RESEARCH ARTICLE

Bianchi type I, Schutz perfect fluid and evolutionary quantum cosmology

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Abstract

We study the classical and quantum cosmology of a universe in which the matter content is a perfect fluid and the background geometry is described by a Bianchi type I metric. To write the Hamiltonian of the perfect fluid we use the Schutz representation, in terms of which, after a particular gauge fixing, we are led to an identification of a clock parameter which may play the role of time for the corresponding dynamical system. In view of the classical cosmology, it is shown that the evolution of the universe represents a late time expansion coming from a big-bang singularity. We also consider the issue of quantum cosmology in the framework of the canonical Wheeler–DeWitt (WDW) equation. It is shown that the Schutz formalism leads to the introduction of a momentum that enters linearly into Hamiltonian. This means that the WDW equation takes the form of a Schrödinger equation for the quantum-mechanical description of the model under consideration. We find the eigenfunctions and with the use of them construct the closed form expressions for the wave functions of the universe. By means of the resulting wave function we evaluate the expectation values and investigate the possibility of the avoidance of classical singularities due to quantum effects. We also look at the problem through Bohmian approach of quantum mechanics and while recovering the quantum solutions, we deal with the reason of the singularity avoidance by introducing quantum potential.

Keywords Bianchi-I model · Quantum cosmology · Perfect fluid · Schutz formalism

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1 Introduction

One of the most important questions in cosmology is that of the initial conditions from which the universe began to evolve. As is well known, standard theories of cosmology based on classical general relativity do not provide an acceptable answer to this question. The main reason is that almost all of these models suffer from the existence of various types of singularities such as big-bang, big-crunch, big-rip and so on. In the presence of any of these singularities, some physical observables take values that are physically unacceptable, which in turn means that their underlying theory is not valid in the vicinity of singularities and thus can not be applied. As to how the physical phenomena should be described near these singularities, the dominant belief implies the development of a concomitant and conducive quantum theory of gravity. These efforts have begun with the works of DeWitt in canonical quantum gravity [1] and so far continue with the more modern approaches such as string theory and loop quantum gravity [2–4]. Based on quantum gravity it would be useful to describe the state of the universe near the classical singularities within the framework of quantum cosmology. Depending on the theory of quantum gravity the formalism of the corresponding quantum cosmology is based on the canonical or loop quantization of the minisuperspace variables and the evolution universe is described by a wave function in this space [5–7].

In this paper we deal with the issue of the classical and quantum cosmology in the framework of the Bianchi type I model. Bianchi models are the most well-known anisotropic and homogeneous space-times whose classical and quantum solutions have been studied in a number of works, see for example [8–14]. On the other hand, an important ingredient in any model theory related to cosmology is the choice of the matter field used to couple with gravity and construct the energy-momentum tensor in Einstein field equations. Inspired by the Weyl principle in cosmology, the most widely used matter source has traditionally been the perfect fluid. So, in the present study, we consider a perfect fluid as the matter content of the universe and describe its evolution in the framework of Schutz formalism [15,16]. In Schutz formulation of the perfect fluid, its four-velocity can be expressed in terms of some thermodynamical potentials. The advantage of using this representation is that a canonical transformation can transform a pair of the fluid's dynamical variable to another conjugate pair (T, p_T) in such a way that the momentum p_T associated to the variable T appears linearly in the Hamiltonian. This means that Schutz representation of perfect fluid can offer a time parameter in terms of dynamical variables of the fluid. Therefore, when canonical quantization we are facing with a Schrödinger like equation in which the wave function, in addition to the dependence on the configuration space variables, has time dependence [17–20]. In this way, it can be seen that by the Schutz formalism we may address the problem of time in quantum cosmology.

In the following, in Sect. 2, after a quick look at some main properties of the Bianchi type I metric, we write the total Hamiltonian based on the Einstein-Hilbert action coupled to a perfect fluid in Schutz representation. Section 3 is devoted to the classical Hamiltonian equations of motion and their solutions. Here, in terms of the above mentioned time parameter, we shall obtain the dynamical behavior of the cosmic scale factors that are the comoving volume function of the universe and its anisotropic

factors. We show that the evolution of the universe represents a late time power law expansion coming from a big-bang singularity. In Sect. 4, we deal with the quantization of the model. By the resulting wave function, we compute the expectation values of the scale factors and show that the evolution of the universe according to the quantum picture is free of classical singularities but in good agreement with classical dynamics in late times of cosmic evolution. Section 5 deals with the Bohmian trajectories of the problem at hand, by means of which we introduce the quantum potential. We will see that the repulsive force associated to the quantum potential may be considered responsible for the elimination of singularity. Finally, we summarize the results in Sect. 6.

2 The model

In this section we make a brief overview of the most important features of the Bianchi type I model and obtain its Lagrangian and Hamiltonian in terms of the ADM variables in $1 + 3$ decomposition. Bianchi type I universe is the simplest anisotropic generalization of the flat FRW space–time whose metric reads as

$$ds^2 = -N^2(t)dt^2 + a^2(t)dx^2 + b^2(t)dy^2 + c^2(t)dz^2. \tag{1}$$

In this metric $N(t)$ is the lapse function and there are three functions $a(t)$, $b(t)$ and $c(t)$, to be determined by the gravitational field equations, and are the scale factors of the corresponding universe in the x, y and z directions respectively. When writing the Lagrangian of this model, the scale factors in different directions are considered as independent variables. The Bianchi I metric as a cosmological setting is the simplest homogeneous and anisotropic model, which becomes the flat FRW metric provided that its scale factors are equal.

