



Static spherically symmetric Einstein-aether models I: perfect fluids with a linear equation of state and scalar fields with an exponential self-interacting potential

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Abstract

We investigate the field equations in the Einstein-aether theory for static spherically symmetric spacetimes and a perfect fluid source and subsequently with the addition of a scalar field (with an exponential self-interacting potential). We introduce more appropriate dynamical variables that facilitate the study of the equilibrium points of the resulting dynamical system and, in addition, we discuss the dynamics at infinity. We study the qualitative properties of the models with a particular interest in their asymptotic behaviour and whether they admit singularities. We also present a number of new solutions.

Keywords Einstein-aether theory · Static spherically symmetric spacetimes · Asymptotic behaviour

Contents

1 Introduction	2
2 Einstein-aether gravity	6
3 Static spherically symmetric spacetime with a perfect fluid	8
4 Stationary comoving aether with perfect fluid and scalar field in static metric	24
5 Discussion	30
A Regularity conditions	40
B Equilibrium points in the finite region of the phase space for a perfect fluid with linear equation of state	43
C Equilibrium points in the finite region of the phase space for the exponential potential	47
References	56

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1 Introduction

Einstein-aether theory [1–9] is an effective field theory that consists of General relativity (GR) coupled to a dynamical time-like unit vector field, the aether. Since both the dynamical aether vector field and the geometric metric tensor characterize the spacetime structure [7], the Lorentz invariance is spontaneously broken by the choice of a preferred frame at each spacetime point (while local rotational symmetry is maintained). Such a Lorentz violation has been proposed to model quantum gravity effects at the microscopic level. In addition, every hypersurface-orthogonal Einstein-aether solution is a solution of the IR limit of Horava–Lifshitz gravity [10–12].

In a recent review some developments of Einstein-aether theory in general and Horava–Lifshitz theory in particular were discussed [13]. This included a discussion of universal horizons and black holes and their thermodynamics, non-relativistic gauge/gravity duality, and the quantization of the theory. The well-posedness of the Cauchy formulation of Einstein-aether theory was recently studied in [14] to ensure the stability of the numerical evolution of the initial value problem, and it was shown that, under suitable conditions on the parameters couplings, the governing equations can be cast into strongly hyperbolic form and even into symmetric hyperbolic form using a first-order formulation in the frame variables. Gravitational plane-waves in Einstein-aether theory were also recently studied, and it was found that [15] the vacuum Einstein-aether theory system of linearly polarized gravitational waves is, in general, overdetermined, and that there are further constraints on the coupling parameters c_i in order to allow arbitrary gravitational plane waves. In GR $c_i = 0$.

Cosmological scenarios in these theories were tested against new observational constraints including updated Cosmic Microwave Background data from Planck and the expansion rates of elliptical and lenticular galaxies, Joint Light-Curve Analysis data for Type Ia supernovae and Baryon Acoustic Oscillations. Using priors on the Hubble parameter and with an alternative parametrization of the equations in which the curvature parameter is considered as a free parameter in the analysis, it was found [16] that the detailed-balance scenario exhibits positive spatial curvature to more than 3σ , whereas for further theory generalizations it was found that there is evidence for positive spatial curvature at 1σ . In general, cosmologically viable extended Einstein-aether theories are known that are compatible with Planck Cosmic Microwave Background temperature anisotropy, polarization, and lensing data [17].

A number of exact solutions and a qualitative analysis of Einstein-aether cosmological models have been presented [18–25]. An emphasis has been placed on whether Lorentz violation affects the inflationary scenario (in particular, in spatially anisotropic cosmological models) in Einstein-aether theory [4–6]. Einstein-aether cosmology has been studied for the FRW metric [26] (including contracting, expanding and bouncing solutions), for the Kantowski–Sachs metric [22,27] and for spatially homogenous metrics [28]. In all cases the matter source was assumed to be coupled to the expansion of the aether field through an exponential potential. The models have been generalized to include an additional scalar field source.

In a recent paper [21] we studied spherically symmetric Einstein aether models with a perfect fluid matter source. We begin by discussing the field equations. In order to perform a dynamical systems analysis it is useful to introduce suitable normal-

ized variables [29–34], which also facilitates their numerical study. We then derive the equilibrium points of the algebraic-differential system in terms of proper normalized variables [21] and analyze their stability. The Einstein-aether static model with a perfect fluid was first introduced in Section 6.1 of [21] utilizing the dynamical variables inspired by [35]. We attempt to find asymptotic expansions for all of the solutions corresponding to the equilibrium points. In particular, explicit known exact spherically symmetric solutions are recovered [35–39] and a number of new solutions with naked singularities or horizons are found and the line elements are presented. We also investigate the dynamics at infinity and we present some numerical results that support our analytic results.

In addition to defining appropriate normalized variables, we also wish to utilize well defined coordinates and to exploit any symmetries of the spacetimes. Since we use qualitative techniques of dynamical systems theory that do not involve actually solving the field equations, some of the problems of coordinate choices and coordinate singularities are avoided. In particular, the local semi-tetrad splitting [40] allows the field equations to be recast in the form of an autonomous system of covariantly defined quantities [41,42].

1.1 The models and spherical symmetry

Spherically symmetric static and stationary solutions are physically important. The evolution equations follow from the Einstein aether action [1,3]. There are extra terms in the Einstein-aether field equations due to the effects of the aether field on the spherically symmetric geometry, and from an additional stress tensor, T_{ab}^{ae} , which depends on a number of dimensionless parameters c_i . In the case of spherical symmetry the aether is hypersurface orthogonal,¹ and so it has vanishing twist so that c_4 can be set to zero without loss of generality [7], leaving a 3-dimensional parameter space. A renormalization of the parameters in the model can be then used to set $8\pi G = 1$, where G defines the effective Newtonian gravitational constant, so that the model can consequently be characterized by only two non-trivial constant parameters. The remaining constraints on the c_i have been summarized in [7,43].

Solutions which involve a static metric coupled to a stationary aether are called “stationary spherical symmetric” models and, in principle, must be treated separately. If the spherically symmetric aether is parallel to the Killing vector, the solutions are referred to as “static aether” solutions (and an explicit solution is known [44]).

Therefore, in Einstein-aether theory, and in contrast to GR, there is an additional spherically symmetric mode corresponding to the radial tilting of the aether. That is, the preferred aether frame can be tilted relative to the CMB rest frame in spherically symmetric models, which adds additional terms to T_{ab}^{ae} , characterized by a so-called hyperbolic tilt angle, α , which measures the boost of the aether relative to this rest frame. The tilt is anticipated to decay in spatially homogeneous models [45,46]. For example, it was shown that to linear order in the anisotropy a Bianchi type I anisotropic

¹ Aether fields are hypersurface orthogonal in the spherically symmetric case and, hence, all solutions of Einstein aether theory will also be solutions of the IR limit of Horava gravity. The converse is not true generally, but it is so for spherically symmetric solutions with a regular center [7].

system (with a positive cosmological constant) relaxes exponentially to the isotropic, de Sitter solution, and that the tilt decays to the future [4]. The dynamics of a tilted aether in a Bianchi I cosmological model without the assumption of a small tilt was studied in [6], and it was found that when the initial hyperbolic tilt angle α (and its time derivative) is sufficiently small, then $\alpha \rightarrow 0$ at late times (consistent with the linearized stability analysis in [4]).

A number of time-independent spherically symmetric solutions and, in particular, black hole solutions, were studied in [44,47], surveyed in [7], and recently revisited in [43]. In general, the dynamics of the perturbations in non-rotating neutron stars and black hole solutions do not differ much from those in GR. Although a fully nonlinear positivity of the energy has been established for spherically symmetric solutions at an instant of time symmetry [9], a comprehensive investigation of the fully nonlinear solutions has not yet been done.

In particular, in Einstein-aether theory there is a 3-parameter family of spherically symmetric static vacuum solutions, since the aether vector and its derivative add 2 extra degrees of freedom at each spacetime point [47]. In the case that we assume asymptotic flatness, for a fixed mass there is then a single parameter family of solutions [7], unlike the the unique Schwarzschild solution in GR. In addition, in GR asymptotic flatness is a result of the vacuum field equations so that the 1-parameter family of local (Schwarzschild) solutions is immediately asymptotically flat. Since the radial aether tilt constitutes an additional local degree of freedom, spherical solutions in aether theory are not necessarily time-independent (even in the stationary case). Spherically symmetric solutions are not generally static, but even in the case of staticity they need not be asymptotically flat.

The model is restricted to a single parameter (the total mass) when the aether is aligned with the time-like Killing field [47]. Therefore, in this case, for a given mass the exterior solution for a static star is the unique “static aether” vacuum solution (depending on the c_i parameters via only one parameter) presented analytically in [47]; it has a global time-like Killing vector, is asymptotically flat, and the affine parameter distance to the singularity is finite along radial null geodesics. Although this static “wormhole” implies an effective negative energy density in the field equations, all solutions in this family actually have a positive total mass. In addition, it was found [48] that such a static aether solution is, in general, linearly stable under precisely the same conditions as flat Minkowski spacetime. In the pure GR limit, $c_1 = 0$, we just have the Schwarzschild solution. However, for small r values the static aether solutions can have quite different behaviour to that of the Schwarzschild solution. More recently, an analytic static spherically symmetric vacuum Einstein-aether solution was obtained numerically [44,47,49,50].

Unlike the case of a singular wormhole, the static solutions have an origin that is regular [47]. It is also well known that no asymptotically flat self-gravitating aether solutions with a regular origin exist [47]; i.e., there are no pure aether stars. In the presence of a perfect fluid, regular asymptotically flat stellar solutions have been shown to exist parameterized (for a given equation of state) by the central pressure (in addition to the vacuum aether parameters). If the central pressure is fixed, then there is only a single parameter that can be further tuned to obtain an asymptotically flat solution. Static aether star solutions with an interior with constant energy density were obtained

numerically in [47] by matching the interior solution to a specific vacuum exterior. The solution inside a fluid star has also been found by numerical integration for more realistic neutron star equations of state [51]. There are small differences from GR in sufficiently compact stars.

Since the Killing vector cannot be time-like on or inside an horizon, the aether cannot be aligned in the case of black holes. Instead, at spatial infinity the aether is at rest but travels in an inward direction at a finite radius. A unique spherical stationary solution from the 1-parameter family of solutions for a given mass is selected if regularity is required at the so-called spin-0 horizon [44,47]. This horizon develops in a regular region of spacetime when a black hole forms under gravitational collapse. Some particular examples of such a collapse producing a nonsingular black hole horizon have been confirmed in numerical simulations of scalar field collapse [52]. Black holes with a nonsingular spin-0 horizon are, in general, very similar to the Schwarzschild solution exterior to the horizon. But in the region interior to the horizon the solutions are typically different by a few percent. However, they do contain a spacelike singularity like the Schwarzschild spacetime. Recently static spherically symmetric, asymptotically flat, regular (non-rotating) black hole solutions in Einstein-aether theory have been studied numerically [43], generalizing the results of [44,47,50]. Quasi-normal modes of black holes in aether theory have also been investigated in [53,54].

This paper is the first of a series of papers devoted to the study of static and stationary Einstein-aether models, and it will be referred hereafter as Paper I. In Paper I here we will study, from the dynamical system point of view, the models and we shall classify the equilibrium points and comment on some particular interesting solutions. We note that the formalism employed facilitates a natural physical interpretation. In some cases the matter configuration is enclosed in a finite radius and the models have an astrophysical application [35,55,56]. In the companion paper [57], referred as the Paper II of the series, we will apply the classical singularity analysis, which is summarized in the so-called ARS algorithm [58–60]. Furthermore, the formulation of the modified Tolman–Oppenheimer–Volkoff (TOV) equations for perfect fluids with linear and polytropic equations of state (EoS) in the Einstein-aether theory is also of interest. The relativistic TOV equations are drastically modified in Einstein-aether theory [57]. The addition of a scalar field, φ , with an exponential or a harmonic potential is also of interest.

In future work in the series we will generalize the work to the conformally static (i.e., timelike selfsimilar) case and to scalar field models with generalized self-interacting potentials. In particular in [61], referred to as paper III of the series, will be studied the general monomial potential [62]:

$$W(\varphi) = \frac{1}{2n}(\mu\varphi)^{2n}, \quad \mu > 0, \quad n = 1, 2, \dots, \quad (1)$$

which contains as a particular case the harmonic potential $W(\varphi) = \frac{1}{2}m^2\varphi^2$.

The plan of the paper follows: the basic definitions of the Einstein-aether gravity are given in Sect. 2. The stability analysis for the static spherically symmetric perfect fluid spacetime are presented in Sect. 3, and the analysis with the additional scalar

field model is discussed in Sect. 4. Finally, in Sect. 5 we discuss our results and draw our conclusions.

2 Einstein-aether gravity

In Einstein-aether theory the action is given by the following expression [7,8]:

$$S = S_{GR} + S_u + S_m, \tag{2}$$

where $S_{GR} = \int d^4x \sqrt{-g} \left(\frac{R}{2}\right)$ is the Einstein–Hilbert term, S_m is the term which corresponds to the matter source and

$$S_u = \int d^4x \sqrt{-g} \left(-K^{ab}{}_{cd} \nabla_a u^c \nabla_b u^d + \lambda (u^c u_c + 1)\right), \tag{3}$$

corresponds to the aether field. λ is a Lagrange multiplier enforcing the time-like constraint on the aether [9], for which we have introduced the coupling [7]

$$K^{ab}{}_{cd} \equiv c_1 g^{ab} g_{cd} + c_2 \delta_c^a \delta_d^b + c_3 \delta_d^a \delta_c^b + c_4 u^a u^b g_{cd}, \tag{4}$$

which depends upon four dimensionless coefficients c_i . Finally u^a is the normalized observer in which $u^a u_a = -1$. For simplicity in the following we redefine the constants, c_i , as follows:

$$c_\theta = c_2 + (c_1 + c_3)/3, \quad c_\sigma = c_1 + c_3, \quad c_\omega = c_1 - c_3, \quad c_a = c_4 - c_1.$$

Variation with respect to the metric tensor in (2) provides the gravitational field equations

$$G_{ab} = T_{ab}^{ae} + T_{ab}^{mat}, \tag{5}$$

in which G_{ab} is the Einstein tensor, T_{ab}^m corresponds to S_m and T_{ab}^{ae} is the aether tensor [52]:

$$\begin{aligned} T_{ab}^{ae} = & 2c_1(\nabla_a u^c \nabla_b u_c - \nabla^c u_a \nabla_c u_b) - 2[\nabla_c(u_a J^c_b) + \nabla_c(u^c J_{(ab)}) - \nabla_c(u_a J_b^c)] \\ & - 2c_4 \dot{u}_a \dot{u}_b + 2\lambda u_a u_b + g_{ab} \mathcal{L}_u, \quad J^a_m = -K^{ab}{}_{mn} \nabla_b u^n, \quad \dot{u}_a = u^b \nabla_b u_a. \end{aligned} \tag{6}$$

In addition, variation with respect to the vector field u^a and the Lagrange multiplier gives us

$$\lambda u_b = \nabla_a J^a_b + c_4 \dot{u}_a \nabla_b u^a, \tag{7a}$$

$$u^a u_a = -1, \tag{7b}$$

where from (7a) we derive the Lagrange multiplier to be

$$\lambda = -u^b \nabla_a J_b^a - c_4 \dot{u}_a \dot{u}^a. \tag{8}$$

Hence the compatibility conditions are

$$0 = h^{bc} \nabla_a J_b^a + c_4 h^{bc} \dot{u}_a \nabla_b u^a. \tag{9}$$

The energy momentum tensor of the matter source in the form of a perfect fluid (with energy density μ , and pressure p) in the 1+3 decomposition with respect to u^a is given by:

$$T_{ab}^m \equiv \mu u_a u_b + p(g_{ab} + u_a u_b), \tag{10}$$

in which $h_{ab} = (g_{ab} + u_a u_b)$ is the projective tensor where $h^{ab} u_b = 0$. We shall use the Eq. (8) as a definition for the Lagrange multiplier, whereas the equation (9) leads to a set of constraints that the aether vector must satisfy.

The theory has additional degrees of freedom (model parameters) in flat space as compared with GR. The theory presents two spin-2 polarizations, as in GR, but also one spin-0 and two spin-1 polarizations. The squared propagation speeds on flat space are, respectively, given by [63]:

$$s_2^2 = \frac{1}{1 - c_1 - c_3} = -\frac{1}{c_\sigma - 1}, \tag{11}$$

$$s_1^2 = \frac{2c_1 - c_1^2 + c_3^2}{2(c_1 - c_4)(1 - c_1 - c_3)} = -\frac{c_\sigma c_\omega - c_\sigma - c_\omega}{2(\beta - 1)(c_\sigma - 1)}, \tag{12}$$

$$s_0^2 = \frac{(c_1 + c_2 + c_3)(2 - c_1 + c_4)}{(c_1 - c_4)(2 + c_1 + c_3 + 3c_2)} = -\frac{(\beta + 1)(3c_\theta + 2c_\sigma)}{3(\beta - 1)(3c_\theta + 2)}, \tag{13}$$

where we have introduced the parameter redefinition $\beta = c_a + 1$.

Stability at the classical and quantum levels requires all of the s_i^2 ($i = 0, 1, 2$) to be positive [9,63]. Ultra-high energy cosmic ray observations requires $s_i^2 > 1 - \mathcal{O}(10^{-15})$ to prevent cosmic rays from losing energy into gravitational modes via Cherenkov-like cascade [64]. Additionally, from constraints on the PPN parameters it follows that $|\alpha_1| \lesssim 10^{-4}$ and $|\alpha_2| \lesssim 10^{-5}$, $\alpha_1 = -\frac{8(c_3^2 - c_1 c_4)}{-c_1^2 + 2c_1 + c_3^2}$, $\alpha_2 = -\frac{(-c_1 + c_4 + 2)(c_1 + 2c_3 - 3 + c_4)(2c_1 - 1 + 3c_2 - 2 + c_3 - c_4)}{c_1 + c_2 + c_3} + \frac{\alpha_1}{2}$ [65]. Combining all the above restrictions we find $c_\omega \approx \mathcal{O}(10^{-15})$, $\beta \approx 1 + \mathcal{O}(10^{-4})$, $c_\sigma \approx 3c_\theta + \mathcal{O}(10^{-4})$. These bounds change if we consider static spherically symmetric curved space or if we change the matter content to include a perfect fluid or scalar field. Therefore, we assume no bounds on the model parameters.

3 Static spherically symmetric spacetime with a perfect fluid

In a static spherically symmetric spacetime with line element

$$ds^2 = -N^2(r)dt^2 + \frac{dr^2}{r^2} + K^{-1}(r)(d\vartheta^2 + \sin^2 \vartheta d\varphi^2), \tag{14}$$

that is, we have fixed the spatial gauge to have $e_1^1(r) \equiv r$, the field equations are [21]:

$$r \frac{dx}{dr} = \frac{\mu + 3p}{2\beta} + 2(\beta - 1)y^2 + 3xy + K, \tag{15a}$$

$$r \frac{dy}{dr} = \frac{\mu + 3p}{2\beta} + 2xy - y^2, \tag{15b}$$

$$r \frac{dp}{dr} = -y(\mu + p), \tag{15c}$$

$$r \frac{dK}{dr} = 2xK. \tag{15d}$$

where $x \equiv \frac{1}{2}r \frac{d \ln(K)}{dr}$, $y \equiv r \frac{d \ln(N)}{dr}$, and p is the pressure of the perfect fluid. Furthermore there exists the constraint equation

$$x^2 = (\beta - 1)y^2 + 2xy + p + K, \tag{16}$$

From (15c) and (15d) we have that $y = -\frac{rp'}{\mu+p}$, $x = \frac{1}{2} \frac{rK'}{K}$, and substituting into (15a), (15b) we find a system of two second-order ordinary differential equations,

$$r^2 \left[\frac{p''}{\mu + p} - \frac{p'\mu'}{(\mu + p)^2} - \frac{p'K'}{K(\mu + p)} - \frac{2(p')^2}{(\mu + p)^2} \right] + \frac{rp'}{\mu + p} + \frac{\mu + 3p}{2\beta} = 0, \tag{17a}$$

$$r^2 \left[\frac{K''}{2K} - \frac{1}{2} \left(\frac{K'}{K} \right)^2 + \frac{3}{2} \frac{p'K'}{K(\mu + p)} - \frac{2(\beta - 1)(p')^2}{(\mu + p)^2} \right] + \frac{rK'}{2K} - K - \frac{\mu + 3p}{2\beta} = 0, \tag{17b}$$

and (16) becomes

$$r^2 \left[(\beta - 1) \left(\frac{p'}{\mu + p} \right)^2 + \frac{p'K'}{K(\mu + p)} - \frac{1}{4} \left(\frac{K'}{K} \right)^2 \right] + p + K = 0, \tag{18}$$

where prime means the derivative with respect r .

3.1 Phase-space evolution

In this section we use the dynamical systems approach for investigating the structure of the whole solution space of (15) [23–25]. With this purpose, we introduce the quantities $\theta = y - x$, $\sigma = y$ as in [35].² The equations then read:

$$r \frac{d\theta}{dr} = -\beta\sigma^2 - \theta^2 + \theta\sigma + p, \tag{19}$$

$$r \frac{d\sigma}{dr} = \frac{-4\beta\theta\sigma + \beta\sigma^2 + \theta^2 - K + \mu_0 + (\eta + 1)p}{2\beta}, \tag{20}$$

$$r \frac{dp}{dr} = -\sigma(\eta p + \mu_0), \tag{21}$$

$$r \frac{dK}{dr} = 2K(\sigma - \theta), \tag{22}$$

where

$$K + p = -\beta\sigma^2 + \theta^2, \tag{23}$$

and we have assumed a linear EoS

$$\mu = \mu_0 + (\eta - 1)p, \tag{24}$$

where the constants μ_0 and η satisfy $\mu_0 \geq 0$, $\eta \geq 1$. The case $\eta = 1$ corresponds to an incompressible fluid with constant energy density, while the case $\mu_0 = 0$ describes a scale-invariant EoS.

Next, we introduce the scale invariant quantities:

$$Q = \frac{\theta}{\sqrt{\frac{\mu_0}{\eta} + \theta^2}}, \quad S = \frac{\sigma}{\sqrt{\frac{\mu_0}{\eta} + \theta^2}}, \quad C = \frac{\eta K}{\mu_0 + \eta\theta^2}, \tag{25}$$

which are more appropriate for describing the dynamics than those used in [21] as it covers new equilibrium points with $Q = S$ (i.e., $x = 0$, where $x \equiv \frac{1}{2} \partial_r \ln(K)$). Furthermore, to define the x -normalized dimensionless variables and the new independent coordinate τ given by $\partial_r(f) = -x \partial_\tau(f)$ in (6.7) of [21], it was presumably assumed that x does not change sign during the whole evolution; but, when x changes sign, the direction of the flow given by the independent variable τ in [21] is lost. For this reason we use below the variables (25) and we introduce a new independent variable λ given by [35]:

$$\frac{dr}{rd\lambda} = \frac{1}{\sqrt{\frac{\mu_0}{\eta} + \theta^2}} = \begin{cases} \sqrt{\frac{\eta(1-Q^2)}{\mu_0}}, & \mu_0 \neq 0 \\ \sqrt{\frac{C}{K}}, & \mu_0 = 0, C \neq 0 \\ \frac{S}{\sigma}, & \mu_0 = 0, C = 0, \end{cases} \tag{26}$$

² Do not confuse these quantities with the usual expansion and shear scalars of homogeneous cosmologies.

which defines unequivocally the flow direction. The “past attractors” ($\lambda \rightarrow -\infty$) corresponds to $r \rightarrow 0$ and the “future attractors” ($\lambda \rightarrow \infty$) corresponds to $r \rightarrow \infty$.

The relation between the variables $\{Q, S, C\}$ and $\{U, V, Y\}$ to be used in the forthcoming paper [57] is

$$U = \frac{C(\eta - 1) - \eta + Q^2 + \beta(\eta - 1)S^2}{C(\eta - 2) - \eta + 2Q^2 - 2QS + S^2(\beta(\eta - 1) + 1)}, \tag{27a}$$

$$V = \frac{(C - (Q - S)^2)(C(\eta - 1) - \eta + Q^2 + \beta(\eta - 1)S^2)}{(C + (Q - S)^2)(C - Q^2 + \beta S^2) + \eta(C - (Q - S)^2)(C + \beta S^2 - 1)}, \tag{27b}$$

$$Y = \frac{C - Q^2 + \beta S^2}{\eta(C + \beta S^2 - 1)}. \tag{27c}$$

We obtain then the evolution equations

$$\frac{dQ}{d\lambda} = (Q^2 - 1)(C + S(2\beta S - Q)), \tag{28a}$$

$$\frac{dS}{d\lambda} = \frac{C(2\beta QS - \eta - 2) + (\beta S^2 - 1)(4\beta QS - \eta - 2Q^2)}{2\beta}, \tag{28b}$$

$$\frac{dC}{d\lambda} = 2C(Q(C + 2\beta S^2 - QS - 1) + S). \tag{28c}$$

We have the useful relations

$$\frac{\mu_0}{\eta} = \frac{(1 - Q^2)K}{C}, \quad p = -\frac{K(C + \beta S^2 - Q^2)}{C}, \quad \frac{C(\mu + p)}{\eta K} = 1 - C - \beta S^2. \tag{29}$$

The Eq. (28) reduce to the system (17) investigated in [35] for $\beta = 1$. Because $\mu_0 \geq 0, \eta \geq 1$, it follows that $-1 \leq Q \leq 1$. Due to $K \geq 0$, it follows that $C \geq 0$. The condition $C + \beta S^2 - Q^2 = 0$ defines the surface of zero-pressure. However, it is not an invariant set of (28), neither $C + \beta S^2 - Q^2 > 0$.³ If we assume that the weak energy condition $p + \mu \geq 0$ is satisfied, then we obtain the subset of the phase space $1 - C - \beta S^2 \geq 0$. Defining $\Omega = 1 - C - \beta S^2$ we obtain

$$\frac{d\Omega}{d\lambda} = \Omega \left[2CQ + S(4\beta QS - \eta - 2Q^2) \right]. \tag{30}$$

Thus $1 - C - \beta S^2 \geq 0$ defines an invariant set. In summary, the Eq. (28) define a flow on the invariant set

$$\left\{ (Q, S, C) : -1 \leq Q \leq 1, C \geq 0, C + \beta S^2 \leq 1 \right\}. \tag{31}$$

³ A set of states $E \subset \mathbb{R}^n$ of a system of differential equations, say (28), is called an invariant set of (28) if for all $\mathbf{x}_0 \in E$ and for all $\lambda \geq 0, \mathbf{x}(\lambda; \mathbf{x}_0) \in E$, where by $\mathbf{x}(\lambda; \mathbf{x}_0)$ we understand the solution of (28) satisfying the initial condition $\mathbf{x}(0; \mathbf{x}_0) = \mathbf{x}_0$, evaluated at λ .

