



New class of solutions in conformally symmetric massless scalar field collapse

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Abstract

In this paper we explicitly show the existence of a class of conformally symmetric solutions of spherically symmetric massless scalar field collapse with a given conformal factor, where the spatial variations of the conformal factor can be neglected in the latter stages of collapse. We obtain an equivalent and simpler form of the Roberts model for a massless scalar field. Our solution becomes self-similar near the central singularity and the critical phenomenon that is observed in self-similar collapse, will also be valid for this new class of solutions.

Keywords Massless scalar field · Conformal symmetries · Gravitational collapse

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1 Introduction

Our intention is to consider gravitational collapse with massless scalar fields in the presence of a conformal symmetry. In particular we seek a connection to the critical phenomena in the latter stages of the collapse process. To this end in this section we provide relevant information relating to the conformal symmetries, massless scalar fields and the structure of this paper.

1.1 Conformal symmetry

Conformal Killing vectors generate constants of the motion for massless scalar particles and generate a Lie algebra in the spacetime manifold. The seminal papers of earlier investigations [1–3] provide general constraints on the kinematical and dynamical quantities. These have enabled studies of conformal symmetries using the $1 + 3$ decomposition in particular geometries under various assumptions [4–7]. Conformal Killing vectors have been studied mainly in highly symmetric spacetimes because the corresponding conformal equations are difficult to integrate. Spherically symmetric spacetimes have been widely considered in different treatments [8–12]. Conformal symmetries have found many applications in relativistic astrophysics and modelling of spherically symmetric compact stars; some examples are given in recent treatments [13–16]. Another advance in studying conformal Killing vector in general is the semi-tetrad $1 + 1 + 2$ decomposition of spacetime; this was completed by Hansraj et al. [17] in the presence of a conformal symmetry. The pp-wave spacetimes generalize plane waves of Minkowski spacetime and provide simple models of gravitational radiation in general relativity, modified gravity theories or theories with scalar fields. Maartens and Maharaj [18] and Keane and Tupper [19] showed that pp-wave spacetimes admit conformal Killing vectors. Scaling properties and conformal symmetries of pp-waves have been studied by Zhang et al. [20] and these can be related to symmetries of massless particles via chrono-projective transformations, which generate new conserved quantities. Conformal symmetries have a wide presence, and it would be useful to also consider its connection to scalar fields.

1.2 Massless scalar field

There has been a lot of work done on the gravitational collapse of massless scalar fields [21–30], in the context of Cosmic Censorship Conjecture (CCC). Scalar fields are the simplest form of fundamental fields satisfying the Klein–Gordon equation, that is derived from a well defined field Lagrangian and therefore constitute a perfect candidate for a fundamental field whose validity will hold at any scale. The observation of scalar fields remains an open question in cosmology. As shown by the detailed numerical works of Choptuik [27], and later by many other authors (see for example [31,32]), the black hole mass threshold in the space of initial data for classes of massless scalar fields show universality and power-law scaling and this implies evidence of critical phenomena. This critical phenomenon directly points towards solutions that are typically self-similar. Therefore we have a scenario where the self-similar solution

is an attractor in the phase space of complete solutions. This critical phenomenon would evolve a smooth initial data to either a black hole or a dispersal, and at the boundary of these two end points lies an infinite time naked singularity that can be visible from infinity. Although this is a direct counterexample for the censorship conjecture, it has been shown rigorously by Christodoulou [23], that these naked singularities are non-generic within the context of a certain parent space.

To mathematically explain the occurrence of a self-similar behaviour in the latter stages of massless scalar field collapse, Brady [33] suggested the presence of certain conformal symmetries in the solutions in general, where the conformal Killing vectors tend to homothetic vectors in the vicinity of the singularity. To look at this phenomenon more transparently we have generated classes of collapsing massless scalar field solutions, with a class of conformal factors, that indeed tend to unity in the vicinity of the central singularity, making the evolution self-similar in the latter stages of collapse. Existence of these solutions indicates the validity of Brady's postulate to some extent.

