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GOLDEN OLDIE

Republication of: Dirac electron in the gravitational field I

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Abstract
This paper has been selected by the Editors of General Relativity and Gravitation for re-publication in the Golden Oldies series. It is one of the papers cited in N. D. Birrell and P. C. W. Davies' book *Quantum Fields in Curved Space* (Cambridge University Press, Cambridge, 1982) on the generalization of the Dirac Equation to a curved spacetime and is notable as containing the first occurrence of the well-known formula $g^{\mu\nu}\nabla_\mu\nabla_\nu + m^2 + R/4$ for the 'square' of the Dirac operator. The original is in German and in a poorly accessible journal: it is presented here in English. The paper is accompanied by an editorial note by B. S. Kay which clarifies the historical development of the topic, and by a brief biography written by M. A. H. MacCallum.

Keywords Schrödinger · Dirac equation · Spin connection · Curved spacetime · Square of Dirac operator

Erwin Schrödinger (Deceased January 4, 1961).

An editorial note to this paper and a biography can be found in this issue preceding this Golden Oldie and online via https://doi.org/10.1007/s10714-019-2625-y.

Original paper: E. Schrödinger "Diracsches Elektron im Schwerefeld I", Sitzungsberichte der Preußischen Akademie der Wissenschaften. Physikalisch-mathematische Klasse, 105–128 (1932). Translated by Claus Kiefer E-mail: kiefer@thp.uni-koeln.de.

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Dirac electron in the gravitational field I

1 Introduction

The unification of the Dirac theory of the electron with the general theory of relativity has already been attempted repeatedly, for example by Wigner [1], Tetrode [2], Fock [3], Weyl [4], Zaycoff [5], Podolsky [6]. Most authors introduce in every world point axes of coordinates and numerically specialized Dirac matrices with respect to them. With this procedure it is a little bit difficult to recognize whether Einstein's idea of teleparallelism, to which reference is partly made, really enters or whether one is independent of it. It is, moreover, necessary to recast the concepts of Riemannian geometry into the less familiar and definitely more complicated form of the "frame components". In order to avoid all of this, it seemed to me desirable, like Tetrode (see also [7]), to rely only on the generalized commutation relations [see equation (2) below]. It turns out that one is led in this way very simply and straightforwardly to the important operators Γ_k, whose trace is the four-potential, and which Fock introduces as the "components of the parallel transport of a spinor"; and one is just as straightforwardly led to the important system of equations [see (8) below], which Fock obtains by a detour through the frame components. By restriction of the admissable reference frames (see Sec. 4 below), which is completely analogous to the usual one in the special theory of relativity, one then introduces the Hermiticities, which are desirable for the interpretation, as well as an *assignment* between tensor *operators* and local c-tensors, which is also completely analogous to the one put up by v. Neumann [8] in the special theory [see equation (57) below]. A principal advantage seems to me that the whole apparatus can be constructed almost entirely by pure operational calculus, without making reference to the ψ-function. I hope that the exact *justification* of this apparatus is not too shocking by its length, for which the author's broad way of writing is partly responsible. Having prepared the ground, with it the *application* and the thinking may turn out to be simple. — I want to declare once and for all my deep indebtedness to the work of my predecessors, but for methodological reasons I ask for the permission to derive everything in a new way, as if it had not yet been found by anyone else.

2 Construction of the metric from fields of matrices

We call the world variables

$$x_0 = ict, \quad x_1 = x, \quad x_2 = y, \quad x_3 = z.$$

The first is always pure imaginary, the other three are real. Dirac's basic idea was to interpret the Euclidean wave operator

$$\frac{\partial^2}{\partial x_0^2} + \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2}$$

as the square of a linear operator

$$\left(\overset{0}{\gamma}_0 \frac{\partial}{\partial x_0} + \overset{0}{\gamma}_1 \frac{\partial}{\partial x_1} + \overset{0}{\gamma}_2 \frac{\partial}{\partial x_2} + \overset{0}{\gamma}_3 \frac{\partial}{\partial x_3}\right)^2$$

where the $\overset{0}{\gamma}_k$ are 4×4 matrices[1], which have to fulfil the condition

$$\overset{0}{\gamma}_i \overset{0}{\gamma}_k + \overset{0}{\gamma}_k \overset{0}{\gamma}_i = 2\delta_{ik}, \tag{1}$$

i.e. the left side is equal to the null matrix or equal to twice the identity matrix, depending on whether $i \neq k$ or $i = k$. By the condition (1), one knows that the $\overset{0}{\gamma}_k$ are exactly determined up to a similarity transformation

$$\overset{0}{\gamma}{}'_k = S^{-1} \overset{0}{\gamma}_k S$$

with an arbitrary non-singular 4×4 transformation matrix S. *This* freedom in the choice of the $\overset{0}{\gamma}_k$ is evident, and one knows, as said, that with it the freedom is exhausted.

Since instead of the wave operator one could have also started from the squared line element:

$$dx_0^2 + dx_1^2 + dx_2^2 + dx_3^2,$$

it seems reasonable to interpret the conditions (1) in such a way that the matrices $\overset{0}{\gamma}_k$, in addition to the other tasks which they are attributed later in the description of the electron, also have the task of describing the world metric, which so far has been assumed Euclidean. If this is not assumed, but instead

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu,$$

one will have to replace (1) by

$$\gamma_i \gamma_k + \gamma_k \gamma_i = 2g_{ik} \tag{2}$$

[2]. The γ_k are functions of space and time, i.e. they are 4×4 matrices, whose elements are functions of the x_i.

In every point P, the equations (2) certainly have solutions for the γ_k if one imagines the g_{ik} *somehow* as given (of course in such a way that they correspond to a non-singular metric). The freedom which still exists for the γ_k, given the g_{ik}, is exactly the same as above for the $\overset{0}{\gamma}_k$, namely: transformation with an arbitrary non-singular matrix S. One recognizes the correctness of these statements by considering one by one the following:

[1] Incidentally, the number of rows is irrelevant for all what follows.

1. The equations (2) can always be solved by 4 suitably chosen linear combinations of an arbitrary Dirac basis system $\overset{0}{\gamma}_k$ — the ansatz leads to conditions for the coefficients that can be met.

2. On the other hand: If one has a system of γ_k for which one knows that it fulfils (2), one can specify 4 linear combinations of these γ_k which fulfil (1) and which thus form a Dirac basis. If one thus has, for example, two systems of solutions γ_k and γ_k' for (2), they can be transformed into a respective Dirac basis by *the same* linear transformation. But these two Dirac bases are certainly related by an S-transformation. The same transformation then also transforms γ_k and γ_k' into each other.

3. It is obvious that *any* S-transformation leaves (2) untouched. — With this, all statements are demonstrated.

A very essential difference between the $\overset{0}{\gamma}_k$ and the γ_k is the following. It is known that there are Hermitian $\overset{0}{\gamma}_k$-systems, but that there are in general no Hermitian γ_k-systems; there are also none where some γ_k are Hermitian and others are skew-Hermitian. This is connected with the well known reality conditions which one has to demand for the g_{ik}: pure imaginary if one and only one index 0 appears, and real otherwise. (One has to recall that the symmetrized product, the anticommutator, of two Hermitian matrices is always Hermitian.) We will later address the Hermiticity questions in more detail and have mentioned them here only to show that for the moment there is not the slightest reason to restrict the transformation S, which is arbitrary in each point, for example to a *unitary* one. Because the γ_k are anyway not Hermitian, one has for now no reason to be concerned about the "conservation of Hermiticity".