In general, Bianchi I space-time is a subset of the class A Bianchi metrics with nine models. They are the most general homogeneous dynamical solutions of the Einstein field equations which admit a three-dimensional isometry group, i.e. their spatially homogeneous sections are invariant under the action of a three-dimensional Lie group. In order to transform the Lagrangian of the dynamical system which corresponds to the Bianchi models to a more manageable form, let us introduce the following change of variables

$$a(t) = e^{u(t)+v(t)+\sqrt{3}w(t)}, \qquad b(t) = e^{u(t)+v(t)-\sqrt{3}w(t)}, \qquad c(t) = e^{u(t)-2v(t)}. \tag{2}$$

In the Misner notation [21], the metrics of all class A Bianchi models can be written in terms of the above variables as

$$ds^2 = -N^2(t)dt^2 + e^{2u(t)}e^{2\beta_{ij}(t)}dx^i dx^j, \tag{3}$$

where $V(t) = e^{3u(t)} = abc$ is the comoving volume of the universe and β_{ij} determines the anisotropic parameters $v(t)$ and $w(t)$ as follows

$$\beta_{ij} = \mathrm{diag}(v + \sqrt{3}w, \, v - \sqrt{3}w, \, -2v). \tag{4}$$

The action of such a structure may be written as (we work in units where $c = \hbar = 8\pi G = 1$)

$$S = \frac{1}{2} \int_M d^4x \sqrt{-g}\mathcal{R} + \int_{\partial M} d^3x \sqrt{h}h_{ij}K^{ij} + \int_M d^4x \sqrt{-g}p, \tag{5}$$

where g is the determinant and $\mathcal{R}$ is the Ricci scalar of the space-time metric (1). Also, K^{ij} is the extrinsic curvature (second fundamental form), which represents how much the spatial space is curved in the way it sits in the space-time manifold, and h_{ij} is the induced metric over the three-dimensional spatial hypersurface, which is the boundary ∂M of the four-dimensional manifold M. The boundary term in (5) will be canceled by the variation of the first term. This is the reason for including a boundary term in the action of a gravitational theory. That such a boundary term is needed is due to the fact that $\mathcal{R}$, the gravitational Lagrangian density contains second derivatives of the metric tensor, a nontypical feature of field theories. The last term of (5) denotes the matter contribution to the total action where p is the pressure of perfect fluid which is linked to its energy density by the equation of state (EoS)

$$p = \omega\rho, \tag{6}$$

where $-1 \leq \omega \leq 1$ is the EoS parameter. In terms of the ADM variables, the gravitational part of the action (5) can be written as [22]

$$S_g = \frac{1}{2} \int dt d^3x \mathcal{L} = \frac{1}{2} \int dt d^3x N \sqrt{h}(K_{ij}K^{ij} - K^2 + R), \tag{7}$$

where h and R are the determinant and Ricci scalar of the spatial geometry h_{ij} respectively, and K represents the trace of K_{ij}. The extrinsic curvature is given by

$$K_{ij} = \frac{1}{2N}\left(N_{i|j} + N_{j|i} - \frac{\partial h_{ij}}{\partial t}\right), \tag{8}$$

where $N_{i|j}$ represents the covariant derivative with respect to h_{ij}. Using (3) and (4) we obtain the nonvanishing components of the extrinsic curvature and its trace as follows

$$\begin{cases} K_{11} = -\frac{1}{N}(\dot{u} + \dot{v} + \sqrt{3}\dot{w})e^{2(u+v+\sqrt{3}w)}, \\ K_{22} = -\frac{1}{N}(\dot{u} + \dot{v} - \sqrt{3}\dot{w})e^{2(u+v-\sqrt{3}w)}, \\ K_{33} = -\frac{1}{N}(\dot{u} - 2\dot{v})e^{2(u-2v)}, \\ K = -3\frac{\dot{u}}{N}, \end{cases} \tag{9}$$

where a dot represents differentiation with respect to t. It is easy to show for the Bianchi-I metric the Ricci scalar of the three dimensional spatial geometry h_{ij} vanishes, $R = 0$. The gravitational part of Lagrangian for the Bianchi-I model may now be written by substituting the above results into action (7), giving

$$\mathcal{L}_g = \frac{3e^{3u}}{N}(-\dot{u}^2 + \dot{v}^2 + \dot{w}^2). \tag{10}$$

The momenta conjugate to the dynamical variables are given by

$$p_u = \frac{\partial \mathcal{L}}{\partial \dot{u}} = -\frac{6}{N}\dot{u}e^{3u}, \quad p_v = \frac{\partial \mathcal{L}}{\partial \dot{v}} = \frac{6}{N}\dot{v}e^{3u}, \quad p_w = \frac{\partial \mathcal{L}}{\partial \dot{w}} = \frac{6}{N}\dot{w}e^{3u}, \tag{11}$$

leading to the following Hamiltonian

$$\mathcal{H}_g = \frac{1}{12}Ne^{-3u}(-p_u^2 + p_v^2 + p_w^2). \tag{12}$$

Now, let us deal with the matter field with which the action of the model is augmented. The matter will come into play in a common way and the total Hamiltonian can be made by adding the matter Hamiltonian to the gravitational part (12). According to Schutzs representation for the perfect fluid, its Hamiltonian can be viewed as (see [17–20] for details)

$$\mathcal{H}_m = N\frac{P_T}{e^{3\omega u}}, \tag{13}$$

where T is a dynamical variable related to the thermodynamical parameters of the perfect fluid and P_T is its conjugate momentum. Finally, we are in a position in which can write the total Hamiltonian $H = \mathcal{H}_g + \mathcal{H}_m$ as

$$H = N\left[\frac{1}{12}e^{-3u}(-p_u^2 + p_v^2 + p_w^2) + P_T e^{-3\omega u}\right]. \tag{14}$$

The setup for constructing the phase space and writing the Lagrangian and Hamiltonian of the model is now complete. In the following sections, we shall deal with classical and quantum cosmologies which can be extracted from a theory with the above mentioned Hamiltonian.