1.3 This paper

The paper is organised as follows: In Sect. 2 we discuss the spherically symmetric massless scalar field and the associated field equations. In Sect. 3 we briefly discuss the equations for the existence of the conformal symmetries in spacetimes that contain both timelike and lightlike matter fields, and transparently show that such symmetry always exists. In the next section we consider the special case of homothety, and by solving the conformal Killing equations and the field equations simultaneously we derive the well known Robert's solution in a straightforward fashion. Finally in Sect. 4, we make a general ansatz for the conformal factor that enables the spacetime to become self-similar near the central singularity. We use this ansatz to simultaneously solve the conformal Killing equations and Einstein field equations to generate a new class of solutions, that become self-similar at the latter stages of collapse. In Sect. 5 we discuss the significance of our result.

Unless otherwise specified, we use natural units ($c = 8\pi G = 1$) throughout this paper, Latin indices run from 0 to 3. The symbol ∇ represents the usual covariant derivative and ∂ corresponds to partial differentiation. We use the $(-, +, +, +)$ signature and the Riemann tensor is defined by

$$R^a{}_{bcd} = \Gamma^a{}_{bd,c} - \Gamma^a{}_{bc,d} + \Gamma^e{}_{bd}\Gamma^a{}_{ce} - \Gamma^e{}_{bc}\Gamma^a{}_{de}, \quad (1)$$

and the Ricci tensor is obtained by contracting the *first* and the *third* indices of the Riemann tensor.

2 Spherical configuration of massless scalar field

As explained beautifully in [34], a massless scalar field $\phi(x^a)$, is the simplest form of a matter field, whose equation of motion can be derived from a covariant well defined

Lagrangian, given by

$$\mathcal{L} = -\frac{1}{2}g^{ab}\phi_{;a}\phi_{;b}. \quad (2)$$

This generates the Euler Lagrangian equation for the field as

$$g^{ab}\phi_{;ab} = 0, \quad (3)$$

and this provides the required wave equation for massless scalar waves. Also we can obtain the energy momentum tensor from the above Lagrangian by considering the change in action induced by the change in metric, which in this case has the form,

$$T_{ab} = \phi_{;a}\phi_{;b} - \frac{1}{2}g_{ab}(\phi_{;c}\phi_{;d}g^{cd}). \quad (4)$$

The above form of the energy momentum tensor for the massless scalar field contains a very interesting phenomenon. We know that the only types of physical matter fields are those whose energy momentum tensor either have one timelike and three spacelike eigenvectors (which includes perfect fluid form of matter) or have double null eigenvectors (which includes null radiation). The former is called a *Type I* matter field while the latter is called a *Type II* matter field. Interestingly the massless scalar field behaves like both types depending on the spacetime geometry. For example, it can be easily checked that a spatially homogeneous scalar field ϕ behaves like a stiff perfect fluid in spherical symmetry. On the other hand, being massless, the scalar waves travel on the null cone, as is evident from the Euler Lagrangian equation. This makes the field behave like a *Type II* matter field too. Therefore, to study the spherically symmetric configuration for a massless scalar field, we consider a general form of the spherically symmetric metric that allows for an arbitrary combination of *Type I* and *Type II* fields, which is given as [35]

$$ds^2 = -e^{2\psi(v,r)} \left[1 - \frac{2m(v,r)}{r} \right] dv^2 + 2\epsilon e^{\psi(v,r)} dvdr + r^2(d\theta^2 + \sin^2\theta d\phi^2), \quad (\epsilon = \pm 1), \quad (5)$$

where $m(v, r)$ is the *Misner–Sharp* mass function that is related to the gravitational energy inside a given radius r [36]. When $\epsilon = +1$, the null coordinate v is the Eddington advanced time, where r is decreasing towards the future along a ray $v = \text{Const}$. This depicts ingoing (or collapsing) null congruence with negative volume expansion. For $\epsilon = -1$, the null coordinate v is the Eddington retarded time and it depicts an outgoing null congruence with positive volume expansion. In this paper, as we focus on gravitational collapse of massless scalar fields, we will keep $\epsilon = +1$ throughout.

