RESEARCH ARTICLE

Gödel spacetime, planar geodesics and the Möbius map

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Abstract

Timelike geodesics on a hyperplane orthogonal to the symmetry axis of the Gödel
spacetime appear to be elliptic-like if standard coordinates naturally adapted to the
cylindrical symmetry are used. The orbit can then be suitably described through an
eccentricity-semi-latus rectum parametrization, familiar from the Newtonian dynam-
ics of a two-body system. However, changing coordinates such planar geodesics all
become explicitly circular, as exhibited by Kundt's form of the Gödel metric. We
derive here a one-to-one correspondence between the constants of the motion along
these geodesics as well as between the parameter spaces of elliptic-like versus circular
geodesics. We also show how to connect the two equivalent descriptions of particle
motion by introducing a pair of complex coordinates in the 2-planes orthogonal to the
symmetry axis, which brings the metric into a form which is invariant under Möbius
transformations preserving the symmetries of the orbit, i.e., taking circles to circles.

Keywords Godel spacetime · Timelike geodesics · Mobius transformations

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Contents

1 Introduction

The geometrical properties and physical aspects of Gödel spacetime [1] which make it the prototype for rotating cosmological models have been investigated in depth in the literature since its discovery, leading to the introduction of several coordinate systems which are better suited for showing its various properties (see, e.g., Refs. [2–7]). Cylindrical-like coordinates are naturally adapted to the cylindrical symmetry of the spacetime about one particular dust particle world line [8]. The dust source particles are at rest with respect to these coordinates, but form a family of twisting world lines, so that the cylindrical symmetry of the spacetime is preserved about every point. Timelike geodesics on a "planar" 2-surface orthogonal to the symmetry axis appear to be closed elliptic-like curves, which are of different types depending on the value of the ratio between the particle's conserved angular momentum (L) and energy (E) [9]. We have recently introduced an eccentricity-semi-latus rectum (e, p) parametrization to describe the motion in close analogy with the Newtonian dynamics of a two-body system [10]. For each class of orbits we have then computed gauge-invariant quantities, like the periastron advance and the precession angle of a test gyroscope, thus providing coordinate-independent information. The shape of the orbit, however, is instead coordinate-dependent. The Kundt form of the Gödel metric [2] makes the orbits of all such planar geodesics explicitly circular.

The goal of this article is to derive a one-to-one correspondence between the constants of the motion and parameter spaces of the planar geodesics in the two different forms of the Gödel metric mentioned above, i.e., the standard cylindrical form and Kundt's form. We will also show how to connect the two equivalent descriptions of particle motion by introducing a pair of complex coordinates in the 2-planes orthogonal to the symmetry axis, which brings the metric into a form which is invariant under Möbius transformations preserving the shape of the orbit, i.e., taking circles to circles.

2 Planar timelike geodesics in the Gödel spacetime

The Gödel spacetime [1,8] is a Petrov type D stationary axisymmetric solution of the nonvacuum Einstein equations with a negative cosmological constant and matter in the form of dust, whose gravitational attraction is balanced by a global rotation.

The stress-energy tensor is then $T = \rho\, u \otimes u$, with constant energy density ρ and unit 4-velocity u of the fluid particles aligned with the time coordinate lines, whereas the cosmological constant has the value $\Lambda = -\omega^2 = -4\pi\rho$. Gödel spacetime is spacetime-homogeneous with five Killing vector fields which allow an infinite number of spatially homogeneous slicing families, each valid within a given neighborhood of some central world line. Changing "spatial" coordinate systems inevitably changes the time slicing family as well.

2.1 Cylindrical-like standard form and elliptic-like curves

Using standard coordinates (t, r, ϕ, z) naturally adapted to the cylindrical symmetry [8], the Gödel metric reads

$$ds^2 = \frac{2}{\omega^2}\left[-dt^2 + dr^2 + \sinh^2 r\,(1 - \sinh^2 r)\,d\phi^2\right.$$
$$\left. +2\sqrt{2}\sinh^2 r\, dt\, d\phi + dz^2\right]. \tag{1}$$

The coordinates here are all dimensionless, while the scaling factor $\omega > 0$ has the dimension of inverse length and describes the vorticity of the dust world lines which rotate in the increasing ϕ direction about the central world line of the coordinate grid. To avoid closed timelike curves one should restrict to the spacetime region $r < r_h$ (i.e., the "physical region"), where the horizon radius

$$r_h = \ln(1 + \sqrt{2}) \approx 0.88137, \qquad \sinh r_h = 1, \quad \cosh r_h = \sqrt{2}, \tag{2}$$

at which the ϕ coordinate circles become null is defined by the condition $g_{\phi\phi} = 0$.

The Killing vectors associated with the spacetime symmetries are

$$\xi_1 = \partial_t, \quad \xi_2 = \partial_\phi, \quad \xi_3 = \partial_z,$$
$$\xi_4 = \sqrt{2}\tanh(r)\cos\phi\,\partial_t - \sin\phi\,\partial_r - \frac{2\cos\phi}{\tanh(2r)}\,\partial_\phi,$$
$$\xi_5 = \sqrt{2}\tanh(r)\sin\phi\,\partial_t + \cos\phi\,\partial_r - \frac{2\sin\phi}{\tanh(2r)}\,\partial_\phi. \tag{3}$$

Introduce the 4-vector

$$\eta = \sqrt{2}\tanh r\,\partial_t - \frac{2}{\tanh(2r)}\,\partial_\phi, \qquad ||\eta||^2 = \frac{1}{\omega^2}(3 - \cosh(4r)), \tag{4}$$

which is spacelike in the region $0 < r < r_* = \frac{1}{4}\text{arccosh}(3) \approx 0.44069$ and timelike for $r_* < r < r_h$. The Killing vectors ξ_4 and ξ_5 thus correspond to a rotation in the plane $r - \phi$

$$\begin{pmatrix} -\xi_5 \\ -\xi_4 \end{pmatrix} = \begin{pmatrix} -\cos\phi & -\sin\phi \\ \sin\phi & -\cos\phi \end{pmatrix} \begin{pmatrix} \partial_r \\ \eta \end{pmatrix}. \tag{5}$$