This phase-space is compact for $\beta \geq 0$ and unbounded for $\beta < 0$. The invariant sets $Q = \pm 1$ corresponds to $\mu_0 = 0$. The expression $(C + \beta S^2 - Q^2) = 0$ defines a surface on the phase space, which refers to the surface of vanishing pressure. This surface, however, is *not* an invariant surface for the flow.

A monotonic function excludes equilibrium points, periodic orbits, recurrent orbits, and homoclinic orbits in its domain. As in [35] we introduce the function

$$Z = \frac{2Q - S}{\sqrt{(2Q - S)^2 + 3(1 - Q^2)}}, \tag{32}$$

which satisfies

$$\frac{dZ}{d\lambda} = -\frac{3(1 - Q^2)(2(2\beta - 1)C + 2(Q - 2\beta S)^2 + \eta S\Omega)}{2\beta(3(1 - Q^2) + (2Q - S)^2)^{3/2}}. \tag{33}$$

Since $C \geq 0, W \geq 0$ it is obvious that (32) is a monotonic decreasing function for $(2\beta - 1) \geq 0$. Furthermore, it is defined everywhere except on the scale-invariant boundaries $Q = \pm 1$. Hence, the “past” ($r \rightarrow 0$) and the “future” ($r \rightarrow \infty$) attractors lie on the $Q = \pm 1$ boundary sets. We also have the auxiliary equations

$$\frac{d \ln N}{d\lambda} = S, \tag{34a}$$

$$\frac{d \ln K}{d\lambda} = -2(Q - S), \tag{34b}$$

$$\frac{d \ln y}{d\lambda} = \frac{-C(\eta + 2) + 2Q(Q - 2\beta S) - \beta\eta S^2 + \eta}{2\beta S}. \tag{34c}$$

The relation between the gravitational potential ϕ (related with the lapse function by $N = e^\phi$), and the matter field is given by

$$\frac{d\phi}{dp} = -\frac{1}{\mu + p}, \quad \mu = \mu_0 + (\eta - 1)p. \tag{35}$$

Hence,

$$e^\phi = e^{c_1}(\mu_0 + \eta p)^{-1/\eta} = \alpha \left(\frac{1 - Q^2}{1 - C - \beta S^2} \right)^{\frac{1}{\eta}}. \tag{36}$$

where α is a freely specifiable constant corresponding to the freedom of scaling the time coordinate in the line element.

The line element expressed in the variables (25) and the dynamical system (28) are invariant under the discrete symmetry

$$(Q, S) \rightarrow (-Q, -S). \tag{37}$$

with a simultaneous reversal of the radial direction $\lambda \rightarrow -\lambda$. With respect to the phase space dynamics this implies that for two points related by this symmetry, say A^+ and A^- , one has the opposite dynamical behavior to the other; that is, if the equilibrium point A^+ is an attractor for a choice of parameters, then A^- is a sink for the same choice of parameters. On the other hand, as both the system and the line element are invariant under (37), a physical solution is represented by two orbits in the phase space. We can, however, without loss of generality, focus upon orbits entering the phase space from the “upper” boundary set $Q = +1$ [35].

3.1.1 Equilibrium points in the finite region of the phase space

The equilibrium points of the system (28) are described in the Appendix B. In Table 1 we summarize the existence and stability conditions of the equilibrium points of physical interest of the system (28). At the relevant equilibrium points we discuss some regularity conditions of the corresponding physical solutions (see Appendix A.1):

1. P_1^+ is a source for $\beta = 1, 1 \leq \eta < 2$. Since the conditions (93) are fulfilled this solution has a regular center as $\lambda \rightarrow -\infty$. Because of $C = 0$ it belongs to the plane-symmetric boundary set. Furthermore, it belongs to the scale invariant boundary $Q = +1$.
2. P_1^- is a sink for $\beta = 1, 1 \leq \eta < 2$. Because of $C = 0$ it belongs to the plane-symmetric boundary set. Furthermore, it belongs to the scale invariant boundary $Q = -1$. Since the conditions (94) are fulfilled this solution is asymptotically flat as $\lambda \rightarrow +\infty$.
3. P_2^+ is a source for $\beta = 1, \eta \geq 1$. Since the first inequality of (93) is not fulfilled, this solution does not have a regular center. Because of $C = 0$ it belongs to the plane-symmetric boundary set. Furthermore, it belongs to the scale invariant boundary $Q = +1$.
4. P_2^- is a sink for $\beta = 1, \eta \geq 1$. Because of $C = 0$ it belongs to the plane-symmetric boundary set. Furthermore, it belongs to the scale invariant boundary $Q = -1$. This solution is not asymptotically flat since the conditions (94) are not fulfilled as $\lambda \rightarrow +\infty$.
5. P_2^+ is a sink for $\beta = -\frac{\eta+2}{4} \leq 1, \eta \geq 1$. Because of $C = 0$ it belongs to the plane-symmetric boundary set. Furthermore, it belongs to the scale invariant boundary $Q = +1$. This solution is not asymptotically flat since the conditions (94) are not fulfilled as $\lambda \rightarrow +\infty$.
6. P_2^- is a source for $\beta = -\frac{\eta+2}{4} \leq 1, \eta \geq 1$. Since the first inequality of (93) is not fulfilled, this solution does not have a regular center as $\lambda \rightarrow -\infty$. Because of $C = 0$ it belongs to the plane-symmetric boundary set. Furthermore, it belongs to the scale invariant boundary $Q = -1$.
7. P_5^+ is a sink for $\eta \geq 1, \beta < 0$. Because of $C = 0$ it belongs to the plane-symmetric boundary set. Furthermore, it belongs to the scale invariant boundary $Q = +1$. This solution is not asymptotically flat since the conditions (94) are not fulfilled as $\lambda \rightarrow +\infty$.
8. P_5^- is a source for $\eta \geq 1, \beta < 0$. Since the conditions (93) are not fulfilled, this solution does not have a regular center as $\lambda \rightarrow -\infty$. Because of $C = 0$ it belongs

Table 1 Eigenvalues, stability and characterization of the equilibrium points of the dynamical system defined by equations (28)

Labels	$(Q, S, C), g_{\mu\nu}$	Existence	Stability
P_1^\pm	$(\pm 1, \pm 1, 0)$ $\left\{ \begin{array}{l} \text{diag} \left(-\bar{N}_0^2 e^{\pm 2\lambda}, \frac{e^{\pm 2\lambda}}{c_1^2}, \bar{K}_0^{-1}, \bar{K}_0^{-1} \sin^2 \vartheta \right), \beta = 1 \\ \text{diag} \left(-\bar{N}_0^2 e^{\pm 2\lambda}, \frac{e^{\pm 11\lambda}}{c_1^2}, \bar{K}_0^{-1}, \bar{K}_0^{-1} \sin^2 \vartheta \right), \beta = \frac{\eta+2}{4} \end{array} \right.$	$\beta = 1$ or $\beta = \frac{\eta+2}{4} \leq 0$.	P_1^+ source for $\beta = 1, 1 \leq \eta < 2$ P_1^- sink for $\beta = 1, 1 \leq \eta < 2$ P_1^\pm saddle for $\beta = 1, \eta > 2$ P_1^\pm saddle for $\beta = \frac{\eta+2}{4} \leq 0$
P_2^\pm	$(\pm 1, \mp 1, 0)$ $\left\{ \begin{array}{l} \text{diag} \left(-\bar{N}_0^2 e^{\pm 2\lambda}, \frac{e^{\pm 6\lambda}}{c_1^2}, \bar{K}_0^{-1} e^{\pm 4\lambda}, \bar{K}_0^{-1} e^{\pm 4\lambda} \sin^2 \vartheta \right), \beta = 1 \\ \text{diag} \left(-\bar{N}_0^2 e^{\pm 2\lambda}, \frac{e^{\mp 11\lambda}}{c_1^2}, \bar{K}_0^{-1} e^{\pm 4\lambda}, \bar{K}_0^{-1} e^{\pm 4\lambda} \sin^2 \vartheta \right), \beta = -\frac{\eta+2}{4} \end{array} \right.$	$\beta = 1$ or $4\beta + \eta + 2 = 0, \beta \leq 1$.	P_2^+ source for $\beta = 1, \eta \geq 1$ P_2^- sink for $\beta = 1, \eta \geq 1$ P_2^+ sink for $\beta = -\frac{\eta+2}{4} \leq 1, \eta \geq 1$ P_2^- source for $\beta = -\frac{\eta+2}{4} \leq 1, \eta \geq 1$
P_3^\pm	$(\pm 1, 0, 1)$ $\text{diag} \left(-\bar{N}_0^2, \bar{K}_0^{-1} e^{\pm 2\lambda}, \bar{K}_0^{-1} e^{\pm 2\lambda}, \bar{K}_0^{-1} e^{\pm 2\lambda} \sin^2 \vartheta \right)$	Always	Saddle
P_4^\pm	$\left(\pm 1, \pm \frac{2}{\eta+2}, 1 - \frac{8\beta}{(\eta+2)^2} \right)$ $\text{diag} \left(-\bar{N}_0^2 e^{\pm \frac{4\lambda}{2+\eta}}, \frac{1}{\bar{K}_0} \left(1 - \frac{8\beta}{(\eta+2)^2} \right)^{\pm \frac{2\eta\lambda}{2+\eta}}, \frac{1}{\bar{K}_0} e^{\pm \frac{2\eta\lambda}{2+\eta}}, \frac{1}{\bar{K}_0} e^{\pm \frac{2\eta\lambda}{2+\eta}} \sin^2 \vartheta \right)$	$0 \leq \beta \leq \frac{1}{8}(\eta+2)^2$	Saddle
P_5^\pm	$\left(\pm 1, \pm \frac{\eta+2}{4\beta}, 0 \right)$ $\text{diag} \left(-\bar{N}_0^2 e^{\pm \frac{\eta+2\lambda}{2\beta}}, \frac{(\eta+2)^2 e^{\pm \frac{\eta(\eta+2)\lambda}{4\beta}}}{16c_1^2 \beta^2}, \bar{K}_0^{-1} e^{\pm 2 \left(1 - \frac{\eta+2}{4\beta} \right) \lambda}, \bar{K}_0^{-1} e^{\pm 2 \left(1 - \frac{\eta+2}{4\beta} \right) \lambda} \sin^2 \vartheta \right)$	$\eta \geq 1, \beta < 0$ or $\eta > 1, 16\beta \geq (\eta+2)^2$	P_5^+ sink for $\eta \geq 1, \beta < 0$ P_5^- source for $\eta \geq 1, \beta < 0$ Saddle otherwise

Table 1 continued

Labels	$(Q, S, C), g_{\mu\nu}$	Existence	Stability
P_6^\pm	$(\pm 1, \pm \frac{1}{\sqrt{\beta}}, 0)$		
	$\text{diag} \left(-\bar{N}_0^2 e^{\pm \frac{2\lambda}{\sqrt{\beta}}}, e^{\pm \frac{2(2\sqrt{\beta}+1)\lambda}{c_1^2\beta}}, \bar{K}_0^{-1} e^{\pm \frac{2(\sqrt{\beta}-1)\lambda}{\sqrt{\beta}}}, \bar{K}_0^{-1} e^{\pm \frac{2(\sqrt{\beta}-1)\lambda}{\sqrt{\beta}}} \sin^2 \vartheta \right)$	$\eta \geq 1, \beta > 0$	P_6^+ source for $\eta \geq 1, 16\beta \geq (\eta + 2)^2$ P_6^- sink for $\eta \geq 1, 16\beta \geq (\eta + 2)^2$ Saddle for $\eta \geq 1, 0 < \beta < \frac{1}{4}$ or $\eta \geq 1, \frac{1}{4} < \beta < \frac{1}{16}(\eta + 2)^2$ Non-hyperbolic otherwise
P_7^\pm	$(\pm 1, \mp \frac{1}{\sqrt{\beta}}, 0)$		
	$\text{diag} \left(-\bar{N}_0^2 e^{\pm \frac{2\lambda}{\sqrt{\beta}}}, e^{\pm \frac{2(1+2\sqrt{\beta})\lambda}{c_1^2\beta}}, \bar{K}_0^{-1} e^{\pm \frac{2(1+\sqrt{\beta})\lambda}{\sqrt{\beta}}}, \bar{K}_0^{-1} e^{\pm \frac{2(1+\sqrt{\beta})\lambda}{\sqrt{\beta}}} \sin^2 \vartheta \right)$	$\eta \geq 1, \beta > 0$	P_7^+ source for $\eta \geq 1, \beta > 0$ P_7^- sink for $\eta \geq 1, \beta > 0$
P_8^\pm	$(\pm \frac{1}{\sqrt{1-\beta}}, \pm \frac{1}{\sqrt{1-\beta}}, \frac{2\beta-1}{\beta-1})$		
	$\text{diag} \left(-\bar{N}_0^2 e^{\pm \frac{2\lambda}{\sqrt{1-\beta}}}, \frac{\eta}{\mu_0} \left(\frac{\beta}{\beta-1} \right), \bar{K}_0^{-1}, \bar{K}_0^{-1} \sin^2 \vartheta \right)$	$\beta \leq 0$	P_8^+ sink for $\beta < 0$ P_8^- source for $\beta < 0$
P_9^\pm	$(\pm 2\sqrt{\beta}, \pm \frac{1}{\sqrt{\beta}}, 0)$		
	$\text{diag} \left(-\bar{N}_0^2 e^{\pm \frac{2\lambda}{\sqrt{\beta}}}, \frac{\eta}{\mu_0} (1 - 4\beta), \bar{K}_0^{-1} e^{\pm \frac{2(2\beta-1)\lambda}{\sqrt{\beta}}}, \bar{K}_0^{-1} e^{\pm \frac{2(2\beta-1)\lambda}{\sqrt{\beta}}} \sin^2 \vartheta \right)$	$\eta \geq 1, 0 < \beta \leq \frac{1}{4}$	Saddle

We use the coordinates $(t, \lambda, \vartheta, \varphi)$. **diag** (...) denotes the diagonal 4×4 matrix. We assume $\eta \geq 1$. We have used the notation $\epsilon = \pm 1$ and c_1, c_2 are integration constants

to the plane-symmetric boundary set. Furthermore, it belongs to the scale invariant boundary $Q = -1$.

9. P_6^+ is a source for $\eta \geq 1, 16\beta \geq (\eta + 2)^2$. It has a regular center as $\lambda \rightarrow -\infty$ only when $\beta = 1$ (i.e., when this point coincides with P_1^+). Otherwise the conditions (93) are not fulfilled, and the solution does not have a regular center as $\lambda \rightarrow -\infty$. Because of $C = 0$ it belongs to the plane-symmetric boundary set. Furthermore, it belongs to the scale invariant boundary $Q = +1$.
10. P_6^- is a sink for $\eta \geq 1, 16\beta \geq (\eta + 2)^2$. Because of $C = 0$ it belongs to the plane-symmetric boundary set. Furthermore, it belongs to the scale invariant boundary $Q = -1$. It is not asymptotically flat as $\lambda \rightarrow +\infty$ unless $\beta = 1$ (i.e., when P_6^- merge with P_1^-).
11. P_7^+ is a source for $\eta \geq 1, \beta > 0$. The conditions (93) are not fulfilled, and the solution does not have a regular center as $\lambda \rightarrow -\infty$. Because of $C = 0$ it belongs to the plane-symmetric boundary set. Furthermore, it belongs to the scale invariant boundary $Q = +1$.
12. P_7^- is a sink for $\eta \geq 1, \beta > 0$. Because of $C = 0$ it belongs to the plane-symmetric boundary set. Furthermore, it belongs to the scale invariant boundary $Q = -1$.
13. P_8^+ is a sink for $\beta < 0$. It is not asymptotically flat as $\lambda \rightarrow +\infty$.
14. P_8^- is a source for $\beta < 0$. It has a regular center as $\lambda \rightarrow -\infty$ if $\eta > 1, \beta < 0, \mu_0 > 0, p_c \geq \frac{\mu_0(6\beta - \beta\eta - 3)}{\beta\eta(\eta - 1)}$.

There are relevant equilibrium points which are saddle points:

1. The equilibrium point P_3^+ represents the Minkowski spacetime in spherical symmetric form. The idea now is to find approximated solutions for the regular orbit near P_3^+ as $\lambda \rightarrow -\infty$. In the limit $\lambda \rightarrow -\infty$ the unstable manifold of P_3^+ provides the necessary mathematical structure for constructing this approximated solution. For this reason we introduce the coordinate transformation

$$Q = \frac{3\beta v_1}{2} - \frac{1}{4}(\eta + 2)v_2 + 1, \quad S = u_1 + v_1, \quad C = 1 - v_2. \tag{38}$$

Applying the Invariant Manifold theorem we find that the local unstable manifold of P_3^+ ,

$\{(u_1, v_2, v_3): u_1 = h(v_1, v_2), h(0, 0) = 0, \frac{\partial h}{\partial v_1}(0, 0) = 0, \frac{\partial h}{\partial v_2}(0, 0) = 0, v_1^2 + v_2^2 \leq \delta\}, \delta > 0$, can be approximated up to third order by the graph

$$u_1 = h(v_1, v_2) \approx \frac{\eta v_1 v_2 (-3\beta + \eta + 2)}{30\beta}. \tag{39}$$

Now, substituting the approximated solutions (found by solving just the linear part of the differential equations along the unstable eigendirections):

$$v_1 = \epsilon_1 e^{2\lambda}, \quad v_2 = \epsilon_2 e^{2\lambda}, \tag{40}$$

where ϵ_1, ϵ_2 are small positive constants, and keeping only the linear terms in ϵ we find the approximated solutions

$$Q = 1 + \frac{3\beta}{2}\epsilon_1 e^{2\lambda} - \frac{1}{4}(\eta + 2)\epsilon_2 e^{2\lambda}, \quad S = \epsilon_1 e^{2\lambda}, \quad C = 1 - \epsilon_2 e^{2\lambda}. \quad (41)$$

Replacing

$$\epsilon_1 = -\frac{2}{3}(\epsilon_1 - \epsilon_2), \quad \epsilon_2 = \frac{4}{2 + \eta}((1 - \beta)\epsilon_1 + \epsilon_2), \quad (42)$$

where ϵ_1 and ϵ_2 are still small constants (we assume they are positive), we find the more familiar equations

$$Q = 1 - \epsilon_1 e^{2\lambda}, \quad S = \frac{2}{3}(\epsilon_2 - \epsilon_1)e^{2\lambda}, \quad C = 1 - \frac{4}{2 + \eta}((1 - \beta)\epsilon_1 + \epsilon_2)e^{2\lambda}, \quad (43)$$

that reproduce equations (27a–27c) of [35] for $\beta = 1$. We see that $\epsilon = \frac{\epsilon_1}{\epsilon_2}$ parametrize a 1-parameter family of regular solutions with an equation of state parameter at the center:

$$\begin{aligned} \frac{p_c}{\mu_c} &= \lim_{\lambda \rightarrow -\infty} \frac{p}{\mu} = \lim_{\lambda \rightarrow -\infty} \frac{\mu - \mu_0}{(\eta - 1)\mu} = \lim_{\lambda \rightarrow -\infty} \frac{C - Q^2 + \beta S^2}{C(\eta - 1) - \eta + Q^2 + \beta(\eta - 1)S^2} \\ &= \frac{2 - \epsilon(2\beta + \eta)}{\epsilon(2\beta(\eta - 1) - 3\eta) + 2(\eta - 1)}. \end{aligned} \quad (44)$$

We see that there exists solutions with a regular center but negative pressure, so that we have to impose the condition⁴

$$\frac{2 - \epsilon(2\beta + \eta)}{\epsilon(2\beta(\eta - 1) - 3\eta) + 2(\eta - 1)} > 0, \quad (45)$$

that is:

- (a) $\eta > 1, \beta \leq -\frac{\eta}{2}, \epsilon > 0$, or
- (b) $\eta > 1, -\frac{\eta}{2} < \beta \leq \frac{3\eta}{2\eta - 2}, 0 < \epsilon < \frac{2}{2\beta + \eta}$, or
- (c) $\eta > 1, \beta > \frac{3\eta}{2\eta - 2}, 0 < \epsilon < \frac{2}{2\beta + \eta}$, or
- (d) $\eta > 1, \beta > \frac{3\eta}{2\eta - 2}, \epsilon > \frac{2\eta - 2}{2\beta\eta - 2\beta - 3\eta}$.

For $C - (Q - S)^2 > 0$, the first and second the Buchdahl conditions are satisfied at the solution as $\lambda \rightarrow -\infty$, if

⁴ This condition is reduced in GR, to $\eta > 1, 0 < \epsilon < \frac{2}{\eta + 2}$, when $\beta = 1$.

$$\frac{(\beta - 1)(\eta + 2)\varepsilon}{\varepsilon(6\beta + \eta - 4) + 2(\eta - 1)} \geq 0, \tag{46}$$

$$\frac{\eta(\eta + 2)\varepsilon(\mu_0 + (\eta - 1)\rho_c)}{\mu_0(\varepsilon(6\beta + \eta - 4) + 2(\eta - 1))} \leq 1. \tag{47}$$

Additionally, taking the limit $\lambda \rightarrow -\infty$ we have

$$\begin{aligned} & \frac{1}{9} \left(7C - 3Q^2 + 3\beta S^2 \right) + 2\sqrt{C(C + 3Q^2 - 3\beta S^2)} - C \\ & + (Q - S)^2 \rightarrow \frac{40}{9} > 0, \end{aligned} \tag{48}$$

such that the third Buchdahl condition is also satisfied. Thus, combining the conditions (45), (46), and (47), we have the conditions for the existence of regular solution at the center associated to P_3^+ .

The quotient, $\frac{\rho_c}{\mu_c}$ in (44) is a gravitational strength parameter. In GR where the parameter $\beta = 1$, the maximal value of the gravitational strength, $\frac{1}{\eta+1}$, is obtained when $\varepsilon_1 = 0$, which corresponds to the subset $Q = 1$. However, in the Einstein-aether theory the parameter β is a freely specifiable parameter, and for $\eta > 1$, $\beta > \frac{3\eta}{2\eta-2} \cdot \frac{\varepsilon_1}{\varepsilon_2} > \frac{2\eta-2}{2\beta\eta-2\beta-3\eta}$, the maximal strength is not $\frac{\rho_c}{\mu_c} = \frac{1}{\eta+1}$ anymore as it is in GR.

2. The equilibrium point P_4^+ generalizes the so called Tolman point (which corresponds to $\beta = 1$), which now is promoted to a 1-parameter solution. This solution exists for $0 \leq \beta \leq \frac{1}{8}(\eta + 2)^2$. The eigenvalues are

$$\begin{aligned} \lambda_1 &= \frac{2\eta}{\eta + 2}, \lambda_2 = -\frac{\eta + 2 + \sqrt{64\beta - 7(\eta + 2)^2}}{2(\eta + 2)}, \\ \lambda_3 &= -\frac{\eta + 2 - \sqrt{64\beta - 7(\eta + 2)^2}}{2(\eta + 2)}. \end{aligned} \tag{49}$$

Notice that $\lambda_2 + \lambda_3 + 1 = 0$.

The eigenvalue λ_1 is always real and positive.

The eigenvalues λ_2, λ_3 are both real and negative for

- (a) $\frac{63}{64} < \beta \leq \frac{9}{8}, 1 < \eta \leq -2 + \frac{8\sqrt{\beta}}{\sqrt{7}}$, or
- (b) $\beta > \frac{9}{8}, 2\sqrt{2}\sqrt{\beta} - 2 < -2 + \eta \leq \frac{8\sqrt{\beta}}{\sqrt{7}}$.

The eigenvalues λ_2, λ_3 are complex conjugated with negative real part for

- (a) $0 < \beta \leq \frac{63}{64}, \eta > 1$, or
- (b) $\beta > \frac{63}{64}, \eta > -2 + \frac{8\sqrt{\beta}}{\sqrt{7}}$.

Following the same method as for the analysis of P_3^+ we can explore approximated solutions related to P_4^+ by constructing the unstable manifold of this equilibrium point.

Case 1:

When λ_2, λ_3 are both reals and negative, that is whenever $\frac{63}{64} < \beta \leq \frac{9}{8}, 1 < \eta \leq -2 + \frac{8\sqrt{\beta}}{\sqrt{7}}$, or $\beta > \frac{9}{8}, 2\sqrt{2}\sqrt{\beta} - 2 < -2 + \eta \leq \frac{8\sqrt{\beta}}{\sqrt{7}}$, we can define the real quantities

$$u = \frac{\lambda_2(\lambda_2 + 1)(1 - Q) ((\eta + 2)^2\lambda_2^2 + (\eta + 2)^2\lambda_2 + 2\eta(\eta + 6))}{(\eta(\lambda_2 - 2) + 2\lambda_2)((\eta + 2)\lambda_2 + 3\eta + 2)}, \tag{50a}$$

$$v_1 = \frac{\lambda_2(\lambda_2 + 2) (2C + \lambda_2^2 + \lambda_2)}{4\lambda_2 + 2} + \frac{\lambda_2(\lambda_2 + 1)(Q - 1) (\eta (\lambda_2^2 + \lambda_2 + 2) + 2(\lambda_2 - 2)(\lambda_2 + 1))}{(2\lambda_2 + 1)((\eta + 2)\lambda_2 + 3\eta + 2)} + \frac{\lambda_2(\lambda_2 + 1) (\lambda_2^2 + \lambda_2 + 2) ((\eta + 2)S - 2)}{4\lambda_2 + 2}, \tag{50b}$$

$$v_2 = -\frac{(\lambda_2^2 - 1) (2C + \lambda_2^2 + \lambda_2)}{4\lambda_2 + 2} + \frac{\lambda_2(\lambda_2 + 1)(Q - 1) (\eta (\lambda_2^2 + \lambda_2 + 2) + 2\lambda_2(\lambda_2 + 3))}{(2\lambda_2 + 1)(\eta(\lambda_2 - 2) + 2\lambda_2)} - \frac{\lambda_2(\lambda_2 + 1) (\lambda_2^2 + \lambda_2 + 2) ((\eta + 2)S - 2)}{4\lambda_2 + 2} \tag{50c}$$

where we have used the relation

$$\beta = \frac{1}{16}(\eta + 2)^2 (\lambda_2^2 + \lambda_2 + 2). \tag{51}$$

Now, applying the Invariant Manifold theorem we find that the local unstable manifold of P_4^+ is

$$\{(u, v_2, v_3): v_1 = h_1(u), v_2 = h_2(u), h_1(0) = 0, h'_1(u) = 0, h_2(0) = 0, h'_2(u) = 0, u \leq \delta\},$$

with $\delta > 0$. Calculating the unstable manifold up to second order in powers of u , neglecting the higher order terms and substituting back to the equations of Q, S, C , in terms of u , through $v_1 = h_1(u), v_2 = h_2(u)$, we obtain that any solution near the unstable manifold of P_4^+ , satisfies

$$Q = 1 - \frac{(\eta + 2)^2(-4\beta + \eta(2\eta + 3) + 2)}{2((4\beta + \eta - 2)((\eta + 2)^2 - 8\beta))} \varepsilon e^{\frac{2\eta\lambda}{\eta+2}}, \tag{52a}$$

$$S = \frac{2}{\eta + 2} - \frac{(\eta + 2)(-8\beta^2 - 2(\eta - 2)\beta + \eta(\eta + 2)^2)}{2(\beta(4\beta + \eta - 2)((\eta + 2)^2 - 8\beta))} \varepsilon e^{\frac{2\eta\lambda}{\eta+2}} + \frac{\eta(\eta + 2)^4(-4\beta + \eta + 2)}{8\beta^2(4\beta + \eta - 2)^2((\eta + 2)^2 - 8\beta)^2(-8\beta + \eta(11\eta + 8) + 4)} \tag{52b}$$

$$\times \left[-32\beta^3 + 4\beta^2(\eta(7\eta - 2) + 8) - \beta(\eta(\eta(2\eta + 33) + 102) + 28) + 8 + \eta(\eta + 2)^2(\eta(\eta + 12) + 4) \right] \varepsilon^2 e^{\frac{4\eta\lambda}{\eta+2}}, \tag{52c}$$

$$C = 1 - \frac{8\beta}{(\eta + 2)^2} + \varepsilon e^{\frac{2\eta}{\eta+2}\lambda} - \left[128\beta^4 - 48\beta^3(\eta(5\eta + 4) + 4) + 8\beta^2(\eta(\eta(7\eta^2 + \eta + 19) + 24) + 12) - \beta(\eta(\eta(\eta(42\eta + 251) + 300) + 80) + 48) + 16 + 2\eta^2(\eta + 2)^3(9\eta + 2) \right] \times \frac{(\eta + 2)^2}{4\beta(4\beta + \eta - 2)^2((\eta + 2)^2 - 8\beta)(-8\beta + \eta(11\eta + 8) + 4)} \varepsilon^2 e^{\frac{4\eta\lambda}{\eta+2}}, \tag{52d}$$

where we have used the original parameters β and η and we have substituted the approximated solution $u_1 = \varepsilon e^{\frac{2\eta}{\eta+2}\lambda}$, that is obtained by integrating the linearized equation along the unstable direction. This expansion is accurate as long as $\lambda \rightarrow -\infty$.