We shall now derive from (2) an important system of differential equations for the γ_k. We imagine the g_{ik} as given and the equations (2) as solved in every point P; solved in such a way that these solutions can be joined together to form four continuous, differentiable fields of matrices, which will obviously be possible. We now proceed from a point P to a neighbouring point P' and form in this sense the complete differential of equation (2),

$$\delta\gamma_i \cdot \gamma_k + \gamma_i \cdot \delta\gamma_k + \delta\gamma_k \cdot \gamma_i + \gamma_k \cdot \delta\gamma_i = 2\frac{\partial g_{ik}}{\partial x_l}\delta x^l. \tag{3}$$

If we now observe the theorem of Ricci, according to which the covariant derivative of the fundamental tensor g_{ik} vanishes identically:

$$g_{ik;l} \equiv \frac{\partial g_{ik}}{\partial x_l} - \Gamma^{\mu}_{kl} g_{i\mu} - \Gamma^{\mu}_{il} g_{\mu k} \equiv 0, \tag{4}$$

the right-hand side of (3) will be equal to the following:

$$2\left(\Gamma^{\mu}_{kl} g_{i\mu} + \Gamma^{\mu}_{il} g_{\mu k}\right)\delta x^l.$$

This value can be attributed to the left-hand side of (3) by setting

$$\delta\gamma_i = \Gamma^{\mu}_{il}\gamma_{\mu}\delta x^l \tag{5}$$

and taking into account (2). I.e. the matrices

$$\gamma_i + \delta\gamma_i = \gamma_i + \Gamma^{\mu}_{il}\gamma_\mu \delta x^l \tag{6}$$

obey equation (2) in point P' if the γ_i obey it in point P.

The ansatz (5) would in general be contradictory if one wanted to apply it to *all* points P' in the neighbourhood of P. For one can convince oneself by a simple calculation that the expression (5) is a total differential if and only if the *curvature* in P vanishes. But according to what has been said above, the γ_i-values in P' — we want to call them $\gamma_i + \delta'\gamma_i$ — can and will differ from our somehow guessed solution ansatz (5) resp. (6) by a similarity transformation, namely, of course, by an infinitely small one if continuity has to be preserved. That is, there must exist an infinitely small matrix ϵ in such a way that

$$\begin{aligned} \gamma_i + \delta'\gamma_i &= (1-\epsilon)(\gamma_i + \delta\gamma_i)(1+\epsilon) = \gamma_i + \delta\gamma_i + \gamma_i\epsilon - \epsilon\gamma_i \\ \text{or} \qquad \delta'\gamma_i &= \Gamma^{\mu}_{il}\gamma_\mu\delta x^l + \gamma_i\epsilon - \epsilon\gamma_i. \end{aligned} \tag{7}$$

In principle, ϵ could assume another, entirely arbitrary, value at any neighbouring point. But if γ_i should have a correct differential quotient with respect to x_l when progressing in direction x_l (i.e. for $\delta x_l \neq 0$, all others $= 0$), ϵ must be proportional to δx_l. The same for any l. If one should then be able to calculate the change of γ_i when progressing in arbitrary direction correctly from its differential quotient, ϵ must be the sum of these four terms. In this way, one arrives at the ansatz

$$\epsilon = -\Gamma_l \delta x^l,$$

in which the Γ_i are four matrices independent of space and time (the minus sign is, of course, completely arbitrary). Inserted into (7), the important system of differential equations announced above follows[2]:

$$\frac{\partial \gamma_i}{\partial x_l} = \Gamma^{\mu}_{il}\gamma_\mu + \Gamma_l\gamma_i - \gamma_i\Gamma_l. \tag{8}$$

We shall express this later as follows: the covariant derivative of the fundamental vectors γ_k vanishes, in full analogy to the theorem of Ricci, equation (4). On the other hand, the source freedom of the four-current is closely related to this system of equations. I want to put particular emphasis on the fact that we have derived it here purely from the conditions on the *metric*, without reference to the ψ-function, for which we had to exploit the freedom in transforming the Dirac matrices. *This* led — and it did it *unavoidably* — to the appearance of the new operators Γ_l, for which we shall see that they are inextricably linked with the four-potential (but they do not form a vector!).

We now investigate, in addition, the necessary conditions for the consistency of the equations (8), namely, that the mixed second differential quotients, when calculated

[2] *In its content*, this agrees with equation (24) in [3]. But the meaning of the symbols here and there is a little bit different. If one wishes to make the two coincide, one should first read our section 5 on Hermiticity!

in two different ways, must coincide. By expressing the first derivatives that appear after differentiation again by (8), one finds:

$$\Phi_{kl}\gamma_i - \gamma_i\Phi_{kl} = R_{kli}^{\cdot\cdot\cdot\mu}\gamma_\mu. \tag{9}$$

Here, $R_{kli}^{\cdot\cdot\cdot\mu}$ is the mixed Riemann curvature tensor in the usual notation (see e.g. Levi-Civita, Der absolute Differentialkalkül, p. 91; Berlin, Springer 1928). Φ_{kl} is an abbreviation that we introduce for the following six matrices, which are antisymmetric in the indices k, l:

$$\Phi_{kl} = \frac{\partial \Gamma_l}{\partial x_k} - \frac{\partial \Gamma_k}{\partial x_l} + \Gamma_l\Gamma_k - \Gamma_k\Gamma_l, \tag{10}$$

which, as it will turn out, stand in close relation to the electromagnetic field. For given γ_i-field, by (8) every Γ_i, and by (9) every Φ_{kl} is fixed up to an added term which is commutable with all γ_i and which is thus a multiple of the identity matrix. From (9), the Φ_{kl} are easily calculable. Besides the γ_i one introduces the contravariant

$$\gamma^i = g^{ik}\gamma_k. \tag{11}$$

Furthermore, one declares

$$s^{\mu\nu} = \frac{1}{2}\left(\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu\right). \tag{12}$$

(The $s^{\mu\nu}$ correspond for $\mu, \nu = 1, 2, 3$ in a certain sense to the spin, for $\mu = 0$, $\nu = 1, 2, 3$ in a certain sense to the velocity. See later.) We remark, in addition, that according to (2) and (11)

$$\gamma_i\gamma^k + \gamma^k\gamma_i = 2\delta_i^k. \tag{13}$$

One now easily finds

$$\gamma_i s^{\mu\nu} - s^{\mu\nu}\gamma_i = 2\left(\delta_i^\mu\gamma^\nu - \delta_i^\nu\gamma^\mu\right). \tag{14}$$

Upon commutation with a γ, the $s^{\mu\nu}$ thus produce again γ. This is exactly what one needs in order to solve (9) with respect to Φ_{kl}. The right-hand side of (9) can, in fact, also be written as $R_{kl,i\mu}\gamma^\mu$, where $R_{kl,i\mu}$ is the *symmetric* Riemann tensor. One then confirms with the C.R. (14) that

$$\Phi_{kl} = -\frac{1}{4}R_{kl,\mu\nu}s^{\mu\nu} + f_{kl}\cdot 1 \tag{15}$$

is the general solution of (9)[3]. f_{kl} is the undetermined multiplier of unity. The f_{kl} will (multiplied by i) assume the role of the electromagnetic field. One recognizes that

[3] In content essentially in accordance with the index-rich frame equations (46), (48) in [3].

although the appearance of these quantities is suggested by the construction of the metric from matrices, it is exactly the f_{kl} which are, for the time being, *not* determined by the γ-field, but left entirely free.

As commutators, the $s^{\mu\nu}$ have trace zero. Therefore,

$$\text{trace } \Phi_{kl} = f_{kl} \cdot \text{trace } 1 = 4 f_{kl}.$$

According to (10) we have, on the other hand,

$$\text{trace } \Phi_{kl} = \frac{\partial}{\partial x_k} (\text{trace } \Gamma_l) - \frac{\partial}{\partial x_l} (\text{trace } \Gamma_k)\,,$$

for differentiation and performance of trace are commutable and the commutator does not contribute to the trace. If one sets, for example,

$$\frac{1}{4} \text{trace } \Gamma_l = \varphi_l,$$

one gets

$$f_{kl} = \frac{\partial \varphi_l}{\partial x_k} - \frac{\partial \varphi_k}{\partial x_l}. \tag{16}$$

The traces of the Γ_l are the four-potential (apart from a factor i).

3 Transformation theory, first part

According to the fundamental principle of general relativity, a *re-labelling* of all points

$$x_k' = x_k'(x_0, x_1, x_2, x_3); \quad k = 0, 1, 2, 3 \tag{17}$$

should not change the form of description. In doing so, the function x_0' should only assume pure imaginary, x_1', x_2', x_3' only assume real values, and the functional determinant should remain positive. We call this a *point substitution*. The g_{ik} then transform as a covariant tensor of second rank.