Near the origin in the polar plane where $\eta \sim \partial_\phi$, this azimuthal rotation of the frame vectors (∂_r, η) aligns $(-\xi_5, -\xi_4)$ with the corresponding Cartesian frame (∂_x, ∂_y) defined in the usual way with respect to (r, ϕ), representing two translational Killing vector fields in that plane, though tilted with respect to the time coordinate hypersurfaces. In fact, when one spatially "translates" the symmetry axis at the origin of these polar coordinates where the time slicing is orthogonal to the time lines, this tilting of these two Killing vectors leads to a new slicing with the same property at the new location.

The geodesic equations are separable and the covariant 4-velocity of a general geodesic has the following form

$$U^\flat = -E\,dt + L\,d\phi + b\,dz + U_r\,dr\,, \tag{6}$$

where

$$U_r = \frac{2}{\omega^2} U^r = \frac{2}{\omega^2} \frac{dr}{d\lambda}\,, \tag{7}$$

with λ denoting an affine parameter and normalization condition $U^\alpha U_\alpha = -\mu^2$, with $\mu^2 = 1, 0, -1$ for timelike, null, spacelike geodesics, respectively. The geodesic equations expressed in terms of the constants of the motion $E = -U \cdot \partial_t$, $L = U \cdot \partial_\phi$ and $b = U \cdot \partial_z$ are

$$\frac{dt}{d\lambda} = -\frac{1}{2}\omega^2 \left(E - \frac{\sqrt{2}X}{\cosh^2 r} \right),$$

$$\left(\frac{dr}{d\lambda} \right)^2 = -\frac{1}{2}\omega^2 \mu^2 - \frac{1}{4}\omega^4 (b^2 + E^2)$$

$$+ \frac{1}{4}\omega^4 \frac{X^2}{\cosh^2 r} - \frac{1}{4}\omega^4 \frac{L^2}{\sinh^2 r}\,,$$

$$\frac{d\phi}{d\lambda} = -\frac{1}{2}\frac{\omega^2}{\sinh^2 r} \left(\sqrt{2}E - \frac{X}{\cosh^2 r} \right),$$

$$\frac{dz}{d\lambda} = \frac{1}{2}\omega^2 b\,, \tag{8}$$

where $X = \sqrt{2}E + L$. These can be integrated analytically. Additional conserved quantities $k_4 = U \cdot \xi_4$ and $k_5 = U \cdot \xi_5$ associated with the Killing vectors ξ_4 and ξ_5 satisfy

$$k_4 \sin\phi + k_5 \cos\phi = \frac{1}{\sinh(2r)}[\sqrt{2}E - (X + L)\cosh(2r)]\,,$$

$$k_4 \cos\phi - k_5 \sin\phi = U^r\,, \tag{9}$$

which define r as an implicit function of ϕ. This rotation of the Killing constants aligns the conserved translation Killing momenta with the polar frame momenta.

2.1.1 Newtonian-like parametrization of planar geodesics

Here we only consider the case $b = 0$ of "planar" timelike geodesics (choosing $\mu = 1$ making $\lambda = \tau$) confined to a constant z hyperplane. The associated 4-velocity is

$$
U = \frac{\omega^2}{2 \cosh^2 r} \left[\sqrt{2} L + E (2 - \cosh^2 r) \right] \partial_t + U^r \partial_r
$$
$$
- \frac{\omega^2}{2 \sinh^2 r \cosh^2 r} \left(E \sqrt{2} \sinh^2 r - L \right) \partial_\phi , \tag{10}
$$

where

$$
(U^r)^2 = \frac{\omega^2}{4 \sinh^2 r \cosh^2 r} \left[-2(\tilde{E}^2 + 1) \cosh^4 r \right.
$$
$$
\left. + 2(3\tilde{E}^2 + 2\tilde{E}\tilde{L} + 1) \cosh^2 r - (2\tilde{E} + \tilde{L})^2 \right] , \tag{11}
$$

having introduced the following rescaling of E and L

$$
\tilde{E} = \frac{\omega}{\sqrt{2}} E \geq 1 , \qquad \tilde{L} = \omega L . \tag{12}
$$

The orbits are elliptic-like and can be classified into three different types according to the ratio between angular momentum and energy [9].

In Ref. [10] we have recently introduced an equivalent parametrization of the orbit in terms of a semi-latus rectum p and eccentricity e, familiar from Newtonian mechanics (but for the change $r \to \cosh^2 r$), i.e.,

$$
\cosh^2 r = \frac{p}{1 + e \cos \chi} , \tag{13}
$$

with $p \geq 1$ and $0 \leq e < 1$ as usual for elliptical orbits, so that $r = r_{(\text{peri})}$ for $\chi = 0$ and $r = r_{(\text{apo})}$ for $\chi = \pi$, i.e.,

$$
\cosh^2 r_{(\text{peri})} = \frac{p}{1 + e} , \qquad \cosh^2 r_{(\text{apo})} = \frac{p}{1 - e} . \tag{14}
$$

The main properties of the different kinds of orbits and the relation between the two parametrizations are summarized below.