Using this solution, we find

$$\begin{aligned} \frac{p_c}{\mu_c} &= \lim_{\lambda \rightarrow -\infty} \frac{C - Q^2 + \beta S^2}{C(\eta - 1) - \eta + Q^2 + \beta(\eta - 1)S^2} \\ &= \lim_{\lambda \rightarrow -\infty} \frac{1}{\eta - 1} - \frac{\varepsilon \left(\eta(\eta + 2)^4(-4\beta + \eta(2\eta + 3) + 2)e^{\frac{2\eta\lambda}{\eta+2}} \right)}{4(\beta(\eta - 1)^2(4\beta + \eta - 2)((\eta + 2)^2 - 8\beta))} + O(\varepsilon^2) \\ &= \frac{1}{\eta - 1} > 0. \end{aligned} \tag{53}$$

Furthermore, the Buchdahl conditions can be expressed as

$$1 \geq \frac{-(\eta - 1)(C + \beta S^2) + \eta - Q^2}{3(C - (Q - S)^2)}, \tag{54}$$

$$1 \leq \frac{\eta(1 - Q^2)(\mu_0 + (\eta - 1)p_c)}{3\mu_0(C - (Q - S)^2)}, \tag{55}$$

$$\frac{1}{9} \left(7C - 3Q^2 + 3\beta S^2 \right) + 2\sqrt{C(C + 3Q^2 - 3\beta S^2)} - C + (Q - S)^2 \geq 0. \tag{56}$$

As $\lambda \rightarrow -\infty$, applying the above conditions we have that the second one is satisfied; and the first and third imply

$$\frac{\beta - \beta\eta}{6\beta - 3(\eta + 1)} \geq 1, \tag{57}$$

$$\frac{4(7\beta + 9)}{9(\eta + 2)^2} + 2\sqrt{\left(4 - \frac{20\beta}{(\eta + 2)^2}\right)\left(1 - \frac{8\beta}{(\eta + 2)^2}\right)} - \frac{4}{\eta + 2} + \frac{4}{9} \geq 0. \tag{58}$$

These conditions are not satisfied for $\beta = 1$ (that is, for GR). But in AE-theory β is a free parameter, such that the above inequalities can be satisfied for $\eta > 1$, $\frac{3\eta+3}{\eta+5} \leq \beta < \frac{\eta+1}{2}$, $64\beta - 7(\eta + 2)^2 \geq 0$.

Case 2:

For the choice $0 < \beta \leq \frac{63}{64}$, $\eta > 1$, or $\beta > \frac{63}{64}$, $\eta > -2 + \frac{8\sqrt{\beta}}{\sqrt{7}}$, the eigenvalues λ_2, λ_3 are complex conjugates with negative real part. Indeed,

$$\Re(\lambda_2) = \Re(\lambda_3) = -\frac{1}{2}, \quad -\Im(\lambda_2) = \Im(\lambda_3) = \frac{\sqrt{7(\eta + 2)^2 - 64\beta}}{2(\eta + 2)}. \tag{59}$$

For the analysis we introduce the parametrization

$$Q = 1 - \frac{(\eta + 2)^2 u(-4\beta + \eta(2\eta + 3) + 2)}{2(4\beta + \eta - 2)(8\beta - (\eta + 2)^2)}, \tag{60a}$$

$$S = \frac{2}{\eta + 2} + \frac{(\eta + 2)u(-8\beta^2 - 2\beta(\eta - 2) + \eta(\eta + 2)^2)}{2\beta(4\beta + \eta - 2)(8\beta - (\eta + 2)^2)} + \frac{(\eta + 2)v_1(5(\eta + 2)^2 - 32\beta)}{16\beta(8\beta - (\eta + 2)^2)} - \frac{(\eta + 2)^2 v_2 \sqrt{7(\eta + 2)^2 - 64\beta}}{16\beta(8\beta - (\eta + 2)^2)}, \tag{60b}$$

$$C = 1 - \frac{8\beta}{(\eta + 2)^2} + u + 2v_1, \tag{60c}$$

where u, v_1, v_2 are reals.

Calculating the unstable manifold up to second order in powers of u , neglecting the higher order terms and substituting back to the equations of Q, S, C , in terms of u , through $v_1 = h_1(u), v_2 = h_2(u)$, we obtain that any solution near the unstable manifold of P_4^+ , satisfies

$$Q = 1 - \frac{(\eta + 2)^2(-4\beta + \eta(2\eta + 3) + 2)}{2(4\beta + \eta - 2)(8\beta - (\eta + 2)^2)} \varepsilon e^{\frac{2\eta\lambda}{\eta+2}}, \tag{61a}$$

$$S = \frac{2}{\eta + 2} + \frac{(\eta + 2)\varepsilon(-8\beta^2 - 2\beta(\eta - 2) + \eta(\eta + 2)^2)e^{\frac{2\eta\lambda}{\eta+2}}}{2\beta(4\beta + \eta - 2)(8\beta - (\eta + 2)^2)} - \frac{(\eta + 2)}{9216\beta^2(7\eta + 2)((\eta + 2)^2 - 8\beta)^2} [2048\beta^3(347\eta + 612) - 32\beta^2(\eta(\eta(8025\eta + 18818) + 7740) - 840) - 4\beta(\eta(\eta(15457\eta + 67932) + 96052) + 49440)(\eta + 2)^2 + 9(\eta(\eta(826\eta + 3043) + 3628) + 1612)(\eta + 2)^4] \varepsilon^2 e^{\frac{4\eta\lambda}{\eta+2}}, \tag{61b}$$

$$C = 1 - \frac{8\beta}{(\eta + 2)^2} + \varepsilon e^{\frac{2\eta\lambda}{\eta+2}} - \frac{(-32\beta^2(6\eta + 53) + 4\beta(\eta(\eta(6\eta + 77) - 126) - 184) + 9(\eta + 2)^2(5\eta + 14))}{72\beta(7\eta + 2)((\eta + 2)^2 - 8\beta)} \varepsilon^2 e^{\frac{4\eta\lambda}{\eta+2}}, \tag{61c}$$

where we have substituted the approximated solution $u_1 = \varepsilon e^{\frac{2\eta}{\eta+2}\lambda}$, that is obtained by integrating the linearized equation along the unstable direction. This expansion is accurate as long as $\lambda \rightarrow -\infty$. At the stable manifold the orbits spiral in and tends asymptotically to the origin with modes $\cos(\frac{\sqrt{7(\eta+2)^2-64\beta}}{2(\eta+2)}\lambda)e^{-\frac{\lambda}{2}}$, $\sin(\frac{\sqrt{7(\eta+2)^2-64\beta}}{2(\eta+2)}\lambda)e^{-\frac{\lambda}{2}}$.

We have the estimates

$$\begin{aligned} \frac{p_c}{\mu_c} &= \lim_{\lambda \rightarrow -\infty} \frac{C - Q^2 + \beta S^2}{C(\eta - 1) - \eta + Q^2 + \beta(\eta - 1)S^2} \\ &= \lim_{\lambda \rightarrow -\infty} \frac{1}{\eta - 1} + \frac{\eta(\eta + 2)^4\varepsilon(-4\beta + 2\eta^2 + 3\eta + 2)e^{\frac{2\eta\lambda}{\eta+2}}}{4\beta(\eta - 1)^2(4\beta + \eta - 2)(-8\beta + \eta^2 + 4\eta + 4)} + O(\varepsilon^2) \\ &= \frac{1}{\eta - 1} > 0. \end{aligned}$$

Furthermore, for $C - (Q - S)^2 > 0$, the Buchdahl conditions reduce to

$$\begin{aligned} \frac{2(\beta - 1)\beta}{-2\beta + \eta + 1} + \beta &\leq 3, \quad 2\beta\mu_0 \geq (\eta + 1)\mu_0, \\ \frac{9\sqrt{40\beta^2 - 13\beta(\eta + 2)^2 + (\eta + 2)^4} + 7\beta + \eta^2 - 5\eta - 5}{2\beta - \eta - 1} &\leq 0, \end{aligned}$$

as $\lambda \rightarrow -\infty$, respectively. That is, when $1 < \eta \leq 1.04725$, $\beta < \frac{1}{64}(7\eta^2 + 28\eta + 28)$, $\mu_0 \leq 0$ or $\eta > 1.04725$, $\beta \leq \frac{3\eta+3}{\eta+5}$, $\mu_0 \leq 0$. We are assuming $\mu_0 \geq 0$, therefore the conditions are fulfilled if $\mu_0 = 0$.

Figure 1 shows the flow of the system (28) for different choices of the parameters β, η . Figure 1a, b represents the scale-invariant ($\mu_0 = 0$) boundary $Q = +1$. In Fig.

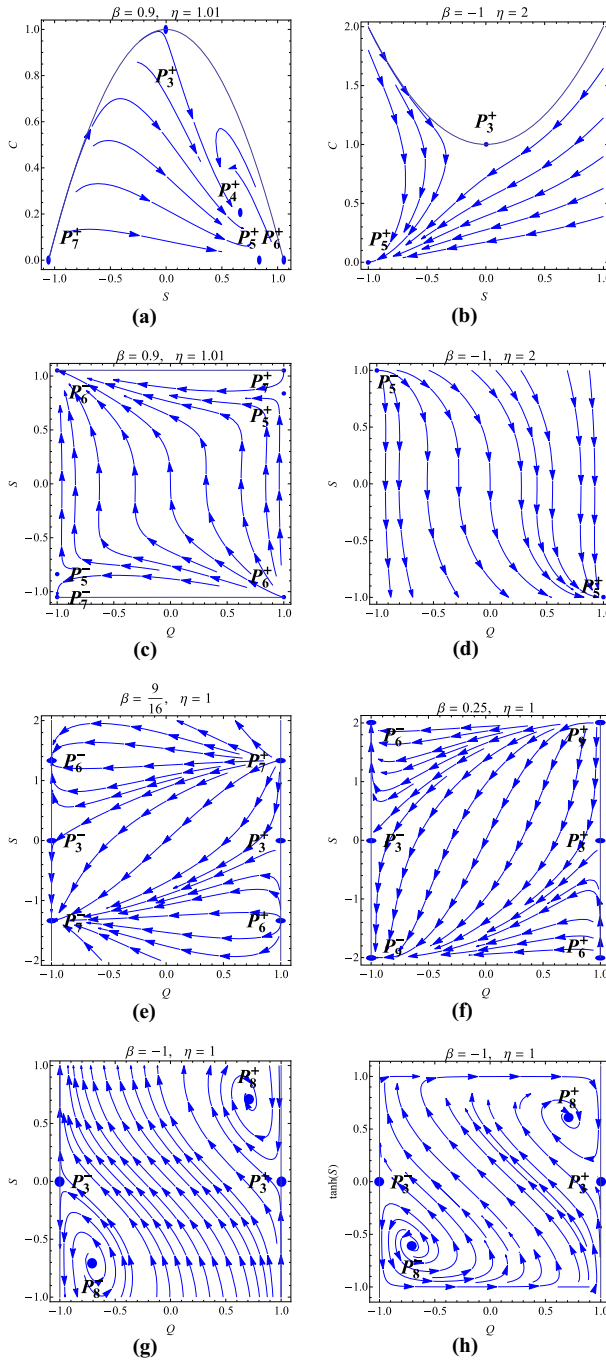


Fig. 1 Streamlines of the system (28) for different choices of the parameters β, η

1a the local attractor is P_4^+ (but it is a saddle point for the 3D dynamical system). At Fig. 1b the attractor is P_5^+ . Figure 1c, d, show the behavior on the plane-symmetric boundary $C = 0$. In Fig. 1c the stable (respectively, unstable) points are P_7^- and P_6^- (respectively, P_7^+ and P_6^+). The P_5^\pm are saddles. In Fig. 1d the attractor is P_5^+ . Figure 1e–h, show the dynamics on the invariant set $C = 1 - \beta S^2$ corresponding to fluids satisfying $\mu + p = 0$. In Fig. 1f P_3^\pm are saddles. The stable (respectively, unstable) points in the physical regions are P_7^- and P_6^- (respectively, P_7^+ and P_6^+). In Fig. 1f is presented the dynamics for $\beta = \frac{1}{4}$ and $\eta = 1$. For these values P_3^\pm are saddles. P_7^- (respectively P_7^+) is a stable (respectively, unstable) node on the physical region. At the bifurcation value $\beta = \frac{1}{4}$, P_6^\pm merges with P_9^\pm and becomes saddles. In the Fig. 1g show the dynamics on the invariant set $C = 1 - \beta S^2$ for $\beta = -1$, $\eta = 1$. The sink is P_8^+ and the source is P_8^- . P_3^\pm are saddles. In Fig. 1h it is shown that the points at infinity ($Q = \pm 1$, $S = \mp\infty$) are saddle. Thus, P_8^+ is a global attractor for this choice of parameters.

3.1.2 Compactification procedure

The variables C, S satisfy $C + \beta S^2 \leq 1$ and $Q \in [-1, 1]$. Therefore, in case of $\beta > 0$ the phase space is compact. However, S, C can be infinite values for $\beta < 0$. Assuming $\beta < 0$, we have $C \leq 1 + |\beta|S^2$, such that we can use the dominant quantity, $1 + |\beta|S^2$, to normalize. By introducing the compact variables, where we have assumed $1 + |\beta|S^2 \neq 0$

$$u = \frac{\sqrt{|\beta|}S}{\sqrt{1 + |\beta|S^2}}, \quad v = \frac{C}{1 + |\beta|S^2}, \tag{62}$$

with

$$(Q, u, v) \in [-1, 1] \times [-1, 1] \times [0, 1],$$

and the radial derivative

$$\frac{df}{d\xi} = \frac{|\beta|}{\sqrt{1 + |\beta|S^2}} \frac{df}{d\lambda}, \tag{63}$$

we obtain the dynamical system

$$\frac{dQ}{d\xi} = (Q^2 - 1) \left[\beta (2u^2 - v) - Qu\sqrt{1 - u^2}\sqrt{|\beta|} \right], \tag{64a}$$

$$\frac{du}{d\xi} = (u^2 - 1) \left[\beta Qu(v - 2) + \frac{1}{2} (\eta + 2Q^2 - (\eta + 2)v) \sqrt{1 - u^2}\sqrt{|\beta|} \right], \tag{64b}$$

$$\frac{dv}{d\xi} = (v - 1)v \left[2\beta Q (u^2 - 1) - (\eta + 2)u\sqrt{1 - u^2}\sqrt{|\beta|} \right]. \tag{64c}$$

The system (64) covers all the equilibrium points in the finite region, with the exception of $P_4^\pm, P_6^\pm, P_7^\pm,$ and P_9^\pm (which do not exist for $\beta < 0$), and incorporates the equilibrium points at infinity. That is, in the new coordinates, the equilibrium points in the finite region are: $P_1^\pm: (Q, u, v) = \left(\pm 1, \pm \frac{\sqrt{|\beta|}}{\sqrt{1+|\beta|}}, 0\right), P_2^\pm: (Q, u, v) = \left(\pm 1, \mp \frac{\sqrt{|\beta|}}{\sqrt{1+|\beta|}}, 0\right), P_3^\pm: (Q, u, v) = (\pm 1, 0, 1), P_5^\pm: (Q, u, v) = \left(\pm 1, \mp \frac{(\eta+2)}{\sqrt{(\eta+2)^2+16|\beta|}}, 0\right), P_8^\pm: (Q, u, v) = \left(\frac{\epsilon}{\sqrt{1+|\beta|}}, \sqrt{\frac{|\beta|}{1+2|\beta|}}\epsilon, 1\right)$. The lines of equilibrium points at infinity are $I_1^\pm: (Q, u, v) = (\pm 1, \pm 1, v_c),$ and $I_2^\pm: (Q, u, v) = (\pm 1, \mp 1, v_c);$ that is, $|S| \rightarrow \infty,$ and $C,$ arbitrary, such that $\frac{C}{1+|\beta|S^2} \rightarrow v_c,$ constant. The eigenvalues are $\pm 2(2 - v_c)\beta, \mp 2(2 - v_c)\beta, 0.$ Therefore, they behave as saddles.

4 Stationary comoving aether with perfect fluid and scalar field in static metric

In this section we investigate a stationary comoving aether with perfect fluid and scalar field in static metric

$$ds^2 = -N^2(r)dt^2 + e_1^1(r)^{-2}dr^2 + e_2^2(r)^{-2}(d\vartheta^2 + \sin^2\vartheta d\varphi^2), \tag{65}$$

and we define $x = \mathbf{e}_1 \ln e_2^2, y = \mathbf{e}_1 \ln N,$ and the differential operator $\mathbf{e}_1 = e_1^1 \partial_r.$

The equations for the variables x, y, p, φ, K are:

$$\mathbf{e}_1(x) = \frac{\mu + 3p}{2\beta} + \mathbf{e}_1(\varphi)^2 - \frac{W(\varphi)}{\beta} + 2(\beta - 1)y^2 + 3xy + K, \tag{66a}$$

$$\mathbf{e}_1(y) = \frac{\mu + 3p}{2\beta} - \frac{W(\varphi)}{\beta} + 2xy - y^2, \tag{66b}$$

$$\mathbf{e}_1(p) = -y(\mu + p), \tag{66c}$$

$$\mathbf{e}_1(\mathbf{e}_1(\varphi)) = -(y - 2x)\mathbf{e}_1(\varphi) + W'(\varphi), \tag{66d}$$

$$\mathbf{e}_1(K) = 2xK, \tag{66e}$$

where $W(\varphi)$ is the scalar field self-interacting potential. The system satisfies the restriction

$$-(x - y)^2 + \beta y^2 + p + \frac{1}{2}\mathbf{e}_1(\varphi)^2 - W(\varphi) + K = 0. \tag{67}$$

Equation (67) is called Gauss constraint, and it corresponds to a first integral of the system (66). This can be proven by applying the differential operator $\mathbf{e}_1(\dots)$ to both sides of (67), and then using the Eq. (66) to eliminate the spatial derivatives. Therefore, by using again the restriction (67) solved for $K,$ we obtain an identity. On the other hand, the aether constraint (9) is identically zero.

As before, we assume an EoS parametrized by (24). We next consider the case of an exponential self-interacting potential; we shall study the case of a harmonic potential $W(\varphi) = \frac{1}{2}m^2\varphi^2$ elsewhere, e.g., as particular example in paper III [61].

4.1 Phase-space evolution: exponential potential $W(\varphi) = W_0e^{-k\varphi}$

For the analysis of the system of Eq. (66) one can use methods to obtain exact solutions. Additionally one can use the dynamical systems approach for investigating the structure of the whole solution space. Using the quantities $\theta = y - x, \sigma = y$ as in [35], the equations read

$$e_1^1 \frac{d\theta}{dr} = -\beta\sigma^2 - \theta^2 + \theta\sigma - W_0e^{-k\varphi} + p - \frac{\Phi^2}{2}, \tag{68a}$$

$$e_1^1 \frac{d\sigma}{dr} = \frac{2\beta\sigma(\sigma - 2\theta) - 2W_0e^{-k\varphi} + \mu_0 + (\eta + 2)p}{2\beta}, \tag{68b}$$

$$e_1^1 \frac{dK}{dr} = -2(\sigma - \theta) \left(\beta\sigma^2 - \theta^2 - W_0e^{-k\varphi} + p + \frac{\Phi^2}{2} \right), \tag{68c}$$

$$e_1^1 \frac{d\varphi}{dr} = \Phi, \tag{68d}$$

$$e_1^1 \frac{d\Phi}{dr} = \Phi(\sigma - 2\theta) - kW_0e^{-k\varphi}, \tag{68e}$$

$$e_1^1 \frac{dp}{dr} = -\sigma(\eta p + \mu_0), \tag{68f}$$

where

$$K + p = -\beta\sigma^2 + \theta^2 + W_0e^{-k\varphi} - \frac{\Phi^2}{2}, \tag{69a}$$

$$\mu = \mu_0 + (\eta - 1)p. \tag{69b}$$

Now we define the radial variable λ

$$e_1^1 \frac{d\lambda}{dr} = \sqrt{\frac{\mu_0}{\eta} + \theta^2}. \tag{70}$$

For convenience, we set one of the metric components as $e_1^1 = r$. This is equivalent to set $\mathbf{e}_1 = \partial_\ell$, where $r = e^\ell$, such that $\ell \rightarrow -\infty$ as $r \rightarrow 0$ and $\ell \rightarrow \infty$ as $r \rightarrow \infty$. In other words, λ unequivocally defines the flow direction. That is, the “past attractors” ($r \rightarrow 0$) correspond to $\lambda \rightarrow -\infty$ and the “future attractors” ($r \rightarrow \infty$) correspond to $\lambda \rightarrow \infty$. Defining the scale invariant quantities:

$$\begin{aligned}
 Q &= \frac{\theta}{\sqrt{\frac{\mu_0}{\eta} + \theta^2}}, \quad S = \frac{\sigma}{\sqrt{\frac{\mu_0}{\eta} + \theta^2}}, \quad C = \frac{\eta K}{\mu_0 + \eta \theta^2}, \quad A_\varphi = \frac{\Phi}{\sqrt{2} \sqrt{\frac{\mu_0}{\eta} + \theta^2}}, \\
 A_W &= \frac{\sqrt{W_0} e^{-\frac{k}{2}\varphi}}{\sqrt{\frac{\mu_0}{\eta} + \theta^2}},
 \end{aligned}
 \tag{71}$$

we obtain the evolution equations

$$\frac{dQ}{d\lambda} = (Q^2 - 1) (2A_\varphi^2 + C + S(2\beta S - Q)), \tag{72a}$$

$$\begin{aligned}
 \frac{dS}{d\lambda} &= \frac{C(2\beta QS - \eta - 2) - 2Q(\beta S^2 - 1)(Q - 2\beta S) - \beta \eta S^2 + \eta}{2\beta} \\
 &\quad \frac{A_W^2 \eta + A_\varphi^2(4\beta QS - \eta - 2)}{2\beta},
 \end{aligned}
 \tag{72b}$$

$$\frac{dC}{d\lambda} = 2C (Q (2A_\varphi^2 + C + 2\beta S^2 - QS - 1) + S), \tag{72c}$$

$$\frac{dA_\varphi}{d\lambda} = A_\varphi (Q (2A_\varphi^2 + C + 2\beta S^2 - QS - 2) + S) - \frac{\sqrt{2}}{2} A_W^2 k, \tag{72d}$$

$$\frac{dA_W}{d\lambda} = \frac{1}{2} A_W (4A_\varphi^2 Q - \sqrt{2} A_\varphi k + 2Q(C + S(2\beta S - Q))). \tag{72e}$$

We have the useful relations

$$\frac{\mu_0}{\eta} = \frac{(1 - Q^2)K}{C}, \tag{73a}$$

$$p = -\frac{K(-A_W^2 + A_\varphi^2 + C + \beta S^2 - Q^2)}{C}, \tag{73b}$$

$$\frac{C(\mu + p)}{\eta K} = A_W^2 - A_\varphi^2 - C - \beta S^2 + 1. \tag{73c}$$

Since $\mu_0 \geq 0, \eta \geq 1$ it follows $-1 \leq Q \leq 1$. Since $K \geq 0$ it follows $C \geq 0$. The condition

$$(-A_W^2 + A_\varphi^2 + C + \beta S^2 - Q^2) = 0$$

defines the surface of zero-pressure. However, it is not an invariant set of (72), neither is

$$(-A_W^2 + A_\varphi^2 + C + \beta S^2 - Q^2) > 0.$$

If we assume that the weak energy condition $p + \mu \geq 0$ is satisfied, then we obtain the subset of the phase space

$$1 + A_W^2 - A_\varphi^2 - C - \beta S^2 \geq 0.$$

Defining $\Omega = 1 + A_W^2 - A_\varphi^2 - C - \beta S^2$ we obtain

$$\frac{d\Omega}{d\lambda} = \Omega \left[4A_\varphi^2 Q + 2Q(C + S(2\beta S - Q)) - \eta S \right]. \tag{74}$$

Thus, $1 + A_W^2 - A_\varphi^2 - C - \beta S^2 \geq 0$ defines an invariant set. Summarizing, the Eq. (72) defines a flow on the invariant set

$$\{(Q, S, C, A_\varphi, A_W) : -1 \leq Q \leq 1, C \geq 0, -A_W^2 + A_\varphi^2 + C + \beta S^2 \leq 1, A_W \geq 0\}. \tag{75}$$

This phase-space is unbounded. The invariant sets $Q = \pm 1$ corresponds to $\mu_0 = 0$. We have the auxiliary equations

$$\frac{d \ln N}{d\lambda} = S, \tag{76a}$$

$$\frac{d \ln K}{d\lambda} = -2(Q - S), \tag{76b}$$

$$\frac{d \ln y}{d\lambda} = -\frac{(A_W^2 + 1)\eta + A_\varphi^2(\eta + 2) + C(\eta + 2) + 4\beta QS + \beta\eta S^2 - 2Q^2}{2\beta S}. \tag{76c}$$

Both the line element expressed in the variables (25) and the dynamical system (72) are invariant under the discrete symmetry

$$(Q, S, A_\varphi, \lambda) \rightarrow (-Q, -S, -A_\varphi, -\lambda). \tag{77}$$

The equilibrium points are discussed in Appendix C.