As long as we impose for the γ_i no other requirement than obeying the equations (2), the question how they have to be transformed under a point substitution cannot at all be answered unambiguously. For after as well as before the point substitution, a similarity transformation with a transformation matrix S that varies from point to point remains entirely free. But we *can* demand that the γ_i are to be transformed as a covariant vector under a pure point substitution, under which (2) at any rate is preserved. One then has to demand the same for the Γ_l, so that (8) is preserved. For the commutator $\Gamma_l \gamma_i - \gamma_i \Gamma_l$ then transforms as a covariant tensor, which is also the case for the rest of the equation, namely,

$$\frac{\partial \gamma_i}{\partial x_l} - \Gamma_{il}^{\mu} \gamma_\mu \tag{18}$$

if γ_i is substituted as a vector; for (18) is, after all, formally the covariant derivative of γ_i. The similarity transformations

$$\gamma_k' = S^{-1}\gamma_k S \tag{19}$$

would then have to be considered as something by themselves, where, as one can easily convince oneself, the Γ_l must be transformed as follows in order to preserve (8):

$$\Gamma_l' = S^{-1}\Gamma_l S - S^{-1}\frac{\partial S}{\partial x_l}, \tag{20}$$

which is *different* from the γ_k. But one would find that after these determinations the following combination of terms, for which we want to introduce the symbol ∇_k,

$$\nabla_k = \frac{\partial}{\partial x_k} - \Gamma_k, \tag{21}$$

first — naturally – transforms as a covariant vector under a pure point substitution (because this holds, of course, for the $\frac{\partial}{\partial x_k}$ alone and was fixed for the Γ_k) and that *secondly* because of (20) the ∇_k transform under an S-transformation exactly in the way that the γ_k transform according to (19),

$$\nabla_k' = S^{-1}\nabla_k S. \tag{22}$$

For the meaning of ∇_k' is, in fact,[4]

$$\nabla_k' = \frac{\partial}{\partial x_k} - \Gamma_k' = \frac{\partial}{\partial x_k} - S^{-1}\Gamma_k S + S^{-1}\frac{\partial S}{\partial x_k}, \tag{23}$$

and one has

$$\frac{\partial}{\partial x_k} = \frac{\partial}{\partial x_k}\cdot S^{-1}S = S^{-1}\frac{\partial}{\partial x_k}S + \frac{\partial S^{-1}}{\partial x_k}S = S^{-1}\frac{\partial}{\partial x_k}S - S^{-1}\frac{\partial S}{\partial x_k}; \tag{24}$$

the last equality holds because of the identity:

$$S^{-1}S \equiv 1; \quad \frac{\partial S^{-1}}{\partial x_k}S + S^{-1}\frac{\partial S}{\partial x_k} \equiv 0.$$

By inserting (24) into (23), one confirms (22).

The Φ_{kl} introduced through (10) would *firstly* behave — naturally — as a covariant tensor under point substitution, *secondly* under an S-transformation analogously to (19),

$$\Phi_{kl}' = S^{-1}\Phi_{kl}S, \tag{25}$$

[4] Translator's note: I have corrected typos in the following two equations.

the latter because of (22) and because they are, according to the definitions (10) and (21), the commutators of the ∇_k:

$$\Phi_{kl} = \nabla_l \nabla_k - \nabla_k \nabla_l. \tag{26}$$

One should add that due to (25) the *traces* of the Φ_{kl}, the f_{kl}, do *not* change under the similarity transformation, whereas those of the Γ_l, which we called φ_l, in fact do, because we do not have for the Γ_l a transformation law analogous to (19) resp. (25), but instead (20).

We have formulated all of this in the "would"-form, because the requirements we have imposed contain the above mentioned arbitrariness: since a point substitution anyhow forces, in general, a *modification* of the γ_i (the old γ_i will, of course, in general no longer obey the equations (2)!), we have for the new selection again a whole *manifold* of γ_i-fields at our disposal, whose members follow from an arbitrary one of them by arbitrary, coordinate-dependent S-transformations. And for the moment none of these members is *intrinsically* distinguished in any way, not even the one selected above.

It is now recommendable, at least for certain purposes, to restrict this freedom of choice to a large extent by using it to satisfy certain Hermiticity aspirations, which are not unavoidable, but which are natural, as is also usually done in the special relativistic Dirac theory. In order to see what can be achieved in this respect, we have to contemplate more closely the eigenvalues of γ_k and their bi-products.

4 Eigenvalues and Hermitization

Since according to (2)

$$\gamma_k \gamma_k = g_{kk}, \quad \text{(no summation!)}$$

γ_k has the eigenvalues $\pm\sqrt{g_{kk}}$, and each of these occur twice, because it has trace zero. The latter is seen if one sets analogously to (12)

$$s_{\mu\nu} = \frac{1}{2}\left(\gamma_\mu \gamma_\nu - \gamma_\nu \gamma_\mu\right).$$

Then one has in analogy to (14)

$$\gamma^i s_{\mu\nu} - s_{\mu\nu}\gamma^i = 2\left(\delta^i_\mu \gamma_\nu - \delta^i_\nu \gamma_\mu\right). \tag{27}$$

Every γ can thus be represented in many ways as a commutator, and a commutator always has trace zero.

Although the γ have only real eigenvalues and thus each single one of them can be made Hermitian by an S-transformation, this can, for example, in general *not* be done *simultaneously*, because according to (2) their symmetric product is equal to $2g_{01} \cdot 1$ and thus is (since g_{01} is pure imaginary) skew-Hermitian.

Let us consider further the products $\gamma_i \gamma^k$, first for $i \neq k$. Their square is [cf. (13)]:

$$\left(\gamma_i \gamma^k\right)^2 = \gamma_i \gamma^k \cdot \gamma_i \gamma^k = -\gamma_i \gamma_i \gamma^k \gamma^k = -g_{ii} g^{kk}. \quad \text{(no summation!)}$$

Thus the eigenvalues are $\pm i\sqrt{g_{ii} g^{kk}}$, and each of them occurs twice, since after all

$$\gamma_i \gamma^k = \frac{1}{2}\left(\gamma_i \gamma^k - \gamma^k \gamma_i\right)$$

as a commutator must have trace zero. — The eigenvalues of $\gamma^k \gamma_i$ are equal and opposite. — On the other hand, one has for $i = k$:

$$\left(\gamma_k \gamma^k\right)^2 = \gamma_k \gamma^k \gamma_k \gamma^k = \gamma_k \left(2 - \gamma_k \gamma^k\right) \gamma^k = 2\gamma_k \gamma^k - g_{kk} g^{kk} \quad \text{(no s.)}$$

$$\left(\gamma_k \gamma^k - 1\right)^2 = 1 - g_{kk} g^{kk}. \quad \text{(no s.)}$$

$\gamma_k \gamma^k - 1$ thus has the eigenvalues $\pm\sqrt{1 - g_{kk} g^{kk}}$, and since it can be written as a commutator:

$$\gamma_k \gamma^k - 1 = \frac{1}{2}\left(\gamma_k \gamma^k - \gamma^k \gamma_k\right), \quad \text{(no s.)}$$

each of them occurs twice. $\gamma_k \gamma^k$ thus has the eigenvalues

$$1 \pm \sqrt{1 - g_{kk} g^{kk}},$$

and each of them occurs twice. For $k = 0$, these values are real, since $g_{00} g^{00} \leq 1$.

Among the 4 matrices

$$\gamma_0 \gamma^0, \quad \gamma_0 \gamma^1, \quad \gamma_0 \gamma^2, \quad \gamma_0 \gamma^3, \tag{28}$$

the first one thus has only real, the three others pure imaginary eigenvalues. They thus just have (apart from a factor i) the reality conditions of a physically reasonable four-vector[5]. It thus seems reasonable to find out whether these four matrices can be

[5] In the Euclidean case, they in fact coincide with the Dirac current vector (apart from a factor i). The difficulty which has prevented us hermitizing the γ_k or the γ^k themselves, namely, that their symmetrized products do not show the needed reality conditions, also does not exist anymore for the matrices (28). For $i \neq k$, one has

$$\gamma_0 \gamma^i \gamma_0 \gamma^k + \gamma_0 \gamma^k \gamma_0 \gamma^i = \gamma_0 (2\delta_0^i - \gamma_0 \gamma^i) \gamma^k + \gamma_0 (2\delta_0^k - \gamma_0 \gamma^k) \gamma^i = 2(\delta_0^i \gamma_0 \gamma^k + \delta_0^k \gamma_0 \gamma^i) - 2 g_{00} g^{ik}.$$

This is indeed *real* if none of the indices i, k is equal to zero, but if $i = 0, k \neq 0$ one has:

$$2\gamma_0 \gamma^k - 2 g_{00} g^{0k}.$$

This has indeed pure imaginary eigenvalues, because we know this for $\gamma_0 \gamma^k$ and because g^{0k} is pure imaginary.

