General Relativity and Gravitation (2020) 52:38
https://doi.org/10.1007/s10714-020-02686-y

RESEARCH ARTICLE

Prospect of Chandrasekhar's limit against modified dispersion relation

Arun Mathew[1] · Malay K. Nandy[1]

Received: 3 October 2019 / Accepted: 7 April 2020 / Published online: 21 April 2020

Abstract
Newtonian gravity predicts the existence of white dwarfs with masses far exceeding the Chandrasekhar limit when the equation of state of the degenerate electron gas incorporates the effect of quantum spacetime fluctuations (via a modified dispersion relation) even when the strength of the fluctuations is taken to be very small. In this paper, we show that this Newtonian "super-stability" does not hold true when the gravity is treated in the general relativistic framework. Employing dynamical instability analysis, we find that the Chandrasekhar limit can be reassured even for a range of high strengths of quantum spacetime fluctuations with the onset density for gravitational collapse practically remaining unaffected.

Keywords Quantum spacetime fluctuations · Non-commutative geometry · Modified dispersion relation · White dwarfs · Dynamical instability

1 Introduction

The Hawking-Wheeler foam [1,2] of quantum space-time fluctuations can be accounted for by a non-commutative spacetime geometry. This modifies the dispersion relation between energy and momentum of any particle. Since a modification in the dispersion relation leads to a modified equation of state (EoS), it is apparent that the stellar structure of white dwarfs governed by its electron degenerate gas would undergo a measurable change if the stiffness of the EoS changes sufficiently. In fact it has been shown [3,4] that white dwarfs with modified EoS can support masses much higher than the Chandrasekhar limit that become "super-stable" when the hydrostatic equilibrium is governed by Newtonian gravity.

✉ Arun Mathew
a.mathew@iitg.ac.in

Malay K. Nandy
mknandy@iitg.ac.in

1 Department of Physics, Indian Institute of Technology Guwahati, Guwahati 781039, India

However the possibility of the existence of excessively massive white dwarfs is unlikely as it is inconsistent with an extensive amount of astronomical observations [5–11]. Although there have been speculations about super-Chandrasekhar white dwarfs, it has been argued [12,13] that they are in fact double degenerate merges of two sub-Chandrasekhar white dwarfs. It is thus extremely important to investigate whether the above-mentioned "super-stability" prevails in general relativity (GR). It may be recalled that general relativity has a profound effect in determining the stability of massive stars against gravitational collapse although it may have an insignificant effect on their stellar structure.

We thus anticipate that a dynamical instability would set in at a critical value of the central density as generally predicted for relativistic stars [14–19]. Following this standard method, we calculate the eigenfrequencies of the normal mode of radial oscillations with respect to various central densities of white dwarfs with the electron degenerate gas treated in the framework of modified dispersion relation. The existence of a vanishing eigenfrequency corresponds to the maximum central density in stable configuration. We identify the onset density of dynamical instability with respect to a parameter characterizing the strength of quantum spacetime fluctuations.

The quantum fluctuations of spacetime would indicate that the space and time coordinates may be treated as operators obeying nontrivial commutation relations. The notion of quantum spacetime was introduced by Snyder [20,21] by regarding the space-time coordinates as Hermitian operators preserving Lorentz invariance and incorporating a characteristic length scale. Heisenberg-like commutation relations between position and momentum do not hold in this framework and they resemble the generalized uncertainty relations explored from different points of view [22–31]. Moreover related viewpoints have emerged from various other considerations about the quantum nature of spacetime [32–35].

In modern times, the mathematical formulation for the quantum nature of spacetime is build upon a non-commutative geometry which is usually taken to be the κ-deformed Poincaré algebra, for example, $[x^i, t] = ix^i/\kappa$ and $[x^i, x^j] = 0$. The noncommutative nature of spacetime leads to a deformation in the dispersion relation of a particle moving in spacetime. Amelino-Camelia and Majid [36] constructed a corresponding Fourier space and obtained the generators of Lorentz transformation leading to a deformed dispersion relation of the form $2\hbar^2\kappa^2\{\cosh(E/\hbar\kappa) - 1\} - p^2c^2e^{-E/\hbar\kappa} = m^2c^4$. Amelino-Camelia [37] further proposed a modified dispersion relation of the form $E^2 - p^2c^2 + f(E, p, m, \ell_P) = m^2c^4$ on the basis of a modified special relativity where both the speed of light c and the Planck length ℓ_P are constrained to be invariant. Magueijo and Smolin [38] constructed a similar modification of special relativity with the condition that the standard theory of relativity is recovered at low energies or large length scales. This required the construction of a modified Lorentz group that acts nonlinearly in the momentum space and it led to the modified invariant $||p||^2 = \frac{\eta^{\mu\nu}p_\mu p_\nu}{(1-\ell_P p_0)^2} = m^2$, where m is the rest mass of the particle. This means that the ideal dispersion relation $E^2 = \mathbf{p}^2+m^2$ is deformed in their modified relativity. The modified dispersion relations can be put into a general form $E^2 - p^2c^2 f(E, p, m, \ell_P) = m^2c^4$ that has been utilised in different contexts. For example, Alexander and Magueijo [39] addressed the horizon and flatness problems in a Friedmann cosmology with a modified dispersion relation of the form $E^2 - p^2c^2 f(E) = 0$, with three different choices: $f(E) = (1+\lambda E)^2$, $f(E) = 4\lambda^2 E^2/(1-e^{-2\lambda E})^2$ and $f(E) = (1+\lambda E)^{2\gamma}$. Bertolami and Zarro [3], considering the form $f(E) = (1 + \lambda E)^2$, employed the deformed dispersion relation $E^2 = p^2c^2(1 + \lambda E)^2 + m^2c^4$ for massive particles. Taking the approximate form $E = \lambda p^2c^2 + \sqrt{p^2c^2 + m^2c^4}$ for low values of λ, they found that the $\mathcal{O}(\lambda)$ correction to the electron degenerate pressure enhances the stability of white dwarfs. The present authors [4] employed the unapproximated dispersion relation and found that white dwarfs of arbitrarily high masses remain stable if the gravitational pull is dictated by Newtonian gravity.

