RESEARCH ARTICLE

Approximate spacetime for neutron stars

Francisco Frutos-Alfaro[1]

Abstract

An approximate realistic metric representing the spacetime of neutron stars is obtained by perturbing the Kerr metric. This metric has five parameters, namely the mass, spin or angular momentum, mass quadrupole, spin octupole and mass hexadecapole. Moreover, a version of the Hartle–Thorne metric containing these parameters is constructed by means of a series transformation between these spacetimes and solving the Einstein field equations. The form of the Pappas metric in Schwarzschild spherical coordinates is found. The three relativistic multipole structures are compared.

Keywords Relativistic multipole moments · Neutron stars · Stationary spacetimes

1 Introduction

Among compact objects are neutron stars (NS). These stars are relativistic rotating objects with high density, and strong gravitational and magnetic fields. The study of NS is relevant to understand the extrem conditions of matter in there, the behaviour of particles around them, and the structure of its spacetime [1,3,12].

The quest to find a realistic spacetime representation for neutron stars (NS) is an important task in astrophysics. Many attemps to obtain this spacetime have been done from approximate metrics until numerical metrics. The first attempt was made by Hartle–Thorne (HT). The relevance of the HT work was that they matched the interior solution with the exterior one [11]. Quevedo and Mashhoon [18] and Manko and Novikov [14] obtained exact solutions with charge and arbitrary mass multipoles. Later, more exact solutions containing other features, for instance magnetic dipole, were found [15]. With the advent of computer technology, the implementations of computer programs to find numerical solutions become a vogue [20]. However, approximate solutions are still important to extract astrophysical information from NS [17]. Moreover, a fourth order HT metric for the exterior of neutron stars was obtained in [21].

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There are several techniques to find solutions of the Einstein field equations (EFE). Among them, the Ernst formalism has played an important role in finding new exact and approximate solutions. This formalism is employed in [14,17,18]. In this contribution, however, we include features like mass quadrupole, spin octupole and mass hexadecapole to the Kerr metric perturbatively. This is achieved easily by means of perturbing the Lewis metric form of the Kerr spacetime [7,8]. The form of the perturbations due to spin octupole and mass hexadecapole has the structure proposed by Ryan [19]. Then, one is certain that these features are introduced in the right manner. This metric has the advantage that it reduces to the Kerr metric which is an exact solution with mass and angular momentum. Moreover, it is simple to implement computationaly.

The HT spacetime does not have spin octupole, mass hexadecapole, and the interactions mass-quadrupole, quadrupole-quadrupole, and spin-quadrupole. Nevertheless, this metric gives excellent results of the inner most stable circular orbit (ISCO) of particles around NS [2]. Adding these interactions, and the spin octupole and the mass hexadecapole to this metric would improve considerably its applicability in computational calculations. One can guess an approximate HT metric with these features from our deformed Kerr spacetime, by finding a transformation between them from the post-linear forms of these metrics and solving the EFE. Both spacetime were tested to be solutions of the EFE by means of REDUCE programs, and these programs are available upon request.

This paper is organized as follows. The perturbation method of the Kerr metric using the Lewis spacetime is discussed in Sect. 2. In Sect. 3, the construction of the HT version is presented via a series transformation for the first order in spin octupole and mass hexadecapole. The inclusion of the interactions of the spin octupole and mass hexadecapole with the mass, spin, quadrupole and with each other is also found in this section. The relativistic multipole structure is found using the Fodor–Hoenselaers–Perjés method [6] in Sect. 4. In Sect. 5, the Pappas metric [17] is compared with the ones presented here. In the last section some conclusions are presented.

2 Generating the metric

The Lewis metric [13] was successfully applied to find approximate solutions of the EFE using the Erez–Rosen metric and the Kerr as seed metrics [7,8]. It is given by

$$ds^2 = -Vdt^2 + 2Wdtd\phi + Xd\rho^2 + Ydz^2 + Zd\phi^2 \tag{1}$$

where the chosen canonical coordinates are $x^1 = \rho$ and $x^2 = z$. The potentials V, W, X, Y and Z are functions of ρ and z with $\rho^2 = VZ + W^2$.

The transformation that leads to the Kerr metric is

$$\rho = \sqrt{\Delta}\sin\theta \quad \text{and} \quad z = (r-m)\cos\theta, \tag{2}$$

where $\Delta = r^2 - 2mr + a^2$, m and a are the mass and the rotational parameter. The angular momentum or spin is given by $\mathcal{S}_1 = J = ma$.