Hermitized (resp. skew-Hermitized) simultaneously. It turns out that this is possible and that in addition a number of other matrices can be simultaneously Hermitized. This goes as follows.

If the metric g_{ik} is *real and positive definite*, the equations (2) can be satisfied by *Hermitian* γ_k, in the same way as the equations (1) can be satisfied by Hermitian $\overset{0}{\gamma}_k$. This I may assume to be known without proof, the only thing involved being the projection of a $\overset{0}{\gamma}_k$-system, which is assumed Hermitian, from rectangular to skew coordinate axes, in which only real coefficients appear as direction cosines. And since the g_{ik} are real in this case, the contravariant γ^k also turn out to be Hermitian; that is, one can also satisfy the contravariant analogues to (2),

$$\gamma^i\gamma^k + \gamma^k\gamma^i = 2g^{ik}, \tag{29}$$

by *Hermitian* γ^k *if* the tensor g^{ik} is real *and* positive definite. This, however, is not the case for our tensor g^{ik}: one can make it the case if one mutilates g^{ik} and simply neglects the "mixed" space-time components g^{0k} for the time being, i.e. sets them to zero. Let

$$\alpha^0, \alpha^1, \alpha^2, \alpha^3 \tag{30}$$

be a Hermitian quadruple of matrices which satifies the equations (29) with the mutilated metric. That is, one demands

$$\alpha^i\alpha^k + \alpha^k\alpha^i = 2g^{ik} \tag{31}$$

if neither or *both* indices i, k are equal to zero, and one demands for $k \neq 0$,

$$\alpha^0\alpha^k + \alpha^k\alpha^0 = 0. \tag{32}$$

Let us now set

$$\gamma^k = \frac{i}{g^{00}}\alpha^0\alpha^k \quad \text{for } k \neq 0 \tag{33}$$

and

$$\gamma^0 = \frac{\alpha^0}{\sqrt{g_{00}g^{00}}} - \frac{1}{g_{00}}\left(g_{01}\gamma^1 + g_{02}\gamma^2 + g_{03}\gamma^3\right). \tag{34}$$

One can convince oneself by calculation that *these* γ^k *obey the unmutilated equations* (29).

Since according to (32) α^0 anticommutes with α^k ($k \neq 0$), $\alpha^0\alpha^k$ is skew for $k \neq 0$ and thus $\gamma^1, \gamma^2, \gamma^3$ are *Hermitian* according to (33). Furthermore, one calculates from (34)

$$\gamma_0 = g_{0k}\gamma^k = \alpha^0\sqrt{\frac{g_{00}}{g^{00}}} = \text{Hermitian.} \tag{35}$$

By our construction of the *contravariant* $\gamma^1, \gamma^2, \gamma^3$ we thus have rendered *Hermitian* at the same time the *covariant* γ_0. — We note also the following Hermiticities: the contravariant purely spatial

$$s^{kl} = \frac{1}{2}\left(\gamma^k\gamma^l - \gamma^l\gamma^k\right) \quad \text{for } k, l = 1, 2, 3 \tag{36}$$

are, as commutators of Hermitian matrices, skew-symmetric. Furthermore, for $k \neq 0$ the $\gamma_0\gamma^k$ and also the $\gamma^k\gamma_0$ are skew, because, already according to (13), γ_0 anticommutes with γ^k ($k \neq 0$). One then finds from (34) and (35) that $\gamma_0\gamma^0$ and $\gamma^0\gamma_0$ are Hermitian. Furthermore, from this it follows very simply by lowering the index that for $k \neq 0$ also $\gamma_0\gamma_k$ and $\gamma_k\gamma_0$, and thus also

$$s_{0k} = \frac{1}{2}\left(\gamma_0\gamma_k - \gamma_k\gamma_0\right)$$

turn out to be skew. But let us emphasize strongly that *nothing* can be said about the *covariant* s_{kl} for $k, l \neq 0$ and also not about the *contravariant* s^{0k}! The same for $\gamma^0, \gamma_1, \gamma_2, \gamma_3$. We summarize once again all statements. According to our construction,

$$\begin{aligned} &\gamma_0, \gamma^1, \gamma^2, \gamma^3, \gamma_0\gamma^0, \gamma^0\gamma_0 \quad \text{are Hermitian;} \\ &\gamma_0\gamma_k, \gamma_k\gamma_0, \gamma_0\gamma^k, \gamma^k\gamma_0, s_{0k}, s^{kl} \quad \text{are skew} \quad (k, l \neq 0). \end{aligned} \tag{37}$$

We now finally want to liberate ourselves from the reference to a particular matrix construction, which has only served as an existence proof. One can easily understand the following: already the requirement that *four* suitably chosen matrices from the ones presented in (37) have the properties stated there — for example, the requirement that $\gamma_0, \gamma^1, \gamma^2, \gamma^3$ be Hermitian — *suffices* to fix the γ-field for given g_{ik} uniquely up to a *unitary* transformation. For there is no *more* freedom at all, given g_{ik}, for the γ-field than the following: transformation with an *arbitrary* matrix. If this transformation is supposed to leave the matrices $\gamma_0, \gamma^1, \gamma^2, \gamma^3$ Hermitian, from which *every* matrix, that is, also every *Hermitian* matrix can be constructed by addition and multiplication[6], the transformation must leave *every Hermitian* matrix Hermitian, i.e., it must be *unitary*. Q.E.D.

From now on we want to admit only such γ-fields — one could also say, only such reference frames —, for which the matrices $\gamma_0, \gamma^1, \gamma^2, \gamma^3$ turn out to be Hermitian. All

[6] First, it is known for Dirac's $\overset{0}{\gamma}_k$ that any matrix can be rationally constructed from them. Then one can conclude the same for the γ_k alone or for the γ^k alone. That we have in the above quadruple γ_0 instead of γ^0 does no harm, because in fact

$$\gamma_0 = g_{00}\gamma^0 + g_{01}\gamma^1 + g_{02}\gamma^2 + g_{03}\gamma^3,$$

from which γ^0 is calculable, since we have certainly $g_{00} \neq 0$.

statements made in (37) then hold automatically. The "admissable" reference system is determined by the metric up to a *unitary* transformation.

It is very comfortable to have reduced the permitted S-transformations by this new requirement to unitary ones, for these are very good-natured and harmless. In general, we do not need to think of them and can proceed as if the γ-field was uniquely determined by the metric. But now, of course, the task arises of determining, when starting from an admissable γ-field and performing a point substitution (17), the transformation law of the γ *more specifically*, namely determining it in such a way that one is led again to an admissable γ-field. The preliminary rule given at the beginning of section 3: to substitute the γ_k as a covariant vector — does not at all obey this condition and does not, of course, correspond to how one proceeds in special relativity, where one does *not at all* substitute the $\overset{0}{\gamma}_k$. In the spirit of section 3 one could say: with every point substitution one must *connect* a fully determined (strictly speaking, determined up to a unitary factor!) S-transformation, which itself, of course, will *not be unitary*, and it is this transformation that has to be determined. One can thus rightfully speak of a *complemented* point substitution. In the next section, we shall perform this task for *infinitely small* point substitutions.