3 Classically cosmological dynamics

The classical dynamics is governed by the Hamiltonian equations, that is

$$\begin{cases} \dot{u} = \{u, H\} = -\frac{Np_u e^{-3u}}{6}, \\ \dot{p}_u = \{p_u, H\} = N\left[3\omega P_T e^{-3\omega u} + \frac{1}{4}e^{-3u}(-p_u^2 + p_v^2 + p_w^2)\right], \\ \dot{v} = \{v, H\} = \frac{Np_v e^{-3u}}{6}, \\ \dot{p}_v = \{p_v, H\} = 0, \\ \dot{w} = \{w, H\} = \frac{Np_w e^{-3u}}{6}, \\ \dot{p}_w = \{p_w, H\} = 0, \\ \dot{T} = \{T, H\} = Ne^{-3\omega u}, \\ \dot{P}_T = \{P_T, H\} = 0. \end{cases} \tag{15}$$

We also have the constraint equation $H = 0$ which comes from the variation with respect to N. Up to this point the cosmological model, in view of the concerning issue of time, has been of course under-determined. Before trying to solve these equations we must decide on a choice of time in the theory. The under-determinacy problem at the classical level may be resolved by using the gauge freedom via fixing the gauge. A glance at the above equations shows that choosing the gauge $N = e^{3\omega u}$, we have

$$N = e^{3\omega u} \Rightarrow \dot{T} = 1 \Rightarrow T = t, \tag{16}$$

which means that variable T may play the role of time in the model. Therefore, the classical equations of motion can be rewritten in the gauge $N = e^{3\omega u}$ as follows

$$\begin{cases} \dot{u} = -\frac{1}{6}e^{-3(1-\omega)u}p_u, \\ \dot{p}_u = 3\omega p_0 + \frac{1}{4}e^{-3(1-\omega)u}(-p_u^2 + p_{0v}^2 + p_{0w}^2), \\ \dot{v} = \frac{1}{6}p_{0v}e^{-3(1-\omega)u}, \\ \dot{w} = \frac{1}{6}p_{0w}e^{-3(1-\omega)u}, \end{cases} \tag{17}$$

in which we have set $p_v = p_{0v} = \text{const}$, $p_w = p_{0w} = \text{const}$ and $P_T = p_0 = \text{const}$. In addition the Hamiltonian constraint $H = 0$, yields

$$\frac{1}{12}e^{-3u}(-p_u^2 + p_v^2 + p_w^2) + P_T e^{-3\omega u} = 0, \tag{18}$$

from which we get

$$- p_u^2 + p_{0v}^2 + p_{0w}^2 = -12p_0 e^{3(1-\omega)u}. \tag{19}$$

By using of this equation, the second equation of the system (17) takes the simple form

$$\dot{p}_u = 3(\omega - 1)p_0, \Rightarrow p_u(t) = 3p_0(\omega - 1)t + p_{0u}, \tag{20}$$

where p_{0u} is an integration constant which can be set to zero without loss of generality, so we set $p_{0u} = 0$.

Before going to solve the rest equations of the system (17), note that we may consider the relation (16) as a gauge fixing condition $G = T - t \approx 0$, the stability of which results the choice $N = e^{3\omega u}$ for the lapse function. Indeed, our dynamical system is a totally constrained classical system in which the canonical coordinates $(\mathbf{q}, \mathbf{p})$, with $\mathbf{q} = (u, v, w, T)$ and $\mathbf{p} = (p_u, p_v, p_w, P_T)$, parameterize its kinematical phase space Γ. In such a system the classical dynamics is given by the constraint $\mathcal{C}(\mathbf{q}, \mathbf{p}) \approx 0$. This means that the phase space variables evolve on a subset of Γ, say $\bar{\Gamma}$, on which this constraint holds. For our problem at hand this constraint is nothing but the Hamiltonian constraint (14). Now, to find the gauge invariant phase space functions, i.e. Dirac observables, we may solve the constraint $\mathcal{C}(\mathbf{q}, \mathbf{p}) \approx 0$. In some special cases by this procedure the constraint takes the form [23,24]

$$\mathcal{C}(q_i, p_i) = \bar{\mathcal{C}}(q_1, \ldots, q_{n-1}; p_1, \ldots, p_{n-1}) + \alpha p_n \approx 0, \tag{21}$$

where $2n$ is the dimension of the phase space. For such a constraint the momentum p_n is a Dirac observable in the sense that $\{p_n, \mathcal{C}\} = 0$, so it is a constant of motion.

On the other hand, we note that the above constraint does not depend on the variable q_n, which in turn, means that it is a linear function of the parameter t, and therefore one may consider it as a suitable clock phase space function in terms of which the evolution of the rest phase space variables can be viewed.

Now, let us return to our problem under consideration and see how the above procedure works. First, note that by the gauge fixing $G = T - t \approx 0$, the Hamiltonian constraint takes the form

$$\mathcal{C}(u, v, w, T; p_u, p_v, p_w, P_T) = \bar{\mathcal{C}}(u; p_u, p_v, p_w) + P_T, \tag{22}$$

where

$$\bar{\mathcal{C}}(u; p_u, p_v, p_w) = \frac{1}{12} e^{-3(1-\omega)u} (-p_u^2 + p_v^2 + p_w^2). \tag{23}$$

It is seen that the constraint is of the form of the above discussed one (21). So, the system de-parameterizes in the sense that the evolution (in terms of the clock parameter T) of all, but the perfect fluid degrees of freedom, can be generated by the time independent gauge fixed Hamiltonian $\bar{\mathcal{C}}$ without referring to the Hamiltonian constraint $H = 0$. In this way, it is not difficult to see the solution (20) is again recovered and thus to continue, we have

$$\dot{u} = \frac{1}{2}(1-\omega) p_0 e^{-3(1-\omega)u} t, \tag{24}$$

where upon integration we obtain

$$u(t) = \ln \left[\frac{3}{4} p_0 (1-\omega)^2 t^2 + C \right]^{\frac{1}{3(1-\omega)}}, \tag{25}$$

with C being an integration constant whose value can be evaluated in terms of the other constants by substitution of the above expression into the constraint Eq. (19) with result: $C = -\frac{1}{12} \frac{p_{0v}^2 + p_{0w}^2}{p_0}$. Therefore, the comoving volume of the universe will be

