \delta\gamma'(\gamma' - \alpha' + 2\beta') + \gamma'(\delta\gamma' - \delta\alpha' + 2\delta\beta') \\ &\quad = -\epsilon(1-a)\phi'\delta\phi' - e^{2\alpha}(2\delta\alpha W + W_\phi\delta\phi), \end{aligned} \tag{14}$$

$$\begin{aligned} &-e^{2(\alpha-\gamma)}\delta\ddot{\alpha} + \delta\gamma'' + 2\delta\beta'' - \delta\alpha'(\gamma' + 2\beta') \\ &\qquad -\alpha'(\delta\gamma' + 2\delta\beta') + 2\delta\gamma\gamma' + 4\beta'\delta\beta' \\ &\quad = -\epsilon(3-a)\phi'\delta\phi' - e^{2\alpha}(2\delta\alpha W + W_\phi\delta\phi), \end{aligned} \tag{15}$$

$$\begin{aligned} &-e^{2(\alpha-\gamma)}\delta\ddot{\beta} + \delta\beta'' + \delta\beta'(\gamma' - \alpha' + 2\beta') \\ &\qquad +\beta'(\delta\gamma' - \delta\alpha' + 2\delta\beta') - 2e^{2(\alpha-\beta)}(\delta\alpha - \delta\beta) \\ &\quad = -\epsilon(1-a)\phi'\delta\phi' - e^{2\alpha}(2\delta\alpha W + W_\phi\delta\phi), \end{aligned} \tag{16}$$

$$\delta\dot{\beta}' + (\beta' - \gamma')\delta\dot{\beta} - \beta'\delta\dot{\alpha} = -\frac{\epsilon}{2}\phi'\delta\dot{\phi}, \tag{17}$$

$$\begin{aligned} &-e^{2(\alpha-\gamma)}\delta\ddot{\phi} + a\delta\phi'' + [\gamma' - a\alpha' + 2\beta']\delta\phi' + [\delta\gamma' - a\delta\alpha' + 2\delta\beta']\phi' \\ &\quad = \epsilon e^{2\alpha}(2\delta\alpha W_\phi + W_{\phi\phi}\delta\phi). \end{aligned} \tag{18}$$

These equations are written in the most general form and contain two kinds of arbitrariness: the choice of the radial coordinate u in the background static metric and the perturbation gauge that fixes the reference frame in perturbed space-time.

3 Master equation and a discrepancy

As in GR, this system possesses only one dynamic degree of freedom connected with the scalar perturbation $\delta\phi$. Accordingly, the perturbation equations can be used to combine the metric and matter perturbations in order to obtain a single "master equation" for $\delta\phi$. This can be achieved most conveniently using the gauge $\delta\beta \equiv 0$. In full similarity with perturbation studies in [10–12], the assumption $\delta\beta \equiv 0$ does not restrict the generality of our consideration since the resulting master equation is in fact gauge-invariant: it only contains the background quantities and the perturbation $\delta\phi$, and the latter represents a certain gauge-invariant quantity (namely, $\delta\phi - \phi'\delta\beta/\beta'$) in the gauge $\delta\beta = 0$.

Under the condition $\delta\beta = 0$, the perturbation equations become

$$
\begin{aligned}
&-e^{2(\alpha-\gamma)}\delta\ddot{\alpha} + \delta\gamma'' + \delta\gamma'(2\gamma' - \alpha' + 2\beta') - \gamma'\delta\alpha' \\
&\quad = -\epsilon(1-a)\phi'\delta\phi' - e^{2\alpha}(2W\delta\alpha + W_\phi\delta\phi), \qquad (19)\\
&-e^{2(\alpha-\gamma)}\delta\ddot{\alpha} + \delta\gamma'' - \delta\alpha'(\gamma' + 2\beta') - (\alpha' - 2\gamma')\delta\gamma' \\
&\quad = -\epsilon(3-a)\phi'\delta\phi' - e^{2\alpha}(2W\delta\alpha + W_\phi\delta\phi), \qquad (20)\\
&\beta'(\delta\gamma' - \delta\alpha') - 2e^{2(\alpha-\beta)}\delta\alpha \\
&\quad = -\epsilon(1-a)\phi'\delta\phi' - e^{2\alpha}(2W\delta\alpha + W_\phi\delta\phi), \qquad (21)\\
&-\beta'\delta\dot{\alpha} = -\frac{\epsilon}{2}\phi'\delta\dot{\phi}, \qquad (22)\\
&a\delta\phi'' + [\gamma' - a\alpha' + 2\beta']\delta\phi' + [\delta\gamma' - a\delta\alpha']\phi' - e^{2(\alpha-\gamma)}\delta\ddot{\phi} \\
&\quad = \epsilon e^{2\alpha}(2W_\phi\delta\alpha + W_{\phi\phi}\delta\phi). \qquad (23)
\end{aligned}
$$