We note that by virtue of the Bianchi identities, the Euler-Lagrangian equation for the massless scalar field is included in the system of the Einstein field equations

$G_{ab} = T_{ab}$ and hence it suffices to solve the following system of equations

$$m_{,v} = 4\pi r^2 T_v^r, \tag{6a}$$

$$m_{,r} = 4\pi r^2 T_v^v, \tag{6b}$$

$$\psi_{,r} = 4\pi r T_r^r. \tag{6c}$$

For the given form of the metric (5), the non-zero components of the energy momentum tensor can be written as

$$T_v^r = \phi_{,v} \left[\phi_{,v} e^{-\psi} + \phi_{,r} \left(1 - \frac{2m}{r} \right) \right], \tag{7a}$$

$$T_v^v = -\frac{1}{2} \phi_{,r}^2 \left(1 - \frac{2m}{r} \right), \tag{7b}$$

$$T_r^r = \phi_{,r}^2. \tag{7c}$$

This gives the following system of Einstein equations:

$$m_{,v} = 4\pi r^2 \phi_{,v}^2 \left[e^{-\psi} + \frac{\phi_{,r}}{\phi_{,v}} \left(1 - \frac{2m}{r} \right) \right], \tag{8a}$$

$$m_{,r} = 2\pi r^2 \phi_{,r}^2 \left(1 - \frac{2m}{r} \right), \tag{8b}$$

$$\psi_{,r} = 4\pi r \phi_{,r}^2. \tag{8c}$$

We may also rewrite (8c) as (using (8b))

$$\psi_{,r} = \frac{2m_{,r}}{r - 2m}, \tag{9}$$

which relates the two unknown metric functions directly. Solutions to these equations, subject to required initial and boundary conditions will completely determine the evolution of a spherically symmetric spacetime containing a massless scalar field as the matter source.

3 Existence of conformal Killing vectors

Any spacetime (with coordinates x^a and metric g_{ab}) is said to possess a *conformal Killing vector* (CKV) ‘ \mathbb{X} ’, if it solves the following conformal Killing equation

$$L_{\mathbb{X}} g_{ab} = S(x^a) g_{ab}. \tag{10}$$

If the components of the vector \mathbb{X} is represented by X^a then the above equation simplifies to the set of equations

$$\nabla_{(a} X_{b)} = S g_{ab}. \tag{11}$$

We know the CKV becomes a Killing vector when $S = 0$. In that case the metric remains invariant when it is Lie dragged along this vector field. The homothetic Killing vector is a special case of the conformal Killing vector if the conformal function S is a non-zero constant. Then without any loss of generality we can always take S to be unity. The spacetimes that possess a homothetic Killing vector are called *self-similar spacetimes*. Henceforth we will denote a CKV as *proper* if S is a non-constant function of the coordinates.

We consider the existence of non-trivial CKV's in the (v, r) subspace of a massless scalar field spacetime. We note that since the spacetime is spherically symmetric, the (θ, ϕ) subspace will have the usual symmetries of a 2-sphere. Therefore a non-trivial symmetry is likely to lie in the (v, r) subspace. Therefore, we look for a CKV of the form

$$\mathbb{X} = A(r, v)\partial_v + B(r, v)\partial_r, \tag{12}$$

where $A(r, v)$ and $B(r, v)$ are unknown functions to be determined by solving the conformal Killing equations (11). We note that due to the spherical symmetry and the form of the CKV chosen, (11) becomes a set of four non-trivial equations. The (θ, θ) component of the above equation is given by $Br = Sr^2$ which simplifies as $(r \neq 0)$

$$B = Sr. \tag{13}$$

The (r, r) component is given by

$$e^\psi A_{;r} = 0. \tag{14}$$

Since $e^\psi \neq 0$, we must have $A_{;r} = 0$ in which case A is a function of v only. We note here that we can always rescale v so that we can set $A(v) = v$ (except in the case that \mathbb{X} is a *Killing vector* in which case A is constant). This will be utilized in the sections that follow to simplify our calculations. Now, we calculate the (r, v) component as (also noting that $e^\psi \neq 0$)