$$V(t) = e^{3u(t)} = V_0 (t^2 - t_*^2)^{\frac{1}{(1-\omega)}}, \tag{26}$$

where $V_0 = \left[\frac{3}{4} p_0 (1-\omega)^2 \right]^{\frac{1}{1-\omega}}$ and $t_*^2 = \frac{p_{0v}^2 + p_{0w}^2}{9 p_0^2 (1-\omega)^2}$. For this solution, the condition $V(t) \geq 0$ separates two sets of solutions, $V_-(t)$ and $V_+(t)$, each of which is valid for $t \leq -t^*$ and $t \geq t^*$ respectively. For the former, we have a contracting universe which decreases its size according to a power law relation and ends its evolution in a singularity at $t = -t^*$, while for the latter, the evolution of the universe begins with a big-bang singularity at $t = t^*$ and then follows the power law expansion at late time of cosmic evolution. The corresponding equations for the anisotropic functions in this case will be

$$\dot{v} = \frac{1}{6} p_{0v} V_0^{-(1-\omega)} (t^2 - t_*^2)^{-1}, \tag{27}$$

and

$$\dot{w} = \frac{1}{6} p_{0w} V_0^{-(1-\omega)} (t^2 - t_*^2)^{-1}, \tag{28}$$

with solutions

$$v(t) = \frac{1}{6t_*} p_{0v} V_0^{-(1-\omega)} \tanh^{-1}\left(\frac{t}{t_*}\right), \qquad w(t) = \frac{1}{6t_*} p_{0w} V_0^{-(1-\omega)} \tanh^{-1}\left(\frac{t}{t_*}\right). \tag{29}$$

It is seen that, unlike the volume factor which was not defined in the time interval $(-t_*, t_*)$, the functions $v(t)$ and $w(t)$ are only well-defined in this interval. Therefore, in general we do not get acceptable physical solutions. However, for some special values of EoS parameter, say for $\omega = \frac{2n-1}{2n}$, $n = 1, 2, \ldots$, the volume takes the form $V(t) = V_0(t^2 - t_*^2)^{2n}$ which is always positive and well-defined also in $(-t_*, t_*)$. For such special occasions, we are facing with a universe that begins its evolution from a singular point at $t = -t_*$ with zero size and high degrees of anisotropy, expands to a isotropic state with maximum volume and then will collapse and eventually end its evolution in a singularity similar to its initial singularity.

At the end of this section let us see what happened if $C = 0$, for which we get $p_{0v} = p_{0w} = 0$. In this case the two last equations of the system (17) give $\dot{v} = \dot{w} = 0$. Therefore, we have

$$V(t) = V_0 t^{\frac{2}{(1-\omega)}}, \qquad v(t) = v_0 = \text{const}, \qquad w(t) = w_0 = \text{const.} \tag{30}$$

In this situation, a look at the scale factors in (2) shows that they take the form

$$a(t) = a_0 e^{u(t)}, \qquad b(t) = b_0 e^{u(t)}, \qquad c(t) = c_0 e^{u(t)}, \tag{31}$$

where $a_0 = e^{v_0 + \sqrt{3}w_0}$, $b_0 = e^{v_0 - \sqrt{3}w_0}$ and $c_0 = e^{-2v_0}$. Under these conditions, by re-definition of the coordinates as $X = a_0 x$, $Y = b_0 y$ and $Z = c_0 z$, the metric will be

$$ds^2 = -N^2(t)dt^2 + e^{2u(t)}(dX^2 + dY^2 + dZ^2), \tag{32}$$

which is nothing but the flat FRW metric. Since expression (30) for $V(t)$ is singular at $t = 0$, we may consider the intervals $-\infty < t \leq 0$ or $0 \leq t < +\infty$ as the domain of variation for the time parameter t. Thus, with $0 \leq t < +\infty$, the above solutions give a flat FRW universe begins to evolve from a big-bang singularity and continually expands according to the power law expression giving in (30).

In the next section we will deal with the quantization of the model described above to see how the presented classical picture can be modified.

4 Quantization of the model

In the previous section we saw that classical model consisted of two categories of solutions (correspond to the cases $C \neq 0$ and $C = 0$), which in general, suffered from the existence of big-bang singularities. In this section we focus our attention on the quantization of the model described above. Quantum cosmology of the Bianchi models are widely investigated in literature, see for instance [25–33] and the references therein. Our starting point is the WDW equation $\hat{H}\Psi(u, v, w, T) = 0$, in which

$\hat{H}$ is the operator version of the Hamiltonian (14) and $\Psi(u, v, w, T)$ is the wave function of the universe. The operator $\hat{H}$ can be constructed by replacing $\hat{q}\Psi = q\Psi$ and $\hat{p}_q\Psi = -i\frac{\partial}{\partial q}\Psi$, for each dynamical variable. Nevertheless, a point to be considered is the issue of the factor-ordering problem when one is going to arrange a quantum mechanical operator equation. In dealing with such Hamiltonians at the quantum level one should be careful when trying to replace the dynamical variables with their quantum operator counterparts, that is, in replacing a variable q and its momentum p_q with their corresponding operators, the ordering considerations should be taken into account. In our present case, this is only important in the first term of the Hamiltonian, since this term includes variables u and p_u that do not commute. Taking these considerations into account, the WDW equation for the Hamiltonian (14) is written as

$$\left[2\frac{\partial^2}{\partial u^2} + 3q\frac{\partial}{\partial u} - 2\frac{\partial^2}{\partial v^2} - 2\frac{\partial^2}{\partial w^2} - 24ie^{3(1-\omega)u}\frac{\partial}{\partial T}\right]\Psi(u, v, w, T) = 0, \quad (33)$$