5 Transformation theory, second part

We start from an admissable γ-field and proceed to primed variables by the infinitely small point substitution

$$x'_k = x_k + \delta x_k \quad \text{or} \quad x_k = x'_k - \delta x_k, \tag{38}$$

which we complement in the sense described above by an infinitely small S-transformation with

$$S = 1 + \Theta; \quad S^{-1} = 1 - \Theta. \tag{39}$$

We shall, as usual, not explicitly denote the change of the variables in the *argument*. The equations between primed and unprimed operators thus do not refer to the *same*, but to *corresponding* values of the argument, i.e. to *the same* point. — We now introduce the abbreviation

$$\frac{\partial \delta x_k}{\partial x_l} = a_l^k. \tag{40}$$

These quantities are pure imaginary if one and only one index is equal to zero, and real otherwise. One then has

$$\begin{aligned} \gamma'_i &= \gamma_i - a_i^l \gamma_l + \gamma_i \Theta - \Theta \gamma_i \\ \gamma'^k &= \gamma^k + a_l^k \gamma^l + \gamma^k \Theta - \Theta \gamma^k. \end{aligned} \tag{41}$$

If one takes the first equation for $i = 0$ and multiplies it from the left into the second equation, one gets (always valid only in quantities of first order):

$$\gamma_0'\gamma'^k = \gamma_0\gamma^k - a_0^l\gamma_l\gamma^k + a_l^k\gamma_0\gamma^l + \gamma_0\gamma^k\Theta - \Theta\gamma_0\gamma^k. \tag{42}$$

We use our freedom of choice for Θ to eliminate resp. to replace the second term on the right-hand side of this equation, which prevents conclusions about Hermiticity. This can be achieved by

$$\Theta = -\frac{1}{2g_{00}}a_0^l\gamma_l\gamma_0. \tag{43}$$

For then one has

$$-2\Theta\gamma_0\gamma^k = a_0^l\gamma_l\gamma^k, \tag{44}$$

and one gets

$$\gamma_0'\gamma'^k = \gamma_0\gamma^k + a_l^k\gamma_0\gamma^l + \gamma_0\gamma^k\Theta + \Theta\gamma_0\gamma^k.$$

One can now convince oneself from our statements (37) that according to (43) Θ is *Hermitian*. Its symmetrized product with $\gamma_0\gamma^k$ is thus Hermitian or skew depending on whether $\gamma_0\gamma^k$ has this property. The same holds for the second term on the right-hand side; it is skew for $k \neq 0$, Hermitian for $k = 0$. Therefore the $\gamma_0'\gamma'^k$ keep the same Hermiticity as the $\gamma_0\gamma^k$. One can show in the same way that γ_0' stays Hermitian, too. With that, the γ'-field is legitimized as "admissable".

Θ is, of course, not unique, but the value given in (43) has, after all, *this* meaning: it is uniquely the *Hermitian* part of the infinitely small matrix to be used. There could be, in addition, an arbitrary infinitely small *skew* part. With some thought one recognizes that it would leave all conclusions unchanged – it corresponds, of course, only to an additional *unitary* transformation! —

We now add the exact definition of a *tensor operator*. *If it is known or fixed that a system of operators*

$$T_{\alpha\beta..}^{\rho\sigma..}$$

transforms under an infinitely small complemented point substitution as a tensor according to the rank indicated by the indices and their positions, but with addition of the commutator

$$T_{\alpha\beta..}^{\rho\sigma..}\Theta - \Theta T_{\alpha\beta..}^{\rho\sigma..}, \tag{45}$$

we want to call the system of operators a tensor operator *of the corresponding rank.*

The following important theorem[7] then holds, which can be found by very easy generalizations of the above conclusions:

Let $T^{\rho\sigma..}_{\alpha\beta..}$ *be a tensor operator and let it be known that in* one *reference frame the operators*

$$\gamma_0 T^{\rho\sigma..}_{\alpha\beta..} \tag{46}$$

are Hermitian or skew, depending on whether zero occurs in the indices $\alpha\beta\cdot\cdot\rho\sigma\cdot\cdot$ *in* even *or* odd *multiples; then this fact remains true in* every *reference frame.* — In this theorem one may, of course, also exchange the words even and odd, i.e. one can take into account the zero in γ_0 or not. But what one must *not* do is concern oneself with the Hermiticity of $T^{\rho\sigma..}_{\alpha\beta..}$ itself, which is completely irrelevant; it is the one of $\gamma_0 T^{\rho\sigma..}_{\alpha\beta..}$ that is relevant! —

One easily confirms that the symbol

$$\nabla_k = \frac{\partial}{\partial x_k} - \Gamma_k$$

introduced in (21) is a vector operator. Γ_k by itself is not, it transforms [with consideration of (20)] under a complemented point substitution obviously as:

$$\Gamma'_k = \Gamma_k - a^i_k \Gamma_i + \Gamma_k \Theta - \Theta \Gamma_k - \frac{\partial \Theta}{\partial x_k}. \tag{47}$$

Here, the last term is surplus, being in conflict with the vector property. The pure differentiator $\frac{\partial}{\partial x_k}$, on the other hand, transforms covariantly in the elementary sense, *without* the Θ-commutator. Taking the two together, these evils compensate, because $\frac{\partial \Theta}{\partial x_k}$ can be interpreted as a commutator of $\frac{\partial}{\partial x_k}$ and Θ. — To talk about "Hermitian" or "skew" makes, of course, no immediate sense for operators such as ∇_k which contain differentiations. For this reason, we have also not included this in the definition of a tensor operator.

If one has *two* tensor operators, one easily confirms by multiplication of their transformation formulae [similarly to what was done above in the transition from (41) to (42)], that one obtains by "writing next to each other", i.e. matrix multiplication, again a tensor if the operator written on the *left* side *does not contain the differential operator*. *Otherwise not*, because it is then not commutable with the substitution coefficients a^k_l. (This is, of course, also not different in standard tensor calculus. Although there $\frac{\partial}{\partial x_k}$ is a vector, one does not, after all, obtain a tensor by the *usual* differentiation of tensor components, but by *covariant* differentiation.) The tensor character of the Φ_{kl} defined by (10) or (26) must be investigated separately. But since we have already seen in section 3 that the Φ_{kl} behave as a tensor under pure point substitution in the elementary sense referred to there, but transform under every S-substitution according to (25), they evidently form also a

[7] The Hermiticity statements make immediate sense only if $T^{\rho\sigma..}_{\alpha\beta..}$ does not contain the differentiator $\frac{\partial}{\partial x_k}$, but is simply a 4×4 matrix with coordinate-dependent elements.

tensor operator under complemented point substitution in the finer sense considered now.

We now want to take care of what may be understood by *covariant differentiation* of a tensor operator. In this, we restrict ourselves to such operators which do *not* contain the *differential operator*, that is, to 4×4 matrices whose elements are functions of coordinates (which does not prevent them having the *form* of differential quotients; for example, Φ_{kl} is allowed, but ∇_k is not). The point is to derive from a tensor operator $T^{\rho\sigma..}_{\alpha\beta..}$ by differentiation with respect to x_λ and addition of suitable complementary terms entities which transform under a complemented point substitution as a tensor operator that is contravariant in $\rho\sigma\cdot\cdot$ and covariant in $\alpha\beta\ldots\lambda$.

We make use of the fact that a complemented point substitution decomposes formally into a pure point substitution and a Θ-transformation, where in the latter one simply adds the commutator with Θ; we use, furthermore, that these two infinitely small transformations are, of course, commutable. Let us now consider the covariant differential quotient in the elementary sense,

$$\frac{\partial T^{\rho\sigma..}_{\alpha\beta..}}{\partial x_\lambda} - \Gamma^{\mu}_{\alpha\lambda} T^{\rho\sigma..}_{\mu\beta..} - + \ldots, \tag{48}$$

this will, of course, transform under a *pure* point substitution as a tensor of rank $^{\rho\sigma..}_{\alpha\beta..\lambda}$. It would only be necessary to show, in addition, that under a Θ-transformation it simply adds the commutator with Θ, as $T^{\rho\sigma..}_{\alpha\beta..}$ itself does. This is true for all terms in the expression stated before except for the first, in which by the Θ-transformation the term

$$\frac{\partial\left(T^{\rho\sigma..}_{\alpha\beta..}\Theta - \Theta T^{\rho\sigma..}_{\alpha\beta..}\right)}{\partial x_\lambda}$$

is added instead of

$$\frac{\partial T^{\rho\sigma..}_{\alpha\beta..}}{\partial x_\lambda} - \Theta\frac{\partial T^{\rho\sigma..}_{\alpha\beta..}}{\partial x_\lambda}.$$

There thus emerges the *surplus* term

$$T^{\rho\sigma..}_{\alpha\beta..}\frac{\partial\Theta}{\partial x_\lambda} - \frac{\partial\Theta}{\partial x_\lambda}T^{\rho\sigma..}_{\alpha\beta..}. \tag{49}$$