The Ansatz for the Lewis potentials to include the spin octupole parameter, S_3 and the mass hexadecapole parameter, M_4, is

$$
\begin{aligned}
V &= V_K \, e^{-2\psi}, \\
W &= W_K + W_{ms}, \\
X &= (X_K + X_{ms}) \, e^{2\chi}, \\
Y &= (Y_K + Y_{ms}) \, e^{2\chi}, \\
Z &= (Z_K + Z_{ms}) \, e^{2\psi},
\end{aligned}
\tag{3}
$$

where the potentials V_K, W_K, X_K, Y_K, Z_K are the Lewis potentials for the Kerr metric. The perturbation terms are W_{ms}, X_{ms}, Y_{ms}, Z_{ms} which include terms with M_4 and S_3, and interaction of these parameters with the other ones.

The functions ψ, χ, W_{ms}, X_{ms}, Y_{ms}, and Z_{ms} are chosen as follows

$$
\begin{aligned}
\psi &= \psi_q + \psi_{ms}, \\
\chi &= \chi_q + \chi_{ms}, \\
W_{ms} &= \left(\xi_0 \frac{Jq}{r^4} + \xi_1 \frac{S_3}{r^3} \right) h_{31} + \xi_2 \frac{m S_3}{r^4} h_{32} \\
&\quad + \xi_3 \frac{q S_3}{r^6} h_{33} + \xi_4 \frac{M_4 S_3}{r^8} h_{34} + \xi_5 \frac{J M_4}{r^6} h_{35}, \\
X_{ms} &= \mu_1 \frac{m M_4}{r^6} h_{41} + \mu_2 \frac{q M_4}{r^8} h_{42} + \mu_3 \frac{M_4^2}{r^{10}} h_{43}, \\
Y_{ms} &= \iota_1 \frac{m M_4}{r^4} h_{51} + \iota_2 \frac{q M_4}{r^6} h_{52} + \iota_3 \frac{M_4^2}{r^8} h_{53} + \iota_4 \frac{S_3^2}{r^6} h_{54}, + \iota_5 \frac{J S_3}{r^4} h_{55}, \\
Z_{ms} &= \left(\zeta_1 \frac{m M_4}{r^4} h_{51} + \zeta_2 \frac{q M_4}{r^6} h_{52} + \zeta_3 \frac{M_4^2}{r^8} h_{53} + \zeta_4 \frac{S_3^2}{r^6} h_{54} + \zeta_5 \frac{J S_3}{r^4} h_{55} \right) \sin^2 \theta,
\end{aligned}
\tag{4}
$$

where

$$
\begin{aligned}
\psi_q &= \frac{q}{r^3} P_2 + 3 \frac{mq}{r^4} P_2, \\
\chi_q &= \frac{q P_2}{r^3} + \frac{1}{3} \frac{mq}{r^4} (5 P_2^2 + 5 P_2 - 1) + \frac{1}{9} \frac{q^2}{r^6} (25 P_2^3 - 21 P_2^2 - 6 P_2 + 2), \\
\psi_{ms} &= \gamma_1 \frac{M_4}{r^5} h_{11} + \gamma_2 \frac{m M_4}{r^6} h_{12} + \gamma_3 \frac{q M_4}{r^8} h_{13} + \gamma_4 \frac{S_3^2}{r^8} h_{14} + \gamma_5 \frac{J S_3}{r^6} h_{15}, \\
\chi_{ms} &= \eta_1 \frac{M_4}{r^5} h_{21} + \eta_2 \frac{m M_4}{r^6} h_{22} + \eta_3 \frac{q M_4}{r^8} h_{23} + \eta_4 \frac{M_4^2}{r^{10}} h_{24} + \eta_5 \frac{S_3^2}{r^8} h_{25} + \eta_6 \frac{J S_3}{r^6} h_{26}.
\end{aligned}
$$

In [10], ψ_q and χ_q were found. The functions h_{ij} are functions of θ only. The first order terms of ψ_{ms}, χ_{ms} and W_{ms} are taken from [19]. The term Jq of W_{ms} represents the interaction of the spin with the quadrupole, which was not considered in [10].