where q is the factor ordering parameter which represents the ambiguity in the ordering of factors u and p_u in the first term of (14). It is clear that there are infinite number of possibilities to choose this parameter. For example $q = 0$ corresponds to no factor ordering, with $q = 1$ the kinetic term of the Hamiltonian takes the form of the Laplacian $-\frac{1}{2}\nabla^2$ of the minisuperspace and $q = -2$ corresponds to the ordering $e^{-3u}p_u^2 = p_u e^{-3u}p_u$. Although in general, the behavior of the wavefunction depends on the chosen factor ordering [34], it can be shown that the factor-ordering parameter will not affect semiclassical calculations in quantum cosmology [35], and so for convenience one usually chooses a special value for it in the special models. With an eye to the constraint Eq. (22), we can see that Eq. (33) is nothing other than $\mathcal{C}\Psi = (\bar{\mathcal{C}} + P_T)\Psi = 0$. In our classical analysis, we saw that the momentum of the perfect fluid P_T, is a constant of motion and hence we have taken its value to be $P_T = p_0 = \text{const.}$ On the other hand our discussion about the sign of the volume function in Eq. (26) shows that the sign of p_0 should, in general, be positive. Therefore, classically, the motion of the system is restricted to those regions of the phase space for which we have $\bar{\mathcal{C}} = \frac{1}{12}e^{-3(1-\omega)u}(-p_u^2 + p_v^2 + p_w^2) < 0$. Now, with separation ansatz

$$\Psi(u, v, w, T) = e^{iET}\psi(u, v, w), \quad (34)$$

for the quantum version of the constraint, we obtain

$$\left[2\frac{\partial^2}{\partial u^2} + 3q\frac{\partial}{\partial u} - 2\frac{\partial^2}{\partial v^2} - 2\frac{\partial^2}{\partial w^2} + 24Ee^{3(1-\omega)u}\right]\psi(u, v, w) = 0. \quad (35)$$

Comparing of this equation with the classical constraint $\mathcal{C} = \bar{\mathcal{C}} + P_T \approx 0$, shows that the separation constant E plays the role of the classically constant P_T and thus when constructing wave packets a superposition over non-negative values of E should be taken (see Eq. 41 below). Note that this choice for the sign of E has in particular the consequence, that the Hamiltonian involved in the WDW equation is given by the

quantization of the classical expression $|\bar{\mathcal{C}}|$, as can be seen directly from Eq. (33) as well as from the fact that $\bar{\mathcal{C}}$ is negative in the classical theory. Separation of variables in the form $\psi(u, v, w) = U(u)V(v)W(w)$ yields the following form for the functions $V(v)$ and $W(w)$:

$$V(v) = e^{\pm k_1 v}, \qquad W(w) = e^{\pm k_2 w}, \tag{36}$$

where k_1 and k_2 are separation constants where without loss of generality, we can choose positive values for them. At this point, let us take a look at the boundary condition on the $V(v)$ and $W(w)$ sector of the wave function. If we assume that for infinite values for the variables v and w, the wave function will be zero, we may take the solutions (36) with upper (lower) sign in the exponential function for $v, w < 0$ ($v, w > 0$). This means that the function $V(v)$ and $W(w)$ can be written as

$$V(v) = e^{-k_1|v|}, \qquad W(w) = e^{-k_2|w|}. \tag{37}$$

Also, for $U(u)$ we get

$$\left[2\frac{d^2}{du^2} + 3q\frac{d}{du} + (24Ee^{3(1-\omega)u} - v^2)\right]U(u) = 0, \tag{38}$$

where $v^2 = 2(k_1^2 + k_2^2)$. The solution of the above equation may be expressed in terms of the Bessel functions J and Y. However, for having well-defined functions in all ranges of the arguments of the Bessel functions, we only keep the function J, in terms of which the solution is

$$U(u) = e^{-\frac{3}{4}qu} J_{\frac{\sqrt{9q^2+8v^2}}{6(1-\omega)}}\left(\frac{4\sqrt{E}e^{\frac{3}{2}(1-\omega)u}}{\sqrt{3}(1-\omega)}\right). \tag{39}$$

Therefore, the eigenfunctions of the WDW equation may be written as

$$\Psi_{E,v}(u, v, w, T) = e^{-\frac{3}{4}qu}e^{iET}e^{-(|v|+|w|)v} J_{\frac{\sqrt{9q^2+8v^2}}{6(1-\omega)}}\left(\frac{4\sqrt{E}e^{\frac{3}{2}(1-\omega)u}}{\sqrt{3}(1-\omega)}\right), \tag{40}$$

in which since the behavior of v and w is similar we have taken $k_1 = k_2 = v/2$ and also the variables v, w are re-scaled as $|v|/2 \to |v|$, $|w|/2 \to |w|$. The wave function of the universe is indeed the general solution of the WDW equation which can be made from the superposition of its eigenfunctions. Thus, we may write the wave function as

$$\Psi(u, v, w, T) = \int_{E=0}^{\infty}\int_{v=0}^{\infty} A(E)C(v)\Psi_{E,v}(u, v, w, T)dEdv, \tag{41}$$

where $A(E)$ and $C(v)$ are some weight functions with the help of which suitable wavepacket will be constructed. Indeed, the wave function in quantum cosmology should be constructed in such a way as to achieve an acceptable match to the classical model. This means that by the above relation, one usually constructs a coherent

wavepacket with good asymptotic behavior in the minisuperspace, peaking in the vicinity of the classical trajectory. So, with the help of equality [36]

$$\int_0^\infty e^{-ar^2} r^{\eta+1} J_\eta(br) dr = \frac{b^\eta}{(2a)^{\eta+1}} e^{-\frac{b^2}{4a}}, \tag{42}$$

the integral over E may be evaluated to yield analytical expression if we choose the weight factor $A(E)$ to be a quasi-Gaussian function as

$$A(r) = \frac{8}{3(1-\omega)^2} r^\eta e^{-\gamma r^2}, \tag{43}$$

where γ is a positive constant and $r = \frac{4\sqrt{E}}{\sqrt{3}(1-\omega)}$. With this, integration over E will be done and we arrive at the following expression for the wave function

$$\Psi(u, v, w, T) = \frac{1}{2a} e^{-\frac{3}{4}qu} e^{-\frac{b^2}{4a}} \int_{v=0}^\infty C(v) e^{-(|v|+|w|)v} \left(\frac{b}{2a}\right)^\eta dv, \tag{44}$$

where $b = e^{\frac{3}{2}(1-\omega)u}$, $a = \gamma - \frac{3}{16}(1-\omega)^2 iT$ and $\eta = \frac{\sqrt{9q^2+8v^2}}{6(1-\omega)}$. A point to be noted is that our choices for $A(E)$ as quasi-Gaussian function also has physical grounds since these types of functions are widely used in quantum mechanics to build localized states. The reason is that these functions have a peak around a particular point of their argument and rapidly decrease as they move away from that point. This makes the wavepacket generated by the above relation (after integration over v), behaves the same way, i.e. is localized around some special values of its arguments.