In this paper, we use the modified dispersion relation $E_{\mathbf{p}}^2 = \mathbf{p}^2c^2(1+\lambda E_{\mathbf{p}})^2+m^2c^4$ that leads to a modified equation of state of the degenerate electron gas. We analyze the stability of white dwarfs by calculating the eigenfrequencies of normal modes of small radial oscillations in the first order of perturbation. We find that general relativity is capable of causing a gravitational collapse even for high strengths of quantum spacetime fluctuations characterized by the parameter $\alpha = \lambda m_e c^2$. However, when this strength is very high ($\alpha > 3.7 \times 10^{-3}$), the quantum space-time fluctuations become strong enough to hold up against a gravitational collapse. A legitimate bound on such parameters occurring in equivalent formulations of quantum spacetime fluctuations indicate an upper bound for α much lower than 3.7×10^{-3} for which we find that the Chandrasekhar critical limit can be realized in the quantum gravitational regime.

The remainder of the paper is organized as follows. In Sect. 2, we present a brief review of dynamical instability in general relativity. In Sect. 3, the dynamical instability is explored for white dwarfs with the equation of state governed by a modified dispersion relation to account for the effect of quantum spacetime fluctuations on the instability. Finally, we conclude the paper in Sect. 4.

2 Dynamical instability in GR: a brief review

In this section we review the dynamical instability of spherically symmetric fluid masses with respect to radial oscillations with the gravity treated general relativistically. The stability of the equilibrium fluid configuration is examined by considering radial perturbation in a spherically symmetric manner. Thus the metric interior to the fluid mass undergoing radial perturbation, given by

$$ds^2 = e^{\nu+\delta\nu}dt^2 - e^{\mu+\delta\mu}dr^2 - r^2(d\theta^2 + \sin^2\theta\, d\phi^2), \tag{1}$$

where $\nu(r)$ and $\mu(r)$ refer to the equilibrium configuration, and $\delta\nu(r,t)$ and $\delta\mu(r,t)$ correspond to a small radial Lagrangian displacement $\zeta(r,t)$ about the equilibrium configuration. Various quantities in the Einstein field equations, such as the pressure and energy density, become dependent on the coordinate time t in addition to the the radial coordinate r. Supposing all perturbations are sinusoidal in the coordinate time, one can express the Lagrangian displacement as $\zeta(r,t) = r^{-2}e^{\nu/2}\psi(r)e^{i\omega t}$ in the first order of perturbation. The equation governing the radial pulsation was obtained by Chandrasekhar [14] which can be expressed in the Sturm–Liouville form [17]

$$\frac{d}{dr}\left(U\frac{d\psi}{dr}\right)+\left(V+\frac{\omega^2}{c^2}W\right)\psi=0, \tag{2}$$

where

$$U(r)= e^{(\mu+3\nu)/2}\frac{\gamma P}{r^2}, \tag{3}$$

$$V(r)= -4\frac{e^{(\mu+3\nu)/2}}{r^3}\frac{dP}{dr}-\frac{8\pi G}{c^4}\frac{e^{3(\mu+\nu)/2}}{r^2}P(P+\varepsilon) \tag{4}$$

$$+\frac{e^{(\mu+3\nu)/2}}{r^2}\frac{1}{P+\varepsilon}\left(\frac{dP}{dr}\right)^2, \tag{5}$$

$$W(r)= \frac{e^{(3\mu+\nu)/2}}{r^2}(P+\varepsilon), \tag{6}$$

with $P(r)$ and $\varepsilon(r)$ are the pressure and energy density of the fluid mass at equilibrium given by the Tolman–Oppenheimer–Volkoff equation [40,41]

$$\frac{dP}{dr}=-\frac{G}{c^2r}(\varepsilon+P)\frac{(m+4\pi Pr^3/c^2)}{(r-2Gm/c^2)} \tag{7}$$

with

$$\frac{dm}{dr}=\frac{4\pi}{c^2}\varepsilon r^2. \tag{8}$$

The above hydrostatic equilibrium render the pressure and energy density to be functions of the radial coordinate r. The equation of state connecting the pressure and energy density lead to the the adiabatic index γ expressed as

$$\gamma=\frac{\varepsilon+P}{P}\left(\frac{dP}{d\varepsilon}\right)_s, \tag{9}$$

making the adiabatic index γ a function of the radial coordinate r.

The admissible radial motion requires the fluid element at the center to have a vanishing displacement ζ (and $d\zeta/dr$ be finite) leading to the condition

$$\psi=0 \quad \text{at} \quad r=0. \tag{10}$$

Furthermore, the solution is also required to satisfy

$$\delta P=-e^{\nu/2}\frac{\gamma P}{r^2}\frac{d\psi}{dr}=0 \quad \text{at} \quad r=R, \tag{11}$$

at the surface, where δP represents the Lagrangian change in pressure and R is the radius of the fluid sphere.