1. Type I: The orbits have $0 < r_{(\text{peri})} \leq r \leq r_{(\text{apo})}$ and go around the origin with monotonically decreasing ϕ opposing the local rotation of the dust source (no turning point in ϕ). The orbit becomes circular for $r_{(\text{peri})} = r_{(\text{apo})}$. The angular momentum is always negative $\tilde{L}_{\min} \leq \tilde{L} < 0$ for fixed values of the energy parameter, with

$$
\tilde{L}_{\min} = -\tilde{E} + \frac{1}{\sqrt{2}} \sqrt{\tilde{E}^2 + 1} . \tag{15}
$$

The relation between $(\tilde{E}, \tilde{L})$ and (p, e) is

$$\tilde{E} = \frac{p + \mathcal{Y}}{\sqrt{(p - \mathcal{Y})^2 - 4p(p - 1)}},$$

$$\tilde{L} = \frac{-2\mathcal{Y}}{\sqrt{(p - \mathcal{Y})^2 - 4p(p - 1)}}, \tag{16}$$

where $\mathcal{Y} = \sqrt{(p - 1)^2 - e^2}$.

The solutions for t and ϕ turn out to be

$$t(\chi) = \frac{1}{\sqrt{2}} \left[\chi - \frac{p + \mathcal{Y}}{\sqrt{1 - e^2}} \arctan(\psi_1) \right],$$

$$\phi(\chi) = -\frac{\chi}{2} - \arctan(\psi_2), \tag{17}$$

with

$$\psi_1 = \sqrt{\frac{1 - e}{1 + e}} \tan \frac{\chi}{2}, \qquad \psi_2 = \sqrt{\frac{p - 1 + e}{p - 1 - e}} \tan \frac{\chi}{2}. \tag{18}$$

2. Type II: The same as type I, with $r_{(\text{peri})} = 0$, and $\cosh^2 r_{(\text{apo})} = 2\tilde{E}^2/(\tilde{E}^2 + 1)$, so that the orbits pass through the origin. These orbits have vanishing angular momentum $\tilde{L} = 0$ corresponding to $p = 1 + e$, with $\tilde{E} = \sqrt{(1 + e)/(1 - 3e)}$. The solutions for t and ϕ turn out to be

$$t(\chi) = \frac{1}{\sqrt{2}} \left[\chi - \sqrt{\frac{1 + e}{1 - e}} \arctan(\psi_1) \right],$$

$$\phi(\chi) = -\frac{\chi}{2}. \tag{19}$$

3. Type III: In this case the ϕ-motion has a turning point in the allowed range of r, preventing the orbit from making a complete circuit about the origin. The angular momentum is always positive $0 < \tilde{L} < L_h$ for fixed values of the energy parameter, with

$$L_h = 2(\tilde{E} - \sqrt{\tilde{E}^2 - 1}). \tag{20}$$

The relation between $(\tilde{E}, \tilde{L})$ and (p, e) is

$$\tilde{E} = \frac{p - \mathcal{Y}}{\sqrt{(p + \mathcal{Y})^2 - 4p(p - 1)}},$$

$$\tilde{L} = \frac{2\mathcal{Y}}{\sqrt{(p + \mathcal{Y})^2 - 4p(p - 1)}}. \tag{21}$$

The solutions for t and ϕ turn out to be

$$t(\chi) = \frac{1}{\sqrt{2}} \left[\chi + \frac{p - \mathcal{y}}{\sqrt{1 - e^2}} \arctan(\psi_1) \right],$$

$$\phi(\chi) = -\frac{\chi}{2} + \arctan(\psi_2). \tag{22}$$

In any case the eccentricity is limited to $e < 1/3$. Furthermore, the relation between the variable χ and the proper time τ is

$$\tan \frac{\chi}{2} = \sqrt{\frac{1+e}{1-e}} \tan(\zeta\tau), \quad \text{or}$$

$$\tau = \frac{1}{\zeta} \arctan \left(\sqrt{\frac{1-e}{1+e}} \tan \frac{\chi}{2} \right), \tag{23}$$

with $\chi = 0 \leftrightarrow \tau = 0$ and

$$\zeta = \frac{\omega\sqrt{\tilde{E}^2 + 1}}{\sqrt{2}}. \tag{24}$$

The allowed parameter spaces for both the $(\tilde{L}, \tilde{E})$ and (p, e) parametrizations are shown in Fig. 1.

Since the spacetime is homogeneous it is enough to consider geodesics of a given kind, e.g., type II orbits passing through the origin, without any loss of generality. One can then switch to a geodesic of a different kind (type I and III) simply by changing the origin of the cylindrical-like coordinate system. In fact, the classification of the orbits depends on the value of the angular momentum parameter $\tilde{L}$ relative to the origin $r = 0$, so that it is always possible to find a new parameter $\tilde{L}'$ with respect to the origin $r' = 0$ of the new coordinate system such that old type II orbits will exhibit the same features as either type I or type III orbits.

The transformation $r \rightarrow r - r_0 = r'$ implies that the new orbital parameters (p', e') are related to the old ones (p, e) by the conditions

$$\cosh^2(r'_{\text{(peri)}} + r_0) = \frac{p}{1 + e}, \quad \cosh^2(r'_{\text{(apo)}} + r_0) = \frac{p}{1 - e}, \tag{25}$$

and

$$\cosh^2 r'_{\text{(peri)}} = \frac{p'}{1 + e'}, \quad \cosh^2 r'_{\text{(apo)}} = \frac{p'}{1 - e'}. \tag{26}$$