4.1.1 Equilibrium points in the finite region of the phase space

We have recovered the previous results for the points $P_1^\pm - P_9^\pm$ (when no scalar field is present). For further details about the derivation of the physical interpretation of the equilibrium points $P_1^\pm - P_9^\pm$ we submit the reader to Appendix B. The equilibrium points with non-trivial scalar field are discussed in Appendix C.

Now, we discuss the more interesting points (in the sense that they have the highest dimensional stable/unstable manifold).

1. P_5^+ is a sink for $\beta < 0, \eta \geq 1$.
2. P_5^- is source for $\beta < 0, \eta \geq 1$.
3. P_8^+ is non-hyperbolic with a 4D stable manifold.
4. P_8^- is non-hyperbolic with a 4D unstable manifold.
5. The following subsets (arcs, or specific equilibrium points) of the line $P_{10}^+(S_c)$ are unstable for the given conditions:

- (a) $S_c < 0, \beta < \frac{1}{S_c^2}, \eta > 1, k < \sqrt{\frac{-2S_c^2+8S_c-8}{\beta S_c^2-1}}$, or
 (b) $S_c < 0, \beta = \frac{1}{S_c^2}, \eta > 1, k \in \mathbb{R}$, or
 (c) $S_c = 0, \beta \in \mathbb{R}, \eta > 1, k < 2\sqrt{2}$, or
 (d) $0 < S_c < \frac{4}{3}, \beta < \frac{1}{S_c^2}, 1 < \eta < \frac{4-2S_c}{S_c}, k < \sqrt{\frac{-2S_c^2+8S_c-8}{\beta S_c^2-1}}$, or
 (e) $0 < S_c < \frac{4}{3}, \beta = \frac{1}{S_c^2}, 1 < \eta < \frac{4-2S_c}{S_c}, k \in \mathbb{R}$.

6. The following subsets (arcs, or specific equilibrium points) of the line $P_{10}^-(S_c)$ are stable for the given conditions:

- (a) $S_c < 0, \beta < \frac{1}{S_c^2}, \eta > 1, k < \sqrt{\frac{-2S_c^2+8S_c-8}{\beta S_c^2-1}}$, or
 (b) $S_c < 0, \beta = \frac{1}{S_c^2}, \eta > 1, k \in \mathbb{R}$, or
 (c) $S_c = 0, \beta \in \mathbb{R}, \eta > 1, k < 2\sqrt{2}$, or
 (d) $0 < S_c < \frac{4}{3}, \beta < \frac{1}{S_c^2}, 1 < \eta < \frac{4-2S_c}{S_c}, k < \sqrt{\frac{-2S_c^2+8S_c-8}{\beta S_c^2-1}}$, or
 (e) $0 < S_c < \frac{4}{3}, \beta = \frac{1}{S_c^2}, 1 < \eta < \frac{4-2S_c}{S_c}, k \in \mathbb{R}$.

7. The following subsets (arcs, or specific equilibrium points) of the line $P_{11}^+(S_c)$ are unstable for the given conditions:

- (a) $S_c < 0, \beta < \frac{1}{S_c^2}, \eta \geq 1, k > -\sqrt{2}\sqrt{-\frac{S_c^2-4S_c+4}{\beta S_c^2-1}}$, or
 (b) $S_c < 0, \beta = \frac{1}{S_c^2}, \eta \geq 1, k \in \mathbb{R}$, or
 (c) $S_c = 0, \beta \in \mathbb{R}, \eta \geq 1, k > -2\sqrt{2}$, or
 (d) $0 < S_c < \frac{4}{3}, \beta < \frac{1}{S_c^2}, 1 \leq \eta < \frac{4-2S_c}{S_c}, k > -\sqrt{2}\sqrt{-\frac{S_c^2-4S_c+4}{\beta S_c^2-1}}$, or
 (e) $0 < S_c < \frac{4}{3}, \beta = \frac{1}{S_c^2}, 1 \leq \eta < \frac{4-2S_c}{S_c}, k \in \mathbb{R}$.

8. The following subsets (arcs, or specific equilibrium points) of the line $P_{11}^-(S_c)$ are stable for the given conditions:

- (a) $S_c < 0, \beta < \frac{1}{S_c^2}, \eta \geq 1, k > -\sqrt{2}\sqrt{-\frac{S_c^2-4S_c+4}{\beta S_c^2-1}}$, or
 (b) $S_c < 0, \beta = \frac{1}{S_c^2}, \eta \geq 1, k \in \mathbb{R}$, or
 (c) $S_c = 0, \beta \in \mathbb{R}, \eta \geq 1, k > -2\sqrt{2}$, or
 (d) $0 < S_c < \frac{4}{3}, \beta < \frac{1}{S_c^2}, 1 \leq \eta < \frac{4-2S_c}{S_c}, k > -\sqrt{2}\sqrt{-\frac{S_c^2-4S_c+4}{\beta S_c^2-1}}$, or
 (e) $0 < S_c < \frac{4}{3}, \beta = \frac{1}{S_c^2}, 1 \leq \eta < \frac{4-2S_c}{S_c}, k \in \mathbb{R}$.

9. P_{12}^+ is non-hyperbolic with a 4D unstable for

- (a) $\beta < 0, \eta \geq 1, k > -\sqrt{2}\sqrt{\frac{(8\beta-\eta-2)^2}{\beta(16\beta-\eta^2-4\eta-4)}}$, or

(b) $\beta > \frac{9}{16}, 1 \leq \eta < 2(2\sqrt{\beta} - 1), k < \sqrt{2}\sqrt{\frac{(8\beta - \eta - 2)^2}{\beta(16\beta - \eta^2 - 4\eta - 4)}}$.

10. P_{12}^- is non-hyperbolic with a 4D stable manifold for

(a) $\beta < 0, \eta \geq 1, k > -\sqrt{2}\sqrt{\frac{(8\beta - \eta - 2)^2}{\beta(16\beta - \eta^2 - 4\eta - 4)}}$, or

(b) $\beta > \frac{9}{16}, 1 \leq \eta < 2(2\sqrt{\beta} - 1), k < \sqrt{2}\sqrt{\frac{(8\beta - \eta - 2)^2}{\beta(16\beta - \eta^2 - 4\eta - 4)}}$.

11. P_{13}^+ is non-hyperbolic with a 4D unstable manifold for

(a) $\beta < 0, \eta \geq 1, k < \sqrt{2}\sqrt{\frac{(8\beta - \eta - 2)^2}{\beta(16\beta - \eta^2 - 4\eta - 4)}}$, or

(b) $\beta > \frac{9}{16}, 1 \leq \eta < 2(2\sqrt{\beta} - 1), k > -\sqrt{2}\sqrt{\frac{(8\beta - \eta - 2)^2}{\beta(16\beta - \eta^2 - 4\eta - 4)}}$.

12. P_{13}^- is non-hyperbolic with a 4D stable manifold for

(a) $\beta < 0, \eta \geq 1, k < \sqrt{2}\sqrt{\frac{(8\beta - \eta - 2)^2}{\beta(16\beta - \eta^2 - 4\eta - 4)}}$, or

(b) $\beta > \frac{9}{16}, 1 \leq \eta < 2(2\sqrt{\beta} - 1), k > -\sqrt{2}\sqrt{\frac{(8\beta - \eta - 2)^2}{\beta(16\beta - \eta^2 - 4\eta - 4)}}$.

13. P_{16}^+ is a source for

(a) $\eta \geq 1, 0 < \beta < \frac{\eta+2}{8}, -\sqrt{\frac{\eta}{\beta}} < k < -2\sqrt{2}\sqrt{\frac{\eta^2}{-16\beta + \eta^2 + 4\eta + 4}}$, or

(b) $\eta \geq 1, 0 < \beta < \frac{\eta+2}{8}, 2\sqrt{2}\sqrt{\frac{\eta^2}{-16\beta + \eta^2 + 4\eta + 4}} < k < \sqrt{\frac{\eta}{\beta}}$.

14. P_{16}^- is a sink for

(a) $\eta \geq 1, 0 < \beta < \frac{\eta+2}{8}, -\sqrt{\frac{\eta}{\beta}} < k < -2\sqrt{2}\sqrt{\frac{\eta^2}{-16\beta + \eta^2 + 4\eta + 4}}$, or

(b) $\eta \geq 1, 0 < \beta < \frac{\eta+2}{8}, 2\sqrt{2}\sqrt{\frac{\eta^2}{-16\beta + \eta^2 + 4\eta + 4}} < k < \sqrt{\frac{\eta}{\beta}}$.

15. P_{18}^+ is a source for

(a) $\eta \geq 1, \beta < 0, k < -\sqrt{2}\sqrt{\frac{4\beta - 1}{\beta}}$, or

(b) $\eta \geq 1, \beta < 0, k > \sqrt{2}\sqrt{\frac{4\beta - 1}{\beta}}$, or

(c) $\eta \geq 1, 0 < \beta \leq \frac{\eta+2}{8}, k < -\sqrt{\frac{\eta}{\beta}}$, or

(d) $\eta \geq 1, 0 < \beta \leq \frac{\eta+2}{8}, k > \sqrt{\frac{\eta}{\beta}}$, or

(e) $\eta \geq 1, \beta > \frac{\eta+2}{8}, k < -\sqrt{2}\sqrt{\frac{4\beta - 1}{\beta}}$, or

(f) $\eta \geq 1, \beta > \frac{\eta+2}{8}, k > \sqrt{2}\sqrt{\frac{4\beta - 1}{\beta}}$.

16. P_{18}^- is a sink for

- (a) $\eta \geq 1, \beta < 0, k < -\sqrt{2}\sqrt{\frac{4\beta-1}{\beta}}$, or
- (b) $\eta \geq 1, \beta < 0, k > \sqrt{2}\sqrt{\frac{4\beta-1}{\beta}}$, or
- (c) $\eta \geq 1, 0 < \beta \leq \frac{\eta+2}{8}, k < -\sqrt{\frac{\eta}{\beta}}$, or
- (d) $\eta \geq 1, 0 < \beta \leq \frac{\eta+2}{8}, k > \sqrt{\frac{\eta}{\beta}}$, or
- (e) $\eta \geq 1, \beta > \frac{\eta+2}{8}, k < -\sqrt{2}\sqrt{\frac{4\beta-1}{\beta}}$, or
- (f) $\eta \geq 1, \beta > \frac{\eta+2}{8}, k > \sqrt{2}\sqrt{\frac{4\beta-1}{\beta}}$.

As can be seen in the Appendix C, all the possible metrics (with nontrivial scalar field) can be written in a compact form as

$$ds^2 = -e^{2a\lambda} dt^2 + e^{2b\lambda} d\lambda^2 + e^{2c\lambda} (d\vartheta^2 + \sin^2 \vartheta d\varphi^2); \tag{78}$$

Hence for $c \neq 0$ always there is a singularity at $l = \infty$. You can make that easily $r = 0$, through the transformation $\exp(2cl) = r^2$. The same for $a = 0$, or $b = 0$ just $c \neq 0$. Now on the other hand the other possible case is for $c = 0$. In that case:

1. We do not have a singularity when: $b < 0$ or when $b > 0$ and $a^2 = ab$.
2. When $b = 0$ and $a^2 \neq ab$ we have a singularity.

5 Discussion

In this paper we have investigated the field equations in the Einstein-aether model in a static spherically symmetric spacetime. The static model with perfect fluid, first introduced in Section 6.1 of [21], has been investigated using more appropriate dynamical variables inspired by [35] with a direct physical interpretation which lead to the system (28), for which we have presented further results. The results of [35] for GR have been extended to the Einstein-aether setup. In particular, we are interested in models which are asymptotically vacuum and asymptotically flat, and which admit singularities. We have found asymptotic expansions for all of the equilibrium points in the finite region. We have shown that the Minkowski spacetime can be given in explicit spherically symmetric form [35] irrespectively on the aether parameter. We have shown that we can have nonregular self-similar perfect fluid solutions like those in [36–38], self-similar plane-symmetric perfect fluid models and Kasner plane-symmetric vacuum solutions [39]. We have discussed the existence of new solutions related with naked singularities or with horizons. The line elements have been presented in explicit form. In addition, we have discussed the dynamics at infinity and presented some numerical results supporting our analytical results. In the next subsection we will summarize all of the sources and sinks in the perfect fluid model. We have also investigated Einstein-aether perfect fluid cosmological models and a scalar field with an exponential self-interaction potential (we shall study the case of a harmonic potential in the Paper III [61]).

In a subsequent paper [57], referred as Paper II, we present a singularity analysis for these models. That is, we study if the gravitational field equations possesses the Painleve property; consequently one can find if an analytic explicit integration can be performed for the field equations. Then, we can apply the classical treatment for the singularity analysis which is summarized in the ARS algorithm. Furthermore, it is of interest the formulation of the modified Tolman–Oppenheimer–Volkoff equations for perfect fluids with linear and polytropic equations of state in the Einstein-aether theory, and the addition of scalar field with exponential or an harmonic potentials. One special application which we are interested in is to use dynamical system tools to determine conditions under which stable stars can form. By using the Tolman–Oppenheimer–Volkoff (TOV) approach [36–38], the relativistic TOV equations are drastically modified in Einstein-aether theory, and we can explore the physical implications of that modification. Then we can construct a 3D dynamical system in compact variables and obtain a picture of the entire solution space for a linear EoS, that can visualized in a geometrical way. This study can be extended to a wide class of EoS, for example polytropic EoS. For higher dimensional systems we still can find information by numerical integrations and the use of projections. The results obtained can be inserted coherently into the physical models, obtaining an appropriate description of the universe both in local and larger scales. More of this analysis can be found in the paper [57].

5.1 Summary of relevant saddles

There are relevant equilibrium points which are saddle points:

1. The equilibrium point P_3^+ represents the Minkowski spacetime in spherical symmetric form. For which we find the more familiar equations

$$Q = 1 - \varepsilon_1 e^{2\lambda}, \quad S = \frac{2}{3}(\varepsilon_2 - \varepsilon_1)e^{2\lambda}, \quad C = 1 - \frac{4}{2 + \eta}((1 - \beta)\varepsilon_1 + \varepsilon_2)e^{2\lambda}, \tag{79}$$

where ε_1 and ε_2 are still small constants (we assume they are positive), that reproduce equations (27a–27c) of [35] for $\beta = 1$. We see that $\varepsilon = \frac{\varepsilon_1}{\varepsilon_2}$ parametrize a 1-parameter family of regular solutions with an equation of state parameter at the center:

$$\begin{aligned} \frac{p_c}{\mu_c} &= \lim_{\lambda \rightarrow -\infty} \frac{p}{\mu} = \lim_{\lambda \rightarrow -\infty} \frac{\mu - \mu_0}{(\eta - 1)\mu} = \lim_{\lambda \rightarrow -\infty} \frac{C - Q^2 + \beta S^2}{C(\eta - 1) - \eta + Q^2 + \beta(\eta - 1)S^2} \\ &= \frac{2 - \varepsilon(2\beta + \eta)}{\varepsilon(2\beta(\eta - 1) - 3\eta) + 2(\eta - 1)}. \end{aligned}$$

The quotient, $\frac{p_c}{\mu_c}$ is a gravitational strength parameter. In GR where the parameter $\beta = 1$, the maximal value of the gravitational strength, $\frac{1}{\eta+1}$, is obtained when $\varepsilon_1 = 0$, which corresponds to the subset $Q = 1$. However, in the Einstein-aether theory the parameter β is a freely specifiable parameter, and for

$\eta > 1, \beta > \frac{3\eta}{2\eta-2}, \frac{\varepsilon_1}{\varepsilon_2} > \frac{2\eta-2}{2\beta\eta-2\beta-3\eta}$, the maximal strength is not $\frac{p_c}{\mu_c} = \frac{1}{\eta+1}$ any-more as it is in GR.

We see that there exists solutions with a regular center but negative pressure, so that we have to impose the condition

$$\frac{2 - \varepsilon(2\beta + \eta)}{\varepsilon(2\beta(\eta - 1) - 3\eta) + 2(\eta - 1)} > 0,$$

that is:

- (a) $\eta > 1, \beta \leq -\frac{\eta}{2}, \varepsilon > 0$, or
- (b) $\eta > 1, -\frac{\eta}{2} < \beta \leq \frac{3\eta}{2\eta-2}, 0 < \varepsilon < \frac{2}{2\beta+\eta}$, or
- (c) $\eta > 1, \beta > \frac{3\eta}{2\eta-2}, 0 < \varepsilon < \frac{2}{2\beta+\eta}$, or
- (d) $\eta > 1, \beta > \frac{3\eta}{2\eta-2}, \varepsilon > \frac{2\eta-2}{2\beta\eta-2\beta-3\eta}$.

This condition is reduced in GR, to $\eta > 1, 0 < \varepsilon < \frac{2}{\eta+2}$, when $\beta = 1$.

For $C - (Q - S)^2 > 0$, the first and second Buchdahl conditions are satisfied at the solution as $\lambda \rightarrow -\infty$, if

$$\frac{(\beta - 1)(\eta + 2)\varepsilon}{\varepsilon(6\beta + \eta - 4) + 2(\eta - 1)} \geq 0,$$

$$\frac{\eta(\eta + 2)\varepsilon(\mu_0 + (\eta - 1)p_c)}{\mu_0(\varepsilon(6\beta + \eta - 4) + 2(\eta - 1))} \leq 1.$$

Additionally, taking the limit $\lambda \rightarrow -\infty$ we have

$$\frac{1}{9} \left(7C - 3Q^2 + 3\beta S^2 \right) + 2\sqrt{C(C + 3Q^2 - 3\beta S^2)} - C$$

$$+ (Q - S)^2 \rightarrow \frac{40}{9} > 0,$$

such that the third Buchdahl condition is also satisfied. Thus, combining these conditions we have the conditions for the existence of regular solution at the center associated to P_3^+ .

2. The equilibrium point P_4^+ generalizes the so called Tolman point (which corresponds to $\beta = 1$), which now is promoted to a 1-parameter solution. This solution exists for $0 \leq \beta \leq \frac{1}{8}(\eta + 2)^2$. Following the same method as for the analysis of P_3^+ we have explored approximated solutions related to P_4^+ by constructing the unstable manifold of this equilibrium point.

Case 1:

When λ_2, λ_3 are both reals and negative, that is whenever $\frac{63}{64} < \beta \leq \frac{9}{8}, 1 < \eta \leq -2 + \frac{8\sqrt{\beta}}{\sqrt{7}}$, or $\beta > \frac{9}{8}, 2\sqrt{2}\sqrt{\beta} - 2 < -2 + \eta \leq \frac{8\sqrt{\beta}}{\sqrt{7}}$, we obtain that any solution near the unstable manifold of P_4^+ , satisfies

$$\begin{aligned}
 Q &= 1 - \frac{(\eta + 2)^2(-4\beta + \eta(2\eta + 3) + 2)}{2((4\beta + \eta - 2)((\eta + 2)^2 - 8\beta))} \varepsilon e^{\frac{2\eta\lambda}{\eta+2}}, \\
 S &= \frac{2}{\eta + 2} - \frac{(\eta + 2)(-8\beta^2 - 2(\eta - 2)\beta + \eta(\eta + 2)^2)}{2(\beta(4\beta + \eta - 2)((\eta + 2)^2 - 8\beta))} \varepsilon e^{\frac{2\eta\lambda}{\eta+2}} + \mathcal{O}(\varepsilon^2)e^{\frac{4\eta\lambda}{\eta+2}}, \\
 C &= 1 - \frac{8\beta}{(\eta + 2)^2} + \varepsilon e^{\frac{2\eta}{\eta+2}\lambda} + \mathcal{O}(\varepsilon^2)e^{\frac{4\eta\lambda}{\eta+2}}.
 \end{aligned}$$

This expansion is accurate as long as $\lambda \rightarrow -\infty$.

Using this solution, we find

$$\begin{aligned}
 \frac{p_c}{\mu_c} &= \lim_{\lambda \rightarrow -\infty} \frac{C - Q^2 + \beta S^2}{C(\eta - 1) - \eta + Q^2 + \beta(\eta - 1)S^2} \\
 &= \lim_{\lambda \rightarrow -\infty} \frac{1}{\eta - 1} - \frac{\varepsilon \left(\eta(\eta + 2)^4(-4\beta + \eta(2\eta + 3) + 2)e^{\frac{2\eta\lambda}{\eta+2}} \right)}{4(\beta(\eta - 1)^2(4\beta + \eta - 2)((\eta + 2)^2 - 8\beta))} + \mathcal{O}(\varepsilon^2) \\
 &= \frac{1}{\eta - 1} > 0.
 \end{aligned}$$

Furthermore, the Buchdahl conditions that can be expressed as

$$\begin{aligned}
 1 &\geq \frac{-(\eta - 1)(C + \beta S^2) + \eta - Q^2}{3(C - (Q - S)^2)}, \\
 1 &\leq \frac{\eta(1 - Q^2)(\mu_0 + (\eta - 1)p_c)}{3\mu_0(C - (Q - S)^2)}, \\
 \frac{1}{9} &\left(7C - 3Q^2 + 3\beta S^2 \right) + 2\sqrt{C(C + 3Q^2 - 3\beta S^2)} - C + (Q - S)^2 \geq 0.
 \end{aligned}$$

And as $\lambda \rightarrow -\infty$, applying the above conditions we have that the second one is satisfied; and the first and third imply

$$\begin{aligned}
 \frac{\beta - \beta\eta}{6\beta - 3(\eta + 1)} &\geq 1, \\
 \frac{4(7\beta + 9)}{9(\eta + 2)^2} + 2\sqrt{\left(4 - \frac{20\beta}{(\eta + 2)^2} \right) \left(1 - \frac{8\beta}{(\eta + 2)^2} \right)} - \frac{4}{\eta + 2} + \frac{4}{9} &\geq 0.
 \end{aligned}$$

These conditions are not satisfied for $\beta = 1$ (that is, for GR). But in AE-theory β is a free parameter, such that the above inequalities can be satisfied for $\eta > 1$, $\frac{3\eta+3}{\eta+5} \leq \beta < \frac{\eta+1}{2}$, $64\beta - 7(\eta + 2)^2 \geq 0$.

Case 2:

For the choice $0 < \beta \leq \frac{63}{64}$, $\eta > 1$, or $\beta > \frac{63}{64}$, $\eta > -2 + \frac{8\sqrt{\beta}}{\sqrt{7}}$, the eigenvalues λ_2, λ_3 are complex conjugates with negative real part. We obtain that any solution near the unstable manifold of P_4^+ satisfies

$$\begin{aligned}
 Q &= 1 - \frac{(\eta + 2)^2(-4\beta + \eta(2\eta + 3) + 2)}{2(4\beta + \eta - 2)(8\beta - (\eta + 2)^2)} \varepsilon e^{\frac{2\eta\lambda}{\eta+2}}, \\
 S &= \frac{2}{\eta + 2} + \frac{(\eta + 2)\varepsilon(-8\beta^2 - 2\beta(\eta - 2) + \eta(\eta + 2)^2)}{2\beta(4\beta + \eta - 2)(8\beta - (\eta + 2)^2)} e^{\frac{2\eta\lambda}{\eta+2}} + \mathcal{O}(\varepsilon^2)e^{\frac{4\eta\lambda}{\eta+2}}, \\
 C &= 1 - \frac{8\beta}{(\eta + 2)^2} + \varepsilon e^{\frac{2\eta\lambda}{\eta+2}} + \mathcal{O}(\varepsilon^2)e^{\frac{4\eta\lambda}{\eta+2}},
 \end{aligned}$$

where we have substituted the approximated solution $u_1 = \varepsilon e^{\frac{2\eta}{\eta+2}\lambda}$, that is obtained by integrating the linearized equation along the unstable direction. This expansion is accurate as long as $\lambda \rightarrow -\infty$. At the stable manifold the orbits spiral in and tend asymptotically to the origin with modes $\cos(\frac{\sqrt{7(\eta+2)^2-64\beta}}{2(\eta+2)}\lambda)e^{-\frac{\lambda}{2}}$, $\sin(\frac{\sqrt{7(\eta+2)^2-64\beta}}{2(\eta+2)}\lambda)e^{-\frac{\lambda}{2}}$.

We have the estimates

$$\begin{aligned}
 \frac{p_c}{\mu_c} &= \lim_{\lambda \rightarrow -\infty} \frac{C - Q^2 + \beta S^2}{C(\eta - 1) - \eta + Q^2 + \beta(\eta - 1)S^2} \\
 &= \lim_{\lambda \rightarrow -\infty} \frac{1}{\eta - 1} + \frac{\eta(\eta + 2)^4\varepsilon(-4\beta + 2\eta^2 + 3\eta + 2)}{4\beta(\eta - 1)^2(4\beta + \eta - 2)(-8\beta + \eta^2 + 4\eta + 4)} e^{\frac{2\eta\lambda}{\eta+2}} + \mathcal{O}(\varepsilon^2) \\
 &= \frac{1}{\eta - 1} > 0.
 \end{aligned}$$

Furthermore, for $C - (Q - S)^2 > 0$, the Buchdahl conditions reduce to

$$\begin{aligned}
 \frac{2(\beta - 1)\beta}{-2\beta + \eta + 1} + \beta &\leq 3, \quad 2\beta\mu_0 \geq (\eta + 1)\mu_0, \\
 \frac{9\sqrt{40\beta^2 - 13\beta(\eta + 2)^2 + (\eta + 2)^4} + 7\beta + \eta^2 - 5\eta - 5}{2\beta - \eta - 1} &\leq 0,
 \end{aligned}$$

as $\lambda \rightarrow -\infty$, respectively.

That is, when $1 < \eta \leq 1.04725$, $\beta < \frac{1}{64}(7\eta^2 + 28\eta + 28)$, $\mu_0 \leq 0$ or $\eta > 1.04725$, $\beta \leq \frac{3\eta+3}{\eta+5}$, $\mu_0 \leq 0$. We are assuming $\mu_0 \geq 0$, therefore the conditions are fulfilled if $\mu_0 = 0$.