Multiplying Eq. (2) from left by ψ and integrating from 0 to R and applying the above boundary conditions, one can obtain the integral

$$J[\psi] = \int_0^R \left\{ U\psi'^2 - V\psi^2 - \frac{\omega^2}{c^2} W\psi^2 \right\} dr. \tag{12}$$

where $\psi' = d\psi/dr$. Minimizing $J[\psi]$ with respect to ψ yields the Sturm–Liouville equation (2), thus providing a variational basis for determining the lowest characteristic eigenfrequency given by

$$\frac{\omega_0^2}{c^2} = \min_{\psi(r)} \frac{\int_0^R \left\{ U\psi'^2 - V\psi^2 \right\} dr}{\int_0^R W\psi^2 dr}, \tag{13}$$

corresponding to the normal mode of oscillation. A sufficient condition for the dynamical instability to set in is that the value of ω_0^2 be negative. Thus a suitable trial function $\psi(r)$ satisfying the boundary conditions and making the right hand side vanish will determine the onset of instability. It can be seen that a power series solution for the above Sturm–Liouville equation (2) about $r = 0$ satisfying the boundary conditions (10) and (11) has a leading order term $\propto r^3$. To determine the eigenfrequency of the fundamental mode we assume a trial function $\psi(r) \propto r^3$ which corresponds to a homologous vibration. It can be shown [42] that this trial function is a sufficiently close approximation to the true eigenfunction of the fundamental mode.

3 Dynamical instability in white dwarfs with modified dispersion relation

It has been reported that when quantum spacetime fluctuations are included in the equation of state of the electron gas, white dwarfs can exist in excessively large masses beyond the Chandrasekhar limit when the gravity is treated in the Newtonian framework although the strength of quantum spacetime fluctuations is taken to be very small [43,44]. However, it is well-known that, in the conventional problem of stability of white dwarfs, a dynamical instability sets in [15,16] when the gravity is treated in the framework of general relativity. It is thus natural to speculate that a similar general relativistic instability may set in when the small effect of quantum spacetime fluctuations is included in the equation of state of the electron gas.

3.1 Modified equation of state

The effect of quantum spacetime fluctuations can be modelled via a modified dispersion relation. We shall take modified dispersion relation [3]

$$E_{\mathbf{p}}^2 = \mathbf{p}^2 c^2 (1 + \lambda E_{\mathbf{p}})^2 + m^2 c^4 \tag{14}$$

which imposes a momentum cutoff at $p_{\max} = (\lambda c)^{-1}$ above which the energy becomes unphysical. For small values of momentum it coincides with the ideal dispersion relation $E_{\mathbf{p}}^2 = \mathbf{p}^2 c^2 + m^2 c^4$, but it deviates strongly for high values of momentum, becoming infinite at the cutoff $p_{\max}$.

Since the electron gas in white dwarfs is completely degenerate, we evaluate the pressure P, internal energy ε_{int}, and mass-density ρ_0 at absolute zero from the grand partition function. Thus we obtain [4] the modified equation of state

$$P = A\tilde{P}(\xi), \quad \rho_0 c^2 = \frac{A}{q}\xi^3, \quad \varepsilon = \rho_0 c^2 + \varepsilon_{\text{int}} = \frac{A}{q}\tilde{\varepsilon}(\xi) \tag{15}$$

where

$$\tilde{P}(\xi) = \xi^3 f(\xi) - 3g(\xi), \qquad \tilde{\varepsilon} = (1-q)\xi^3 + 3qg(\xi), \tag{16}$$

$$\xi = \frac{p_F}{m_e c}, \quad A = \frac{8\pi m_e^4 c^5}{3h^3}, \quad q = \frac{m_e}{\mu_e m_u} = 2.74297 \times 10^{-4}, \tag{17}$$

$$f(\xi) = \left[\alpha\xi^2 + \sqrt{(1-\alpha^2)\xi^2 + 1}\right]\left[1 - \alpha^2\xi^2\right]^{-1}, \tag{18}$$

and

$$\begin{aligned} g(\xi) = \frac{1}{\alpha^4}\Bigg[& 2\tanh^{-1}\alpha\xi + \tanh^{-1}\frac{\xi(1-\alpha^2)}{\alpha + \sqrt{1+(1-\alpha^2)\xi^2}} \\ & - \frac{(2-\alpha^2)}{2\sqrt{1-\alpha^2}}\sinh^{-1}\xi\sqrt{1-\alpha^2}\Bigg] \\ & - \frac{\xi}{3\alpha^3}\left[3 + \alpha^2\xi^2 + \frac{3\alpha}{2}\sqrt{1+(1-\alpha^2)\xi^2}\right] \end{aligned} \tag{19}$$

with $\alpha = \lambda m_e c^2$.

Employing the relativistic expression $\gamma = \frac{\tilde{\varepsilon}+q\tilde{P}}{\tilde{P}}\left(\frac{d\tilde{P}}{d\tilde{\varepsilon}}\right)_s$ for the adiabatic index γ, and using Eqs. (15-19), we obtain

$$\gamma = \frac{\tilde{\varepsilon}(\xi) + q\tilde{P}(\xi)}{\tilde{P}(\xi)}\left(\frac{\xi}{3}\right)\left(1 - q + q\frac{dg}{d\xi}\right)^{-1}\left(\frac{df}{d\xi}\right) \tag{20}$$

for the electron gas with the modified equation of state.