In fact, the orbit with old orbital parameters (p, e) is shifted along the radial direction. However, one can recover the usual relation (14) by introducing a new pair of orbital parameters (p', e') with respect to the new coordinate system. For instance, the new parameters associated with old type II orbits having $p = 1 + e$ turn out to be
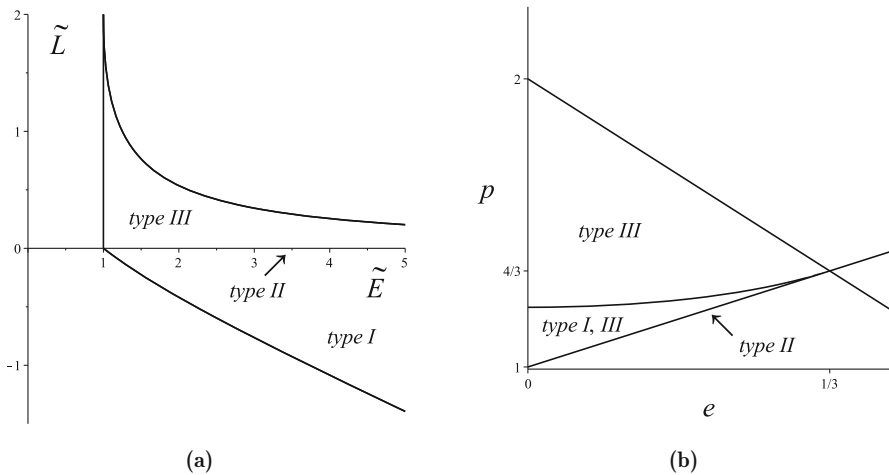
Fig. 1 The allowed parameter spaces $(\tilde{L}, \tilde{E})$ and (p, e) for planar timelike geodesics in the physical region $r < r_h$ are shown in panels (a) and (b), respectively. Type II orbits have $\tilde{L} = 0$, corresponding to the line $p = 1 + e$ where $r_{(\mathrm{peri})} = 0$ [(see Eq. (14)]. The type I ($\tilde{L}_{\min} \leq \tilde{L} < 0$) and III ($0 < \tilde{L} < L_h$) orbits are folded over onto each other into the triangular region between the lines $p = 1 + e$ and $p = 2(1 - e)$ where $r_{(\mathrm{apo})} = r_h$. The upper boundary for type I orbits is implicitly defined by the vanishing of the denominator in Eq. (16), for which these geodesics become null

$$p' = \cosh^2 r_0 \frac{(1 + 6e + e^2)\cosh(2r_0) - 4\sqrt{e}(1 + e)\sinh(2r_0) + (1 - e)^2}{(1 + e)^2 \cosh(2r_0) - 2\sqrt{e}(1 + e)\sinh(2r_0) + (1 - e)^2},$$

$$e' = 2\sqrt{e}\frac{2\sqrt{e}\cosh(2r_0) - (1 + e)\sinh(2r_0)}{(1 + e)^2 \cosh(2r_0) - 2\sqrt{e}(1 + e)\sinh(2r_0) + (1 - e)^2}, \tag{27}$$

which reduce to $p' = \cosh^2 r_0$ and $e' = 0$ for $e = 0$, corresponding to a circular orbit with radius r_0, as for type I orbits in the same limiting case of vanishing eccentricity.

2.2 Kundt's form and circles

The cylindrical standard form (1) of the metric is related to Kundt's form [2]

$$ds^2 = -\left(dT + \frac{1}{\omega Y}dX\right)^2 + \frac{1}{2\omega^2 Y^2}(dX^2 + dY^2) + dZ^2, \tag{28}$$

by the transformation

$$\omega T = \sqrt{2}t - 2\arctan\left(e^{-2r}\tan\frac{\phi}{2}\right) + \phi,$$

$$\sqrt{2}\omega(X + iY) = i\frac{1 - \tanh r e^{-i\phi}}{1 + \tanh r e^{-i\phi}}, \qquad Z = \frac{\sqrt{2}}{\omega}z. \tag{29}$$

We notice the following Killing vectors

$$\Xi_1 = \partial_T , \qquad \Xi_2 = \partial_X , \qquad \Xi_3 = X\partial_X + Y\partial_Y , \qquad \Xi_4 = \partial_Z , \tag{30}$$

which are combinations of those associated with the cylindrical-like metric (see Eq. (3))

$$
\begin{aligned}
\Xi_1 &= \frac{\omega}{\sqrt{2}}\partial_t = \frac{\omega}{\sqrt{2}}\xi_1 , \\
\Xi_2 &= \omega(\partial_t - \sqrt{2}\partial_\phi) + \sqrt{2}\omega\left[\frac{1}{\sqrt{2}}\tanh(r)\cos\phi\,\partial_t - \frac{1}{2}\sin\phi\,\partial_r - \frac{\cos\phi}{\tanh(2r)}\partial_\phi\right] \\
&= \omega\left(\xi_1 - \sqrt{2}\xi_2 + \frac{1}{\sqrt{2}}\xi_4\right) , \\
\Xi_3 &= -\frac{1}{\sqrt{2}}\tanh(r)\sin\phi\,\partial_t - \frac{1}{2}\cos\phi\,\partial_r + \frac{\sin\phi}{\tanh(2r)}\partial_\phi = -\frac{1}{2}\xi_5 , \\
\Xi_4 &= \frac{\omega}{\sqrt{2}}\partial_z = \frac{\omega}{\sqrt{2}}\xi_3 .
\end{aligned}
\tag{31}
$$

When expressed in this coordinate system the azimuthal Killing vector $\xi_2 = \partial_\phi$ becomes

$$\xi_2 = \frac{1}{\omega}(1 - \sqrt{2}\omega Y)\partial_T - \frac{\sqrt{2}}{4\omega}[1 + 2\omega^2(X^2 - Y^2)]\partial_X - \sqrt{2}\omega XY\partial_Y = \Xi_5 . \tag{32}$$