Equation (22) is easily integrated in t leading to

$$
\delta\alpha = \eta(u)\delta\phi + \xi(u), \qquad (24)
$$

where $\xi(u)$ is an arbitrary function of the radial coordinate. The function $\xi(u)$ corresponds to a possible static perturbation and can be added to the static solution, thus slightly changing the background without leaving the set of background solutions. The function $\eta(u)$ in (24) is defined as

$$
\eta = \epsilon\phi'/(2\beta'). \qquad (25)
$$

On the other hand, the difference of (19) and (20) gives

$$
\delta\gamma' = \eta\delta\phi' - \eta'\delta\phi. \qquad (26)
$$

Ignoring static perturbations (that is, putting $\xi(u) \equiv 0$), a substitution of (24) and (26) into (23) leads to the final master equation

$$
\begin{aligned}
&-e^{2(\alpha-\gamma)}\delta\ddot{\phi} + a\delta\phi'' + \Big[2\beta' + \gamma' - a\alpha' + \eta(1-a)\phi'\Big]\delta\phi' \\
&\quad + \Big[-(1+a)\eta'\phi' - \epsilon e^{2\alpha}\big(2\eta W_\phi + W_{\phi\phi}\big)\Big]\delta\phi = 0, . \qquad (27)
\end{aligned}
$$

whose analysis for particular solutions of the background equations should lead to definite conclusions on their stability or instability.

However, $\delta\gamma'$ may be alternatively calculated from (21), which gives

$$2\beta'\delta\gamma' = \epsilon a\phi'\delta\phi' + e^{2\alpha}\Big[2(e^{-2\beta} - W)\delta\alpha - W_\phi\delta\phi\Big]. \tag{28}$$

Comparing (26) and (28) and using the background equations, we obtain the relation

$$(1-a)\phi'\delta\phi' = (1-a)\big[\phi'' - \alpha'\phi' + \eta\phi'^2\big]\delta\phi \tag{29}$$

(the calculation is most conveniently carried out using the harmonic radial coordinate, such that $\alpha = 2\beta + \gamma$). We immediately see that at $a = 1$, that is, in GR, this equation becomes an identity, thus confirming that the stability study in GR remains consistent. However, with $a \neq 1$, in Rastall's theory, Eq. (29) is nontrivial, and its integration in u gives

$$\delta\phi = F(t)\phi' e^{\alpha + Q(u)}, \qquad Q(u) = \int \eta(u)\phi'(u)du, \tag{30}$$

where $F(t)$ is an arbitrary function of time, which only should be small, i.e., $O(\varepsilon)$.

Now, we can substitute (30) to the master equation (27) and observe the usual separation of variables: the quantity $\ddot{F}/F$ will be equal to a certain combination of functions taking part in the background solution, hence this combination is equal to some separation constant. Such a condition, is, in general, not satisfied by the background solution, which makes the whole stability study inconsistent. The only possibility to avoid this inconsistency is to put $F(t) = \text{const}$, i.e., there is no time-dependent linear perturbation. Moreover, if we have a static $\delta\phi \neq 0$, we can assume that we are dealing with a slightly shifted background relative to which $\delta\phi = 0$.

How will the situation change if we try to consider high-order perturbations? From the field Eqs. (5)–(9) it is clear that in the order $O(\varepsilon^2)$ the perturbation equations will contain the same expressions with second-order perturbations as those which previously emerged for first-order ones in the order $O(\varepsilon)$, plus certain quadratic combinations of these first order quantities. Since, however, the latter were found to be zero, we conclude that in the order $O(\varepsilon^2)$ the equations are identical to those in $O(\varepsilon)$ and again have only a trivial solution. This picture will be repeated in all higher orders, and we have to conclude that no small time-dependent perturbations of the system under study are possible. In this sense, the background solutions are stable.

We will confirm the above observation using two known background solutions of Rastall theory as examples. Using one of them, we will demonstrate that invoking a nonzero $\xi(u)$ in (24) does not solve the consistency problem.