5.2 Summary of sources and sinks

5.2.1 Perfect fluid

For this analysis we have used the formulation $\{Q, S, C\}$ given by the model (28), which represents the evolution of a perfect fluid has the EoS $\mu = \mu_0 + (\eta - 1)p$, $\eta > 1$ in the static Einstein-aether theory. We have found the following summary of sources/sinks:

1. P_1^+ is a source for $\beta = 1, 1 \leq \eta < 2$. Since the conditions (93) are fulfilled this solution has a regular center as $\lambda \rightarrow -\infty$. Because of $C = 0$ it belongs to the plane-symmetric boundary set. Furthermore, it belongs to the scale invariant boundary $Q = +1$.
2. P_1^- is a sink for $\beta = 1, 1 \leq \eta < 2$. Because of $C = 0$ it belongs to the plane-symmetric boundary set. Furthermore, it belongs to the scale invariant boundary $Q = -1$. Since the conditions (94) are fulfilled this solution is asymptotically flat as $\lambda \rightarrow +\infty$.
3. P_2^+ is a source for $\beta = 1, \eta \geq 1$. Since the first inequality of (93) is not fulfilled, this solution does not have a regular center. Because of $C = 0$ it belongs to the plane-symmetric boundary set. Furthermore, it belongs to the scale invariant boundary $Q = +1$.
4. P_2^- is a sink for $\beta = 1, \eta \geq 1$. Because of $C = 0$ it belongs to the plane-symmetric boundary set. Furthermore, it belongs to the scale invariant boundary $Q = -1$. This solution is not asymptotically flat since the conditions (94) are not fulfilled as $\lambda \rightarrow +\infty$.
5. P_2^+ is a sink for $\beta = -\frac{\eta+2}{4} \leq 1, \eta \geq 1$. Because of $C = 0$ it belongs to the plane-symmetric boundary set. Furthermore, it belongs to the scale invariant boundary $Q = +1$. This solution is not asymptotically flat since the conditions (94) are not fulfilled as $\lambda \rightarrow +\infty$.
6. P_2^- is a source for $\beta = -\frac{\eta+2}{4} \leq 1, \eta \geq 1$. Since the first inequality of (93) is not fulfilled, this solution does not have a regular center as $\lambda \rightarrow -\infty$. Because of $C = 0$ it belongs to the plane-symmetric boundary set. Furthermore, it belongs to the scale invariant boundary $Q = -1$.
7. P_5^+ is a sink for $\eta \geq 1, \beta < 0$. Because of $C = 0$ it belongs to the plane-symmetric boundary set. Furthermore, it belongs to the scale invariant boundary $Q = -1$. Furthermore, it belongs to the scale invariant boundary $Q = +1$. This solution is not asymptotically flat since the conditions (94) are not fulfilled as $\lambda \rightarrow +\infty$.
8. P_5^- is a source for $\eta \geq 1, \beta < 0$. Since the conditions (93) are not fulfilled, this solution does not have a regular center as $\lambda \rightarrow -\infty$. Because of $C = 0$ it belongs to the plane-symmetric boundary set. Furthermore, it belongs to the scale invariant boundary $Q = -1$.
9. P_6^+ is a source for $\eta \geq 1, 16\beta \geq (\eta + 2)^2$. It has a regular center as $\lambda \rightarrow -\infty$ only when $\beta = 1$ (i.e., when this point coincides with P_1^+). Otherwise the conditions (93) are not fulfilled, and the solution does not have a regular center as $\lambda \rightarrow -\infty$. Because of $C = 0$ it belongs to the plane-symmetric boundary set. Furthermore, it belongs to the scale invariant boundary $Q = +1$.
10. P_6^- is a sink for $\eta \geq 1, 16\beta \geq (\eta + 2)^2$. Because of $C = 0$ it belongs to the plane-symmetric boundary set. Furthermore, it belongs to the scale invariant boundary $Q = -1$. It is not asymptotically flat as $\lambda \rightarrow +\infty$ unless $\beta = 1$ (i.e., when P_6^- merge with P_1^-).
11. P_7^+ is a source for $\eta \geq 1, \beta > 0$. The conditions (93) are not fulfilled, and the solution does not have a regular center as $\lambda \rightarrow -\infty$. Because of $C = 0$ it belongs to the plane-symmetric boundary set. Furthermore, it belongs to the scale invariant boundary $Q = +1$.

- 12. P_7^- is a sink for $\eta \geq 1, \beta > 0$. Because of $C = 0$ it belongs to the plane-symmetric boundary set. Furthermore, it belongs to the scale invariant boundary $Q = -1$.
- 13. P_8^+ is a sink for $\beta < 0$. It is not asymptotically flat as $\lambda \rightarrow +\infty$.
- 14. P_8^- is a source for $\beta < 0$. It has a regular center as $\lambda \rightarrow -\infty$ if $\eta > 1, \beta < 0, \mu_0 > 0, p_c \geq \frac{\mu_0(6\beta - \beta\eta - 3)}{\beta\eta(\eta - 1)}$.

5.2.2 Perfect fluid plus a scalar field with exponential potential

On the other hand, we have taken a natural extension of the previous analysis, as in the General Relativistic case [55], by studying the model (72), which corresponds to a stationary comoving aether with perfect fluid and scalar field with exponential potential in a static metric. And we have presented the following summary of sources/sinks:

- 1. P_5^+ is a sink for $\beta < 0, \eta \geq 1$.
- 2. P_5^- is source for $\beta < 0, \eta \geq 1$.
- 3. P_8^+ is non-hyperbolic with a 4D stable manifold.
- 4. P_8^- is non-hyperbolic with a 4D unstable manifold.
- 5. The following subsets (arcs, or specific equilibrium points) of the line $P_{10}^+(S_c)$ are unstable for the given conditions:

- (a) $S_c < 0, \beta < \frac{1}{S_c^2}, \eta > 1, k < \sqrt{\frac{-2S_c^2 + 8S_c - 8}{\beta S_c^2 - 1}}$, or
- (b) $S_c < 0, \beta = \frac{1}{S_c^2}, \eta > 1, k \in \mathbb{R}$, or
- (c) $S_c = 0, \beta \in \mathbb{R}, \eta > 1, k < 2\sqrt{2}$, or
- (d) $0 < S_c < \frac{4}{3}, \beta < \frac{1}{S_c^2}, 1 < \eta < \frac{4 - 2S_c}{S_c}, k < \sqrt{\frac{-2S_c^2 + 8S_c - 8}{\beta S_c^2 - 1}}$, or
- (e) $0 < S_c < \frac{4}{3}, \beta = \frac{1}{S_c^2}, 1 < \eta < \frac{4 - 2S_c}{S_c}, k \in \mathbb{R}$.

- 6. The following subsets (arcs, or specific equilibrium points) of the line $P_{10}^-(S_c)$ are stable for the given conditions:

- (a) $S_c < 0, \beta < \frac{1}{S_c^2}, \eta > 1, k < \sqrt{\frac{-2S_c^2 + 8S_c - 8}{\beta S_c^2 - 1}}$, or
- (b) $S_c < 0, \beta = \frac{1}{S_c^2}, \eta > 1, k \in \mathbb{R}$, or
- (c) $S_c = 0, \beta \in \mathbb{R}, \eta > 1, k < 2\sqrt{2}$, or
- (d) $0 < S_c < \frac{4}{3}, \beta < \frac{1}{S_c^2}, 1 < \eta < \frac{4 - 2S_c}{S_c}, k < \sqrt{\frac{-2S_c^2 + 8S_c - 8}{\beta S_c^2 - 1}}$, or
- (e) $0 < S_c < \frac{4}{3}, \beta = \frac{1}{S_c^2}, 1 < \eta < \frac{4 - 2S_c}{S_c}, k \in \mathbb{R}$.

- 7. The following subsets (arcs, or specific equilibrium points) of the line $P_{11}^+(S_c)$ are unstable for the given conditions:

- (a) $S_c < 0, \beta < \frac{1}{S_c^2}, \eta \geq 1, k > -\sqrt{2} \sqrt{-\frac{S_c^2 - 4S_c + 4}{\beta S_c^2 - 1}}$, or
- (b) $S_c < 0, \beta = \frac{1}{S_c^2}, \eta \geq 1, k \in \mathbb{R}$, or

- (c) $S_c = 0, \beta \in \mathbb{R}, \eta \geq 1, k > -2\sqrt{2}$, or
- (d) $0 < S_c < \frac{4}{3}, \beta < \frac{1}{S_c^2}, 1 \leq \eta < \frac{4-2S_c}{S_c}, k > -\sqrt{2}\sqrt{-\frac{S_c^2-4S_c+4}{\beta S_c^2-1}}$, or
- (e) $0 < S_c < \frac{4}{3}, \beta = \frac{1}{S_c^2}, 1 \leq \eta < \frac{4-2S_c}{S_c}, k \in \mathbb{R}$.

8. The following subsets (arcs, or specific equilibrium points) of the line $P_{11}^-(S_c)$ are stable for the given conditions:

- (a) $S_c < 0, \beta < \frac{1}{S_c^2}, \eta \geq 1, k > -\sqrt{2}\sqrt{-\frac{S_c^2-4S_c+4}{\beta S_c^2-1}}$, or
- (b) $S_c < 0, \beta = \frac{1}{S_c^2}, \eta \geq 1, k \in \mathbb{R}$, or
- (c) $S_c = 0, \beta \in \mathbb{R}, \eta \geq 1, k > -2\sqrt{2}$, or
- (d) $0 < S_c < \frac{4}{3}, \beta < \frac{1}{S_c^2}, 1 \leq \eta < \frac{4-2S_c}{S_c}, k > -\sqrt{2}\sqrt{-\frac{S_c^2-4S_c+4}{\beta S_c^2-1}}$, or
- (e) $0 < S_c < \frac{4}{3}, \beta = \frac{1}{S_c^2}, 1 \leq \eta < \frac{4-2S_c}{S_c}, k \in \mathbb{R}$.

9. P_{12}^+ is non-hyperbolic with a 4D unstable for

- (a) $\beta < 0, \eta \geq 1, k > -\sqrt{2}\sqrt{\frac{(8\beta-\eta-2)^2}{\beta(16\beta-\eta^2-4\eta-4)}}$, or
- (b) $\beta > \frac{9}{16}, 1 \leq \eta < 2(2\sqrt{\beta}-1), k < \sqrt{2}\sqrt{\frac{(8\beta-\eta-2)^2}{\beta(16\beta-\eta^2-4\eta-4)}}$.

10. P_{12}^- is non-hyperbolic with a 4D stable manifold for

- (a) $\beta < 0, \eta \geq 1, k > -\sqrt{2}\sqrt{\frac{(8\beta-\eta-2)^2}{\beta(16\beta-\eta^2-4\eta-4)}}$, or
- (b) $\beta > \frac{9}{16}, 1 \leq \eta < 2(2\sqrt{\beta}-1), k < \sqrt{2}\sqrt{\frac{(8\beta-\eta-2)^2}{\beta(16\beta-\eta^2-4\eta-4)}}$.

11. P_{13}^+ is non-hyperbolic with a 4D unstable manifold for

- (a) $\beta < 0, \eta \geq 1, k < \sqrt{2}\sqrt{\frac{(8\beta-\eta-2)^2}{\beta(16\beta-\eta^2-4\eta-4)}}$, or
- (b) $\beta > \frac{9}{16}, 1 \leq \eta < 2(2\sqrt{\beta}-1), k > -\sqrt{2}\sqrt{\frac{(8\beta-\eta-2)^2}{\beta(16\beta-\eta^2-4\eta-4)}}$.

12. P_{13}^- is non-hyperbolic with a 4D stable manifold for

- (a) $\beta < 0, \eta \geq 1, k < \sqrt{2}\sqrt{\frac{(8\beta-\eta-2)^2}{\beta(16\beta-\eta^2-4\eta-4)}}$, or
- (b) $\beta > \frac{9}{16}, 1 \leq \eta < 2(2\sqrt{\beta}-1), k > -\sqrt{2}\sqrt{\frac{(8\beta-\eta-2)^2}{\beta(16\beta-\eta^2-4\eta-4)}}$.

13. P_{16}^+ is a source for

- (a) $\eta \geq 1, 0 < \beta < \frac{\eta+2}{8}, -\sqrt{\frac{\eta}{\beta}} < k < -2\sqrt{2}\sqrt{\frac{\eta^2}{-16\beta+\eta^2+4\eta+4}}$, or

$$(b) \eta \geq 1, 0 < \beta < \frac{\eta+2}{8}, 2\sqrt{2}\sqrt{\frac{\eta^2}{-16\beta+\eta^2+4\eta+4}} < k < \sqrt{\frac{\eta}{\beta}}.$$

14. P_{16}^- is a sink for

$$(a) \eta \geq 1, 0 < \beta < \frac{\eta+2}{8}, -\sqrt{\frac{\eta}{\beta}} < k < -2\sqrt{2}\sqrt{\frac{\eta^2}{-16\beta+\eta^2+4\eta+4}}, \text{ or}$$

$$(b) \eta \geq 1, 0 < \beta < \frac{\eta+2}{8}, 2\sqrt{2}\sqrt{\frac{\eta^2}{-16\beta+\eta^2+4\eta+4}} < k < \sqrt{\frac{\eta}{\beta}}.$$

15. P_{18}^+ is a source for

$$(a) \eta \geq 1, \beta < 0, k < -\sqrt{2}\sqrt{\frac{4\beta-1}{\beta}}, \text{ or}$$

$$(b) \eta \geq 1, \beta < 0, k > \sqrt{2}\sqrt{\frac{4\beta-1}{\beta}}, \text{ or}$$

$$(c) \eta \geq 1, 0 < \beta \leq \frac{\eta+2}{8}, k < -\sqrt{\frac{\eta}{\beta}}, \text{ or}$$

$$(d) \eta \geq 1, 0 < \beta \leq \frac{\eta+2}{8}, k > \sqrt{\frac{\eta}{\beta}}, \text{ or}$$

$$(e) \eta \geq 1, \beta > \frac{\eta+2}{8}, k < -\sqrt{2}\sqrt{\frac{4\beta-1}{\beta}}, \text{ or}$$

$$(f) \eta \geq 1, \beta > \frac{\eta+2}{8}, k > \sqrt{2}\sqrt{\frac{4\beta-1}{\beta}}.$$

16. P_{18}^- is a sink for

$$(a) \eta \geq 1, \beta < 0, k < -\sqrt{2}\sqrt{\frac{4\beta-1}{\beta}}, \text{ or}$$

$$(b) \eta \geq 1, \beta < 0, k > \sqrt{2}\sqrt{\frac{4\beta-1}{\beta}}, \text{ or}$$

$$(c) \eta \geq 1, 0 < \beta \leq \frac{\eta+2}{8}, k < -\sqrt{\frac{\eta}{\beta}}, \text{ or}$$

$$(d) \eta \geq 1, 0 < \beta \leq \frac{\eta+2}{8}, k > \sqrt{\frac{\eta}{\beta}}, \text{ or}$$

$$(e) \eta \geq 1, \beta > \frac{\eta+2}{8}, k < -\sqrt{2}\sqrt{\frac{4\beta-1}{\beta}}, \text{ or}$$

$$(f) \eta \geq 1, \beta > \frac{\eta+2}{8}, k > \sqrt{2}\sqrt{\frac{4\beta-1}{\beta}}.$$

5.3 Universal horizons

In the Einstein-aether theory there are spherical black hole solutions formed by gravitational collapse for all viable parameter values of the theory. However, due to the Lorentz-violating nature of the theory, these solutions are quite different from the standard black holes in GR, since the broken Lorentz invariance completely modifies the causal structure of gravity, and the Killing horizon does not capture the notion of the causal boundary. Indeed, Lorentz-violating theories now admit superluminal excitations, which can cross the Killing horizon and escape to spatial infinity. In some particular Lorentz-violating theories, like the Einstein-aether theory, the static, spherically-symmetric, black hole solutions contain a special hypersurface called the “universal horizon” that acts as a genuine causal boundary because it traps all excitations, even those which could be traveling at arbitrarily high velocities [43,66]. Consequently, still there is a causally disconnected region in black hole solutions but

now being bounded by a universal horizon not far inside the metric horizon, so that a notion of black hole persists [43,67].

For studying the causal structure of spacetimes with a causally preferred foliation, a framework was developed that allows for rigorously defined concepts such as black/white holes and also formalizes the notion of a universal horizon introduced previously in the simpler setting of static and spherically symmetric geometries [67]. The question of what happens to the universal horizon in the extremal limit, where no such region exists any longer, has also been investigated [68]. In addition, Hawking radiation has been found to be associated with the universal horizon. These absolute causal boundaries are not Killing horizons but still obey a first law of black hole mechanics [69] and must consequently have an entropy if they do not violate a generalized second law. At these horizons, the Hawking radiation is thermal with a temperature proportional to its surface gravity. The viability of the first law (and hence a thermodynamical interpretation) has been studied for several known exact universal horizon solutions [70] and calculations do, indeed, appear to predict the emission of a thermal flux [71].

Therefore, there are absolute causal boundaries in gravitational theories with broken Lorentz invariance, in which there exists a surface located at a finite $r = r_{uh}$ called a universal horizon (and which always lies inside the Killing horizon) which acts like a one-way membrane, so that particles even with infinitely large speed cannot escape from it once they are inside it. In stationary spacetimes it has been shown that the universal horizon can be characterized by the local coordinate and gauge invariant condition

$$u_a \xi^a = 0 \text{ at } r = r_{uh}, \tag{80}$$

where ξ^a denotes the asymptotically time-like Killing vector associated with stationarity and u_a is the four-velocity of the aether [67,72]. Since u_a is time-like by definition, the condition $u_a \xi^a = 0$ can only be satisfied in the region of the spacetime where ξ^a is spacelike.

Unfortunately, the gauge and coordinates used in the qualitative analysis in this paper are not well suited for studying the possible existence of a universal horizon unless, due to a topological pathology, it is located at $r = 0$ ($\lambda \rightarrow -\infty$) or $r \rightarrow \infty$ ($\lambda \rightarrow -\infty$) and characterized by one of the equilibrium points studied earlier. Here we are interested in $N(r_{uh}) = 0$ at finite $r = r_{uh}$. Assuming that N is analytic at $r = r_{uh}$, we can write $N = (r - r_{uh})^n f(r)$, where $f(r) = a^2 + b(r - r_{uh}) + \dots$ close to $r = r_{uh}$, so that

$$y = r \frac{N'}{N} = \frac{n^2 r_{uh}}{(r - r_{uh})} + \dots \rightarrow \infty \tag{81}$$

as $r \rightarrow r_{uh}$. We can study the behavior of N in terms of the variables $\sigma = y$ and $\theta = y - x$ subject to the constraint (67) or the normalized variables defined by eqn. (25), e.g.,

$$S = \frac{\sigma}{\sqrt{\frac{\mu_0}{\eta} + \theta^2}} = \frac{1}{\sqrt{\frac{\mu_0}{\eta y^2} + \left(1 - \frac{x}{y}\right)^2}}, \tag{82}$$

and subject to the constraint

$$1 - C - \beta S^2 \geq 0, \tag{83}$$

where $C \geq 0$, so that Q, S, C are all bounded when $\beta > 0$. Combining all of the restrictions we find as $r \rightarrow r_{uh}$:

1. If $x \rightarrow cy$: then $(c - 1)^2 - \beta > 0$. If $c = 1$, then β is necessarily negative and $S \rightarrow \infty$. If $c \neq 1$, then $S \rightarrow S_0$, where S_0 is defined in terms of the other parameters, and $p + K$ diverges (and assuming that p does not diverges at an horizontal horizon, this implies that $K \rightarrow \infty$).
2. If $x \gg y$: $x^2 \simeq p + K$ diverges and $S \rightarrow 0$ (and C bounded with $C < 1$); note that equilibrium point P_3 has $S = 0, C = 1$.
3. If $y \gg x$: $S \rightarrow 1$, and $\beta \leq 1$; for the GR value $\beta = 1, C \rightarrow 0$.

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A Regularity conditions

In this appendix are summarized some regularity conditions that must satisfy the relevant physical solutions, especially if they are expected to be used as star models.

A.1 Perfect fluid with linear equation of state

A.1.1 Conditions for regularity at the origin and asymptotic flatness

Using the coordinate change $(t, r) \rightarrow (t, \rho)$, where ρ is a new radial coordinate, such that the line element

$$ds^2 = -N^2(r)dt^2 + r^{-2}dr^2 + K^{-1}(r)(d\vartheta^2 + \sin^2 \vartheta d\varphi^2), \tag{84}$$

becomes

$$ds^2 = -N(r(\rho))^2 dt^2 + \frac{d\rho^2}{1 - \frac{2m(\rho)}{\rho}} + \rho^2(d\vartheta^2 + \sin^2 \vartheta d\varphi^2), \tag{85}$$

we have the identifications

$$K = \rho^{-2}, \tag{86}$$

$$\frac{dr}{r} = \frac{d\rho}{\sqrt{1 - \frac{2m(\rho)}{\rho}}}, \tag{87}$$

where $m(\rho)$ denotes the mass up to the radius ρ .

Then, we define the Misner–Sharp mass [73]

$$\mathcal{M}(\rho) := \frac{m(\rho)}{\rho} = \frac{C - (Q - S)^2}{2C}. \tag{88}$$

As a first approach, we impose regularity at the center, that is, as $\lambda \rightarrow -\infty$, by extrapolating the conditions for relativistic stars as given by the Buchdahl inequalities [74,75], which in units where $8\pi G = 1$ are expressed as

$$\mathcal{M} \geq \frac{1}{6}\rho^2\mu, \quad \mathcal{M} \leq \frac{1}{6}\rho^2\mu_c, \quad \mathcal{M} \leq \frac{2}{9} \left(1 - \frac{3}{4}\rho^2p + \sqrt{1 + \frac{3}{4}\rho^2p} \right), \tag{89}$$

where μ_c is the energy density at the center of the star and ρ is a radial variable. To find the generalized regularity conditions we have to integrate the full equations which determine the star’s structure and the geometry in the static spherically symmetric Einstein-aether theory for a perfect fluid starting from the center $\rho = 0$ with central density μ_c , out to the surface $\rho = \rho_*$ where the pressure vanishes. That is, we have to consider the boundary conditions

$$p(0) = p(\rho_c) = p_c, \quad m(0) = 0, \quad e^{2\phi(\rho_*)} = 1 - 2\frac{m(\rho_*)}{\rho_*} \tag{90}$$

and follow the same strategy to find estimates for the mass as in [74,75]. Because we have assumed the linear equation of state $\mu = \mu_0 + (\eta - 1)p$, the energy density at the surface of zero pressure is μ_0 . The central energy density and central pressure are related through $\mu_c = \mu_0 + (\eta - 1)p_c$.

Notice the additional relations for the radial coordinate;

$$\rho^2 = \frac{\eta(1 - Q^2)}{\mu_0 C}, \tag{91}$$

and for the matter energy density:

$$\mu := \mu_0 + (\eta - 1)p = \frac{\mu_0 (\eta - Q^2 - (\eta - 1)(C + \beta S^2))}{\eta (1 - Q^2)}. \tag{92}$$

1. Thus, the Buchdahl conditions (89) can be expressed in terms of the variables Q, S, C , as

$$\frac{\eta - Q^2 - (\eta - 1)(C + \beta S^2)}{3} \leq C - (Q - S)^2$$

$$\leq \min \left\{ \frac{\eta(1 - Q^2)(\mu_0 + (\eta - 1)p_c)}{3\mu_0}, \frac{1}{9} \left((7C - 3Q^2 + 3\beta S^2) + 2\sqrt{C(C + 3Q^2 - 3\beta S^2)} \right) \right\}, \tag{93}$$

as $\lambda \rightarrow -\infty$, where p_c is the central pressure of the star.

2. Asymptotic flatness as $\rho \rightarrow +\infty$ or, equivalently, as $\lambda \rightarrow +\infty$:

$$\lim_{\lambda \rightarrow +\infty} [C - (Q - S)^2] = 0, \quad \lim_{\lambda \rightarrow +\infty} [Q^2 - C - \beta S^2] = 0. \tag{94}$$

The first condition corresponds to $\lim_{\lambda \rightarrow +\infty} \mathcal{M} = 0$. The second condition implies from (36) that

$$\lim_{\lambda \rightarrow +\infty} e^\phi = \alpha,$$

and that the surface of zero pressure is reached. The constant α is absorbed by a time redefinition. This means that asymptotically we obtain the Minkowski metric.

3. From the relations (29), vacuum ($\mu_0 = 0, \mu = 0, p = 0$) corresponds to

$$Q^2 = 1, \quad C + \beta S^2 = 1. \tag{95}$$

A.1.2 Stars

To obtain physically reasonable spherically symmetric models with non-negative pressure one matches each interior solution with the exterior Schwarzschild vacuum solution

$$ds^2 = - \left(1 - \frac{2M}{\rho} \right) dt^2 + \frac{d\rho^2}{\left(1 - \frac{2M}{\rho} \right)} + \rho^2(d\vartheta^2 + \sin^2\vartheta d\varphi^2). \tag{96}$$

when the radius, $\rho_* = \sqrt{\frac{\eta(1-Q_*^2)}{\mu_0(Q_*^2 - \beta S_*^2)}}$, where the pressure becomes zero.⁵ When the radio ρ_* is reached, $\frac{1-Q_*^2}{1-C_*-\beta S_*^2} = 1, e^\phi = \alpha$. This fixes $\alpha = \sqrt{1 - 2M/R} = \sqrt{\frac{(Q_* - S_*)^2}{Q_*^2 - \beta S_*^2}}$, where M is the total mass of the star as given by

$$M = \frac{S_*(2Q_* - (\beta + 1)S_*)}{2(Q_*^2 - \beta S_*^2)} \sqrt{\frac{\eta(1 - Q_*^2)}{\mu_0(Q_*^2 - \beta S_*^2)}}. \tag{97}$$

⁵ In our set up, the solutions in their way from $Q = +1$ to $Q = -1$, all intersect the surface of vanishing pressure $C + \beta S^2 - Q^2 = 0$ at an interior point (Q_*, S_*, C_*) .

The interior solution (evaluated at the surface of zero pressure)

$$ds^2 = -\frac{(Q_* - S_*)^2}{Q_*^2 - \beta S_*^2} dt^2 + \frac{Q_*^2 - \beta S_*^2}{(Q_* - S_*)^2} d\rho^2 + \rho^2(d\vartheta^2 + \sin^2 \vartheta d\varphi^2). \tag{98}$$

is matched at $\rho = \rho_*$ with the static vacuum spacetime described by (96).