We *remove* it by adding in (48) as a completion the commutator

$$T^{\rho\sigma..}_{\alpha\beta..}\Gamma_\lambda - \Gamma_\lambda T^{\rho\sigma..}_{\alpha\beta..}.$$

In this way, we arrive at the final *definition for the covariant differentiation of a tensor operator*:

$$T^{\rho\sigma..}_{\alpha\beta..;\lambda} = \frac{\partial T^{\rho\sigma..}_{\alpha\beta..}}{\partial x_\lambda} - \Gamma^{\mu}_{\alpha\lambda} T^{\rho\sigma..}_{\mu\beta..} - + \ldots + T^{\rho\sigma..}_{\alpha\beta..}\Gamma_\lambda - \Gamma_\lambda T^{\rho\sigma..}_{\alpha\beta..}. \tag{50}$$

Proof According to (47), the added term behaves as follows: under a pure point transformation, as a tensor of the desired rank; under a Θ-transformation, it adds, firstly, its commutator with Θ and, secondly, it removes the surplus (49). With this, the proof that (50) is a tensor is completed. One can write (50) also in the form:

$$T^{\rho\sigma..}_{\alpha\beta..;\lambda} = \nabla_\lambda T^{\rho\sigma..}_{\alpha\beta..} - T^{\rho\sigma..}_{\alpha\beta..}\nabla_\lambda - \Gamma^{\mu}_{\alpha\lambda} T^{\rho\sigma..}_{\mu\beta..} - +, \tag{51}$$

which differs from the elementary formula only by the appearance of the differentiator ∇_λ instead of the simple $\frac{\partial}{\partial x_k}$. □

One now recognizes that the important system of differential equations (8), which we have met already at the beginning of our investigations, expresses nothing more than the vanishing of the covariant derivatives of the metric vector γ_k. This is in full analogy to the theorem of Ricci, which states the same for the metric tensor g_{ik}. Exactly the same holds, by the way, for every tensor derived from the γ_k by multiplication and addition with *constant* coefficients, e.g. γ^k, $s_{\mu\nu}$, $s^{\mu\nu}$ etc. All of these have the covariant derivative zero. This is an immediate consequence of the equations (8).

6 Interpretation by the ψ-spinor

The restriction of the γ-fields to what we called "admissable" will be felt to be especially comfortable if the interpretation of the operators is based on a four-component ψ-function, a so-called spinor, on which they act. If a system of equations

$$T^{\rho\sigma..}_{\alpha\beta..}\psi = 0 \tag{52}$$

has to remain valid in *any* reference frame if it holds in *one* frame, one must demand that ψ, as an invariant of an S-transformation, transforms under a pure point substitution as follows:

$$\psi' = S^{-1}\psi. \tag{53}$$

The first is self-evident. And during an S-transformation it follows indeed from (52) by multiplication from the left by S^{-1} that

$$S^{-1}T^{\rho\sigma..}_{\alpha\beta..}SS^{-1}\psi = T'^{\rho\sigma..}_{\alpha\beta..}\psi' = 0.$$

For a complemented infinitely small point substitution one will thus have to set

$$\psi' = \psi - \Theta\psi, \tag{54}$$

where Θ is the Hermitian matrix (43). Since now ∇_k is a vector operator, there follows among other things: if the four numbers

$$\nabla_k \psi = \frac{\partial \psi}{\partial x_k} - \Gamma_k \psi \tag{55}$$

vanish in *one* reference frame, they do so in every frame. It is appropriate to call them *covariant derivatives* of the spinor ψ.

From the operators (q-numbers), one gets the ordinary numbers (c-numbers), which according to taste and mode of expression can be interpreted phyically as position probability, density of electricity, current density, transition probability etc., as follows: one applies the corresponding operator A to a spinor ψ: $A\psi$, and then forms the so-called Hermitian inner product of the two spinors ψ and $A\psi$, i.e. one multiplies the first component of the conjugate complex ψ^* with the first of $A\psi$, the second of ψ^* with the second of $A\psi$ etc. and then adds these 4 products. For this, we want to write briefly[8]

$$\psi^* A \psi. \tag{56}$$

If A does *not* contain the differential operator $\frac{\partial}{\partial x_k}$, but is only a 4×4 matrix with coordinate dependent elements, one can also say: one inserts the components of ψ^* and ψ as arguments into the bilinear form constructed from this matrix.

Only if the matrix is Hermitian (resp. skew) will the c-number (56) always be real (resp. purely imaginary), as is necessary for the components of c-tensors which one wants to interpret physically. We have now seen in section 5: if $T^{\rho\sigma..}_{\alpha\beta..}$ is a tensor operator, under an admissable transformation (i.e. under a *complemented* point substitution) one will not at all preserve the Hermiticity of *its* components, but instead those of $\gamma_0 T^{\rho\sigma..}_{\alpha\beta..}$. The reality conditions needed for a physical tensor of rank $^{\rho\sigma..}_{\alpha\beta..}$ are thus not at all preserved by, for example, the c-numbers $\psi^* T^{\rho\sigma..}_{\alpha\beta..} \psi$, but by the c-numbers

$$\mathrm{T}^{\rho\sigma..}_{\alpha\beta..} = \psi^* \gamma_0 T^{\rho\sigma..}_{\alpha\beta..} \psi. \tag{57}$$

We now want to show that it is also *them* that really transform as a c-tensor of rank $^{\rho\sigma..}_{\alpha\beta..}$ and thus have to be counted as the physical interpretation of the tensor operators $T^{\rho\sigma..}_{\alpha\beta..}$. This is because one finds, when performing the complemented point substitution (38), (40), first the following:

$$\begin{aligned}
\mathrm{T}^{\rho\sigma..\prime}_{\alpha\beta..} &= (\psi^* - \Theta^*\psi^*)(\gamma_0 - a^l_0\gamma_l + \gamma_0\Theta - \Theta\gamma_0) \\
&\qquad (T^{\rho\sigma..}_{\alpha\beta..} - a^l_\alpha T^{\rho\sigma..}_{l\beta..} - + \ldots)(\psi - \Theta\psi) \\
&= \mathrm{T}^{\rho\sigma..}_{\alpha\beta..} - \Theta^*\psi^*\gamma_0 T^{\rho\sigma..}_{\alpha\beta..}\psi - \psi^*\Theta\gamma_0 T^{\rho\sigma..}_{\alpha\beta..}\psi - a^l_0\psi^*\gamma_l T^{\rho\sigma..}_{\alpha\beta..}\psi - \\
&\qquad - a^l_\alpha \mathrm{T}^{\rho\sigma..}_{l\beta..} - + \cdots
\end{aligned} \tag{58}$$

[8] In this way of writing, the order does *not* matter. $A\varphi B\chi$ means *the same* as $B\chi A\varphi$, namely *always*: first component of $A\varphi$ times first of $B\chi$ plus second of $A\varphi$ times second of $B\chi$ plus etc.

(Two terms containing Θ have cancelled each other, namely the one arising from $-\Theta\psi$ and the one arising from $\gamma_0\Theta$; terms of second order in Θ and a_k^l are, of course, suppressed.) The second, third, and fourth terms on the right-hand side cancel each other, for: the second and the third are equal to each other, because Θ^* may be transferred under "transposition" (exchange of rows and columns) to the other factor and in this way becomes Θ because it is Hermitian. Furthermore, one has from (43)

$$-2\psi^*\Theta\gamma_0 T_{\alpha\beta..}^{\rho\sigma..}\psi = \frac{1}{g_{00}} a_0^l \psi^*\gamma_l\gamma_0\gamma_0 T_{\alpha\beta..}^{\rho\sigma..}\psi = a_0^l\psi^*\gamma_l T_{\alpha\beta..}^{\rho\sigma..}\psi,$$

which thus cancels against the fourth term, as stated. One thus obtains for the c-tensor (57)

$$\mathrm{T}_{\alpha\beta..}^{\rho\sigma..\prime} = \mathrm{T}_{\alpha\beta..}^{\rho\sigma..} - a_\alpha^l \mathrm{T}_{l\beta..}^{\rho\sigma..} - + \ldots, \tag{59}$$

the usual substitution formula, Q.E.D. — One should note explicitly that in this proof the operator $T_{\alpha\beta..}^{\rho\sigma..}$ itself does not need to be moved nor to be commuted with a a_k^l. The proof thus also still holds, i.e. $\mathrm{T}_{\alpha\beta..}^{\rho\sigma..}$ even *then* transforms as a c-tensor, if $T_{\alpha\beta..}^{\rho\sigma..}$ contains the differential operator $\frac{\partial}{\partial x_k}$. Only the Hermiticity statements then make no immediate sense for the local tensor components.