$$A\psi_{;v} + B\psi_{;r} - 2S + A_{;v} + B_{;r} = 0. \tag{15}$$

Finally, the (v, v) component is calculated as (again noting that $e^\psi \neq 0$)

$$-\left(1 - \frac{2m}{r}\right) [A\psi_{;v} + B\psi_{;r} - S + A_{;v}] + \frac{m_{;v}A}{r} + e^{-\psi} B_{;v} - \frac{mB}{r^2} + \frac{m_{;r}B}{r} = 0. \tag{16}$$

Substituting (13) into (15) we can rewrite (15) as

$$A\psi_{;v} + B\psi_{;r} - S + A_{;v} = \frac{B}{r} - B_{;r}, \tag{17}$$

and upon inserting (17) into (16) and simplifying we obtain

$$\left(1 - \frac{2m}{r}\right) B_{;r} + e^{-\psi} B_{;v} + \frac{B}{r} \left(m_{;r} + \frac{m}{r} - 1\right) + \frac{m_v A}{r} = 0. \tag{18}$$

We treat (18) as an equation in m . The important point to note here is that the function $B(v, r)$ obeys a first order partial differential equation, and hence the existence of a solution is guaranteed. A choice of the function S provides an expression for B which allows for the simplification of (18). This then allows us to solve for the mass function m , and the function ψ as will be demonstrated in subsequent sections.

4 An example: Homothety and Robert’s solution

To transparently illustrate, how to solve the field equations along with the conformal Killing equations simultaneously, we derive here the well known example of the solution originally presented by Roberts [37], which is a self-similar solution and a subclass of Brady’s solution [33]. The Roberts solution in its original form is given in double null coordinates and has a complicated form. Since this is the case of homothety, the function S in (13) is a non-zero constant. Without loss of generality we may set $S = 1$, which, from (13), gives $B = r$. Thus we have that $B_{,r} = 1$ and $B_{,v} = 0$, and we can write (18) (noting we have set $A = v$)

$$m_{,r} + \frac{v}{r}m_{,v} - \frac{m}{r} = 0, \tag{19}$$

whose general solution is given by

$$m = rf \left(\frac{v}{r} \right). \tag{20}$$

We shall set $x = v/r$ ($f = f(x)$) so that $m = rf$ and

$$m_{,r} = f - xf_{,x}. \tag{21}$$

Now using (8b) and (8c) we get

$$m_{,r} = \frac{1}{2}r\psi_{,r} \left(1 - \frac{2m}{r} \right). \tag{22}$$

A self-similar metric has the property that for any constant c , we have $g_{ab}(cv, cr) = g_{ab}(v, r)$. This definitely implies that $\psi(v, r) = \psi(x)$ and hence $\psi_{,r} = -\psi_{,x}(x/r)$. This allows us to write the RHS of (21) as

$$f - xf_{,x} = -\frac{1}{2}x\psi_{,x}(1 - 2f). \tag{23}$$

Also, from (8b), we get (assuming $\phi = \phi(x)$, that makes Robert’s solution a subclass of Brady’s solution)

$$\frac{f - xf_{,x}}{1 - 2f} = 2\pi\phi_{,x}^2x^2. \tag{24}$$

Now $m_{,v} = f_{,x}$ and $\phi_{,v} = (\phi_{,x}/r)$. Therefore Eq. (8a) becomes

$$f_{,x} = 4\pi\phi_{,x}^2 [e^{-\psi} - x(1 - 2f)],$$

which upon using (24), simplifying and solving for $f_{,x}$ gives

$$f_{,x} = \frac{2f}{x} \left[\frac{2xf + e^{-\psi} - x}{(2xf + 2e^{-\psi} - x)} \right]. \quad (25)$$