B Equilibrium points in the finite region of the phase space for a perfect fluid with linear equation of state

We now try to find some asymptotic expansions for all the equilibrium points of (28). By convenience, we introduce the radial rescaling $r = e^\ell$, such that $\ell \rightarrow -\infty$ as $r \rightarrow 0$ and $\ell \rightarrow \infty$ as $r \rightarrow \infty$. Hence, Eq. (84) becomes

$$ds^2 = -N^2(\ell)dt^2 + d\ell^2 + K^{-1}(\ell)(d\vartheta^2 + \sin^2 \vartheta d\varphi^2), \tag{99}$$

The equilibrium points of the system (28) are

1. $P_1^\pm: (Q, S, C) = (\pm 1, \pm 1, 0)$ exist for $\beta = 1$ or $\beta = \frac{\eta+2}{4} \leq 0$. For $\beta = 1, 1 \leq \eta < 2$, P_1^+ (respectively, P_1^-) is a source (respectively, a sink); for $\beta = 1, \eta > 2$, P_1^\pm are saddles and for $\beta = \frac{\eta+2}{4} \leq 0$, P_1^\pm are saddles. On substitution of the values of Q and S into (34a), (34b) and integration we obtain $N = \bar{N}_0 e^{\pm\lambda}$ and $K = \bar{K}_0 = \text{constant}$. After we evaluate the values of Q, S, C in (26) and (34c), it follows that $\frac{d\ell}{d\lambda} = \pm \frac{1}{y}$ and $\frac{dy}{d\lambda} = \begin{cases} \mp 1, & \beta = 1 \\ \mp \frac{\eta}{2}, & \beta = \frac{\eta+2}{4} \end{cases}$.

Then, $y(\lambda) = \begin{cases} c_1 e^{-\lambda\epsilon}, & \beta = 1 \\ c_1 e^{-\frac{1}{2}\eta\lambda\epsilon}, & \beta = \frac{\eta+2}{4} \end{cases}$, $\ell(\lambda) = \begin{cases} \frac{e^{\lambda\epsilon}}{c_1} + c_2, & \beta = 1 \\ \frac{2e^{\frac{\eta\lambda\epsilon}{2}}}{c_1\eta} + c_2, & \beta = \frac{\eta+2}{4} \end{cases}$, where

$\epsilon = \pm 1$ and c_1, c_2 are constants of integration. For $\beta = 1$ the metric becomes $ds^2 = -\bar{N}_0^2 e^{\pm 2\lambda} dt^2 + \frac{e^{\pm 2\lambda}}{c_1^2} d\lambda^2 + \bar{K}_0^{-1} (d\vartheta^2 + \sin^2 \vartheta d\varphi^2)$

or $ds^2 = -\bar{N}_0^2 c_1^2 \rho_\pm^2 dt^2 + d\rho_\pm^2 + \bar{K}_0^{-1} (d\vartheta^2 + \sin^2 \vartheta d\varphi^2)$ under the coordinate transformation $\rho_\pm = \frac{e^{\pm\lambda}}{c_1}$. They correspond to P_1^\pm in [35], which are the Kasner's plane-symmetric vacuum solutions [39]. For $\beta = \frac{\eta+2}{4} \leq 0$ the metric becomes $ds^2 = -\bar{N}_0^2 e^{\pm 2\lambda} dt^2 + \frac{e^{\pm \eta\lambda}}{c_1^2} d\lambda^2 + \bar{K}_0^{-1} (d\vartheta^2 + \sin^2 \vartheta d\varphi^2)$ or

$ds^2 = -\bar{N}_0^2 c_1^{\frac{4}{\eta}} \rho_\pm^{\frac{4}{\eta}} dt^2 + \frac{4}{\eta^2} d\rho_\pm^2 + \bar{K}_0^{-1} (d\vartheta^2 + \sin^2 \vartheta d\varphi^2)$ under the coordinate transformation $\rho_\pm = \frac{e^{\pm \frac{\eta}{2}\lambda}}{c_1}$. In this case the Ricci scalar is $R = \frac{(\eta-2)}{\rho_\pm^2} + 2\bar{K}_0$. Thus

for $\eta \neq 2$ there is a naked singularity at $\rho_\pm = 0^+$.

2. $P_2^\pm: (Q, S, C) = (\pm 1, \mp 1, 0)$ exist for $\beta = 1$ or $4\beta + \eta + 2 = 0, \beta \leq 1$. For $\beta = 1, \eta \geq 1$, P_2^+ (respectively, P_2^-) is a source (respectively, a sink). For $4\beta + \eta + 2 = 0, \beta \leq 1, \eta \geq 1$, P_2^+ (respectively, P_2^-) is a sink (respectively, a source). On substitution of the values of Q and S into (34a), (34b) and integration we obtain $N = \bar{N}_0 e^{\mp\lambda}$ and $K = \bar{K}_0 e^{\mp 4\lambda}$. After we evaluate the values of Q, S, C

in (26) and (34c), it follows that $\frac{d\ell}{d\lambda} = \mp \frac{1}{y}$ and $\frac{dy}{d\lambda} = \begin{cases} \mp 3, & \beta = 1 \\ \pm \frac{\eta}{2}, & 4\beta + \eta + 2 = 0 \end{cases}$.

Then, $y(\lambda) = \begin{cases} c_1 e^{-3\lambda\epsilon}, & \beta = 1 \\ c_1 e^{\frac{\eta\lambda\epsilon}{2}}, & 4\beta + \eta + 2 = 0 \end{cases}$,

$\ell(\lambda) = \begin{cases} c_2 - \frac{e^{3\lambda\epsilon}}{3c_1}, & \beta = 1 \\ \frac{2e^{-\frac{1}{2}\eta\lambda\epsilon}}{c_1\eta} + c_2, & 4\beta + \eta + 2 = 0 \end{cases}$, where $\epsilon = \pm 1$ and c_1, c_2 are constants

of integration. For $\beta = 1$ the metric becomes $ds^2 = -\bar{N}_0^2 e^{\mp 2\lambda} dt^2 + \frac{e^{\pm 6\lambda}}{c_1^2} d\lambda^2 + \bar{K}_0^{-1} e^{\pm 4\lambda} (d\vartheta^2 + \sin^2 \vartheta d\varphi^2)$ or $ds^2 = -\frac{\bar{N}_0^2}{\rho_{\pm}} dt^2 + \frac{\rho_{\pm}}{4c_1^2} d\rho_{\pm}^2 + \frac{\rho_{\pm}^2}{\bar{K}_0} (d\vartheta^2 + \sin^2 \vartheta d\varphi^2)$

under the coordinate transformation $\rho_{\pm} = e^{\pm 2\lambda}$. The Ricci Scalar becomes $R = \frac{2\bar{K}_0}{\rho_{\pm}^2}$. Thus at $\rho_{\pm} = 0^+$ we have a singularity. The equilibrium points P_2^{\pm}

correspond to P_2^{\pm} in [35], which are the Kasner's plane-symmetric vacuum solutions [39]. For $4\beta + \eta + 2 = 0$ the metric becomes $ds^2 = -\bar{N}_0^2 e^{\mp 2\lambda} dt^2 +$

$\frac{e^{\mp \eta\lambda}}{c_1^2} d\lambda^2 + \bar{K}_0^{-1} e^{\pm 4\lambda} (d\vartheta^2 + \sin^2 \vartheta d\varphi^2)$ or $ds^2 = -\frac{\bar{N}_0^2}{\rho_{\pm}} dt^2 + \frac{\rho_{\pm}^{-\frac{(4+\eta)}{2}}}{4c_1^2} d\rho_{\pm}^2 +$

$\frac{\rho_{\pm}^2}{\bar{K}_0} (d\vartheta^2 + \sin^2 \vartheta d\varphi^2)$ under the coordinate transformation $\rho_{\pm} = e^{\pm 2\lambda}$. The Ricci

Scalar becomes $R = \frac{2\bar{K}_0}{\rho_{\pm}^2} - \frac{3}{4} c_1^2 (\eta + 6) \rho_{\pm}^{\eta/2}$. Thus at $\rho_{\pm} = 0^+$ and at $\rho_{\pm} = +\infty$ we have singularities.

- $P_3^{\pm}: (Q, S, C) = (\pm 1, 0, 1)$, always exist and are saddles. When we substitute the values of Q and S into (34a), (34b) and integrate, we obtain $N = \bar{N}_0 = \text{constant}$ and $K = \bar{K}_0 e^{\mp 2\lambda}$. From (26) and the definition of C it follows that $d\ell^2 = K^{-1} d\lambda^2$. Thus the line element (14) becomes $ds^2 = -\bar{N}_0^2 dt^2 + \bar{K}_0^{-1} e^{\pm 2\lambda} d\lambda^2 + \bar{K}_0^{-1} e^{\pm 2\lambda} (d\vartheta^2 + \sin^2 \vartheta d\varphi^2)$. Defining $\rho_{\pm} = \frac{e^{\pm \lambda}}{\sqrt{\bar{K}_0}}$, we get $ds^2 = -\bar{N}_0^2 dt^2 + d\rho_{\pm}^2 + \rho_{\pm}^2 (d\vartheta^2 + \sin^2 \vartheta d\varphi^2)$, which corresponds to Minkowski spacetime on explicitly spherically symmetric form [35]. These points are the analogues of P_3^{\pm} investigated in [35].

- $P_4^{\pm}: (Q, S, C) = \left(\pm 1, \pm \frac{2}{\eta+2}, 1 - \frac{8\beta}{(\eta+2)^2}\right)$ exist for $\eta \geq 1, 0 \leq \beta \leq \frac{1}{8}(\eta + 2)^2$ and are saddles. When we substitute the values of Q and S into (34a), (34b) and integrate, we obtain $N = \bar{N}_0 e^{\pm \frac{2\lambda}{2+\eta}}$ and $K = \bar{K}_0 e^{\mp \frac{2\eta\lambda}{2+\eta}}$. From (26) and the definition of C it follows that $d\ell^2 = \frac{1}{\bar{K}_0} \left(1 - \frac{8\beta}{(\eta+2)^2}\right) e^{\pm \frac{2\eta\lambda}{2+\eta}} d\lambda^2$. Thus the line element (14) becomes $ds^2 = -\bar{N}_0^2 e^{\pm \frac{4\lambda}{2+\eta}} dt^2 + \frac{1}{\bar{K}_0} \left(1 - \frac{8\beta}{(\eta+2)^2}\right) e^{\pm \frac{2\eta\lambda}{2+\eta}} d\lambda^2 + \frac{1}{\bar{K}_0} e^{\pm \frac{2\eta\lambda}{2+\eta}} (d\vartheta^2 + \sin^2 \vartheta d\varphi^2)$. Defining $\rho_{\pm} = \frac{1}{\sqrt{\bar{K}_0}} e^{\pm \frac{\eta\lambda}{2+\eta}}, T = \bar{N}_0 \bar{K}_0^{\frac{1}{\eta}} t$, we obtain $ds^2 = -\rho_{\pm}^{\frac{4}{\eta}} dT^2 + \left(\frac{(\eta+2)^2 - 8\beta}{\eta^2}\right) d\rho_{\pm}^2 + \rho_{\pm}^2 (d\vartheta^2 + \sin^2 \vartheta d\varphi^2)$. For $\beta = 1$ these points are the analogues of P_4^{\pm} investigated in [35], which corresponds to a nonregular self-similar perfect fluid solution discussed in [36–38]. One interest-

ing feature is that following the ARS algorithm, the dominant terms found in the previous part of this Section correspond to the points P_4^\pm .

- $P_5^\pm: (Q, S, C) = \left(\pm 1, \pm \frac{\eta+2}{4\beta}, 0\right)$ exist for $\eta \geq 1, \beta < 0$ or $\eta > 1, 16\beta \geq (\eta+2)^2$. P_5^+ (respectively, P_5^-) is a sink (respectively, a source) for $\eta \geq 1, \beta < 0$. Otherwise they are saddles. When we substitute the values of Q and S into (34a), (34b) and integrate, we obtain $N = \bar{N}_0 e^{\pm \frac{(\eta+2)\lambda}{4\beta}}$ and $K = \bar{K}_0 e^{\mp 2\left(1 - \frac{\eta+2}{4\beta}\right)\lambda}$. From (26) and the definition of S it follows that $\frac{d\ell}{d\lambda} = \pm \frac{\eta+2}{4\beta y}$. On the other hand, when we evaluate the values of Q, S, C in (34c), it follows that $\frac{dy}{d\lambda} = \mp \frac{\eta(\eta+2)}{8\beta} y$. Then we have $y(\lambda) = c_1 e^{-\frac{\eta(\eta+2)\lambda\epsilon}{8\beta}}$, $\ell(\lambda) = \frac{2e^{\frac{\eta(\eta+2)\lambda\epsilon}{8\beta}}}{c_1 \eta} + c_2$, where $\epsilon = \pm 1$ and c_1, c_2 are constants of integration. The metric (14) becomes $ds^2 = -\bar{N}_0^2 e^{\pm \frac{(\eta+2)\lambda}{2\beta}} dt^2 + \frac{(\eta+2)^2 e^{\pm \frac{\eta(\eta+2)\lambda}{4\beta}}}{16c_1^2 \beta^2} d\lambda^2 + \bar{K}_0^{-1} e^{\pm 2\left(1 - \frac{\eta+2}{4\beta}\right)\lambda} (d\vartheta^2 + \sin^2 \vartheta d\varphi^2)$. On the introduction of $\rho_\pm = e^{\pm \left(1 - \frac{\eta+2}{4\beta}\right)\lambda}$ the line element becomes $-\bar{N}_0^2 \rho_\pm^{\frac{2(\eta+2)}{4\beta - \eta - 2}} dt^2 + \frac{(\eta+2)^2}{2c_1^2 (4\beta - \eta - 2)^2} \rho_\pm^{-\frac{16\beta - \eta^2 - 6\eta + 24}{2(4\beta - \eta - 2)}} d\rho_\pm^2 + \frac{\rho_\pm^2}{\bar{K}_0} (d\vartheta^2 + \sin^2 \vartheta d\varphi^2)$. For $\beta = 1$ these solutions are the analogues of P_5^\pm investigated in [35], which correspond to self-similar plane-symmetric perfect fluid models. For $\eta \geq 1, 16\beta \geq (\eta+2)(\eta+4)$ the exponent of the tt component is positive and the exponent of the $\rho_\pm \rho_\pm$ component is negative. Thus the singularity has an horizon at $\rho_\pm = 0^+$.

- $P_6^\pm: (Q, S, C) = \left(\pm 1, \pm \frac{1}{\sqrt{\beta}}, 0\right)$. They exist for $\eta \geq 1, \beta > 0$. P_6^+ (respectively, P_6^-) is a source (respectively, a sink) for $\eta \geq 1, 16\beta \geq (\eta+2)^2$. They are saddles for $\eta \geq 1, 0 < \beta < \frac{1}{4}$ or $\eta \geq 1, \frac{1}{4} < \beta < \frac{1}{16}(\eta+2)^2$. Otherwise they are non-hyperbolic. When we substitute the values of Q and S into (34a), (34b) and integrate, we obtain $N = \bar{N}_0 e^{\pm \frac{\lambda}{\sqrt{\beta}}}$ and $K = \bar{K}_0 e^{\mp \frac{2(\sqrt{\beta}-1)\lambda}{\sqrt{\beta}}}$. From (26) and the definition of S it follows that $\frac{d\ell}{d\lambda} = \pm \frac{1}{y\sqrt{\beta}}$. On the other hand, when we evaluate the values of Q, S, C in (34c), it follows that $\frac{dy}{d\lambda} = \pm \frac{1-2\sqrt{\beta}}{\sqrt{\beta}} y$. Then $y(\lambda) = c_1 e^{\left(-\frac{1}{\sqrt{\beta}} - 2\right)\lambda\epsilon}$, $\ell(\lambda) = c_2 - \frac{e^{\left(\frac{1}{\sqrt{\beta}} + 2\right)\lambda\epsilon}}{c_1 (2\sqrt{\beta} + 1)}$, where $\epsilon = \pm 1$ and c_1, c_2 are constants of integration. The line element (14) becomes $ds^2 = -\bar{N}_0^2 e^{\pm \frac{2\lambda}{\sqrt{\beta}}} dt^2 + \frac{e^{\pm \frac{2(2\sqrt{\beta}+1)\lambda}{\sqrt{\beta}}}}{c_1^2 \beta} d\lambda^2 + \bar{K}_0^{-1} e^{\pm \frac{2(\sqrt{\beta}-1)\lambda}{\sqrt{\beta}}} (d\vartheta^2 + \sin^2 \vartheta d\varphi^2)$. Under the transformation $\rho_\pm = e^{\pm \frac{(\sqrt{\beta}-1)\lambda}{\sqrt{\beta}}}$, the metric becomes $ds^2 = -\bar{N}_0^2 \rho_\pm^{\frac{2(1+\sqrt{\beta})}{\beta-1}} dt^2 + \frac{2(2+\beta+3\sqrt{\beta})}{c_1^2 (1+\beta-2\sqrt{\beta})} d\rho_\pm^2 + \frac{\rho_\pm^2}{\bar{K}_0} (d\vartheta^2 + \sin^2 \vartheta d\varphi^2)$. Because the exponents of the tt and $\rho_\pm \rho_\pm$ components are of the same sign for $\beta > 0$, $\rho_\pm = 0^+$ is a naked singularity. For $\beta = 1$ these points correspond to P_1^\pm in [35], which are Kasner's plane-symmetric vacuum solutions [39].

7. $P_7^\pm: (Q, S, C) = \left(\pm 1, \mp \frac{1}{\sqrt{\beta}}, 0\right)$ exist for $\eta \geq 1, \beta > 0$. P_7^+ (respectively, P_7^-) is a source (respectively, a sink) for $\eta \geq 1, \beta > 0$. When we substitute the values of Q and S in (34a), (34b) and integrate, we obtain $N = \bar{N}_0 e^{\mp \frac{\lambda}{\sqrt{\beta}}}$ and $K = \bar{K}_0 e^{\mp \frac{2(1+\sqrt{\beta})\lambda}{\sqrt{\beta}}}$. From (26) and the definition of S it follows that $\frac{d\ell}{d\lambda} = \mp \frac{1}{y\sqrt{\beta}}$. On the other hand, when we evaluate the values of Q, S, C in (34c), it follows that $\frac{dy}{d\lambda} = \mp \frac{1+2\sqrt{\beta}}{\sqrt{\beta}} y$. Then $y(\lambda) = c_1 e^{(-\frac{1}{\sqrt{\beta}}-2)\lambda\epsilon}$, $\ell(\lambda) = c_2 - \frac{e^{(\frac{1}{\sqrt{\beta}}+2)\lambda\epsilon}}{c_1(2\sqrt{\beta}+1)}$, where $\epsilon = \pm 1$. The line element (14) becomes $ds^2 = -\bar{N}_0^2 e^{\mp \frac{2\lambda}{\sqrt{\beta}}} dt^2 + \frac{e^{\pm \frac{2(1+2\sqrt{\beta})\lambda}{\sqrt{\beta}}}}{c_1^2 \beta} d\lambda^2 + \bar{K}_0^{-1} e^{\pm \frac{2(1+\sqrt{\beta})\lambda}{\sqrt{\beta}}} (d\vartheta^2 + \sin^2 \vartheta d\varphi^2)$. Under the change of variables $\rho_\pm = e^{\pm \frac{(1+\sqrt{\beta})\lambda}{\sqrt{\beta}}}$, the metric becomes $-\bar{N}_0^2 \rho_\pm^{\frac{2(1-\sqrt{\beta})}{\beta-1}} dt^2 + \frac{\rho_\pm^{\frac{2(\beta-\sqrt{\beta})}{\beta-1}}}{c_1^2(1+\beta+2\sqrt{\beta})} d\rho_\pm^2 + \frac{\rho_\pm^2}{\bar{K}_0} (d\vartheta^2 + \sin^2 \vartheta d\varphi^2)$. As the exponents of the tt and $\rho_\pm \rho_\pm$ components are both negative for $\beta > 0$, $\rho_\pm = 0^+$ is a naked singularity. For $\beta = 1$ these points correspond to P_2^\pm in [35] and these are the Kasner's plane-symmetric vacuum solutions [39].
8. $P_8^\pm: (Q, S, C) = \left(\pm \frac{1}{\sqrt{1-\beta}}, \pm \frac{1}{\sqrt{1-\beta}}, \frac{2\beta-1}{\beta-1}\right)$. They exist for $\beta \leq 0$. P_8^+ (respectively, P_8^-) is a sink (respectively, a source) for $\beta < 0$. When we substitute the values of Q and S into (34a), (34b) and integrate, we obtain $N = \bar{N}_0 e^{\pm \frac{\lambda}{\sqrt{1-\beta}}}$, and $K = \bar{K}_0 = \text{constant}$. From (26) and the definition of Q it follows that $d\ell^2 = \frac{\eta}{\mu_0} \left(\frac{\beta}{\beta-1}\right) d\lambda^2$. The metric becomes $ds^2 = -\bar{N}_0^2 e^{\pm \frac{2\lambda}{\sqrt{1-\beta}}} dt^2 + \frac{\eta}{\mu_0} \left(\frac{\beta}{\beta-1}\right) d\lambda^2 + \bar{K}_0^{-1} (d\vartheta^2 + \sin^2 \vartheta d\varphi^2)$. These are regular solutions with constant curvature $R = 2 \left(\frac{\mu_0}{\beta\eta} + \bar{K}_0\right)$ for $\beta < 0$.
9. $P_9^\pm: (Q, S, C) = \left(\pm 2\sqrt{\beta}, \pm \frac{1}{\sqrt{\beta}}, 0\right)$ exist for $\eta \geq 1, 0 < \beta \leq \frac{1}{4}$. They are saddles for $\eta \geq 1, 0 < \beta \leq \frac{1}{4}$ and non-hyperbolic for $\beta = \frac{1}{4}$ (numerically it is the saddle in Fig. 1f). When we substitute the values of Q and S into (34a), (34b) and integrate, we obtain $N = \bar{N}_0 e^{\pm \frac{\lambda}{\sqrt{\beta}}}$ and $K = \bar{K}_0 e^{\mp \frac{2(2\beta-1)\lambda}{\sqrt{\beta}}}$. For $\beta < \frac{1}{4}$ it follows from (26) and the definition of Q that $d\ell^2 = \frac{\eta}{\mu_0} (1-4\beta)d\lambda^2$. The line element (14) becomes $ds^2 = -\bar{N}_0^2 e^{\pm \frac{2\lambda}{\sqrt{\beta}}} dt^2 + \frac{\eta}{\mu_0} (1-4\beta)d\lambda^2 + \bar{K}_0^{-1} e^{\pm \frac{2(2\beta-1)\lambda}{\sqrt{\beta}}} (d\vartheta^2 + \sin^2 \vartheta d\varphi^2)$. Under the change of variables $\rho_\pm = \frac{1}{\sqrt{\bar{K}_0}} e^{\pm \frac{(2\beta-1)\lambda}{\sqrt{\beta}}}$, $T = \bar{N}_0 \bar{K}_0^{\frac{1}{2(2\beta-1)}} t$, the metric becomes $ds^2 = -\rho_\pm^{\frac{2}{2\beta-1}} dT^2 + \frac{\eta(1-4\beta)\beta}{\mu_0(2\beta-1)^2} \frac{d\rho_\pm^2}{\rho_\pm^2} + \rho_\pm^2 (d\vartheta^2 + \sin^2 \vartheta d\varphi^2)$. Because the exponents of the TT and $\rho_\pm \rho_\pm$ components are both negative for $0 < \beta < \frac{1}{4}$, $\rho_\pm = 0^+$ is a naked singularity.

C Equilibrium points in the finite region of the phase space for the exponential potential

The system admits the equilibrium points $P_1^\pm - P_9^\pm$ discussed before in the invariant set $A_\varphi = A_W = 0$. For further details about the physical interpretation of the equilibrium points $P_1^\pm - P_9^\pm$ we submit the reader to Appendix B, where we have represented the line elements of their corresponding cosmological solutions. The stability conditions change slightly due to the new axis A_φ, A_W .

1. P_1^\pm as discussed in the previous section.
2. P_2^\pm as discussed in the previous section.
3. $P_3^\pm := (\pm 1, 0, 1, 0, 0)$. They always exist and are saddle since the eigenvalues are: $\mp 1, \mp 1, \pm 1, \pm 2, \pm 2$.
4. $P_4^\pm := \left(\pm 1, \pm \frac{2}{2+\eta}, 1 - \frac{8\beta}{(2+\eta)^2}, 0, 0\right)$. Exists for $0 \leq \beta \leq \frac{1}{8}(\eta + 2)^2, \eta \geq 1$.

The eigenvalues are $\pm \frac{\eta}{\eta+2}, \pm \frac{2\eta}{\eta+2}, \mp 1, \mp \frac{\eta+2+\sqrt{64\beta-7(\eta+2)^2}}{2(\eta+2)}, \mp \frac{\eta+2-\sqrt{64\beta-7(\eta+2)^2}}{2(\eta+2)}$.

This point is a saddle (at least two eigenvalues have different signs).

5. $P_5^\pm := \left(\pm 1, \pm \frac{2+\eta}{4\beta}, 0, 0, 0\right)$. Exist for $\eta \geq 1, \beta < 0$ or $\eta \geq 1, \beta \geq \frac{1}{16}(\eta + 2)^2$.
The eigenvalues are $\pm \frac{\eta(\eta+2)}{8\beta}, \pm \frac{(\eta+2)^2-8\beta}{4\beta}, \pm \frac{(\eta+2)^2-16\beta}{8\beta}, \pm \frac{(\eta+2)^2-16\beta}{8\beta}, \pm \frac{\eta(\eta+2)}{4\beta}$.
 P_5^+ (respectively, P_5^-) is a sink (respectively, a source), for $\beta < 0, \eta \geq 1$. Otherwise, they are saddles.

6. P_6^\pm as discussed in the previous section are not isolated anymore, and belong to the lines of equilibrium points P_{10}^\pm, P_{11}^\pm as we will see below. This is different to the results in Appendix B.
7. P_7^\pm as discussed in the previous section are not isolated anymore, and belong to the lines of equilibrium points P_{10}^\pm, P_{11}^\pm as we will see below. This a different to the results in Appendix B.

8. $P_8^\pm := \left(\pm \frac{1}{\sqrt{1-\beta}}, \pm \frac{1}{\sqrt{1-\beta}}, \frac{2\beta-1}{\beta-1}, 0, 0\right)$. They exist for $\eta \geq 1, \beta \leq 0$. The eigenvalues are $0, \mp \frac{1}{\sqrt{1-\beta}}, \mp \frac{1-\sqrt{16\beta-7}}{2\sqrt{1-\beta}}, \mp \frac{1+\sqrt{16\beta-7}}{2\sqrt{1-\beta}}, \mp \frac{\eta}{\sqrt{1-\beta}}$. These points are non-hyperbolic. For P_8^+ (respectively, P_8^-) and given $\beta < 0$ (we have assumed $\eta \geq 1$), there are two negative (respectively, positive) eigenvalues, and two complex eigenvalues with negative (respectively, positive) real parts. So P_8^+ (respectively, P_8^-) has a 4D stable (respectively, unstable) manifold.