For the following it will be convenient to extend formula (55) to the case when one does not have a spinor, but instead its complex-conjugate. The complex-conjugate of (55) would read

$$\frac{\partial\psi^*}{\partial x_k *} - \Gamma_k^*\psi^*,$$

but this would for $k = 0$ ($x_0 = ict$!) in the Euclidean case *not* become the usual derivative, but its negative, which would be very inconvenient. We are thus, unfortunately, forced to change the sign for $k = 0$ and to define the covariant derivative of ψ^* as

$$\nabla_k\psi^* = \frac{\partial\psi^*}{\partial x_k} \mp \Gamma_k^*\psi^* \tag{60}$$

(upper sign for $k = 1, 2, 3$; lower sign for $k = 0$.) We now want, in addition, to investigate the covariant derivative of the c-tensor (57), which, as we expect, will be somehow connected with the one of the tensor operator defined in (50). One first finds:

$$\begin{aligned}\mathrm{T}_{\alpha\beta..;\lambda}^{\rho\sigma..} &= \frac{\partial\mathrm{T}_{\alpha\beta..}^{\rho\sigma..}}{\partial x_\lambda} - \Gamma_{\lambda\alpha}^{\mu}\mathrm{T}_{\mu\beta..}^{\rho\sigma..} - + \ldots = \\ &= \frac{\partial\psi^*}{\partial x_\lambda}\gamma_0 T_{\alpha\beta..}^{\rho\sigma..}\psi + \psi^*\frac{\partial\gamma_0}{\partial x_\lambda}T_{\alpha\beta..}^{\rho\sigma..}\psi + \psi^*\gamma_0\frac{\partial T_{\alpha\beta..}^{\rho\sigma..}}{\partial x_\lambda}\psi + \psi^*\gamma_0 T_{\alpha\beta..}^{\rho\sigma..}\frac{\partial\psi}{\partial x_\lambda} - \\ &- \Gamma_{\lambda\alpha}^{\mu}\psi^*\gamma_0 T_{\alpha\beta..}^{\rho\sigma..}\psi - + \ldots .\end{aligned}$$

One can now extend the four derivatives in this equation to *covariant* derivatives according to (60), (8), (50), (55), in which the one of γ_0 vanishes. In this way, one obtains

$$\mathrm{T}^{\rho\sigma..}_{\alpha\beta..;\lambda} = \nabla_k \psi^* \gamma_0 T^{\rho\sigma..}_{\alpha\beta..} \psi + \psi^* \gamma_0 T^{\rho\sigma..}_{\alpha\beta..;\lambda} \psi + \psi^* \gamma_0 \mathrm{T}^{\rho\sigma..}_{\alpha\beta..} \nabla \psi \tag{61}$$

plus a *remainder*, for which it has to be shown now that it vanishes. This remainder is

$$\begin{aligned}\text{remainder} = {} & \pm \Gamma^*_\lambda \psi^* \gamma_0 T^{\rho\sigma..}_{\alpha\beta..} \psi + \\ & + \psi^* [\Gamma^\mu_{0\lambda} \gamma_\mu T^{\rho\sigma..}_{\alpha\beta..} + (\Gamma_\lambda \gamma_0 - \underline{\gamma_0 \Gamma_\lambda}) T^{\rho\sigma..}_{\alpha\beta..} - \underline{\gamma_0 T^{\rho\sigma..}_{\alpha\beta..} \Gamma_\lambda + \gamma_0 \Gamma_\lambda T^{\rho\sigma..}_{\alpha\beta..}}] \psi + \\ & + \underline{\psi^* \gamma_0 T^{\rho\sigma..}_{\alpha\beta..} \Gamma_\lambda \psi}.\end{aligned}$$

The underlined terms cancel each other. $\pm \Gamma^*_\lambda$ is transferred to the other factor as $\pm \Gamma^\dagger_\lambda$.[9] There is still

$$\begin{aligned}\text{remainder} &= \psi^* A T^{\rho\sigma..}_{\alpha\beta..} \psi \text{ with} \\ A &= \Gamma^\mu_{0\lambda} \gamma_\mu + (\Gamma_\lambda \pm \Gamma^\dagger_\lambda) \gamma_0.\end{aligned}$$

The proof will be completed if we can show that

$$\frac{1}{2g_{00}} A \gamma_0 \equiv \frac{1}{2} (\Gamma_\lambda \pm \Gamma^\dagger_\lambda) + \frac{1}{2g_{00}} \Gamma^\mu_{0\lambda} \gamma_\mu \gamma_0 \tag{62}$$

vanishes. (For from this one has $A \equiv 0$, because γ_0 has the non-vanishing eigenvalues $\pm\sqrt{g_{00}}$. If $A = 0$, the "remainder" vanishes and equation (61) will be proved.)

The operator (62) now is in the case of the upper sign, valid for $\lambda = 1, 2, 3$, the *Hermitian*, in the case of the lower sign the *skew-Hermitian* part of

$$\Gamma_\lambda + \frac{1}{2g_{00}} \Gamma^\mu_{0\lambda} \gamma_\mu \gamma_0. \tag{63}$$

It can be recognized without too much effort that this operator, if commuted with the according to (37) *Hermitian* matrices $\gamma_0, \gamma^1, \gamma^2, \gamma^3$ gives for $\lambda = 1, 2, 3$ *only Hermitian*, for $\lambda = 0$ *only skew-Hermitian* results. For this reason, its Hermitian (resp. for $\lambda = 0$ its skew-Hermitian) part must at any rate be *commutable* with $\gamma_0, \gamma^1, \gamma^2, \gamma^3$, thus must be a multiple of unity. In other words, the parts which are supposed to vanish reduce to

$$\begin{aligned}\text{real part trace } \left(\Gamma_\lambda + \frac{1}{2g_{00}} \Gamma^\mu_{0\lambda} \gamma_\mu \gamma_0\right) \text{ for } \lambda = 1, 2, 3 \\ \text{imaginary part trace } \left(\Gamma_0 + \frac{1}{2g_{00}} \Gamma^\mu_{00} \gamma_\mu \gamma_0\right).\end{aligned}$$

[9] With the dagger † we shall denote the transposed and complex-conjugate matrix, as is almost always (unfortunately, only *almost*) the case.

Since trace $\gamma_\mu \gamma_0 = 4 g_{\mu 0}$ and

$$g_{\mu 0} \Gamma^{\mu}_{0\lambda} = \Gamma_{0,0\lambda} = \frac{1}{2} \frac{\partial g_{00}}{\partial x_\lambda} \quad \text{for } \lambda = 0, 1, 2, 3$$

the question is thus whether one really has

$$\begin{aligned} \text{real part trace } \ \Gamma_\lambda &= -\frac{\partial \lg g_{00}}{\partial x_\lambda} \quad \text{for } \lambda = 1, 2, 3; \\ \text{imaginary part trace } \ \Gamma_\lambda &= -\frac{\partial \lg g_{00}}{\partial x_0} ? \end{aligned} \tag{64}$$

It now turns out that we have promised too much. This is because we *cannot prove* these equations, the reason being that the Γ_k were, after all, originally introduced and so far exclusively applied in such a manner that *only their commutators* with other matrices play a role, for which their *traces* are completely irrelevant. These play a role for the first time in the covariant derivative of the *spinor*, equation (55) and (60), of which we just make use for the first time in the equation (61) that we want to prove. We can only prove that we are free to define the corresponding trace parts by (64). And this is indeed the case. On the one hand, it certainly holds in *one* reference frame, because the right-hand sides of (64) possess the necessary reality. On the other hand, one can show from (47) and (43) that the decree once imposed is invariant under admissable transformations — I suppress the proof.

By this decree, the covariant derivative of the spinor is made precise in a desired way. But the decree is, in fact, desired also in another way. If the trace parts in question *cannot* be described as the derivatives of one and the same function ($-\lg g_{00}$), they would generate *pure imaginary* electromagnetic field strengths in the traces of the Φ_{kl}. This is avoided in this way. — The *real part* of trace Γ_0 and the *imaginary parts* of trace Γ_λ ($\lambda = 1, 2, 3$), from which the *real* field strength follow, still remain *free*.