The timelike geodesic 4-velocity vector of the metric (28) is re-expressed as

$$
\begin{aligned}
U = &-\frac{\sqrt{2}}{C}\left(\frac{Y'}{2} - Y\right)\partial_T + \frac{\sqrt{2}}{C}\omega Y(Y' - Y)\,\partial_X \\
&+\frac{\sqrt{2}}{C}\omega Y(X - X')\,\partial_Y + \frac{B}{C}\partial_Z ,
\end{aligned}
\tag{33}
$$

i.e.,

$$U^\flat = -\frac{Y'}{\sqrt{2}C}dT - \frac{1}{\sqrt{2}C\omega}dX + \frac{(X - X')}{\sqrt{2}C\omega Y}dY + \frac{B}{C}dZ , \tag{34}$$

with normalization condition ($U^\alpha U_\alpha = -1$) equivalent to

$$(X - X')^2 + (Y - Y')^2 = \frac{1}{2}(Y')^2 - B^2 - C^2 \equiv R^2 , \tag{35}$$

where X', Y', C and B are constants associated with the above Killing vectors

$$\Xi_1 \cdot U = -\frac{Y'}{\sqrt{2}C}, \qquad \Xi_2 \cdot U = -\frac{1}{\sqrt{2}\omega C},$$

$$\Xi_3 \cdot U = -\frac{X'}{\sqrt{2}\omega C}, \qquad \Xi_4 \cdot U = \frac{B}{C},$$

$$\Xi_5 \cdot U = \frac{1}{4\omega^2 C}[1 - 2\sqrt{2}\omega Y' + 2\omega^2((X')^2 + (Y')^2 - R^2)]. \tag{36}$$

Eq. (35) suggests the following parametric equations for the orbit

$$X(\tau) = X' + R\cos\alpha(\tau), \qquad Y(\tau) = Y' + R\sin\alpha(\tau), \qquad Z(\tau) = \frac{B}{C}\tau, \tag{37}$$

which imply

$$\frac{d\alpha(\tau)}{d\tau} = \frac{\sqrt{2}}{C}\omega(Y' + R\sin\alpha(\tau)), \qquad \frac{dT(\tau)}{d\tau} = \frac{\sqrt{2}}{C}\left(\frac{Y'}{2} + R\sin\alpha(\tau)\right), \tag{38}$$

with solution

$$\alpha(\tau) = 2\arctan\left(\frac{Y'\tan(\Omega\tau)}{\kappa - R\tan(\Omega\tau)}\right), \qquad \omega T(\tau) = \alpha(\tau) - \frac{Y'}{\sqrt{2}C}\omega\tau, \tag{39}$$

where

$$\kappa = \sqrt{(Y')^2 - R^2}, \qquad \Omega = \frac{\omega}{\sqrt{2}C}\kappa, \tag{40}$$

and initial conditions have been chosen so that $T(0) = 0 = \alpha(0)$.

Let us now consider planar geodesics corresponding to $B = 0$. Applying the coordinate transformation (29) to the particle's 4-velocity then yields the following relation between the constants of motion

$$E = \frac{Y'}{\omega C}, \qquad L = \frac{C}{2} + \frac{1}{C}\left[\left(\frac{Y'}{2} - \frac{1}{\sqrt{2}\omega}\right)^2 + \frac{(X')^2}{2} - \frac{1}{4\omega^2}\right], \tag{41}$$

which imply

$$\omega X' = \pm\left[-\frac{1}{2} - (\tilde{E}^2 + 1)\tilde{C}^2 + 2(\tilde{E} + \tilde{L})\tilde{C}\right]^{1/2}$$

$$\equiv \pm\sqrt{\tilde{E}^2 + 1}\sqrt{(\tilde{C} - \tilde{C}_1)(\tilde{C}_2 - \tilde{C})},$$

$$\omega Y' = \sqrt{2}\tilde{E}\tilde{C},$$

$$\omega R = \tilde{C}\sqrt{\tilde{E}^2 - 1}, \tag{42}$$

with $\tilde{C} = \omega C > 0$ and $\tilde{C}_1 \le \tilde{C} \le \tilde{C}_2$. Because Y' is always positive, the centers cannot be located along the X-axis. In contrast, they may lie on the Y-axis if either $\tilde{C} = \tilde{C}_1$ (with $\omega Y'_1 = \sqrt{2}\tilde{E}\tilde{C}_1$) or $\tilde{C} = \tilde{C}_2$ (with $\omega Y'_2 = \sqrt{2}\tilde{E}\tilde{C}_2$). Furthermore, the radius of the orbit is such that $R_1 \le R \le R_2$.

Let us study the locus of the centers for type II orbits, for fixed values of $\tilde{C}$. Substituting $\tilde{L} = 0$ and $\tilde{E} = \sqrt{(1+e)/(1-3e)}$ in Eq. (42) gives

$$\tilde{C}_1 = \frac{\sqrt{1-3e}}{2(1-e)}(\sqrt{1+e} - \sqrt{2e}), \qquad \tilde{C}_2 = \frac{\sqrt{1-3e}}{2(1-e)}(\sqrt{1+e} + \sqrt{2e}). \quad (43)$$

Their behavior as functions of e is shown in Fig. 2. Therefore

$$Y'_1 = \frac{\sqrt{1+e}}{\sqrt{2}(1-e)}(\sqrt{1+e} - \sqrt{2e}), \qquad Y'_2 = \frac{\sqrt{1+e}}{\sqrt{2}(1-e)}(\sqrt{1+e} + \sqrt{2e}), \quad (44)$$

which become $Y'_1 = 1/\sqrt{2} = Y'_2$ and $Y'_1 = \sqrt{2} - 1$, $Y'_2 = \sqrt{2} + 1$ in the two limiting cases $e \to 0$ and $e \to 1/3$, respectively. The locus of the centers is shown in Fig. 3.

The radius of the orbit is

$$\omega R = 2\tilde{C}\sqrt{\frac{e}{1-3e}}, \quad (45)$$

so that its limiting values are

$$\omega R_1 = \frac{\sqrt{e}}{1-e}(\sqrt{1+e} - \sqrt{2e}), \qquad \omega R_2 = \frac{\sqrt{e}}{1-e}(\sqrt{1+e} + \sqrt{2e}). \quad (46)$$

The orbit intersects the Y−axis at $\omega Y = 1/\sqrt{2}$ for every value of $\tilde{C}$ and e and at $\omega Y = 2\omega Y' - 1/\sqrt{2}$. Fig. 4 shows the circles corresponding to type II orbits for a fixed value of $\tilde{C}$ and different values of the eccentricity e.