3.2 Stability analysis

The Einstein field equation for the static interior Schwarzschild metric can be solved [40,41] to obtain the metric components

$$e^{-\mu} = 1 - 2q\frac{\tilde{m}}{\eta} \tag{21}$$

and

$$e^{\nu} = \left(1 - 2q\frac{\tilde{M}}{\eta_R}\right) \times \exp\left\{-2q\int_0^{\xi} \frac{\frac{d\tilde{P}}{d\xi}}{(\tilde{\varepsilon} + q\tilde{P})}\, d\xi\right\}, \tag{22}$$

where $\tilde{m} = m/m_0$ and $\eta = r/r_0$ are dimensionless variables with

$$r_0 = \frac{qc^2}{\sqrt{4\pi AG}} \quad \text{and} \quad m_0 = \frac{q^2c^4}{\sqrt{4\pi A}}\frac{1}{G^{3/2}}. \tag{23}$$

The dimensionless quantities $\tilde{M} = M/m_0$ and $\eta_R = R/r_0$ correspond to the mass M and radius R of the fluid sphere. Using Eqs. (16-19), Eq. (22) can be simplified to

$$e^{\nu} = \left(1 - 2q\frac{\tilde{M}}{\eta_R}\right)\left(\frac{1}{1 - q + qf(\xi)}\right)^2. \tag{24}$$

The dependence of the above field quantities on the radial coordinate η correspond to the Tolman–Oppenheimer–Volkoff equation [40,41] of hydrostatic equilibrium expressed here as

$$\frac{d\tilde{P}}{d\eta} = \left(\frac{df}{d\xi}\right)\xi^3\frac{d\xi}{d\eta} = -\left(\frac{\tilde{\varepsilon} + q\tilde{P}}{\eta}\right)\left(\frac{\tilde{m} + q\tilde{P}\eta^3}{\eta - 2q\tilde{m}}\right) \tag{25}$$

with the mass equation

$$\frac{d\tilde{m}}{d\eta} = \tilde{\varepsilon}\eta^2. \tag{26}$$

The functions $U(\eta)$, $V(\eta)$ and $W(\eta)$ in the Sturm–Liouville equation (2) are readily obtained from Eqs. (3-6) employing Eqs. (21), (24) and (25). In consequence, Eq. (13) yields the eigenfrequency

$$\omega_0^2 = \left(\frac{qc^2}{r_0^2}\right)\frac{\mathcal{I} + \mathcal{J}}{\mathcal{K}}, \tag{27}$$

where

$$\mathcal{I} = \int_0^{\eta_R} e^{(\mu+3\nu)/2}\frac{\gamma\tilde{P}}{\eta^2}\psi_0'^2 d\eta, \tag{28}$$

$$\mathcal{J} = \int_0^{\eta_R} \frac{e^{(\mu+3\nu)/2}}{\eta^2}\left[\frac{4}{\eta}\frac{d\tilde{P}}{d\eta} + 2qe^{\mu}\tilde{P}(\tilde{\varepsilon} + q\tilde{P}) - \frac{q}{\tilde{\varepsilon} + q\tilde{P}}\left(\frac{d\tilde{P}}{d\eta}\right)^2\right]\psi_0^2 d\eta, \tag{29}$$

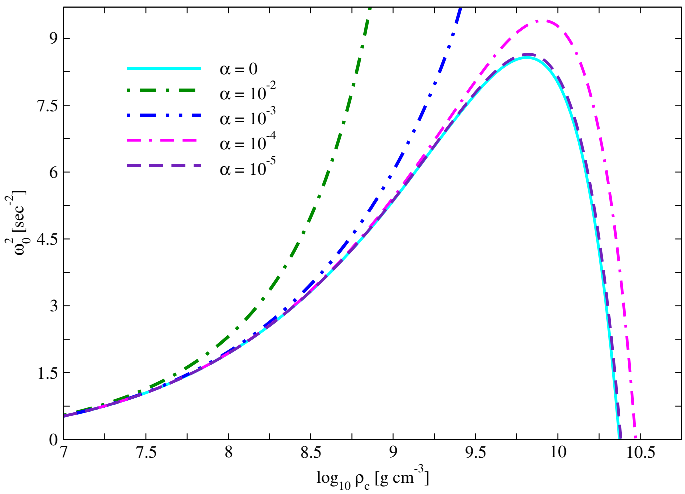

Fig. 1 Eigenfrequency for normal modes against central density of relativistic white dwarfs for various values of the parameter α that characterises the strength of quantum spacetime fluctuations. The scale of this plot cannot accommodate the cases $\alpha = 10^{-2}$ and 10^{-3}. It will be evident from Fig. 2 that the case $\alpha = 10^{-3}$ gives a zero eigenfrequency solution whereas for the case $\alpha = 10^{-2}$, the curve reaches a terminal point without giving a zero eigenfrequency solution

and

$$\mathcal{K} = \int_0^{\eta_R} e^{(3\mu+\nu)/2} \frac{\tilde{\varepsilon} + q\tilde{P}}{\eta^2} \psi_0^2 d\eta, \tag{30}$$

with ψ_0 the eigenfunction associated with the fundamental mode that minimizes the right-hand side of Eq. (13).

As stated earlier, we make the choice $\psi_0 = \eta^3$ as a trial function and evaluate the integrals $\mathcal{I}$, $\mathcal{J}$, and $\mathcal{K}$ given by Eqs. (28-30) for different values of the central Fermi momentum ξ_c. The corresponding eigenfrequencies are obtained from Eq. (27). Figure 1 displays the eigenfrequency against the central density ρ_c [related to ξ_c through $\rho = (A/qc^2)\{(1-q)\xi^3 + 3qg(\xi)\}$] for different strengths of quantum spacetime fluctuations parametrized by α, namely, $\alpha = 10^{-2}$, 10^{-3}, 10^{-4}, and 10^{-5}, including the ideal case ($\alpha = 0$).

We observe from Fig. 1 that for $\alpha = 10^{-5}$, the characteristic eigenfrequencies are close to the ideal values. As the strength is increased to $\alpha = 10^{-4}$, the eigenfrequencies depart from the ideal values but they follow a trend similar to the ideal case, and gravitational instability can set in dictated by general relativity by virtue of the existence of a vanishing eigenfrequency and a corresponding critical central density ρ_c^*. This signifies the dominance of gravitational pull determined by general relativity over the effect of quantum spacetime fluctuations on the equation of state.