9. $P_9^\pm := \left(\pm 2\sqrt{\beta}, \pm \frac{1}{\sqrt{\beta}}, 0, 0, 0\right)$. Exist for $\eta \geq 1, 0 < \beta \leq \frac{1}{4}$.
The eigenvalues are $0, \pm \frac{2(1-2\beta)}{\sqrt{\beta}}, \pm \frac{1-4\beta}{\sqrt{\beta}}, \pm \frac{1-4\beta}{\sqrt{\beta}}, \mp \frac{\eta}{\sqrt{\beta}}$. The equilibrium point is saddle for $\eta \geq 1, 0 < \beta < \frac{1}{4}$, and non-hyperbolic when $\beta = \frac{1}{4}$.

Now, let's discuss the new equilibrium points, due to the extra coordinates A_φ and A_W related to the scalar field. They are:

10. Line of equilibrium points $P_{10}^\pm(S_c) := \left(\pm 1, \pm S_c, 0, \pm \sqrt{1 - \beta S_c^2}, 0\right)$, where we have explicitly shown the dependence on the parameter S_c of the lines. They exist for $\beta \leq 0, S_c \in \mathbb{R}$ or $\beta > 0, -\frac{1}{\sqrt{\beta}} \leq S_c \leq \frac{1}{\sqrt{\beta}}$.

Eigenvalues: $0, \pm 2, \pm 2(2 - S_c), \pm \left(2 - S_c - \frac{k\sqrt{1-\beta S_c^2}}{\sqrt{2}} \right), \pm (4 - S_c(2 + \eta))$.

These lines cover the points P_7^\pm and P_6^\pm in the previous section for the choices $S_c = -\frac{1}{\sqrt{\beta}}$ and $S_c = \frac{1}{\sqrt{\beta}}$ respectively. Since $Q^2 = 1$ ($\mu_0 = 0$), it follows from the definition of λ that the metric can be written as $ds^2 = -N^2 dt^2 + \frac{S_c^2}{y^2} d\lambda^2 + K^{-1} (d\vartheta^2 + \sin^2 \vartheta d\varphi^2)$. When we substitute the values of Q, S, C, A_φ and A_W into (76a), (76b), and (76c) and integrating, we get $N = \bar{N}_0 e^{\pm S_c \lambda}, K = \bar{K}_0 e^{\mp 2(1-S_c)\lambda}, y = \bar{y}_0 e^{\mp (2-S_c)\lambda}$. Thus, the metric can be written as $ds^2 = -\bar{N}_0 e^{\pm 2S_c \lambda} dt^2 + \frac{S_c^2}{\bar{y}_0^2} e^{\pm 2(2-S_c)\lambda} d\lambda^2 + \bar{K}_0^{-1} e^{\pm 2(1-S_c)\lambda} (d\vartheta^2 + \sin^2 \vartheta d\varphi^2)$.

The following subsets (arcs, or specific equilibrium points) of the line $P_{10}^+(S_c)$ (respectively, $P_{10}^-(S_c)$) are unstable (respectively, stable) for the given conditions:

- (a) $S_c < 0, \beta < \frac{1}{S_c^2}, \eta > 1, k < \sqrt{\frac{-2S_c^2+8S_c-8}{\beta S_c^2-1}}$, or
- (b) $S_c < 0, \beta = \frac{1}{S_c^2}, \eta > 1, k \in \mathbb{R}$, or
- (c) $S_c = 0, \beta \in \mathbb{R}, \eta > 1, k < 2\sqrt{2}$, or
- (d) $0 < S_c < \frac{4}{3}, \beta < \frac{1}{S_c^2}, 1 < \eta < \frac{4-2S_c}{S_c}, k < \sqrt{\frac{-2S_c^2+8S_c-8}{\beta S_c^2-1}}$, or
- (e) $0 < S_c < \frac{4}{3}, \beta = \frac{1}{S_c^2}, 1 < \eta < \frac{4-2S_c}{S_c}, k \in \mathbb{R}$.

11. Line of equilibrium points $P_{11}^\pm(S_c) := (\pm 1, \pm S_c, 0, \mp \sqrt{1 - \beta S_c^2}, 0)$, where we have explicitly shown the dependence on the parameter S_c of the lines. They exist for $\beta \leq 0, S_c \in \mathbb{R}$ or $\beta > 0, -\frac{1}{\sqrt{\beta}} \leq S_c \leq \frac{1}{\sqrt{\beta}}$. Eigenval-

ues: $0, \pm 2, \pm 2(2 - S_c), \pm \left(2 - S_c + \frac{k\sqrt{1-\beta S_c^2}}{\sqrt{2}} \right), \pm (4 - S_c(2 + \eta))$. These lines cover the points P_7^\pm and P_6^\pm in the previous section for the choices $S_c = -\frac{1}{\sqrt{\beta}}$ and $S_c = \frac{1}{\sqrt{\beta}}$ respectively. When we substitute the values of Q, S, C, A_φ and A_W into (76a), (76b), and (76c) and integrating, we get $N = \bar{N}_0 e^{\pm S_c \lambda}, K = \bar{K}_0 e^{\mp 2(1-S_c)\lambda}, y = \bar{y}_0 e^{\mp (2-S_c)\lambda}$. Thus, the metric can be written as $ds^2 = -\bar{N}_0 e^{\pm 2S_c \lambda} dt^2 + \frac{S_c^2}{\bar{y}_0^2} e^{\pm 2(2-S_c)\lambda} d\lambda^2 + \bar{K}_0^{-1} e^{\pm 2(1-S_c)\lambda} (d\vartheta^2 + \sin^2 \vartheta d\varphi^2)$.

The following subsets (arcs, or specific equilibrium points) of the line $P_{11}^+(S_c)$ (respectively, $P_{11}^-(S_c)$) are unstable (respectively, stable) for the given conditions:

- (a) $S_c < 0, \beta < \frac{1}{S_c^2}, \eta \geq 1, k > -\sqrt{2} \sqrt{-\frac{S_c^2-4S_c+4}{\beta S_c^2-1}}$, or
- (b) $S_c < 0, \beta = \frac{1}{S_c^2}, \eta \geq 1, k \in \mathbb{R}$, or
- (c) $S_c = 0, \beta \in \mathbb{R}, \eta \geq 1, k > -2\sqrt{2}$, or
- (d) $0 < S_c < \frac{4}{3}, \beta < \frac{1}{S_c^2}, 1 \leq \eta < \frac{4-2S_c}{S_c}, k > -\sqrt{2} \sqrt{-\frac{S_c^2-4S_c+4}{\beta S_c^2-1}}$, or
- (e) $0 < S_c < \frac{4}{3}, \beta = \frac{1}{S_c^2}, 1 \leq \eta < \frac{4-2S_c}{S_c}, k \in \mathbb{R}$.

12. $P_{12}^{\pm} := \left(\pm 1, \pm \frac{2+\eta}{4\beta}, 0, \pm \frac{\sqrt{16\beta-(\eta+2)^2}}{4\sqrt{\beta}}, 0 \right)$. These exist for $\eta \geq 1, \beta < 0$; or $\eta \geq 1, \beta \geq \frac{1}{16}(\eta+2)^2$. The eigenvalues are: $0, \pm 2, \mp \frac{-16\beta+2(\eta+2)+\sqrt{2}\sqrt{\beta}k\sqrt{16\beta-(\eta+2)^2}}{8\beta}, \mp \frac{-8\beta+\eta+2}{2\beta}, \mp \frac{(\eta+2)^2-16\beta}{4\beta}$. Substituting the values of Q, S, C, A_φ and A_W into (76a), (76b), and (76c) and integrating, we get $N = \bar{N}_0 e^{\pm \frac{(\eta+2)\lambda}{4\beta}}, K = \bar{K}_0 e^{\mp \frac{\lambda(4\beta-\eta-2)}{2\beta}}, y = \bar{y}_0 e^{\mp \frac{\eta\lambda(-8\beta+\eta(\eta+4)+4)}{4\beta(\eta+2)}}$. It follows from the definition of λ that the metric can be written as $ds^2 = -\bar{N}_0^2 e^{\pm \frac{(\eta+2)\lambda}{2\beta}} dt^2 + \frac{(2+\eta)^2}{16\bar{y}_0^2 \beta^2} e^{\pm \frac{\eta\lambda(-8\beta+\eta(\eta+4)+4)}{2\beta(\eta+2)}} d\lambda^2 + \bar{K}_0^{-1} e^{\pm \frac{\lambda(4\beta-\eta-2)}{2\beta}} (d\vartheta^2 + \sin^2 \vartheta d\varphi^2)$. The equilibrium point P_{12}^+ (respectively, P_{12}^-) has a 4D unstable (respectively, stable) manifold for

- (a) $\beta < 0, \eta \geq 1, k > -\sqrt{2} \sqrt{\frac{(8\beta-\eta-2)^2}{\beta(16\beta-\eta^2-4\eta-4)}}$, or
- (b) $\beta > \frac{9}{16}, 1 \leq \eta < 2(2\sqrt{\beta}-1), k < \sqrt{2} \sqrt{\frac{(8\beta-\eta-2)^2}{\beta(16\beta-\eta^2-4\eta-4)}}$.

13. $P_{13}^{\pm} := \left(\pm 1, \pm \frac{2+\eta}{4\beta}, 0, \mp \frac{\sqrt{16\beta-(\eta+2)^2}}{4\sqrt{\beta}}, 0 \right)$. These exist for $\eta \geq 1, \beta < 0$; or $\eta \geq 1, \beta \geq \frac{1}{16}(\eta+2)^2$. The eigenvalues are: $0, \pm 2, \pm \frac{16\beta-2(\eta+2)+\sqrt{2}\sqrt{\beta}k\sqrt{16\beta-(\eta+2)^2}}{8\beta}, \mp \frac{-8\beta+\eta+2}{2\beta}, \mp \frac{(\eta+2)^2-16\beta}{4\beta}$. Substituting the values of Q, S, C, A_φ and A_W into (76a), (76b), and (76c) and integrating, we get $N = \bar{N}_0 e^{\pm \frac{(\eta+2)\lambda}{4\beta}}, K = \bar{K}_0 e^{\mp \frac{\lambda(4\beta-\eta-2)}{2\beta}}, y = \bar{y}_0 e^{\mp \frac{\eta\lambda(-8\beta+\eta(\eta+4)+4)}{4\beta(\eta+2)}}$. It follows from the definition of λ that the metric can be written as $ds^2 = -\bar{N}_0^2 e^{\pm \frac{(\eta+2)\lambda}{2\beta}} dt^2 + \frac{(2+\eta)^2}{16\bar{y}_0^2 \beta^2} e^{\pm \frac{\eta\lambda(-8\beta+\eta(\eta+4)+4)}{2\beta(\eta+2)}} d\lambda^2 + \bar{K}_0^{-1} e^{\pm \frac{\lambda(4\beta-\eta-2)}{2\beta}} (d\vartheta^2 + \sin^2 \vartheta d\varphi^2)$. These points are non-hyperbolic (one zero eigenvalue). The equilibrium point P_{13}^+ (respectively, P_{13}^-) has a 4D unstable (respectively, stable) manifold for

- (a) $\beta < 0, \eta \geq 1, k < \sqrt{2} \sqrt{\frac{(8\beta-\eta-2)^2}{\beta(16\beta-\eta^2-4\eta-4)}}$, or
- (b) $\beta > \frac{9}{16}, 1 \leq \eta < 2(2\sqrt{\beta}-1), k > -\sqrt{2} \sqrt{\frac{(8\beta-\eta-2)^2}{\beta(16\beta-\eta^2-4\eta-4)}}$.

14. $P_{14}^{\pm} := \left(\pm 1, \pm \frac{k\sqrt{\beta(k^2-8)+2+4}}{\beta k^2+2}, 0, \mp \frac{\sqrt{2}(\sqrt{\beta(k^2-8)+2-2\beta k})}{\beta k^2+2}, 0 \right)$. These exist for

- (a) $\beta < 0, -\sqrt{2} \sqrt{\frac{4\beta-1}{\beta}} \leq k < -\sqrt{2} \sqrt{-\frac{1}{\beta}}, \eta \geq 1$, or
- (b) $\beta < 0, -\sqrt{2} \sqrt{-\frac{1}{\beta}} < k < \sqrt{2} \sqrt{-\frac{1}{\beta}}, \eta \geq 1$, or
- (c) $\beta < 0, \sqrt{2} \sqrt{-\frac{1}{\beta}} < k \leq \sqrt{2} \sqrt{\frac{4\beta-1}{\beta}}, \eta \geq 1$, or
- (d) $0 \leq \beta \leq \frac{1}{4}, k \in \mathbb{R}, \eta \geq 1$ or

- (e) $\beta > \frac{1}{4}, k \leq -\sqrt{2}\sqrt{\frac{4\beta-1}{\beta}}, \eta \geq 1$, or
- (f) $\beta > \frac{1}{4}, k \geq \sqrt{2}\sqrt{\frac{4\beta-1}{\beta}}, \eta \geq 1$.

The eigenvalues are: $0, 0, \pm 2, \mp \frac{2k(\sqrt{\beta(k^2-8)+2-2\beta k})}{\beta k^2+2}, \pm \frac{-4\eta-(\eta+2)k\sqrt{\beta(k^2-8)+2+4\beta k^2}}{\beta k^2+2}$. Substituting the values of Q, S, C, A_φ and A_W into (76a), (76b), and (76c) and integrating, we get $N = \bar{N}_0 \exp\left(\pm \frac{\lambda(k\sqrt{\beta(k^2-8)+2+4})}{\beta k^2+2}\right), K = \bar{K}_0 \exp\left(\mp \frac{2k\lambda(\beta k - \sqrt{\beta(k^2-8)+2-2})}{\beta k^2+2}\right), y = \bar{y}_0 \exp\left(\mp \frac{k\lambda(2\beta k - \sqrt{\beta(k^2-8)+2})}{\beta k^2+2}\right)$. It follows from the definition of λ that the metric

can be written as $ds^2 = -\bar{N}_0^2 e^{\pm \frac{2\lambda(k\sqrt{\beta(k^2-8)+2+4})}{\beta k^2+2}} dt^2 + \frac{1}{\bar{y}_0^2} \left(\frac{4+k\sqrt{\beta(k^2-8)+2}}{2+\beta k^2}\right)^2 e^{\pm \frac{2k\lambda(2\beta k - \sqrt{\beta(k^2-8)+2})}{\beta k^2+2}} d\lambda^2 + \bar{K}_0^{-1} e^{\pm \frac{2k\lambda(\beta k - \sqrt{\beta(k^2-8)+2-2})}{\beta k^2+2}} (d\vartheta^2 + \sin^2 \vartheta d\varphi^2)$. The equilibrium points are non-hyperbolic (two zero eigenvalues). The equilibrium point P_{14}^+ (respectively, P_{14}^-) has a 3D unstable (respectively, stable) manifold in the following cases:

- (a) $\beta > \frac{9}{16}, \eta \geq 2(4\beta - 1), k < -2\sqrt{2}\sqrt{\frac{\eta^2}{-16\beta+\eta^2+4\eta+4}}$, or
- (b) $\frac{3}{8} < \beta < \frac{9}{16}, \eta \geq 2(4\beta - 1), k < -2\sqrt{2}\sqrt{\frac{\eta^2}{-16\beta+\eta^2+4\eta+4}}$, or
- (c) $\beta < -\frac{1}{4}, \eta \geq \frac{2(4\beta-1)}{4\beta+1}, -\sqrt{2}\sqrt{\frac{4\beta-1}{\beta}} \leq k < -2\sqrt{2}\sqrt{\frac{\eta^2}{-16\beta+\eta^2+4\eta+4}}$, or
- (d) $-\frac{1}{4} \leq \beta < 0, \eta \geq 1, -\sqrt{2}\sqrt{-\frac{1}{\beta}} < k < -2\sqrt{2}\sqrt{\frac{\eta^2}{-16\beta+\eta^2+4\eta+4}}$, or
- (e) $0 \leq \beta \leq \frac{3}{8}, \eta \geq 1, k < -2\sqrt{2}\sqrt{\frac{\eta^2}{-16\beta+\eta^2+4\eta+4}}$, or
- (f) $\beta < -\frac{1}{4}, 1 \leq \eta < \frac{2(4\beta-1)}{4\beta+1}, -\sqrt{2}\sqrt{-\frac{1}{\beta}} < k < -2\sqrt{2}\sqrt{\frac{\eta^2}{-16\beta+\eta^2+4\eta+4}}$, or
- (g) $\beta = \frac{9}{16}, \eta = 1, k \geq \frac{2\sqrt{10}}{3}$, or
- (h) $\beta = \frac{9}{16}, 1 < \eta < \frac{5}{2}, \frac{2\sqrt{10}}{3} \leq k < 2\sqrt{2}\sqrt{\frac{\eta^2}{\eta^2+4\eta-5}}$, or
- (i) $\beta = \frac{9}{16}, 1 \leq \eta < \frac{5}{2}, k \leq -\frac{2\sqrt{10}}{3}$, or
- (j) $\beta = \frac{9}{16}, \eta \geq \frac{5}{2}, k < -2\sqrt{2}\sqrt{\frac{\eta^2}{\eta^2+4\eta-5}}$, or
- (k) $\beta > \frac{9}{16}, 1 \leq \eta \leq 2(2\sqrt{\beta} - 1), k \geq \sqrt{2}\sqrt{\frac{4\beta-1}{\beta}}$, or
- (l) $\beta > \frac{9}{16}, 2(2\sqrt{\beta} - 1) < \eta < 2(4\beta - 1), \sqrt{2}\sqrt{\frac{4\beta-1}{\beta}} \leq k < 2\sqrt{2}\sqrt{\frac{\eta^2}{-16\beta+\eta^2+4\eta+4}}$, or

- (m) $\beta > \frac{9}{16}, 2(2\sqrt{\beta} - 1) < \eta < 2(4\beta - 1), k \leq -\sqrt{2}\sqrt{\frac{4\beta-1}{\beta}}$, or
- (n) $\beta > \frac{9}{16}, 1 \leq \eta \leq 2(2\sqrt{\beta} - 1), k \leq -\sqrt{2}\sqrt{\frac{4\beta-1}{\beta}}$, or
- (o) $\beta < -\frac{1}{4}, \eta \geq \frac{2(4\beta-1)}{4\beta+1}, \sqrt{2}\sqrt{-\frac{1}{\beta}} < k \leq \sqrt{2}\sqrt{\frac{4\beta-1}{\beta}}$, or
- (p) $\beta < 0, \eta \geq 1, \sqrt{2}\sqrt{-\frac{1}{\beta}} < k \leq \sqrt{2}\sqrt{\frac{4\beta-1}{\beta}}$, or
- (q) $-\frac{1}{4} \leq \beta < 0, \eta \geq 1, -\sqrt{2}\sqrt{\frac{4\beta-1}{\beta}} \leq k < -\sqrt{2}\sqrt{-\frac{1}{\beta}}$, or
- (r) $\beta < -\frac{1}{4}, 1 \leq \eta < \frac{2(4\beta-1)}{4\beta+1}, -\sqrt{2}\sqrt{\frac{4\beta-1}{\beta}} \leq k < -\sqrt{2}\sqrt{-\frac{1}{\beta}}$, or
- (s) $\frac{3}{8} < \beta < \frac{9}{16}, 1 \leq \eta < 2(4\beta - 1), \sqrt{2}\sqrt{\frac{4\beta-1}{\beta}} \leq k < 2\sqrt{2}\sqrt{\frac{\eta^2}{-16\beta+\eta^2+4\eta+4}}$, or
- (t) $\frac{3}{8} < \beta < \frac{9}{16}, 1 \leq \eta < 2(4\beta - 1), k \leq -\sqrt{2}\sqrt{\frac{4\beta-1}{\beta}}$.

15. $P_{15}^{\pm} := \left(\pm 1, \pm \frac{4-k\sqrt{\beta(k^2-8)+2}}{2+\beta k^2}, 0, \pm \frac{\sqrt{2}(2\beta k + \sqrt{\beta(k^2-8)+2})}{2+\beta k^2}, 0 \right)$. These exist for

- (a) $\beta < 0, -\sqrt{2}\sqrt{\frac{4\beta-1}{\beta}} \leq k < -\sqrt{2}\sqrt{-\frac{1}{\beta}}, \eta \geq 1$, or
- (b) $\beta < 0, -\sqrt{2}\sqrt{-\frac{1}{\beta}} < k < \sqrt{2}\sqrt{-\frac{1}{\beta}}, \eta \geq 1$, or
- (c) $\beta < 0, \sqrt{2}\sqrt{-\frac{1}{\beta}} < k \leq \sqrt{2}\sqrt{\frac{4\beta-1}{\beta}}, \eta \geq 1$, or
- (d) $0 \leq \beta \leq \frac{1}{4}, \eta \geq 1, k \in \mathbb{R}$, or
- (e) $\beta > \frac{1}{4}, k \leq -\sqrt{2}\sqrt{\frac{4\beta-1}{\beta}}, \eta \geq 1$, or
- (f) $\beta > \frac{1}{4}, k \geq \sqrt{2}\sqrt{\frac{4\beta-1}{\beta}}, \eta \geq 1$.

The eigenvalues are: $0, 0, \pm 2, \pm \frac{2k(\sqrt{\beta(k^2-8)+2}+2\beta k)}{\beta k^2+2}, \pm \frac{-4\eta+(\eta+2)k\sqrt{\beta(k^2-8)+2}+4\beta k^2}{\beta k^2+2}$. Substituting the values of Q, S, C, A_φ and A_W into (76a), (76b), and (76c) and integrating, we get $N = \bar{N}_0 \exp\left(\pm \frac{\lambda(4-k\sqrt{\beta(k^2-8)+2})}{\beta k^2+2}\right), K = \bar{K}_0 \exp\left(\mp \frac{2k\lambda(\beta k + \sqrt{\beta(k^2-8)+2}-2)}{\beta k^2+2}\right), y = \bar{y}_0 \exp\left(\mp \frac{k\lambda(2\beta k + \sqrt{\beta(k^2-8)+2})}{\beta k^2+2}\right)$. It follows from the definition of λ that the metric

can be written as $ds^2 = -\bar{N}_0^2 e^{\pm \frac{2\lambda(4-k\sqrt{\beta(k^2-8)+2})}{\beta k^2+2}} dt^2 + \frac{1}{y_0^2} \left(\frac{4-k\sqrt{\beta(k^2-8)+2}}{2+\beta}\right)^2 e^{\pm \frac{2k\lambda(2\beta k + \sqrt{\beta(k^2-8)+2})}{\beta k^2+2}} d\lambda^2 + \bar{K}_0^{-1} e^{\pm \frac{2k\lambda(\beta k + \sqrt{\beta(k^2-8)+2}-2)}{\beta k^2+2}} (d\vartheta^2 + \sin^2 \vartheta d\varphi^2)$. The equilibrium points are non-hyperbolic (two zero eigenvalues). The equilibrium

point P_{15}^+ (respectively, P_{15}^-) has a 3D unstable (respectively, stable) manifold in the following cases:

- (a) $\beta < -\frac{1}{4}$, $1 \leq \eta < \frac{2(4\beta-1)}{4\beta+1}$, $2\sqrt{2}\sqrt{\frac{\eta^2}{-16\beta+\eta^2+4\eta+4}} < k < \sqrt{2}\sqrt{-\frac{1}{\beta}}$, or
- (b) $\beta < -\frac{1}{4}$, $1 \leq \eta < \frac{2(4\beta-1)}{4\beta+1}$, $-\sqrt{2}\sqrt{\frac{4\beta-1}{\beta}} \leq k < -\sqrt{2}\sqrt{-\frac{1}{\beta}}$, or
- (c) $\beta < -\frac{1}{4}$, $\eta \geq \frac{2(4\beta-1)}{4\beta+1}$, $-\sqrt{2}\sqrt{\frac{4\beta-1}{\beta}} \leq k < -\sqrt{2}\sqrt{-\frac{1}{\beta}}$, or
- (d) $\beta < -\frac{1}{4}$, $1 \leq \eta < \frac{2(4\beta-1)}{4\beta+1}$, $\sqrt{2}\sqrt{-\frac{1}{\beta}} < k \leq \sqrt{2}\sqrt{\frac{4\beta-1}{\beta}}$, or
- (e) $\beta < -\frac{1}{4}$, $\eta \geq \frac{2(4\beta-1)}{4\beta+1}$, $2\sqrt{2}\sqrt{\frac{\eta^2}{-16\beta+\eta^2+4\eta+4}} < k \leq \sqrt{2}\sqrt{\frac{4\beta-1}{\beta}}$, or
- (f) $-\frac{1}{4} \leq \beta < 0$, $\eta \geq 1$, $2\sqrt{2}\sqrt{\frac{\eta^2}{-16\beta+\eta^2+4\eta+4}} < k < \sqrt{2}\sqrt{-\frac{1}{\beta}}$, or
- (g) $-\frac{1}{4} \leq \beta < 0$, $\eta \geq 1$, $-\sqrt{2}\sqrt{\frac{4\beta-1}{\beta}} \leq k < -\sqrt{2}\sqrt{-\frac{1}{\beta}}$, or
- (h) $-\frac{1}{4} \leq \beta < 0$, $\eta \geq 1$, $\sqrt{2}\sqrt{-\frac{1}{\beta}} < k \leq \sqrt{2}\sqrt{\frac{4\beta-1}{\beta}}$, or
- (i) $0 \leq \beta \leq \frac{3}{8}$, $\eta \geq 1$, $k > 2\sqrt{2}\sqrt{\frac{\eta^2}{-16\beta+\eta^2+4\eta+4}}$, or
- (j) $\frac{3}{8} < \beta < \frac{9}{16}$, $1 \leq \eta < 2(4\beta - 1)$, $k \geq \sqrt{2}\sqrt{\frac{4\beta-1}{\beta}}$, or
- (k) $\frac{3}{8} < \beta < \frac{9}{16}$, $1 \leq \eta < 2(4\beta - 1)$, $-2\sqrt{2}\sqrt{\frac{\eta^2}{-16\beta+\eta^2+4\eta+4}} < k \leq -\sqrt{2}\sqrt{\frac{4\beta-1}{\beta}}$, or
- (l) $\frac{3}{8} < \beta < \frac{9}{16}$, $\eta \geq 2(4\beta - 1)$, $k > 2\sqrt{2}\sqrt{\frac{\eta^2}{-16\beta+\eta^2+4\eta+4}}$, or
- (m) $\beta = \frac{9}{16}$, $\eta = 1$, $k \leq -\frac{2\sqrt{10}}{3}$, or
- (n) $\beta = \frac{9}{16}$, $1 \leq \eta < \frac{5}{2}$, $k \geq \frac{2\sqrt{10}}{3}$, or
- (o) $\beta = \frac{9}{16}$, $1 < \eta < \frac{5}{2}$, $-2\sqrt{2}\sqrt{\frac{\eta^2}{\eta^2+4\eta-5}} < k \leq -\frac{2\sqrt{10}}{3}$, or
- (p) $\beta = \frac{9}{16}$, $\eta \geq \frac{5}{2}$, $k > 2\sqrt{2}\sqrt{\frac{\eta^2}{\eta^2+4\eta-5}}$, or
- (q) $\beta > \frac{9}{16}$, $2(2\sqrt{\beta} - 1) < \eta < 2(4\beta - 1)$, $k \geq \sqrt{2}\sqrt{\frac{4\beta-1}{\beta}}$, or
- (r) $\beta > \frac{9}{16}$, $1 \leq \eta \leq 2(2\sqrt{\beta} - 1)$, $k \geq \sqrt{2}\sqrt{\frac{4\beta-1}{\beta}}$, or
- (s) $\beta > \frac{9}{16}$, $2(2\sqrt{\beta} - 1) < \eta < 2(4\beta - 1)$, $-2\sqrt{2}\sqrt{\frac{\eta^2}{-16\beta+\eta^2+4\eta+4}} < k \leq -\sqrt{2}\sqrt{\frac{4\beta-1}{\beta}}$, or
- (t) $\beta > \frac{9}{16}$, $1 \leq \eta \leq 2(2\sqrt{\beta} - 1)$, $k \leq -\sqrt{2}\sqrt{\frac{4\beta-1}{\beta}}$, or
- (u) $\beta > \frac{9}{16}$, $\eta \geq 2(4\beta - 1)$, $k > 2\sqrt{2}\sqrt{\frac{\eta^2}{-16\beta+\eta^2+4\eta+4}}$.

$$16. P_{16}^{\pm} := \left(\pm 1, \pm \frac{(\eta+2)k^2}{2(\eta^2+2\beta k^2)}, 0, \pm \frac{\eta(\eta+2)k}{2\sqrt{2}(\eta^2+2\beta k^2)}, \frac{\sqrt{\eta}\sqrt{\eta+2}\sqrt{k^2((\eta+2)^2-16\beta)-8\eta^2}}{2\sqrt{2}(\eta^2+2\beta k^2)} \right).$$

These exist for

- (a) $\eta \geq 1, \beta = 0, k \geq \frac{2\sqrt{2}\eta}{\eta+2}$, or
- (b) $\eta \geq 1, \beta = 0, k \leq -\frac{2\sqrt{2}\eta}{\eta+2}$, or
- (c) $\eta \geq 1, \beta < 0, -\sqrt{\frac{-\eta^2}{\beta}} < k \leq -2\sqrt{2}\sqrt{\frac{\eta^2}{-16\beta+\eta^2+4\eta+4}}$, or
- (d) $\eta \geq 1, \frac{\eta+2}{8} \leq \beta < \frac{1}{16}(\eta+2)^2, k = \frac{2\eta}{\sqrt{\frac{1}{2}(\eta+2)^2-8\beta}}$, or
- (e) $\eta \geq 1, \frac{\eta+2}{8} \leq \beta < \frac{1}{16}(\eta+2)^2, k = -\frac{2\eta}{\sqrt{\frac{1}{2}(\eta+2)^2-8\beta}}$, or
- (f) $\eta \geq 1, \beta < 0, 2\sqrt{2}\sqrt{\frac{\eta^2}{-16\beta+\eta^2+4\eta+4}} \leq k < \sqrt{\frac{-\eta^2}{\beta}}$, or
- (g) $\eta \geq 1, 0 < \beta < \frac{\eta+2}{8}, -\sqrt{\frac{\eta}{\beta}} \leq k \leq -2\sqrt{2}\sqrt{\frac{\eta^2}{-16\beta+\eta^2+4\eta+4}}$, or
- (h) $\eta \geq 1, 0 < \beta < \frac{\eta+2}{8}, 2\sqrt{2}\sqrt{\frac{\eta^2}{-16\beta+\eta^2+4\eta+4}} \leq k \leq \sqrt{\frac{\eta}{\beta}}$.