3 Möbius transformations

A Möbius transformation $T_{(a,b,c,d)}$ of the complex plane $\mathbb{C}$ has the fractional-linear form

$$T_{(a,b,c,d)}(z) = \frac{az+b}{cz+d}, \quad (47)$$

where $a, b, c, d, z \in \mathbb{C}$ are complex numbers and $ad - bc \ne 0$. The set of all Möbius transformations is a group under composition. The transformations $S_{(a,b)}(z) = T_{(a,b,0,1)}(z) = az + b$ (setting $d = 1$ without loss of generality since one can always take $a \to a/d$, $b \to b/d$) form the subgroup of similarities, whereas the transformation $T_{(0,1,1,0)}(z) = 1/z$ is termed an inversion. Every Möbius transformation $T_{(a,b,c,d)}$ can be interpreted as a nonunique combination of a similarity and an inversion.

Fig. 2 The behavior of the quantities $\tilde{C}_1$ and $\tilde{C}_2$ as functions of e is shown for orbits of type II
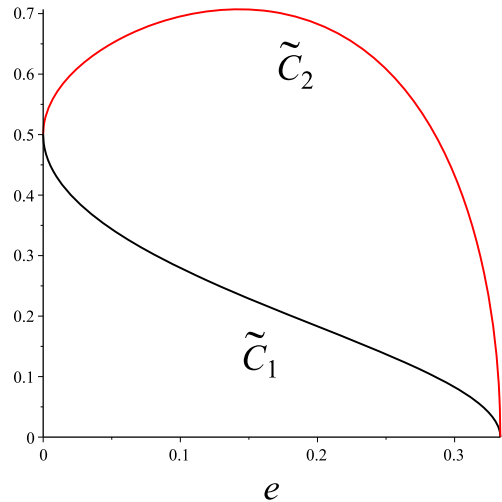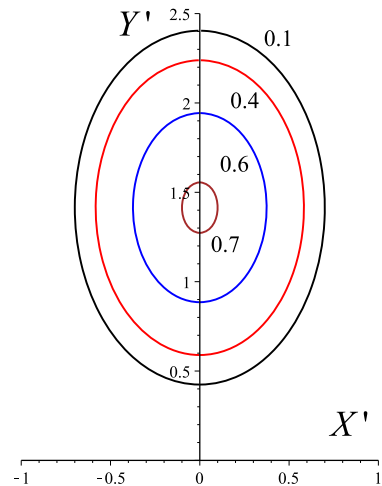

Fig. 3 The locus of the centers for type II orbits is shown for selected values of $\tilde{C} = [0.1, 0.4, 0.6, 0.7]$

It is well known that there exists an isomorphism between the Lorentz group SO(3, 1) that preserves the orientation of space and the group SL(2, $\mathbb{C}$) of conformal transformations of the 2-dimensional sphere, under which Lorentz transformations correspond to conformal transformations of this sphere (see, e.g., Ref. [11] and references therein), represented by complex 2×2 matrices of unit determinant. The basic geometric property of Möbius transformations is that they preserve angles between curves and map circles (and straight lines) into circles (and straight lines).

Fig. 4 The circles corresponding to type II orbits are shown for $\tilde{C} = 0.4$ and selected values of the eccentricity $e = [0.03, 0.1, 0.2, 0.25, 0.3]$. The dotted curve is the locus of the centers of Fig. 3. Only positive values of X' are taken, for simplicity [(see Eq. (42)]
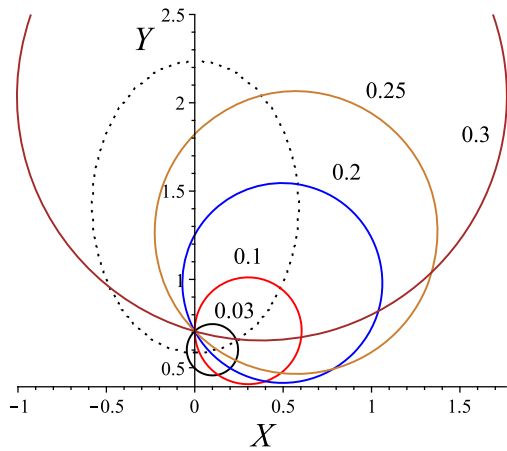
3.1 Flat spacetime

Consider the Minkowski metric expressed in cylindrical coordinates

$$ds^2 = -dt^2 + dr^2 + r^2 d\phi^2 + dz^2 , \tag{48}$$

simply derived from the standard Cartesian inertial coordinate form

$$ds^2 = -dt^2 + dx^2 + dy^2 + dz^2 , \tag{49}$$

using the coordinate transformation $x = r \cos \phi$, $y = r \sin \phi$ in the x-y plane. Identifying this real plane with the complex plane with the further transformation $\zeta = x + iy = re^{i\phi}$, the inverse transformation is

$$r = (\zeta \bar{\zeta})^{1/2} , \qquad \phi = \frac{1}{2i} \ln \left(\frac{\zeta}{\bar{\zeta}} \right) , \tag{50}$$

which leads to the following expression for the metric (48)

$$ds^2 = -dt^2 + d\zeta d\bar{\zeta} + dz^2 , \tag{51}$$

which remains invariant under the following complex linear transformation

$$\zeta' = \sigma \zeta + \beta , \qquad \sigma = e^{i\theta} , \ \theta \in \mathbb{R} \tag{52}$$

representing a combined rotation and translation of the x-y plane as long as σ is a unit complex number. A pure rotation (set $\beta = 0$) corresponds to a counterclockwise rotation $\phi \to \phi + \theta$ by the angle θ in this plane, or

$$(x, y) \to (x \cos \theta - y \sin \theta, x \sin \theta + y \cos \theta) . \tag{53}$$