However, for higher strengths of α, such as $\alpha = 10^{-3}$ and 10^{-2}, the scale of the ordinate in Fig. 1 is not adequate to analyze their behaviors. For an adequate analysis of the situation, we show a log-log plot in Fig. 2. It is clear from Fig. 2

that, for $\alpha > 3.7 \times 10^{-3}$, the curves follow a trend completely different from the ideal case because they reach terminal points at non-zero eigenfrequencies and thus zero eigenfrequency solutions do not exist. This signifies the non-existence of any gravitational instability or collapse and the white dwarfs remain "super-stable". The central densities at which the curves terminate are higher than Chandrasekhar's value of 2.3×10^{10} g cm^{-3}. From this super-stability, we conclude that quantum spacetime fluctuations are sufficiently strong for $\alpha > 3.7 \times 10^{-3}$ so that the gravitational pull determined by general relativity is incapable of bringing about any instability.

On the other hand, for $\alpha \leq 3.7 \times 10^{-3}$, the trends of the eigenfrequencies are similar to that of the ideal case and general relativistic instabilities can set in due to the existence of vanishing eigenfrequencies of the normal mode. This obviously means that, for $\alpha \leq 3.7 \times 10^{-3}$, the gravitational pull determined by general relativity is strong enough to bring about an instability or collapse. Thus the value $\alpha = 3.7 \times 10^{-3}$ marks a transition point for strength of quantum gravitational fluctuations competing against gravity pull that leads to the general relativistic instability.

Since the general relativistic instability (corresponding to the zero eigenfrequency) occurs at a critical central density ρ_c^* determined by the parameter α, it is worth studying the behavior of the corresponding critical values of the central Fermi momenta ξ_c^* with respect to the parameter α. We see from the right-hand part of Fig. 3 that as the strength of α is increased from 10^{-5} to 10^{-4}, the critical value ξ_c^* (or equivalently ρ_c^*) remains approximately constant. In fact, our calculation shows that there is negligible variation in the value of ξ_c^* in the range $0 < \alpha < 10^{-5}$ (which is also evident from Fig. 1 from the near-coincidence of the two ρ_c^* values). Thus in the range $0 < \alpha < 10^{-4}$, we expect that general relativistic instability would yield nearly the same critical masses. In this range, we find $\rho_c^* = 2.3 - 2.9 \times 10^{10}$ g cm^{-3} which is in the vicinity of Chandrasekhar's value of 2.3×10^{10} g cm^{-3} [16]. This indicates that Chandrasekhar's general relativistic critical mass of 1.42 $M_\odot$ is negligibly affected in this range of the parameter α.

We thus see that the effect of general relativity is robust enough to cause an instability against the effect of quantum spacetime fluctuations even for strengths such as $\alpha = 10^{-4}$. Experimental bounds on such parameters occurring in equivalent formulations of quantum spacetime fluctuations is available in the literature. For example, Das and Vagenas [45] discussed various experimental bounds on the generalised uncertainty parameter β_0. Taking the bound $\beta_0 \sim 10^{34}$, which is a legitimate upper bound coming from the electroweak theory, it translates to $\alpha \sim 10^{-6}$ in our case. Thus $\alpha = 10^{-4}$ is in fact a large value for the strength of quantum spacetime fluctuations. If the strength of α is increased beyond 10^{-4}, we see from Fig. 3 that the critical central Fermi momentum ξ_c^* also increases. However, after reaching a maximum it eventually falls off due to increased role of gravitating pressure in comparison to the gravitating mass. Finally, the curve approaches the line $\xi = \alpha^{-1} = \xi_{\max}$ until it makes an intersection at $\alpha = 3.7 \times 10^{-3}$, the maximum strength of α for the existence of a vanishing eigenfrequency and hence a gravitational collapse. This intersection is shown in the inset of Fig. 3 by an open circle where $\xi_c^* = 2.7 \times 10^2$. It is obvious that the curve cannot cross the line $\xi = \alpha^{-1}$, having reached the maximum value $\xi_c^* = \alpha^{-1}$.

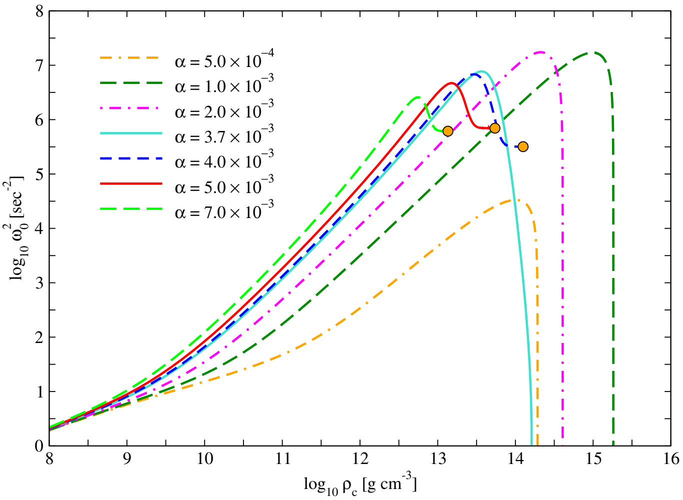

Fig. 2 Eigenfrequency for normal modes versus central density of relativistic white dwarfs for various strengths of α higher than those in Fig. 1. For $\alpha \leq 3.7 \times 10^{-3}$, the curves indicate the existence of zero eigenfrequency solutions leading to gravitational collapse. For $\alpha > 3.7 \times 10^{-3}$, the curves reach terminal points and zero eigenfrequency solutions do not exist excluding the possibility of gravitational collapse