Now, let us return to the superposition over v which as seen from (44) finding a function $C(v)$ that leads to an analytical expression for the wave function is not an easy task. However, notice that the integrand in (44) only has a significant amount for the small values of v and becomes smaller and smaller as v increases. Under this condition, we may take the wave function as being represented by relation (44) with the integral to be truncated at a suitable value of v displaying this property. So, if we assume that the superposition over v is taken over some values of v in the range $v << q$, we have $\eta \simeq \frac{q}{2(1-\omega)}$ and the integral (44) takes the form

$$\Psi(u, v, w, T) = \frac{1}{(2a)^{\eta+1}} e^{-\frac{b^2}{4a}} \int_{v=0}^\sigma C(v) e^{-(|v|+|w|)v} dv, \tag{45}$$

where σ is a positive constant. Now, we may use the equality [36]

$$\int_0^\sigma v^{\alpha-1} e^{-pv} dv = p^{-\alpha} \gamma(\alpha, \sigma p), \qquad \text{Re}(\alpha) > 0, \tag{46}$$

where $\gamma(a, x)$ is the incomplete Gamma function

$$\gamma(\alpha, z) = \int_0^z e^{-t} t^{\alpha-1} dt, \qquad \text{Re}(\alpha) > 0,$$

to take the weight factor $C(\nu)$ as $C(\nu) = \nu^{\alpha-1}$. At the first look, it seems that this form for the function $C(\nu)$ (which may look rather ad hoc) is chosen in such a way that yields an analytical expression for the wave function. However, note that for small values of ν, we have $\mu^2 = 1 - \nu^{\alpha-1} > 0$, so we may write $e^{-\mu^2} = e^{-(1-\nu^{\alpha-1})} \simeq 1 - \mu^2 = \nu^{\alpha-1}$. This shows that the in the domain of ν, for which we are going to perform the superposition (45), the function $C(\nu) = \nu^{\alpha-1}$ may be approximated with a Gaussian function whose physical grounds are described above. This lead us to

$$\Psi(u, v, w, T) = \frac{1}{(2a)^{\eta+1}} e^{-\frac{b^2}{4a}} (|v| + |w|)^{-\alpha} \gamma(\alpha, \sigma(|v| + |w|)). \quad (47)$$

Finally, in terms of the volume $V = e^{3u}$ the wave function reads as (we rescale $\frac{3}{16}T \to t$ and $\frac{V^{1-\omega}}{4} \to V^{1-\omega}$)

$$\Psi(V, v, w, t) = \frac{\mathcal{N}}{\left[\gamma - (1-\omega)^2 it\right]^{\eta+1}}$$
$$\times \exp\left[-\frac{V^{1-\omega}}{\gamma - (1-\omega)^2 it}\right] (|v| + |w|)^{-\alpha} \gamma(\alpha, \sigma(|v| + |w|)), \quad (48)$$

where $\mathcal{N}$ is a normalization factor. Now, with this wave function we are interested to see how the evolution of the dynamical variables is predicted in the framework of the quantum model. In fact, what we expect from a consistent model of quantum cosmology is that its late time predictions are in agreement with the classical model, but should be separated from classical solutions in the early times of cosmic evolution where classical singularities occur. To see how this procedure works in our model, let us evaluate the time dependence of the expectation value of a dynamical variable q as

$$\langle q \rangle(t) = \frac{\langle \Psi | q | \Psi \rangle}{\langle \Psi | \Psi \rangle}. \quad (49)$$

To calculate the expectation values, note that the WDW Eq. (33) is like a Schrödinger equation $i\partial\Psi/\partial T = H\Psi$, in which the Hamiltonian operator is Hermitian with respect to the inner product

$$\langle \Phi | \Psi \rangle = \int_{(V,v,w)} V^{1-\omega} \Phi^* \Psi \, dV \, dv \, dw. \quad (50)$$

Now, we may write the the expectation value for the volume as

$$\langle V \rangle(t) = \frac{\int_{V=0}^{\infty} \int_{v=-\infty}^{\infty} \int_{w=-\infty}^{\infty} V^{1-\omega} \Psi^* V \Psi \, dV \, dv \, dw}{\int_{V=0}^{\infty} \int_{v=-\infty}^{\infty} \int_{w=-\infty}^{\infty} V^{1-\omega} \Psi^* \Psi \, dV \, dv \, dw}, \quad (51)$$

which yields

$$\langle V \rangle(t) = V_0 \left[\gamma^2 + (1-\omega)^4 t^2 \right]^{\frac{1}{1-\omega}}. \tag{52}$$

Also, the expectation values of the anisotropic functions read as

$$\langle v \rangle(t) = \frac{\int_{V=0}^{\infty} \int_{v=-\infty}^{\infty} \int_{w=-\infty}^{\infty} V^{1-\omega} \Psi^* v \Psi dV dv dw}{\int_{V=0}^{\infty} \int_{v=-\infty}^{\infty} \int_{w=-\infty}^{\infty} V^{1-\omega} \Psi^* \Psi dV dv dw},$$
$$\text{similar expression for } \langle w \rangle(t), \tag{53}$$

with the result

$$\langle v \rangle(t) = \frac{\int_{v,w=-\infty}^{\infty} v \left[(|v|+|w|)^{-\alpha} \gamma \left(\alpha, \sigma(|v|+|w|) \right) \right]^2 dv dw}{\int_{v,w=-\infty}^{\infty} \left[(|v|+|w|)^{-\alpha} \gamma \left(\alpha, \sigma(|v|+|w|) \right) \right]^2 dv dw} = 0,$$
$$\text{similarly: } \langle w \rangle(t) = 0. \tag{54}$$