The eigenvalues are: $\mp \frac{8\eta^2+k^2(16\beta-(\eta+2)^2)}{4(\eta^2+2\beta k^2)}, \mp \frac{4\eta^2+k^2(8\beta-(\eta+2)^2)}{2(\eta^2+2\beta k^2)}, \pm \frac{\eta(\eta+2)k^2}{2(\eta^2+2\beta k^2)},$
 $\mp \frac{\sqrt{\beta}(8\eta^2+k^2(16\beta-(\eta+2)^2))-\sqrt{8\eta^2+k^2(16\beta-(\eta+2)^2)}\sqrt{8\eta^2(\beta+\eta+2)+\beta k^2(16\beta-(\eta+2)(9\eta+2))}}{8\sqrt{\beta}(\eta^2+2\beta k^2)},$
 $\mp \frac{\sqrt{\beta}(8\eta^2+k^2(16\beta-(\eta+2)^2))+\sqrt{8\eta^2(\beta+\eta+2)+\beta k^2(16\beta-(\eta+2)(9\eta+2))}\sqrt{8\eta^2+k^2(16\beta-(\eta+2)^2)}}{8\sqrt{\beta}(\eta^2+2\beta k^2)}.$

Substituting the values of Q, S, C, A_φ and A_W into (76a), (76b), and (76c) and integrating, we get $N = \bar{N}_0 e^{\pm \frac{(\eta+2)k^2\lambda}{2(2\beta k^2+\eta^2)}}, K = \bar{K}_0 e^{\mp \frac{\lambda(k^2(4\beta-\eta-2)+2\eta^2)}{2\beta k^2+\eta^2}}, y = \bar{y}_0 e^{\mp \frac{\eta(\eta+2)k^2\lambda}{4(2\beta k^2+\eta^2)}}.$ From the definition of λ the metric can be written as

$$ds^2 = -\bar{N}_0^2 e^{\pm \frac{(\eta+2)k^2\lambda}{2\beta k^2+\eta^2}} dt^2 + \frac{1}{\bar{y}_0^2} \left(\frac{k^2(2+\eta)}{4\beta k^2+2\eta^2} \right)^2 e^{\pm \frac{\eta(\eta+2)k^2\lambda}{2(2\beta k^2+\eta^2)}} d\lambda^2 + \bar{K}_0^{-1} e^{\pm \frac{\lambda(k^2(4\beta-\eta-2)+2\eta^2)}{2\beta k^2+\eta^2}} (d\vartheta^2 + \sin^2 \vartheta d\varphi^2).$$

P_{16}^+ (respectively, P_{16}^-) is a source (respectively, a sink) for

- (a) $\eta \geq 1, 0 < \beta < \frac{\eta+2}{8}, -\sqrt{\frac{\eta}{\beta}} < k < -2\sqrt{2}\sqrt{\frac{\eta^2}{-16\beta+\eta^2+4\eta+4}}$, or
- (b) $\eta \geq 1, 0 < \beta < \frac{\eta+2}{8}, 2\sqrt{2}\sqrt{\frac{\eta^2}{-16\beta+\eta^2+4\eta+4}} < k < \sqrt{\frac{\eta}{\beta}}$.

It is a saddle for

- (a) $\eta \geq 1, \beta < 0, -\frac{\sqrt{-\frac{\eta^2}{\beta}}}{\sqrt{2}} < k < -2\sqrt{2}\sqrt{\frac{\eta^2}{-16\beta+\eta^2+4\eta+4}}$ or
- (b) $\eta \geq 1, \beta < 0, 2\sqrt{2}\sqrt{\frac{\eta^2}{-16\beta+\eta^2+4\eta+4}} < k < \frac{\sqrt{-\frac{\eta^2}{\beta}}}{\sqrt{2}}$.

non-hyperbolic otherwise.

17. $P_{17}^\pm := \left(\pm 1, \pm \frac{(\eta+2)k^2}{2(\eta^2+2\beta k^2)}, 0, \pm \frac{\eta(\eta+2)k}{2\sqrt{2}(\eta^2+2\beta k^2)}, -\frac{\sqrt{\eta}\sqrt{\eta+2}\sqrt{k^2((\eta+2)^2-16\beta)-8\eta^2}}{2\sqrt{2}(\eta^2+2\beta k^2)} \right)$.
 These exist for

- (a) $\eta \geq 1, \beta < 0, k < -\frac{\sqrt{-\frac{\eta^2}{\beta}}}{\sqrt{2}}$, or
- (b) $\eta \geq 1, \beta < 0, k = -2\sqrt{2}\sqrt{\frac{\eta^2}{-16\beta+\eta^2+4\eta+4}}$, or
- (c) $\eta \geq 1, \beta < 0, k = 2\sqrt{2}\sqrt{\frac{\eta^2}{-16\beta+\eta^2+4\eta+4}}$, or
- (d) $\eta \geq 1, \beta < 0, k > \frac{\sqrt{-\frac{\eta^2}{\beta}}}{\sqrt{2}}$, or
- (e) $\eta \geq 1, 0 \leq \beta < \frac{1}{16}(\eta+2)^2, k = -2\sqrt{2}\sqrt{\frac{\eta^2}{-16\beta+\eta^2+4\eta+4}}$, or
- (f) $\eta \geq 1, 0 \leq \beta < \frac{1}{16}(\eta+2)^2, k = 2\sqrt{2}\sqrt{\frac{\eta^2}{-16\beta+\eta^2+4\eta+4}}$.

Eigenvalues: $\mp \frac{8\eta^2+k^2(16\beta-(\eta+2)^2)}{4(\eta^2+2\beta k^2)}, \mp \frac{4\eta^2+k^2(8\beta-(\eta+2)^2)}{2(\eta^2+2\beta k^2)}, \pm \frac{\eta(\eta+2)k^2}{2(\eta^2+2\beta k^2)},$
 $\mp \frac{\sqrt{\beta}(8\eta^2+k^2(16\beta-(\eta+2)^2))-\sqrt{8\eta^2+k^2(16\beta-(\eta+2)^2)}\sqrt{8\eta^2(\beta+\eta+2)+\beta k^2(16\beta-(\eta+2)(9\eta+2))}}{8\sqrt{\beta}(\eta^2+2\beta k^2)},$
 $\mp \frac{\sqrt{\beta}(8\eta^2+k^2(16\beta-(\eta+2)^2))+\sqrt{8\eta^2(\beta+\eta+2)+\beta k^2(16\beta-(\eta+2)(9\eta+2))}\sqrt{8\eta^2+k^2(16\beta-(\eta+2)^2)}}{8\sqrt{\beta}(\eta^2+2\beta k^2)}.$

Substituting the values of Q, S, C, A_φ and A_W into (76a), (76b), and (76c) and integrating, we get $N = \bar{N}_0 e^{\pm \frac{(\eta+2)k^2\lambda}{4\beta k^2+2\eta^2}}, K = \bar{K}_0 e^{\mp \frac{\lambda(k^2(4\beta-\eta-2)+2\eta^2)}{2\beta k^2+\eta^2}}, y = \bar{y}_0 e^{\mp \frac{\eta(\eta+2)k^2\lambda}{4(2\beta k^2+\eta^2)}}.$ It follows from the definition of λ that the metric can be written as

$$ds^2 = -\bar{N}_0^2 e^{\pm \frac{(\eta+2)k^2\lambda}{2\beta k^2+\eta^2}} dt^2 + \frac{1}{\bar{y}_0^2} \left(\frac{k^2(2+\eta)}{4\beta k^2+2\eta^2} \right)^2 e^{\pm \frac{\eta(\eta+2)k^2\lambda}{2(2\beta k^2+\eta^2)}} d\lambda^2 + \bar{K}_0^{-1} e^{\pm \frac{\lambda(k^2(4\beta-\eta-2)+2\eta^2)}{2\beta k^2+\eta^2}} (d\vartheta^2 + \sin^2 \vartheta d\varphi^2).$$

They are a saddle for

- (a) $\eta \geq 1, \beta < 0, k < -\frac{\sqrt{-\frac{\eta^2}{\beta}}}{\sqrt{2}}$, or

(b) $\eta \geq 1, \beta < 0, k > \frac{\sqrt{-\frac{\eta^2}{\beta}}}{\sqrt{2}}$.

non-hyperbolic, otherwise.

18. $P_{18}^{\pm} := \left(\pm 1, \pm \frac{1}{2\beta}, 0, \pm \frac{k}{2\sqrt{2}}, \frac{\sqrt{\beta(k^2-8)+2}}{2\sqrt{2}\sqrt{\beta}} \right)$.

These exist for

- (a) $\eta \geq 1, 0 < \beta \leq \frac{1}{4}, k \in \mathbb{R}$, or
- (b) $\eta \geq 1, \beta > \frac{1}{4}, k \leq -\sqrt{2}\sqrt{\frac{4\beta-1}{\beta}}$, or
- (c) $\eta \geq 1, \beta > \frac{1}{4}, k \geq \sqrt{2}\sqrt{\frac{4\beta-1}{\beta}}$.

The eigenvalues are: $\pm \frac{k^2}{2}, \pm \frac{-4\beta+\beta k^2+2}{2\beta}, \pm \frac{-8\beta+\beta k^2+2}{4\beta}, \pm \frac{-8\beta+\beta k^2+2}{4\beta}, \pm \frac{\beta k^2-\eta}{2\beta}$.

Substituting the values of Q, S, C, A_φ and A_W into (76a), (76b), and (76c) and integrating, we get $N = \bar{N}_0 e^{\pm \frac{\lambda}{2\beta}}, K = \bar{K}_0 e^{\mp \frac{(2\beta-1)\lambda}{\beta}}, y = \bar{y}_0 e^{\mp \frac{1}{4}k^2\lambda}$. It follows from the definition of λ that the metric can be written as $ds^2 = -\bar{N}_0^2 e^{\pm \frac{\lambda}{\beta}} dt^2 + \frac{1}{4\bar{y}_0^2 \beta^2} e^{\pm \frac{k^2\lambda}{2}} d\lambda^2 + \bar{K}_0^{-1} e^{\pm \frac{(2\beta-1)\lambda}{\beta}} (d\vartheta^2 + \sin^2 \vartheta d\varphi^2)$.

P_{18}^+ (respectively, P_{18}^-) is a source (respectively, a sink) for

- (a) $\eta \geq 1, \beta < 0, k < -\sqrt{2}\sqrt{\frac{4\beta-1}{\beta}}$, or
- (b) $\eta \geq 1, \beta < 0, k > \sqrt{2}\sqrt{\frac{4\beta-1}{\beta}}$, or
- (c) $\eta \geq 1, 0 < \beta \leq \frac{\eta+2}{8}, k < -\sqrt{\frac{\eta}{\beta}}$, or
- (d) $\eta \geq 1, 0 < \beta \leq \frac{\eta+2}{8}, k > \sqrt{\frac{\eta}{\beta}}$, or
- (e) $\eta \geq 1, \beta > \frac{\eta+2}{8}, k < -\sqrt{2}\sqrt{\frac{4\beta-1}{\beta}}$, or
- (f) $\eta \geq 1, \beta > \frac{\eta+2}{8}, k > \sqrt{2}\sqrt{\frac{4\beta-1}{\beta}}$.

It is non-hyperbolic for

- (a) $\eta \geq 1, 0 < \beta \leq \frac{1}{4}, k \in \left\{ -\sqrt{\frac{\eta}{\beta}}, 0, \sqrt{\frac{\eta}{\beta}} \right\}$, or
- (b) $\eta \geq 1, \frac{1}{4} < \beta < \frac{\eta+2}{8}, k \in \left\{ -\sqrt{\frac{\eta}{\beta}}, -\sqrt{2}\sqrt{\frac{4\beta-1}{\beta}}, \sqrt{2}\sqrt{\frac{4\beta-1}{\beta}}, \sqrt{\frac{\eta}{\beta}} \right\}$, or
- (c) $\eta \geq 1, \beta \geq \frac{\eta+2}{8}, k \in \left\{ -\sqrt{2}\sqrt{\frac{4\beta-1}{\beta}}, \sqrt{2}\sqrt{\frac{4\beta-1}{\beta}} \right\}$

Otherwise, they are saddle.

19. $P_{19}^{\pm} := \left(\pm 1, \pm \frac{2}{\beta k^2+2}, \frac{\beta(k^2-4)+2}{\beta k^2+2}, \pm \frac{\sqrt{2}\beta k}{\beta k^2+2}, \sqrt{-\frac{2\beta}{\beta k^2+2}} \right)$.

There exist for

- (a) $\eta \geq 1, \beta = 0, k \in \mathbb{R}$, or
- (b) $\eta \geq 1, \beta < 0, -\sqrt{2}\sqrt{-\frac{1}{\beta}} < k < \sqrt{2}\sqrt{-\frac{1}{\beta}}$.

For $\beta = 0$ these points reduces to P_8^\pm . Substituting the values of Q, S, C, A_φ and A_W into (76a), (76b), and (76c) and integrating, we get $N = \bar{N}_0 e^{\pm \frac{2\lambda}{\beta k^2+2}}, K = \bar{K}_0 e^{\mp \frac{2\beta k^2 \lambda}{\beta k^2+2}}, y = \bar{y}_0 e^{\mp \frac{\beta k^2 \lambda}{\epsilon(\beta k^2+2)}}$. Since $Q^2 = 1, C \neq 0$, it follows from the definition of λ that the metric can be written as

$$ds^2 = -\bar{N}_0^2 e^{\pm \frac{4\lambda}{\beta k^2+2}} dt^2 + \bar{K}_0^{-1} e^{\pm \frac{2\beta k^2 \lambda}{\beta k^2+2}} \left[\left(\frac{\beta(k^2-4)+2}{\beta k^2+2} \right) d\lambda^2 + (d\vartheta^2 + \sin^2 \vartheta d\varphi^2) \right].$$

For $\beta \neq 0$, the eigenvalues are

$$\mp 1, \pm \frac{2\beta k^2}{\beta k^2+2}, \pm \frac{2(\beta k^2-\eta)}{\beta k^2+2}, \mp \frac{\beta k^2+2+\sqrt{64\beta-7\beta^2 k^4+32\beta^2 k^2-28\beta k^2-28}}{2(\beta k^2+2)},$$

$$\mp \frac{\beta k^2+2-\sqrt{64\beta-7\beta^2 k^4+32\beta^2 k^2-28\beta k^2-28}}{2(\beta k^2+2)}.$$

Hence, the point P_{19}^+ (respectively, P_{19}^-) behaves as a saddle.

References

1. Jacobson, T., Mattingly, D.: Phys. Rev. D **64**, 024028 (2001)
2. Eling, C., Jacobson, T., Mattingly, D. [arXiv:gr-qc/0410001](https://arxiv.org/abs/gr-qc/0410001)
3. Donnelly, W., Jacobson, T.: Phys. Rev. D **82**, 064032 (2010)
4. Kanno, S., Soda, J.: Phys. Rev. D **74**, 063505 (2006)
5. Zlosnik, T.G., Ferreira, P.G., Starkman, G.D.: Phys. Rev. D **75**, 044017 (2007)
6. Carruthers, I., Jacobson, T.: Phys. Rev. D **83**, 024034 (2011)
7. Jacobson, T.: PoS QG-PH **020** (2007)
8. Carroll, S.M., Lim, E.A.: Phys. Rev. D **70**, 123525 (2004)
9. Garfinkle, D., Jacobson, T.: Phys. Rev. Lett. **107**, 191102 (2011)
10. Horava, P.: Phys. Rev. D **79**, 084008 (2009)
11. Jacobson, T.: Phys. Rev. D **81**, 101502 (2010)
12. Jacobson, T.: Phys. Rev. D **89**, 081501 (2014)
13. Wang, A.: Int. J. Mod. Phys. D **26**(07), 1730014 (2017)
14. Sarbach, O., Barausse, E., Preciado-López, J.A.: [arXiv:1902.05130](https://arxiv.org/abs/1902.05130) [gr-qc]
15. Oost, J., Bhattacharjee, M., Wang, A.: Gen. Relativ. Gravit. **50**(10), 124 (2018)
16. Nilsson, N.A., Czuchry, E.: Phys. Dark Univ. **23**, 100253 (2019)
17. Trinh, D., Pace, F., Battye, R.A., Bolliet, B.: Phys. Rev. D **99**(4), 043515 (2019)
18. Barrow, J.D.: Phys. Rev. D **85**, 047503 (2012)
19. Sandin, P., Alhulaimi, B., Coley, A.: Phys. Rev. D **87**(4), 044031 (2013)
20. Alhulaimi, B., Coley, A., Sandin, P.: J. Math. Phys. **54**, 042503 (2013)
21. Coley, A.A., Leon, G., Sandin, P., Latta, J.: JCAP **1512**, 010 (2015)
22. Latta, J., Leon, G., Paliathanasis, A.: JCAP **1611**(11), 051 (2016)
23. Lefschetz, S.: Differential Equations: Geometric Theory. Dover, New York (1977)
24. Lynch, S.: Dynamical Systems with Applications Using Maple. Birkhauser, Boston (2010)
25. Perko, L.: Differential Equations and Dynamical Systems, 3rd edn. Springer, New York (2001)
26. Campista, M., Chan, R., da Silva, M.F.A., Goldoni, O., Satheeshkumar, V.H., da Rocha, J.F.V.: [arXiv:1807.07553](https://arxiv.org/abs/1807.07553) [gr-qc]
27. Van Den Hoogen, R.J., Coley, A.A., Alhulaimi, B., Mohandas, S., Knighton, E., O’Neil, S.: JCAP **1811**(11), 017 (2018)
28. Alhulaimi, B., Van Den Hoogen, R.J., Coley, A.A.: JCAP **1712**(12), 045 (2017)

29. Wainwright, J., Ellis, G.F.R. (eds.): *Dynamical Systems in Cosmology*. Cambridge University Press, Cambridge (1997)
30. Coley, A.A.: *Dynamical systems and cosmology* (Astrophysics and Space Science Library. 291. ISBN 1-4020-1403-1)
31. Copeland, E.J., Liddle, A.R., Wands, D.: *Phys. Rev. D* **57**, 4686 (1998)
32. Ferreira, P.G., Joyce, M.: *Phys. Rev. Lett.* **79**, 4740 (1997)
33. Chen, X., Gong, Y., Saridakis, E.N.: *JCAP* **0904**, 001 (2009)
34. Xu, C., Saridakis, E.N., Leon, G.: *JCAP* **1207**, 005 (2012)
35. Nilsson, U.S., Uggla, C.: *Ann. Phys.* **286**, 278 (2001)
36. Tolman, R.C.: *Phys. Rev.* **55**, 364 (1939)
37. Oppenheimer, J.R., Volkoff, G.M.: *Phys. Rev.* **55**, 374 (1939)
38. Misner, C.W., Zapolsky, H.S.: *Phys. Rev. Lett.* **12** (1964) 635 [Erratum: *Phys. Rev. Lett.* **13** (1964) 122.]
39. Kasner, Edward: *Trans. Am. Math. Soc.* **27**, 101–105 (1925)
40. Ganguly, A., Gannouji, R., Goswami, R., Ray, S.: *Class. Quantum Gravity* **32**(10), 105006 (2015)
41. Clarkson, C.A., Barrett, R.K.: *Class. Quantum Gravity* **20**, 3855 (2003)
42. Clarkson, C.: *Phys. Rev. D* **76**, 104034 (2007)
43. Barausse, E., Jacobson, T., Sotiriou, T.P.: *Phys. Rev. D* **83**, 124043 (2011)
44. Eling, C., Jacobson, T.: *Class. Quantum Gravity* **23**, 5643 (2006)
45. Coley, A.A., Hervik, S.: *Class. Quantum Gravity* **22**, 579 (2005)
46. Coley, A.A., Hervik, S., Lim, W.C.: *Class. Quantum Gravity* **23**, 3573 (2006)
47. Eling, C., Jacobson, T.: *Class. Quantum Gravity* **23**, 5625 (2006)
48. Seifert, M.D.: *Phys. Rev. D* **76**, 064002 (2007)
49. Gao, C., Shen, Y.-G. [arXiv:1301.7122](https://arxiv.org/abs/1301.7122) [gr-qc]
50. Tamaki, T., Miyamoto, U. [arXiv:0709.1011](https://arxiv.org/abs/0709.1011) [gr-qc]
51. Eling, C., Jacobson, T., Miller, M.Coleman: *Phys. Rev. D* **76**, 042003 (2007)
52. Garfinkle, D., Eling, C., Jacobson, T.: *Phys. Rev. D* **76**, 024003 (2007)
53. Konoplya, R.A., Zhidenko, A.: *Phys. Lett. B* **644**, 186 (2007)
54. Konoplya, R.A., Zhidenko, A.: *Phys. Lett. B* **648**, 236 (2007)
55. Nilsson, U.S., Uggla, C.: *Ann. Phys.* **286**, 292 (2001)
56. Carr, B.J., Coley, A.A., Goliath, M., Nilsson, U.S., Uggla, C.: *Class. Quantum Gravity* **18**, 303 (2001)
57. Leon, G., Coley, A., Paliathanasis, A.: Static spherically symmetric Einstein-aether models II: integrability and the modified Tolman–Oppenheimer–Volkoff approach. [arXiv:1906.05749](https://arxiv.org/abs/1906.05749) [gr-qc]
58. Ablowitz, M.J., Ramani, A., Segur, H.: *Lettere al Nuovo Cimento* **23**, 333 (1978)
59. Ablowitz, M.J., Ramani, A., Segur, H.: *J. Math. Phys.* **21**, 715 (1980)
60. Ablowitz, M.J., Ramani, A., Segur, H.: *J. Math. Phys.* **21**, 1006 (1980)
61. Leon, G., Latta, J., Millano, A.: Static spherically symmetric Einstein-aether models III: perfect fluids plus a scalar field with an harmonic potential (**in preparation**)
62. Alho, A., Hell, J., Uggla, C.: *Class. Quantum Gravity* **32**(14), 145005 (2015)
63. Jacobson, T., Mattingly, D.: *Phys. Rev. D* **70**, 024003 (2004)
64. Elliott, J.W., Moore, G.D., Stoica, H.: *JHEP* **0508**, 066 (2005)
65. Foster, B.Z., Jacobson, T.: *Phys. Rev. D* **73**, 064015 (2006)
66. Cropp, B., Liberati, S., Mohd, A., Visser, M.: *Phys. Rev. D* **89**, 064061 (2014)
67. Franchini, N., Saravani, M., Sotiriou, T.P.: *Phys. Rev. D* **96**, 104044 (2017)
68. Bhattacharyya, J., Colombo, M., Sotiriou, T.P.: *Class. Quantum Gravity* **33**, 235003 (2016)
69. Berglund, P., Bhattacharyya, J., Mattingly, D.: *Phys. Rev. D* **85**, 124019 (2012)
70. Liberati, S., Pacilio, C.: *Phys. Rev. D* **96**, 104060 (2017)
71. Berglund, P., Bhattacharyya, J., Mattingly, D.: *Phys. Rev. Lett.* **110**(7), 071301 (2013)
72. Lin, K., Mukohyama, S., Wang, A., Zhu, T.: *Phys. Rev. D* **95**(12), 124053 (2017)
73. Misner, C.W., Sharp, D.H.: *Phys. Rev.* **136**, B571 (1964)
74. Buchdahl, H.A.: *Phys. Rev.* **116**, 1027 (1959)
75. Hartle, J.B.: *Phys. Rep.* **46**, 201–247 (1978)