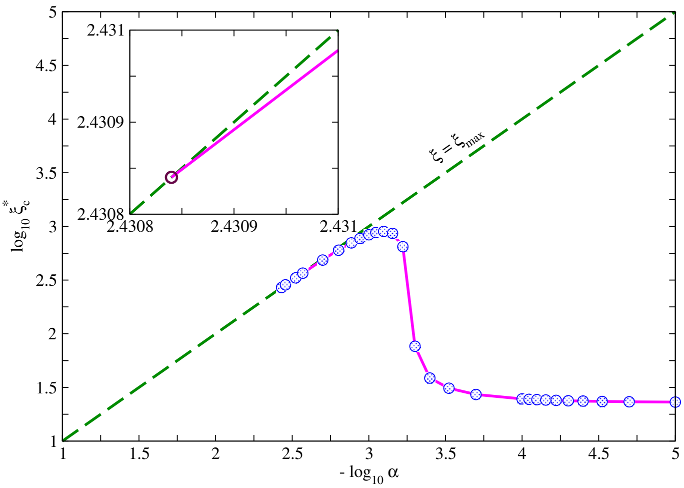

Fig. 3 Critical value of the central Fermi momentum for the onset of gravitational instability versus the parameter α. The dashed straight line represents $\xi = \xi_{\max} = \alpha^{-1}$. The inset shows the intersection of the two curves at $\alpha = 3.7 \times 10^{-3}$

4 Conclusion

The modified dispersion relation is one of the scenarios in which the effect of quantum gravity is phenomenologically taken into account. We expect the effect of quantum gravity to have some signature when the density of the matter is very high. In this context it is worthwhile to recall that the standard Chandrasekhar limit of 1.44 $M_\odot$ is approached when the density of the electron degenerate gas in white dwarfs approaches infinity in the framework of Newtonian gravity. The problem thus calls for taking account of the effect of quantum gravity at very high densities of the electron degenerate gas. When this notion is followed and the electron gas is treated via a modified dispersion relation making the equation of state more stiff than the ideal one, it is found that white dwarfs become "super-stable" and higher masses beyond the Chandrasekhar limit are possible [4] when the gravity is treated in the Newtonian framework. However, as noted in the introduction and in more detail below, white dwarfs are most likely to exist below the Chandrasekar limit. It is thus extremely important to resolve this paradoxical situation.

It is known from Chandrasekhar's study [14,16] that a dynamical instability sets in when the gravity is treated general relativistically. Consequently it is natural to ask the question whether general relativity would be capable of reassuring the Chandrasekar limit when the effect of quantum space-time fluctuations is included in the equation of state. Motivated by this query, we analyzed the problem of stability of white dwarfs governed by general relativity and incorporating quantum space-time fluctuations in the electron degenerate gas via a modified equation of state.

To analyze the stability, we followed the standard methodology of perturbations generating radial pulsations in spherically symmetric white dwarfs and calculated the corresponding eigenfrequencies of the radial oscillations. The corresponding eigenvalue equation is in the Sturm–Liouville form whose eigenfrequencies can be related to a variational principle. With an appropriate trial function for the Lagrangian displacement, we calculated the eigenfrequencies for various strengths of the quantum space-time fluctuations parametrized by α.

We find that, for large values of α such that $\alpha > 3.7 \times 10^{-3}$, white dwarfs remain "super-stable" as they do not exhibit any zero eigenfrequency in the normal mode. Such white dwarfs can support maximum masses determined by the maximum values of the central density ρ_c where the curves terminate as shown in Fig. 2. These values of ρ_c are higher than the critical density obtained by Chandrasekhar suggesting the possibility of white dwarfs of masses higher than Chandrasekar's general relativistic value of 1.42 $M_\odot$. However, these cases are unlikely because we do not expect that the strength α of spacetime fluctuations to be as large as or higher than 0.0037, as discussed earlier on the basis of bounds on similar parameters. It may however be noted that there have been observations of some type Ia SNe events (SN 2003fg, SN 2006gz, SN 2007if, SN 2009dc) [12,13,46–48] which produced a high amount of ^{56}**Ni** ranging from 1.2 $M_\odot$ to 1.7 $M_\odot$. This suggested that the progenitors of these SNe events had masses in the super-Chandrasekhar range from 2.2 $M_\odot$ to 2.8 $M_\odot$. However the unusually low and slowly declining Silicon velocity in SN 2006gz indicated that it is a double degenerate (DD) merger of two sub-Chandrasekar white dwarfs as argued by Hicken et al. [12]. Moreover, Silverman et al. [13] argued, basis on simulations, that SN 2009dc was most likely due to the merger of two white dwarfs. Chen and Li [49] considered a single-degenerate white dwarf with differential rotation and accreting matter slowly from a normal companion. They predicted that white dwarf masses in excess of 1.7 $M_\odot$ are very much unlikely. On the other hand, in the presence of a strong magnetic field of the order of $\sim 10^{15}$ Gauss, Das and Mukhopadhyay [50] predicted that a white dwarf can support a mass of 2.3–2.6 $M_\odot$ due to the existence of Landau levels. However, pointing to various disagreements among the existing SNe Ia models, van Kerkwijk [51] argued that SNe Ia events generally take place due to the merger of two carbon-oxygen white dwarfs. Thus, for normal white dwarfs (without rotation or magnetic field), a super-Chandrasekhar mass seems to be unlikely.

For smaller strengths of quantum spacetime fluctuations, such that $\alpha \leq 3.7 \times 10^{-3}$, we find that general relativity is capable of bringing about an instability at finite central densities ρ_c^* because of the existence of a vanishing eigenfrequency in the normal mode as shown in Fig. 2. This signifies that gravity governed by general relativity is strong enough to cause a gravitational collapse against the effect of quantum spacetime fluctuations on the equation of state. It is important to note that general relativity is robust enough to bring about a gravitational collapse even for high strengths of quantum spacetime fluctuations such as 10^{-3} or 10^{-4}.