It is important whether the quantum solutions are capable of solving the singularity of the classical models. We see that the quantum wave function (48) gives the evolution of the expectation value of the volume factor as a nonsingular (never vanishing) bouncing function (52), which its late time behavior coincides to the late time behavior of the classical solution (26) and (30), that is $V(t) \sim t^{\frac{2}{1-\omega}}$. Another feature of the quantum picture is that it also predicts a zero expectation value for the anisotropy of the corresponding universe. This means that unlike classical models in which the expansion of the universe was anisotropic (and in some cases even with infinite amounts), the quantization of the system predicts an isotropic expansion. In summary, as a result of the quantization of this cosmological model, the two classical solutions presented in the previous section are united in a single non-singular picture.

5 Bohmian trajectories

In the previous section, we saw that quantization of the model based on the canonical quantization procedure led to the replacement of the classical singular solutions with a non-singular bouncing picture. Now, we might ask what causes the bounce? It is clear that the answer to this question should be sought in the emergence of quantum effects that show themselves when the size of the universe is very small. To deal with this question, we are going to use the ontological (Bohmian) interpretation of quantum mechanics [37–39]. In this formulation of the quantum mechanics one writes the wave function in the polar form $\Psi(\mathbf{q}, t) = \Omega(\mathbf{q}, t) e^{iS(\mathbf{q},t)}$, where Ω and S are some real functions of the configuration space variables and time. For the wavefunction (48) these function take the form

$$\Omega(V, v, w, t) = \left[\gamma^2 + (1-\omega)^4 t^2 \right]^{-\frac{\eta+1}{2}} \exp \left(-\frac{\gamma}{\gamma^2 + (1-\omega)^4 t^2} V^{1-\omega} \right) f(v, w), \tag{55}$$

$$S(V, v, w, t) = (\eta + 1) \arctan \frac{(1 - \omega)^2 t}{\gamma} - \frac{(1 - \omega)^2 t}{\gamma^2 + (1 - \omega)^4 t^2} V^{1-\omega}, \tag{56}$$

where $f(v, w) = (|v| + |w|)^{-\alpha} \gamma (\alpha, \sigma(|v| + |w|))$. In the Bohm-de Broglie interpretation of quantum mechanics the central equations come from substituting the above mentioned polar form of the wave function into the Schrödinger (or WDW) equation. By this procedure one gets a continuity equation and the modified Hamilton–Jacobi equation as, (see [40,41] for details)

$$H\left(q_i, p_i = \frac{\partial S}{\partial q_i}\right) + \mathcal{Q} = 0, \tag{57}$$

where $\mathcal{Q}$ is the quantum potential which in our case a simple algebra gives it as

$$\mathcal{Q} = \frac{1}{V^{1-\omega} \Omega} \left[18 V^2 \frac{\partial^2 \Omega}{\partial V^2} + 9(q + 2) V \frac{\partial \Omega}{\partial V} - 2 \left(\frac{\partial^2 \Omega}{\partial v^2} + \frac{\partial^2 \Omega}{\partial w^2} \right) \right]. \tag{58}$$

The dynamical behavior of the variables will be determined from the key equation $p_i = \frac{\partial S}{\partial q_i}$. To see how this mechanism works, first note that with $N = e^{3\omega u} = V^\omega$ the Lagrangian (10) takes the form $\mathcal{L}_g = -\frac{1}{3} \frac{\dot{V}^2}{V^{\omega+1}} + \cdots$, from which we obtain $p_V = -\frac{2}{3} \frac{\dot{V}}{V^{\omega+1}}$. Therefore, from (56), the equation $p_V = \frac{\partial S}{\partial V}$ gives us

$$\frac{2}{3} \frac{\dot{V}}{V^{\omega+1}} = \frac{(1 - \omega)^3 t}{\gamma^2 + (1 - \omega)^4 t^2} V^{-\omega}, \tag{59}$$

from which, after integration we obtain the Bohmian representation of the volume factor as (considering that we have already re-scaled t and V when writing the wavefunction 48)

$$V(t) \sim \left[\gamma^2 + (1 - \omega)^4 t^2 \right]^{\frac{1}{1-\omega}}. \tag{60}$$

Also, since $\frac{\partial S}{\partial v} = \frac{\partial S}{\partial w} = 0$, from (11) we arrive at

$$\dot{v} = \dot{w} = 0 \Rightarrow v, w = \text{const.} \tag{61}$$

These expressions have exactly the same dynamical behavior that we had previously obtained from the quantum cosmological model. As is clear, here too, the bouncing (from a non-zero value) behavior near the classical singularity is the main property of the volume function and the anisotropy functions such as the quantum model have a constant value. A point to be emphasized here is that, in addition to preventing the classical singularities, the appearance of a bounce in the quantum and Bohemian models, from this aspect is also interesting that it predicts the existence of a minimum size when the universe is evolving. This is important because we know that the existence of a minimal length in nature is an idea supported by all candidates of quantum gravity.