Using (25) in (23) and solving for $\psi_{,x}$ we get

$$\psi_{,x} = -\frac{2f}{2xf - x + 2e^{-\psi}}, \quad (26)$$

and solving (25) and (26) simultaneously we obtain

$$f = \frac{\alpha x}{2\beta} \left[2\sqrt{\alpha x^2 + \beta} - x \right], \quad (27a)$$

$$\psi = -\frac{1}{2} \ln(\alpha x^2 + \beta). \quad (27b)$$

Noting that $m = rf(x)$ we write m and ψ in terms of v and r as

$$m(v, r) = \frac{\alpha v}{2\beta r} \left[2\sqrt{\alpha v^2 + \beta r^2} - v \right], \quad (28a)$$

$$\psi(v, r) = -\frac{1}{2} \ln \left[\frac{\alpha v^2 + \beta r^2}{r^2} \right]. \quad (28b)$$

The above gives the complete solution for the spherically symmetric self-similar spacetime metric with massless scalar field as the matter source. We note that both Roberts and Brady obtained the self-similar solutions using double null coordinates, which is better suited for null fluids. However, as we have shown, one can obtain the same solution using the general metric for arbitrary combination of Type I and Type II matter fields (with a single null coordinate) and using the equations for conformal symmetries. This enables us to directly solve for the mass function, which determines the trapped regions in the spacetime. Our treatment has the advantage of being simpler and more transparent.

5 A new class of solutions with proper conformal symmetry

Extensive numerical works by Choptuik [27] and Abrahams and Evans [32], clearly shows that the numerical evolution of massless scalar field collapse behaves in a self-similar manner in the vicinity of the central singularity. Although no solid theoretical proof for the above observation is obtained as yet, we may try to look for one in the context of conformal symmetry. For example, if these spacetimes indeed possess a

conformal symmetry whose conformal factor tends to a constant non-zero value in the vicinity of the central singularity, then in that region the spacetime will evolve in a self-similar fashion. In this section, we generate a class of solutions that have the above property. In other words, these solutions have proper conformal symmetry which tends to homothety in the vicinity of the central singularity at $r = 0$. To start with, we look for existence of the conformally symmetric solutions that have the following conformal factor

$$S = 1 + r^n f(r), \tag{29}$$

where $f(r)$ is a C^2 function of r at $r = 0$. It is very clear that for any $n \geq 2$ and for $r \ll 1$ (during the latter stages of collapse), the spacetime will indeed become self-similar for these classes of solutions if they exist. Also we would like to emphasise here that a non-dynamic conformal factor doesn't imply a non-dynamic spacetime, as it is very clear with the existence of a dynamic self-similar solution.

We can now immediately solve for the CKV component $B(v, r)$ by using (29) in (13), and we have

$$B(v, r) = Sr = r + r^{(n+1)} f(r), \tag{30}$$

and from which we have $B_{,r} = 1 + (n + 1)r^n f(r) + r^{(n+1)} f'(r)$ and $B_{,v} = 0$. Now substituting (30) and the expressions for $B_{,v}$ and B_r in (18) we obtain (after some simplification)

$$\begin{aligned} &\left(1 - \frac{2m(r, v)}{r}\right) \left(1 + \frac{r^{n+1}(n + 1)f(r)}{r} + r^{n+1} f'(r)\right) \\ &+ \frac{(r + r^{n+1} f(r)) \left(m_{,r} + \frac{m(r, v)}{r} - 1\right)}{r} + \frac{m_{,v} v}{r} = 0. \end{aligned} \tag{31}$$

The solution for the mass function that allows for such a conformal symmetry is given as

$$m(v, r) = P(r) [Q(r) + F(\chi)], \tag{32}$$

where we have defined

$$\chi = v \exp \left[- \int \frac{dr}{r^{(n+1)} f(r) + r} \right], \tag{33}$$

and the function $F(\chi)$ is an arbitrary function of integration. The functions $P(r)$ and $Q(r)$ are given as follows

$$P(r) = \exp \left[\int \frac{2r^{n+1} \left(\frac{d}{dr} f(r)\right) + 1 + f(r) (2n + 1) r^n}{r (f(r)r^n + 1)} dr \right], \tag{34}$$

and

$$Q(r) = - \int \frac{\exp \left[\int \frac{2r^{n+1} \left(\frac{d}{dr} f(r) \right) + 1 + f(r)(2n+1)r^n}{r(f(r)r^{n+1})} dr \right] \left(r^n n f(r) + r^{n+1} \left(\frac{d}{dr} f(r) \right) \right)}{f(r)r^{n+1} + 1} . \tag{35}$$