We have also seen from Figs. 1 and 3 that in the range $0 < \alpha < 10^{-4}$, general relativistic instability yields comparable critical central densities, namely, $\rho_c^* = 2.3 - 2.9 \times 10^{10}$ g cm^{-3}. This range is in the vicinity of Chandrasekhar's value of 2.3×10^{10} g cm^{-3} [16]. This indicates that the stellar structure of relativistic white dwarfs is hardly affected in this range of the parameter α where the critical mass is about 1.42 $M_\odot$.

We may recall that when the gravity is treated in the Newtonian framework, masses far exceeding the Chandrasekhar limit are found to be "super-stable" even for very low values of α. It is thus obvious that while Newtonian gravity is unable to dominate over the stiffness of the equation of state generated by quantum spacetime fluctuations, general relativity does possess the capacity to overcome the stiffness of the equation of state that can lead to a gravitational collapse.

Thus even for a high value of α, such as 10^{-4} or 10^{-5}, the onset density ρ_c^* for gravitational collapse is practically unaffected (with respect to the ideal case) when the gravity is treated general relativistically in spite of the effect of quantum spacetime fluctuations opposing gravitational collapse. This is of direct relevance to the core collapse supernovae where the degenerate core of the progenitors are found to have a mass of about 1.4 $M_\odot$. However it may be recalled that there may be no clear distinction between the gravitational core collapse and fast β-capture that may occur nearly simultaneously at the onset, effectively making no difference in the impending supernova explosion [18]. Thus our study suggests that when the inevitable effect of quantum spacetime fluctuations is included in the process, the situation practically remains indistinguishable from the ideal core collapse scenario.

Acknowledgements Arun Mathew is indebted to the Ministry of Human Resource Development, Government of India, for financial assistance through a doctoral fellowship.

References

1. Wheeler, J.A.: Gravitational Collapse and the Creation and Annihilation of Matter. In: DeWitte, B.S., DeWitte, C.M. (eds.) Relativity Groups and Topology. Gordan and Breach, Routledge (1964)
2. Hawking, S.: Nuclear Phys. B **144**(2), 349 (1978). https://doi.org/10.1016/0550-3213(78)90375-9

3. Bertolami, O., Zarro, C.A.D.: Phys. Rev. D **81**, 025005 (2010). https://doi.org/10.1103/PhysRevD.81.025005
4. Mathew, A., Nandy, M.K.: Res. Astron. Astrophys. **18**(12), 151 (2018). https://doi.org/10.1088/1674-4527/18/12/151
5. Shipman, H.L.: Astrophys. J. **177**, 723 (1972). https://doi.org/10.1086/151752
6. Shipman, H.L.: Astrophys. J. **213**, 138 (1977). https://doi.org/10.1086/155138
7. Shipman, H.L.: Astrophys. J. **228**, 240 (1979). https://doi.org/10.1086/156841
8. Vennes, S., Thejll, P.A., Galvan, R.G., Dupuis, J.: Astrophys. J. **480**(2), 714 (1997). https://doi.org/10.1086/303981
9. Marsh, M.C., Barstow, M.A., Buckley, D.A., Burleigh, M.R., Holberg, J.B., Koester, D., O'Donoghue, D., Penny, A.J., Sansom, A.E.: Mon. Not. R. Astron. Soc. **286**(2), 369 (1997). https://doi.org/10.1093/mnras/286.2.369
10. Vennes, S.: Astrophys. J. **525**(2), 995 (1999). https://doi.org/10.1086/307949
11. Kilic, M., Prieto, C.A., Brown, W.R., Koester, D.: Astrophys. J. **660**(2), 1451 (2007). https://doi.org/10.1086/514327
12. Hicken, M., Garnavich, P.M., Prieto, J.L., Blondin, S., DePoy, D.L., Kirshner, R.P., Parrent, J.: Astrophys. J. Lett. **669**(1), L17 (2007)
13. Silverman, J.M., Ganeshalingam, M., Li, W., Filippenko, A.V., Miller, A.A., Poznanski, D.: Monthly Not. R. Astron. Soc. **410**(1), 585 (2011). https://doi.org/10.1111/j.1365-2966.2010.17474.x
14. Chandrasekhar, S.: Phys. Rev. Lett. **12**, 114 (1964). https://doi.org/10.1103/PhysRevLett.12.114
15. Chandrasekhar, S.: Astrophys. J. **140**, 417 (1964). https://doi.org/10.1086/147938
16. Chandrasekhar, S., Tooper, R.F.: Astrophys. J **139**, 1396 (1964). https://doi.org/10.1086/147883
17. Bardeen, J.M., Thorne, K.S., Meltzer, D.W.: Astrophys. J. **145**, 505 (1966). https://doi.org/10.1086/148791
18. Wheeler, J.C., Hansen, C.J., Cox, J.P.: ApJL **2**, 253 (1968)
19. Thorne, K.S.: Comments Astrophys. Space Phys. **1**, 12 (1969)
20. Snyder, H.S.: Phys. Rev. **71**, 38 (1947). https://doi.org/10.1103/PhysRev.71.38
21. Snyder, H.S.: Phys. Rev. **72**, 68 (1947). https://doi.org/10.1103/PhysRev.72.68
22. Mead, C.A.: Phys. Rev. **135**, B849 (1964). https://doi.org/10.1103/PhysRev.135.B849
23. Veneziano, G.: Euro. Phys. Lett. **2**(3), 199 (1986)
24. Padmanabhan, T.: Class. Quantum Grav. **4**(4), L107 (1987). https://doi.org/10.1088/0264-9381/4/4/007
25. Amati, D., Ciafaloni, M., Veneziano, G.: Phys. Lett. B **216**(1), 41 (1989). https://doi.org/10.1016/0370-2693(89)91366-X
26. Konishi, K., Paffuti, G., Provero, P.: Phys. Lett. B **234**(3), 276 (1990). https://doi.org/10.1016/0370-2693(90)91927-4
27. Greensite, J.: Phys. Lett. B **255**(3), 375 (1991). https://doi.org/10.1016/0370-2693(91)90781-K
28. Maggiore, M.: Phys. Lett. B **304**(1), 65 (1993). https://doi.org/10.1016/0370-2693(93)91401-8
29. Kempf, A., Mangano, G., Mann, R.B.: Phys. Rev. D **52**, 1108 (1995). https://doi.org/10.1103/PhysRevD.52.1108
30. Adler, R.J., Santiago, D.I.: Mod. Phys. Lett. A **14**(20), 1371 (1999). https://doi.org/10.1142/S0217732399001462
31. Pedram, P.: Phys. Lett. B **714**(2), 317 (2012). https://doi.org/10.1016/j.physletb.2012.07.005
32. Rovelli, C.: Class. Quantum Grav. **8**(2), 297 (1991). http://stacks.iop.org/0264-9381/8/i=2/a=011
33. Rovelli, C.: Class. Quantum Grav. **8**(2), 317 (1991). http://stacks.iop.org/0264-9381/8/i=2/a=012
34. Amelino-Camelia, G.: Mod. Phys. Lett. A **09**(37), 3415 (1994). https://doi.org/10.1142/S0217732394003245
35. Amelino-Camelia, G.: Mod. Phys. Lett. A **11**(17), 1411 (1996). https://doi.org/10.1142/S0217732396001417
36. Amelino-Camelia, G., Majid, S.: Int. J. Mod. Phys. A **15**(27), 4301 (2000). https://doi.org/10.1142/S0217751X00002779
37. Amelino-Camelia, G.: Int. J. Mod. Phys. D **11**(01), 35 (2002). https://doi.org/10.1142/S0218271802001330
38. Magueijo, J.a., Smolin, L.: Phys. Rev. Lett. **88**, 190403 (2002). https://doi.org/10.1103/PhysRevLett.88.190403
39. Alexander, S., Magueijo, J.: arXiv:hep-th/0104093v2 (2004)
40. Tolman, R.C.: Phys. Rev. **55**, 364 (1939). https://doi.org/10.1103/PhysRev.55.364