To understand the origin of the singularity avoidance, let us evaluate the quantum potential in terms of the volume and anisotropic factors. With the help of the Eq. (60) the function Ω takes the form

$$\Omega(V, v, w) = V^{-\frac{1}{4}(q - 2\omega + 2)} (|v| + |w|)^{-\alpha} \gamma\left(\alpha, \sigma(|v| + |w|)\right), \qquad (62)$$

by means of which and Eq. (58) we arrive at the following expression for the quantum potential

$$
\begin{aligned}
\mathcal{Q}(V, v, w) \\
&= -\frac{9}{8}(q - 2\omega + 2)(q + 2\omega - 2)V^{\omega - 1} \\
&\quad + V^{\omega - 1}\frac{e^{-\sigma(v+w)}\left[4(\sigma(v+w))^{\alpha}(\alpha + \sigma(v+w) + 1) - 4\alpha(\alpha + 1)e^{\sigma(v+w)}\gamma(\alpha, \sigma(v+w))\right]}{(v+w)^2\gamma(\alpha, \sigma(v+w))}.
\end{aligned}
$$
$$(63)$$

As this relation shows, the quantum potential goes to zero when the volume function is large (note that $\omega - 1 < 0$) which is an expected behavior since in this regime the universe evolves classically and so the quantum effects can be ignored. On the other hand, by reducing the volume function, the quantum potential increases and here is where the quantum effects are important in cosmology. Indeed, in this regime a huge repulsive force may be produced which can be extracted from the quantum potential as $\overrightarrow{\mathcal{F}} = -\nabla\mathcal{Q}$, with components

$$\mathcal{F}_V = -\frac{\partial\mathcal{Q}}{\partial V} = -\frac{(9/8)(1 - \omega)(q - 2\omega + 2)(q + 2\omega - 2) - (1 - \omega)h(v, w)}{V^{2-\omega}}, \qquad (64)$$

$$\mathcal{F}_{(v, w)} = -\frac{\partial\mathcal{Q}}{\partial(v, w)} = -\frac{e^{-2\sigma(v+w)}}{V^{1-\omega}}g(v, w), \qquad (65)$$

where $h(v, w)$ is the fraction appearing in the second term of (63) and $g(v, w) = \partial_v h / e^{-2\sigma(v+w)}$. From the above expressions it is clear that in the vicinity of classical singularity (for small values of V) this force becomes large and its repulsive nature prevents the volume to evolve to zero size but instead a bounce will occur. In summary, the above discussion shows that quantum potential and consequently quantum effects are important when the universe is in its early stages and in the late time of cosmic evolution, i.e. in the limit of the large scale factors these effects can be neglected, see Fig. 1. Therefore, asymptotically the classical behavior is recovered.

6 Summary

In this paper we have studied the Bianchi type I cosmological model with a perfect fluid as its matter content. To write the Hamiltonian of the corresponding dynamical system we have used the Schutz' formalism for perfect fluid, the advantages of which is that it allows us to introduce a time parameter in terms of the one of the fluid's
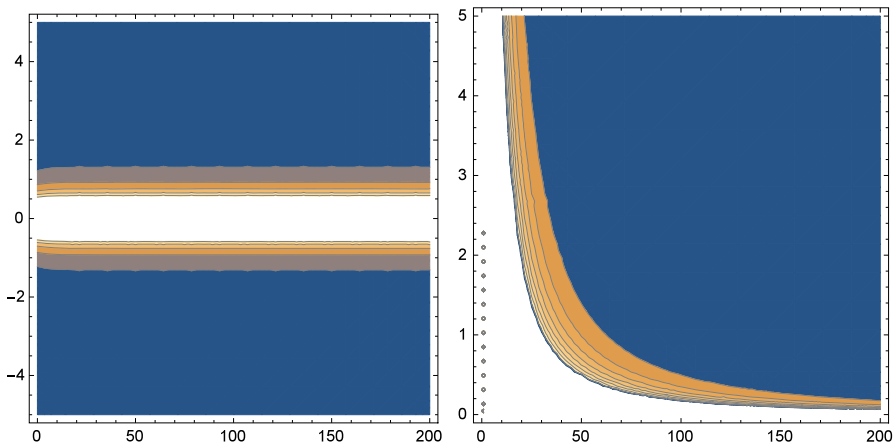
Fig. 1 The contour plots of $\mathcal{F}_V$ (left) and $\mathcal{F}_{(v,w)}$ (right). The figures are plotted for the numerical values: $\omega = -1, q = 1, \sigma = 1$ and $\alpha = 2$. As the figures show, these quantities get their significant amounts (white area) near the classical singularity. For $\mathcal{F}_V$ (left), to better see this issue, we have expanded the V-axis to negative values, although we know that V is positive

thermodynamical variable. Indeed, since the momentum conjugate to this variable appears linearly in the fluid's Hamiltonian, this formalism helps one to have an eye to the problem of time when quantizing the system.

Based on this time gauge, we constructed the Hamiltonian equations of motion, solving of them led us to two classes of classical solution. The first one gave us unacceptable physical solutions except for special values of EoS parameter in the form $\omega = \frac{2n-1}{2n}$. The evolution of volume factor in this case is such that, from a big-bang singularity it begins an power law expansion until reaches a maximum value and then its contraction phase starts and ends in a singularity. The evolution of the universe in this case is anisotropic with an infinite of anisotropy in the singular points. Finally, by another set of classical solutions, we obtained an expanding volume factor with singularity at $t = 0$ in which the volume of the universe is zero. While the universe expands its anisotropy remains constant which by a re-scaling of the coordinates we showed that this case can be viewed as a flat FRW universe.

We then have dealt with quantization of the cosmological setting. Here, because of our specific choice of time gauge, the WDW equation took the form of a Schrödinger equation allowed us to have a time dependent wave function. We showed that the WDW equation can be separated and its eigenfunctions can be obtained in terms of known special functions. By finding suitable weight function we constructed the superposition of the eigenfunctions which led us to an appropriate wave function. By means of the obtained wave function we got the expectation values of the volume and anisotropic functions. Based on the results, we verified that while the quantum solutions recover the late time behavior of classical models, they predict a non-singular bouncing universe in the early times of cosmic evolutions. Therefore, the avoidance of classical singularities and the recovery of the late time classical dynamics are important features of our quantum model.

In the last part of the article, we returned to the quantum picture by using of the Bohmian approach to quantum mechanics. It is shown that the resulting expressions for the dynamical variables of the system are the same as those previously obtained from the canonical quantization of the model. However, the use of the Bohmian trajectories has helped us to understand the origin of the avoidance of singularity by a repulsive force due to the existence of the quantum potential. We saw that near the classical singularities the quantum potential gets a huge amount and thus the repulsive nature of its corresponding force prevents the universe to reach the singularity. This is while this force tends to zero at late times which means that the quantum effects are negligible there and the universe behaves classically.

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