We can also use the Eq. (9), to solve for the metric function $\psi(r, v)$, and the general solution is then given by

$$\psi(v, r) = W(r) + G(\chi). \tag{36}$$

where

$$W(r) = - \left(\int \frac{r^{n+1} \left(\frac{d}{dr} f(r) \right) + f(r)r^n (-1 + n)}{r + r^{n+1} f(r)} dr \right), \tag{37}$$

and the function $G(\chi)$ is another arbitrary function of integration. We would like to point out here that the freedom of arbitrariness of the functions $F(\chi)$ and $G(\chi)$ no longer remains, once we invoke the Einstein field equation

$$\psi(v, r)_{,r} = \frac{2m(v, r)_{,r}}{r - 2m(v, r)}. \tag{38}$$

Noting that

$$\chi_{,r} = - \frac{\chi}{r^{(n+1)} f(r) + r}, \tag{39}$$

the above field equation gives the constraint for the functions $F(\chi)$ and $G(\chi)$ as

$$W_{,r} - 2 \frac{(PQ)_{,r} + P_{,r}F}{r - 2(PQ + PF)} = \frac{\chi}{r^{(n+1)} f(r) + r} \left(G_{,\chi} - 2 \frac{F_{,\chi}}{r - 2(PQ + PF)} \right). \tag{40}$$

Any arbitrary functions F and G that satisfy the above differential constraint will then solve the system of field equations. It can be easily seen that the above class of solutions becomes the self-similar Robert solution when we put $f = 0$. Thus this class of solutions with proper conformal symmetry generalises the class of self-similar solutions. Also these solutions have the property as observed by Choptuik in his numerical simulations: these tend to homothety in the vicinity of the central singularity.

6 Discussion

Lake [36] conjectured the existence of an open set in the vicinity of the singularity in the spacetime manifold, where any continual collapse of realistic matter fields would behave in a self-similar fashion. Although there is no rigorous analytical proof for this claim, robust numerical studies for massless scalar field collapse indeed points in this direction. More interestingly, a recent study of gravitational collapse of generalised Vaidya spacetimes with the most general possible conformal symmetry also showed

the existence of such open set, specially in the transition zone of naked singularity and black hole phases [38]. These observations definitely warrant detailed studies of conformal symmetries that are associated with the spacetimes of fundamental matter fields.

In this work, we explicitly showed the existence of classes of solutions of spherically symmetric massless scalar field collapse, that possess a given form of conformal symmetry, where the spatial variations of the conformal factor can be neglected in the latter stages of collapse. It is interesting to observe that we obtain an equivalent form of the Roberts [37] model for a massless scalar field using the conformal symmetry approach. Our form is simpler and does not require double null coordinates; we have instead a conformal symmetry. Needless to say, these new class of solutions have the same properties that was observed by Choptuik, that is, these solutions become homothetic in the vicinity of the central singularity. Also, we transparently showed how the existence of conformal symmetries helps to integrate the field equations together with the conformal Killing equations, to find new and physically interesting solutions.