The Minkowski spacetime timelike geodesics are straight lines representing the world lines of massive test particles of mass $m > 0$. If they are parametrized by the proper time τ, assuming that they pass through the point (x_0, y_0, z_0) of the above inertial coordinate system at $\tau = 0$, these take the form

$$t = E\tau, \qquad x - x_0 = p_x\tau, \qquad y - y_0 = p_y\tau, \qquad z - z_0 = p_z\tau, \qquad (54)$$

with the normalization condition

$$E^2 = m^2 + p^2, \qquad p^2 = p_x^2 + p_y^2 + p_z^2 \tag{55}$$

for the associated energy and momentum per unit mass. The momentum 1-form (per unit mass) along the geodesic is

$$P = -E dt + p_a dx^a. \tag{56}$$

Their representation in cylindrical coordinates centered now at (x_0, y_0, z_0) becomes

$$t = E\tau, \qquad r = p_\perp \tau, \qquad \phi = \arctan\left(\frac{p_y}{p_x}\right), \qquad z - z_0 = p_z\tau, \tag{57}$$

where

$$p_\perp = \sqrt{p_x^2 + p_y^2}. \tag{58}$$

We limit our considerations to orbits on a constant z hyperplane by assuming $p_z = 0$. The fact that ϕ is constant along the motion allows for the parametrization

$$p_x = p_\perp \cos\phi \qquad p_y = p_\perp \sin\phi, \qquad \phi = \text{const.} \tag{59}$$

or equivalently

$$\zeta = re^{i\phi} = (p_x + ip_y)\tau, \qquad |\zeta| = p_\perp \tau. \tag{60}$$

The effect of the transformation (52) is then

$$\zeta' = e^{i\theta}(p_x + ip_y)\tau + \beta = (p_x' + ip_y')\tau + \beta, \tag{61}$$

rotating the momentum directly analogous to Eq. (53)

$$(p_x', p_y') = (\cos\theta \, p_x - \sin\theta \, p_y, \sin\theta \, p_x + \cos\theta \, p_y). \tag{62}$$

Applying the Möebius transformation (47) to the straight lines (54), they are mapped to circles (or again straight lines). In fact, the line (54) for $p_z = 0$ and $z = z_0$, i.e.,

$$p_y x - p_x y - p_y x_0 + p_x y_0 = 0, \tag{63}$$

can be written as

$$\bar{\alpha}\zeta - \alpha\bar{\zeta} - 2i(-p_y x_0 + p_x y_0) = 0 , \tag{64}$$

where

$$\alpha = p_x + i p_y . \tag{65}$$

The effect of the map $T_{(0,1,1,0)}(\zeta) = 1/\zeta$ to Eq. (64) is

$$\frac{\bar{\alpha}}{\zeta} - \frac{\alpha}{\bar{\zeta}} - 2i(-p_y x_0 + p_x y_0) = 0 , \tag{66}$$

i.e.,

$$\bar{\alpha}\bar{\zeta} - \alpha\zeta - 2i(-p_y x_0 + p_x y_0)\zeta\bar{\zeta} = 0 , \tag{67}$$

which is the equation of a circle if $-p_y x_0 + p_x y_0 \neq 0$ (or a straight line again if $-p_y x_0 + p_x y_0 = 0$)

$$|\zeta - \zeta_c|^2 = \rho^2 , \tag{68}$$

where

$$\zeta_c = -i\frac{\alpha}{2(-p_y x_0 + p_x y_0)} , \qquad \rho = |\zeta_c| . \tag{69}$$

These circles, however, are no longer geodesics.

3.2 Möbius form of the Gödel metric: taking circles to circles

Analogous to the Minkowski spacetime introduction of a complex variable $\zeta = r e^{i\phi}$ in the "polar plane", define

$$\zeta = \tanh r \, e^{i\phi} , \tag{70}$$

with inverse map

$$r = \operatorname{arctanh}((\zeta\bar{\zeta})^{1/2}) = \ln\left(\frac{1 + (\zeta\bar{\zeta})^{1/2}}{1 - (\zeta\bar{\zeta})^{1/2}}\right)^{1/2} , \qquad \phi = \frac{1}{2i}\ln\left(\frac{\zeta}{\bar{\zeta}}\right) . \tag{71}$$

which transforms the cylindrical-like standard metric (1) into the form

$$ds^2 = \frac{2}{\omega^2}\left\{ -\left[dt - \frac{i(\zeta d\bar{\zeta} - \bar{\zeta}d\zeta)}{\sqrt{2}(1 - \zeta\bar{\zeta})}\right]^2 + \frac{d\bar{\zeta}d\zeta}{(1 - \zeta\bar{\zeta})^2} + dz^2 \right\} . \tag{72}$$

This derivation needs the differentials

$$d\zeta = e^{i\phi}[(1 - \tanh^2 r)dr + i \tanh r \, d\phi],$$
$$\bar{\zeta}d\zeta = \tanh r(1 - \tanh^2 r)dr + i\bar{\zeta}\zeta d\phi, \tag{73}$$

leading to

$$d\phi = \frac{\bar{\zeta}d\zeta - \zeta d\bar{\zeta}}{2i\bar{\zeta}\zeta}, \qquad dr = \frac{\bar{\zeta}d\zeta + \zeta d\bar{\zeta}}{2(\zeta\bar{\zeta})^{1/2}(1 - \zeta\bar{\zeta})}. \tag{74}$$

Similarly, the complex map

$$\omega T = \sqrt{2}t - i \ln(\zeta), \qquad X = \frac{1}{\zeta} + \bar{\zeta}, \qquad Y = i\left(\frac{1}{\zeta} - \bar{\zeta}\right), \qquad \omega Z = \sqrt{2}z, \tag{75}$$

brings Kundt's form of the metric (28) into the same Möbius form (72).