41. Oppenheimer, J.R., Volkoff, G.M.: Phys. Rev. **55**, 374 (1939). https://doi.org/10.1103/PhysRev.55.374
42. Meltzer, D.W., Thorne, K.S.: Astrophys. J **145**, 514 (1966). https://doi.org/10.1086/148792
43. Rashidi, R.: Ann. Phys. (N. Y.) **374**, 434 (2016). https://doi.org/10.1016/j.aop.2016.09.005
44. Mathew, A., Nandy, M.K.: Ann. Phys. (N. Y.) **393**, 184 (2018). https://doi.org/10.1016/j.aop.2018.04.008
45. Das, S., Vagenas, E.C.: Phys. Rev. Lett. **101**, 221301 (2008). https://doi.org/10.1103/PhysRevLett.101.221301
46. Howell, A., Sullivan, M., Nugent, P.E., Ellis, R.S., Conley, A.J., Le Borgne, D., Carlberg, R.G., Guy, J., Balam, D., Basa, S., Fouchez, D., Hook, I.M., Hsiao, E.Y., Neill, J.D., Pain, R., Perrett, K.M., Pritchet, C.J.: Nature **443**(7109), 308 (2006). https://doi.org/10.1038/nature05103
47. Yamanaka, M., Kawabata, K.S., Kinugasa, K., Tanaka, M., Imada, A., Maeda, K., Nomoto, K., Arai, A., Chiyonobu, S., Fukazawa, Y., Hashimoto, O., Honda, S., Ikejiri, Y., Itoh, R., Kamata, Y., Kawai, N., Komatsu, T., Konishi, K., Kuroda, D., Miyamoto, H., Miyazaki, S., Nagae, O., Nakaya, H., Ohsugi, T., Omodaka, T., Sakai, N., Sasada, M., Suzuki, M., Taguchi, H., Takahashi, H., Tanaka, H., Uemura, M., Yamashita, T., Yanagisawa, K., Yoshida, M.: Astrophys. J. Lett. **707**(2), L118 (2009)
48. Scalzo, R.A., Aldering, G., Antilogus, P., Aragon, C., Bailey, S., Baltay, C., Bongard, S., Buton, C., Childress, M., Chotard, N., Copin, Y., Fakhouri, H.K., Gal-Yam, A., Gangler, E., Hoyer, S., Kasliwal, M., Loken, S., Nugent, P., Pain, R., Pécontal, E., Pereira, R., Perlmutter, S., Rabinowitz, D., Rau, A., Rigaudier, G., Runge, K., Smadja, G., Tao, C., Thomas, R.C., Weaver, B., Wu, C.: Astrophys. J. **713**(2), 1073 (2010)
49. Chen, W.C., Li, X.D.: Astrophys. J. **702**(1), 686 (2009)
50. Das, U., Mukhopadhyay, B.: Phys. Rev. D **86**, 042001 (2012). https://doi.org/10.1103/PhysRevD.86.042001
51. van Kerkwijk, M.H.: Philos. Trans. R. Soc. A **371**(1), 20120236 (2013)