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References

1. Maartens, R., Mason, D.P., Tsamparlis, M.: *J. Math. Phys.* **27**, 2987 (1986)
2. Mason, D.P., Maartens, R.: *J. Math. Phys.* **28**, 2182 (1987)
3. Coley, A.A., Tupper, B.O.J.: *Class. Quantum Gravity* **11**, 2553 (1994)
4. Hall, G.S., Steele, J.D.: *J. Math. Phys.* **32**, 1847 (1991)
5. Tupper, B.O.J.: *Class. Quantum Gravity* **13**, 1679 (1996)
6. Tupper, B.O.J., Keane, A.J., Hall, G.S., Coley, A.A., Carot, J.: *Class. Quantum Gravity* **20**, 801 (2003)
7. Tupper, B.O.J., Keane, A., Carot, J.: *Class. Quantum Gravity* **29**, 145016 (2003)
8. Maartens, R., Maharaj, S.D., Tupper, B.O.J.: *Class. Quantum Gravity* **12**, 2577 (1995)
9. Moopanar, S., Maharaj, S.D.: *Int. J. Theor. Phys.* **49**, 1878 (2010)
10. Moopanar, S., Maharaj, S.D.: *J. Eng. Math.* **82**, 125 (2013)
11. Wagh, S.M., Govender, K.S.: *Gen. Relativ. Gravit.* **38**, 1253 (2006)
12. Apostolopoulos, P.S., Tsamparlis, M.: *Class. Quantum Gravity* **18**, 3775 (2001)
13. Shee, D., Rahaman, F., Guha, B.K., Ray, R.: *Astrophys. Space Sci.* **361**, 167 (2016)
14. Banerjee, A., Banerjee, S., Hansraj, S., Ovgun, A.: *Eur. Phys. J. Plus* **132**, 150 (2017)
15. Herrera, L., Di Prisco, A.: *Int. J. Mod. Phys. D* **27**, 1750176 (2018)
16. Singh, K.N., Bhar, P., Rahaman, F., Pant, N.: *J. Phys. Commun.* **2**, 015002 (2018)
17. Hansraj, C., Goswami, R., Maharaj, S.D.: *Gen. Relativ. Gravit.* **52**, 63 (2020)
18. Maartens, R., Maharaj, S.D.: *Class. Quantum Gravity* **8**, 503 (1991)
19. Keane, A.J., Tupper, B.O.J.: *Class. Quantum Gravity* **21**, 2037 (2004)
20. Zhang, P.M., Cariglia, M., Elbistan, M., Horvathy, P.A.: *J. Math. Phys.* **61**, 022502 (2020)
21. Wyman, M.: *Phys. Rev. D* **24**, 839 (1981)
22. Xanthopoulos, B.C., Zannias, T.: *Phys. Rev. D* **40**, 2564 (1989)
23. Christodoulou, D.: *Commun. Pure Appl. Math.* **44**, 339 (1991)
24. Christodoulou, D.: *Commun. Pure Appl. Math.* **46**, 1131 (1993)
25. Christodoulou, D.: *Ann. Math.* **140**, 607 (1994)
26. Christodoulou, D.: *Ann. Math.* **149**, 183 (1999)
27. Choptuik, M.W.: *Phys. Rev. Lett.* **70**, 9 (1993)
28. Giambo, R.: *Class. Quantum Gravity* **22**, 2295 (2005)

29. Gundlach, C.: Phys. Rev. Lett. **75**, 3214 (1995)
30. Gundlach, C., Martin-Garcia, J.M.: Living Rev. Relativ. **10**, 5 (2007)
31. Hamade, R.S., Stewart, J.M.: Class. Quantum Gravity **13**, 493 (1996)
32. Abrahams, A.M., Evans, C.R.: Phys. Rev. Lett. **70**, 2980 (1993)
33. Brady, P.R.: Phy. Rev. D **51**, 4168 (1995)
34. Hawking, S.W., Ellis, G.F.R.: The Large Scale Structure of Spacetime. Cambridge University Press, Cambridge (1973)
35. Barrabes, C., Israel, W.: Phys. Rev. D **43**, 1129 (1991)
36. Lake, K.: Phys. Rev. Lett. **68**, 3129 (1992)
37. Roberts, M.D.: Gen. Relativ. Gravit. **21**, 907 (1989)
38. Ojako, S.O., Goswami, R., Maharaj, S.D., Narain, R.: Class. Quantum Gravity **37**, 055005 (2020)

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