We must, in addition, take a look at the pure *unitary* transformations which besides the complemented point substitutions are also still admissable in themselves. The only remaining thing to be said is that such a desired unitary transformation must, of course, also be applied to ψ according to the prescription (53). It is then completely irrelevant and harmless. In particular, the components of the c-tensors (57) are *completely* insensitive to it; this also holds for the trace parts that were fixed in (64).

The essential results of this section are:

1. The determination of the transformation law (54) and the covariant derivative (55) for the spinor.
2. The assignment of the c-tensor components to the tensor operator according to (57) and the proof that they really transform as usual tensor components *of the same rank*.
3. The presentation of a relatively simple formula (61) for calculating the covariant derivative of a c-tensor; a formula which mainly is of interest because it demands for its validity the, in principle welcome,

4. *normalization* of that trace part of Γ_λ that *without* normalization would lead to the appearance of pure imaginary electromagnetic field strengths.

7 The Dirac equation

The operator $\gamma^k \nabla_k$ is an invariant which one can suitably call "absolute value of the gradient". The generalized Dirac equation demands[10]

$$\gamma^k \nabla_k = \mu \psi, \tag{68}$$

where μ is a universal constant,

$$\mu = \frac{2\pi mc}{h}.$$

Let us call the c-vector belonging to γ^k after the assignment (57) iS^k, that is,

$$iS^k = \psi^* \gamma_0 \gamma^k \psi. \tag{69}$$

Since the covariant derivative of the *operator* γ^k vanishes, the one of S^k reduces according to (61) to[11]

$$iS^k_{;\lambda} = \nabla_\lambda \psi^* \gamma_0 \gamma^k \psi + \psi^* \gamma_0 \gamma^k \nabla_\lambda \psi.$$

10 But one could be tempted to "symmetrize" and to take as the left-hand side of (68)

$$\frac{1}{2}(\gamma^k \nabla_k + \nabla_k \gamma^k). \tag{65}$$

Footnote 10 continued
But this term can be rewritten. The vanishing of the covariant derivative of γ^k states:

$$\nabla_l \gamma^k - \gamma^k \nabla_l = -\Gamma^k_{l\mu} \gamma^\mu.$$

Contraction leads to

$$\nabla_k \gamma^k - \gamma^k \nabla_k = -\Gamma^k_{k\mu} \gamma^\mu = -\frac{\partial \lg \sqrt{g}}{\partial x_\mu} \gamma^\mu. \tag{66}$$

Therefore,

$$\frac{1}{2}(\gamma^k \nabla_k + \nabla_k \gamma^k) = \gamma^k \nabla_k - \frac{1}{2} \frac{\partial \lg \sqrt{g}}{\partial x_k} \gamma^k = g^{\frac{1}{4}} g^k \nabla_k g^{-\frac{1}{4}}. \tag{67}$$

This is *not* an invariant operator, about which we may not be surprised. Namely, $\nabla_k \gamma^k$ is not one and also has no duty to be one. For we have also emphasized above that a product of two tensor operators is only then definitely a tensor operator if the *left* factor does not contain the differentiator. As a matter of fact, the use of the ansatz (65) would anyway come to the same thing, one would only have to use $g^{-\frac{1}{4}} \psi$ instead of ψ, that is, one would have to transform $g^{-\frac{1}{4}} \psi$ as a spinor. We thus keep the ansatz (68).

11 Translator's note: I have corrected a typo in this equation.

If one forms by contraction the covariant divergence:

$$iS^{\lambda}_{;\lambda} = \nabla_\lambda \psi^* \gamma_0 \gamma^\lambda \psi + \psi^* \gamma_0 \gamma^\lambda \nabla_\lambda \psi,$$

the first summand is the negative complex-conjugate of the second[12], but this one is according to (68)

$$\mu \psi^* \gamma_0 \psi,$$

and thus is real, because γ_0 is Hermitian. Therefore,

$$S^{\lambda}_{;\lambda} = 0. \tag{70}$$

In this way, the source freedom of the *four-current*, which according to our assignment (57) belongs as a c-vector to the *contravariant metric vector*, follows from the Dirac equation and the fundamental equations (8) (cf. [3], p. 267).

We now want to square the Dirac equation in order to compare the result with the one familiar from the special theory (for brevity, ψ will be suppressed):

$$\gamma^k \nabla_k \gamma^l \nabla_l = \mu^2. \tag{71}$$

One replaces the first two factors by equation (66) (in the footnote) and uses that one has according to (2) and (12)

$$\gamma^k \gamma^l = g^{kl} + s^{kl}. \tag{72}$$

This leads to

$$\nabla_k (g^{kl} + s^{kl}) \nabla_l + \frac{\partial \lg \sqrt{g}}{\partial x_\mu} \gamma^\mu \gamma^l \nabla_l = \mu^2.$$

From the vanishing of the covariant derivative of s^{kl} follows

$$\nabla_k s^{kl} - s^{kl} \nabla_k = -\frac{\partial \lg \sqrt{g}}{\partial x_\mu} s^{\mu l}.$$

This leads to [after using again (72)]:

$$\nabla_k g^{kl} \nabla_l + s^{kl} \nabla_k \nabla_l + \frac{\partial \lg \sqrt{g}}{\partial x_\mu} g^{\mu l} \nabla_l = \mu^2.$$

[12] The *Hermitian* $\gamma_0 \gamma^0$ is applied as $(\gamma_0 \gamma^0)^*$ to the first factor, the skew $\gamma_0 \gamma^\lambda$, $\lambda \neq 0$, as $-(\gamma_0 \gamma^\lambda)^*$. In return, ∇_0 gets a change of sign, ∇_λ, $\lambda \neq 0$, not. Compare the remarks to equation (60) made in the text above as well as the comment to equation (56).

For the second term one finds from (26) and because of the antisymmetry of the s^{kl} that it is equal to $-\frac{1}{2}s^{kl}\Phi_{kl}$. The first and the third (in which one replaces μ by k) combine to the generalized Laplace operator; one then finally gets:

$$\frac{1}{\sqrt{g}}\nabla_k\sqrt{g}g^{kl}\nabla_l - \frac{1}{2}s^{kl}\Phi_{kl} = \mu^2. \tag{73}$$

It is of interest to insert here for Φ_{kl} the expression (15) found much earlier. In this, the invariant

$$\frac{1}{8}R_{kl,\mu\nu}s^{kl}s^{\mu\nu}$$

appears. Due to the symmetry of the covariant Riemann curvature tensor in the first and second index pair this is also equal to

$$\frac{1}{16}R_{kl,\mu\nu}(s^{kl}s^{\mu\nu} + s^{\mu\nu}s^{kl}).$$

If one now — something that I do not want to carry out in extenso — really calculates the symmetrized products of the s^{kl} and then makes use of the known cyclic symmetry

$$R_{kl,\mu\nu} + R_{l\mu,k\nu} + R_{\mu k,l\nu} = 0,$$

one finally gets

$$\frac{1}{8}R_{kl,\mu\nu}s^{kl}s^{\mu\nu} = -\frac{1}{4}g^{k\mu}g^{l\nu}R_{kl,\mu\nu} = -\frac{R}{4},$$

where R is the *invariant curvature*. Consequently, inserting Φ_{kl} from (15) into (73) gives the following:

$$\frac{1}{\sqrt{g}}\nabla_k\sqrt{g}g^{kl}\nabla_l - \frac{R}{4} - \frac{1}{2}f_{kl}s^{kl} = \mu^2. \tag{74}$$

In the third term on the left-hand side one recognizes the familiar influence of the field strength on the spin tensor, where f_{kl} is the pure trace part already removed from Φ_{kl}, which can well be called field strength in the proper sense and which is, as mentioned several times, still completely undetermined by the metric.

The second term seems to me to be of considerable theoretical interest. It is, however, too small by many, many powers of ten to be able to *replace*, for example, the term on the right-hand side. For μ is the reciprocal Compton wavelength, about 10^{11}cm^{-1}. At least it seems significant that one naturally meets in the generalized theory a term at all similar to the enigmatic mass term (see also [9]).

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