4 Special solutions

4.1 $a = -1, V = 0$

In this case, the solution has been obtained using the quasiglobal radial coordinate such that $\alpha = -\gamma$, and the background equation are

$$\gamma'' + 2\gamma'(\gamma' + \beta') = -\epsilon\phi'^2, \tag{31}$$

$$\gamma'' + 2\beta'' + 2\gamma'(\gamma' + \beta') + 2\beta'^2 = -2\epsilon\phi'^2, \tag{32}$$

$$\beta'' + 2\beta'(\gamma' + \beta') - e^{-2(\gamma+\beta)} = -\epsilon\phi'^2, \tag{33}$$

$$-\phi'' + 2\beta'\phi' = 0. \tag{34}$$

The background solutions are possible only for $\epsilon = -1$ and are given by (denoting now the radial coordinates by x)

$$ds^2 = A(x)dt^2 - \frac{dx^2}{A(x)} - \sqrt{\frac{3}{2C^2}}\frac{d\Omega^2}{x}, \tag{35}$$

$$A(x) = K/x - (C/\sqrt{6})\,x^3, \tag{36}$$

$$\phi = \sqrt{3/2}\ln x + \phi_0, \tag{37}$$

where ϕ_0, C and K are integration constants. The metric has the same properties as in the k-essence case studied in Ref. [6], with a horizon at $x = \left(\frac{\sqrt{6}K}{C}\right)^{1/4}$ and a singular spatial infinity $x \to \infty$. Unlike [6], however, the scalar field is here defined as to span from minus to plus infinity.

The master Eq. (27) now reduces to

$$e^{-4\gamma}\delta\ddot{\phi} + \delta\phi'' + \frac{2}{x}\delta\phi' = 0. \tag{38}$$

Two different expressions for $\delta\gamma'$, obtained as described above, are

$$\delta\gamma' = \sqrt{3/2}\delta\phi' = -\sqrt{3/2}\delta\phi' - \sqrt{6}e^{2\alpha-2\beta}\delta\phi. \tag{39}$$

Substituting the background solution, we are able to integrate the last equality in x, obtaining

$$\delta\phi = F(t)\psi(x), \qquad \psi(x) = (K - (C/\sqrt{6})x^4)^{-1/2}. \tag{40}$$

Substituting (40) into (38) and separating the variables, we obtain

$$-\frac{\ddot{F}}{F} = \frac{\psi''}{\psi} + \frac{2}{x}\frac{\psi'}{\psi} = \text{const.} \tag{41}$$

It is easy to verify that the last equation does not hold for $\psi(x)$ given by (40).

4.2 $a = 0, V = \Lambda = \text{const}$

We are using here the the harmonic radial coordinate, such that $\alpha = 2\beta + \gamma$. The background equations are

$$\gamma'' = -\frac{\epsilon}{2}\phi'^2 - 3e^{2\alpha}\Lambda, \tag{42}$$

$$\gamma'' + 2\beta'' - \alpha'^2 + \gamma'^2 + 2\beta'^2 = -\frac{3}{2}\epsilon\phi'^2 - 3e^{2\alpha}\Lambda, \tag{43}$$

$$\beta'' - e^{2(\gamma+\beta)} = -\frac{\epsilon}{2}\phi'^2 - 3e^{2\alpha}\Lambda, \tag{44}$$

$$\alpha'\phi' = 0. \tag{45}$$

The last equation implies $\alpha = \text{const}$, which can be rescaled to $\alpha = 0$ by a trivial redefinition of the radial coordinate. The solution has the form

$$ds^2 = \frac{9b^4}{\cosh^4 bu}dt^2 - du^2 - \frac{\cosh^2 bu}{3b^2}d\Omega^2, \tag{46}$$

$$\phi' = \pm b\sqrt{6 - \frac{4}{\cosh^2 bu}}, \quad \epsilon = -1, \quad b = \sqrt{\Lambda}. \tag{47}$$

Equation (23) for scalar field perturbations now reads

$$e^{-2\gamma}\delta\ddot{\phi} - \phi'\delta\gamma' = 0. \tag{48}$$

Two alternative expressions for $\delta\gamma'$ are (26) and, as given by (28) after using the background solution,

$$\delta\gamma' = \frac{3}{2}\phi'\delta\phi. \tag{49}$$

Comparing (26) and (49), we obtain

$$\eta\delta\phi' - \eta'\delta\phi = \frac{3}{2}\phi'\delta\phi. \tag{50}$$

We can integrate this equation, finding the spatial behavior of $\delta\phi$:

$$\delta\phi = F(t)\frac{\phi'}{\beta'}e^{-3\beta}, \qquad F(t) = \text{arbitrary function.} \tag{51}$$

Substituting it to the master equation (48) with $\delta\gamma'$ given by (49), we arrive at

$$\frac{2}{3}\frac{\ddot{F}}{F} = \phi'^2 e^{-4\beta} = \text{const}. \tag{52}$$

The latter equality does not hold for our background solution.