Successively, the second change of variables

$$t = t' + \frac{i\sqrt{2}}{2}\ln\left(\frac{\bar{\alpha}_0 - \bar{\zeta}'\beta_0}{\alpha_0 - \zeta'\bar{\beta}_0}\right), \qquad \zeta = \frac{\beta_0 - \zeta'\bar{\alpha}_0}{-\alpha_0 + \zeta'\bar{\beta}_0},$$
$$\bar{\zeta} = \frac{\bar{\beta}_0 - \bar{\zeta}'\alpha_0}{-\bar{\alpha}_0 + \bar{\zeta}'\beta_0}, \qquad z = z, \tag{76}$$

leaves the metric formally invariant, i.e.,

$$ds^2 = \frac{2}{\omega^2}\left\{-\left[dt' - \frac{i(\zeta'd\bar{\zeta}' - \bar{\zeta}'d\zeta')}{\sqrt{2}(1 - \zeta'\bar{\zeta}')}\right]^2 + \frac{d\bar{\zeta}'d\zeta'}{(1 - \zeta'\bar{\zeta}')^2} + dz^2\right\}, \tag{77}$$

as already discussed in Ref. [12]. Planar timelike geodesics satisfy the equations

$$\dot{t} = \frac{\omega^2\mathcal{E}}{2} + \frac{i(\zeta\dot{\bar{\zeta}} - \bar{\zeta}\dot{\zeta})}{\sqrt{2}(1 - \zeta\bar{\zeta})},$$
$$\dot{\zeta}\dot{\bar{\zeta}} = \frac{\omega^2}{2}\left(\frac{\omega^2\mathcal{E}^2}{2} - 1\right)(1 - \zeta\bar{\zeta})^2,$$
$$2\zeta\dot{\bar{\zeta}}^2 = (1 - \zeta\bar{\zeta})(i\sqrt{2}\omega^2\mathcal{E}\dot{\zeta} - \ddot{\zeta}), \tag{78}$$

where a dot denotes derivative with respect to the proper time τ and $\mathcal{E}$ is the conserved energy per unit mass of the particle. Circular geodesics $\zeta = Re^{-i\Omega\tau}$ then exist, with

$$R = \frac{\sqrt{\omega^2\mathcal{E}^2 + 2} - \sqrt{2}\omega\mathcal{E}}{\sqrt{\omega^2\mathcal{E}^2 - 2}}, \qquad \Omega = \omega\sqrt{\omega^2\mathcal{E}^2 + 2}, \tag{79}$$

and

$$t = \left(\frac{\Omega}{\sqrt{2}} - \frac{1}{2}\omega^2 \mathcal{E} \right) \tau \,. \tag{80}$$

4 Concluding remarks

Despite the existence of pathological regions where time travel is allowed, the Gödel spacetime has attracted a great deal of attention in the literature. The appearance of closed timelike curves (CTCs) can be avoided, e.g., by introducing extra matter sources like massless scalar fields [13] or even by including corrections from string theory [14]. This has led to several generalizations of the Gödel spacetime within a class of solutions of general relativity and extended theories of gravity known as Gödel-type metrics, the main feature of which is the possibility of eliminating the presence of CTCs (see, e.g., Ref. [15]). Furthermore, although the Gödel spacetime is not in any way a realistic model of aspects of our actual Universe given its rotation and lack of expansion, it has been widely used to investigate the effects of spacetime global vorticity. Its very simple mathematical structure has also allowed analytical studies of geodesic motion [3,6,9,16] as well as stability analysis against a variety of perturbations [17–19].

Planar timelike geodesics of the Gödel spacetime, i.e., those confined to a hyperplane orthogonal to the symmetry axis, have been studied extensively in the literature using different coordinate systems. Among them is the frequently used choice of cylindrical-like coordinates naturally adapted to the spacetime symmetries. Particle trajectories viewed in these coordinates appear to be elliptic-like and their features can be conveniently studied through an eccentricity-semi-latus rectum (p, e) parametrization familiar (though slightly generalized) from Newtonian mechanics. Such deformed ellipses are of three different kinds, depending on the chosen range of orbital parameters, and correspond to closed curves which either go around the origin (type I) or pass through it (type II) or are centered away from it (type III). Type I orbits may reduce to circles for vanishing eccentricity. However, since the Gödel spacetime is homogeneous, it must look the same from every point. Therefore, the shape of geodesics about the origin is topologically equivalent to that about any other point chosen as the origin of a new cylindrical-like coordinate system.

The best way to show this feature is to pass to Kundt's form of the Gödel metric, for which planar geodesics all become circular. We have established here a one-to-one correspondence between the conserved quantities as well as the parameter space of elliptic-like versus circular geodesics: different kinds of elliptic-like orbits turn out to correspond to circles with different loci of the center as well as different radii depending on the allowed range of orbital parameters p and e. We have also shown how to connect the two equivalent descriptions of particle motion by introducing a pair of complex coordinates in the 2-planes orthogonal to the symmetry axis, which brings the metric into a form which is invariant under Möbius transformations preserving the symmetries of the orbit, i.e., taking circles to circles.

We expect that the present analysis will be useful in investigating features of geodesic motion in more general Gödel-type spacetimes which share the same isome-

tries as the Gödel spacetime. Furthermore, the description of geodesic motion as well as the link between elliptic-like and circular geodesics discussed here can be used to study gravitational perturbations induced by a particle moving along a geodesic path within the gravitational self-force scheme developed for black holes. In fact, there are only a few perturbative studies of the Gödel solution, which account for the effects of the global rotation on the evolution of scalar density perturbations [17,18], and more general (vector and tensor) perturbations [19]. We will leave this question for future work.

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