We can ask if we can cure this problem by considering the "integration constant" $\xi(u)$ that appears in (24). Then the two expressions for $\delta\gamma'$ become

$$\delta\gamma' = \eta\delta\phi' - \eta'\delta\phi - \xi' = \frac{3}{2}\phi'\delta\phi - 3\beta'\xi. \tag{53}$$

This relation is integrated giving

$$\delta\phi = \eta e^{-3\beta}[F(t) + H(u)], \qquad H(u) = \int \frac{e^{3\beta}}{\eta^2}(\xi' - 3\beta'\xi)du, \tag{54}$$

where $H(u)$ is actually an arbitrary function of u due to arbitrariness of ξ. Inserting this $\delta\phi$ to the master equation, we obtain

$$\ddot{F} - \frac{3}{2}\phi'^2 e^{-4\beta}[F(t) + H(u)] - 6\beta'^2 e^{-\beta}\xi = 0. \tag{55}$$

Differentiating (55) with respect to u, we obtain the equation

$$(-3\phi'\phi'' + 6\phi'^2\beta')[F(t) + H(u)] - \frac{3}{2}\phi'^2 H'(u) - 6\beta' e^{3\beta}[2\beta''\xi - \beta'(\beta'\xi - \xi')] = 0. \tag{56}$$

It is clear from this equation that F must be a constant. Hence, it is a differential equation for functions of u making it possible to calculate $\xi(u)$ in terms of the background functions. As could be expected, the assumption $\xi(u) \neq 0$ leads to a purely static perturbation of the background, and time-dependent perturbations turn out to be impossible. Thus invoking $\xi(u)$ does not solve the consistency problem for time-dependent perturbations.

5 Conclusion

We have performed a stability analysis of the exact black hole-type solutions found in the context of the Rastall theory of gravity in the presence of a self-interacting scalar field, which were originally reported in Ref. [5]. The corresponding metrics are the same as those found in the context of k-essence theories [6]. This coincidence of the metrics found in so different contexts (but with quite different behaviors of the corresponding scalar fields) was discussed in detail in Ref. [7]. The perturbation analysis of the k-essence solutions was carried out in [8], and it was concluded that they are unstable. Hence, it was natural to perform a similar analysis for the corresponding Rastall solutions since the equivalence with the k-essence solutions may not be preserved at the perturbative level.

A surprising result is that such a stability analysis in the Rastall case is inconsistent: linear time-dependent spherically symmetric perturbations simply cannot exist. In this sense, the solutions may be said to be stable under such perturbations. The reason for our conclusion is that in the Rastall theory different combinations of the perturbed equations lead to different "master" equations for the scalar field perturbation $\delta\phi$.

This problem does not exist for k-essence solutions. We cannot attribute this feature to a wrong choice of the coordinates or the perturbation gauge since, as discussed, e.g., in [10–12], the perturbation method employed here leads to a gauge-invariant master equation.

We must stress that we have followed the standard procedure of studying the first-order (linear) perturbation equations. No other hypothesis has been made. The only possibility to avoid the inconsistency is to set the linear perturbations equal to zero and to study non-linear perturbations. However, the same inconsistency will then appear again and again: for example, the second order perturbations equations contain the same previous expressions with second-order perturbations and products of first-order perturbations which should be zero according to the first-order analysis, and so we get the same equations as in the first order. The whole perturbative scheme turns out to be inconsistent with time-dependent perturbations.

Very probably the roots of the inconsistency detected for Rastall black hole-type solutions come from the absence of a Lagrangian formulation of this theory. It must be remarked that a similar inconsistency was found in the cosmological context, also with a scalar field as a matter source. But, for the cosmological solutions the inconsistency has been cured by introducing ordinary baryonic matter. In the static, spherically symmetric case studied in [5] such an extension is less obvious.

There have been some attempts to find a Lagrangian formulation for the Rastall theory, see, e.g., [13,14], but the resulting theories were not completely equivalent to the Rastall one, or require a completely new geometric framework: using these formulations, we, strictly speaking, go away from the original context of the Rastall theory.

The results reported here may point at some restrictions inherent to the applicability of the Rastall theory, and even maybe of any non-Lagrangian theory. We hope to extend the present analysis in order to try to answer this question in our future works.

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