From this Ansatz, the perturbative terms can be determined by solving the EFE. The functions h_{ij} are combinations of Legendre polynomials, $P_n(\cos\theta)$, $n = 1, \ldots, 8$, and an associated Legendre polynomial, $P_3^1(\cos\theta) = (5P_2 + 1)\sin\theta$. After solving the EFE, the functions h_{ij} are given by

$$h_{11} = P_4,$$

$$h_{12} = P_4,$$

$$h_{13} = P_7,$$

$$h_{14} = \frac{7}{64} + \frac{7}{48}P_2 + \frac{63}{352}P_4 + \frac{175}{528}P_6 + \frac{16}{429}P_7,$$

$$h_{15} = -\frac{3}{4}P_2 - P_4,$$

$$h_{21} = P_4,$$

$$h_{22} = \frac{45}{11}P_4,$$

$$h_{23} = P_7,$$

$$h_{24} = -\left(\frac{5}{99} + \frac{625}{9009}P_2 + \frac{90}{1001}P_4 + \frac{250}{1683}P_6 + \frac{12250}{24453}P_8\right),$$

$$h_{25} = -\frac{287}{1728} - \frac{91}{528}P_2 - \frac{7}{416}P_4 + \frac{175}{144}P_6 + \frac{16}{429}P_7, \qquad (5)$$

$$h_{26} = \frac{5}{4}P_2 - 3P_4,$$

$$h_{31} = P_3^1\sin\theta,$$

$$h_{32} = \frac{5}{2}P_2 - \frac{8}{63}P_3 - \frac{5}{2}P_4 + \frac{8}{63}P_5,$$

$$h_{33} = -\frac{1}{10} - \frac{1}{6}P_2 - \frac{27}{55}P_4 + \frac{25}{33}P_6,$$

$$h_{34} = -\frac{5}{33}P_2 - \frac{25}{143}P_4 - \frac{35}{66}P_6 + \frac{245}{286}P_8,$$

$$h_{35} = \frac{20}{11}(P_4 - P_6),$$

$$h_{41} = P_3,$$

$$h_{42} = -\frac{24}{7}P_2 + \frac{600}{77}P_4 + \frac{300}{143}P_6,$$

$$h_{43} = P_4,$$

$$h_{51} = \frac{2}{33} - \frac{2}{5}P_1 + \frac{10}{33}P_2 + \frac{2}{5}P_3 + \frac{6}{11}P_4,$$

$$h_{52} = -\frac{3}{26} + \frac{47}{26}P_2P_4 - \frac{303}{182}P_2 - \frac{5}{182}P_4,$$

$$h_{53} = \frac{33191}{4157010} + \frac{33191}{831402}P_2 + \frac{165259}{692835}P_4 + \frac{882}{3553}P_6 + \frac{882}{2717}P_8,$$

$$h_{54} = \frac{125293}{247104} + \frac{35315}{82368}P_2 - \frac{581}{9152}P_4 - \frac{6125}{6336}P_6,$$

$$h_{55} = 3(P_4 - P_2).$$

The constants are found to be

$$\gamma_1 = 1, \quad \gamma_2 = 5, \quad \gamma_3 = \gamma_4 = \gamma_5 = 1,$$
$$\eta_1 = \eta_2 = \eta_3 = \eta_4 = \eta_5 = \eta_6 = 1,$$
$$\xi_0 = 1, \quad \xi_1 = \frac{7}{12}, \quad \xi_2 = \xi_3 = \xi_4 = \xi_5 = 1,$$
$$\mu_1 = 4, \quad \mu_2 = \mu_3 = 1,$$
$$\iota_1 = \iota_2 = \iota_3 = \iota_4 = \iota_5 = 1,$$
$$\zeta_1 = \zeta_2 = \zeta_3 = \zeta_4 = \zeta_5 = -1.$$

The new metric potentials are

$$V = \frac{1}{\Sigma^2} \left[\Delta - a^2 \sin^2 \theta \right] e^{-2\psi},$$
$$W = -\frac{2Jr}{\Sigma^2} \sin^2 \theta + W_{ms},$$
$$X = \left(\frac{\Sigma^2}{\Delta} + X_{ms} \right) e^{2\chi}, \tag{6}$$
$$Y = \left(\Sigma^2 + Y_{ms} \right) e^{2\chi},$$
$$Z = \left(\frac{1}{\Sigma^2} \left[(r^2 + a^2)^2 - a^2 \Delta \sin^2 \theta \right] - Y_{ms} \right) e^{2\psi} \sin^2 \theta,$$

where $\Sigma^2 = r^2 + a^2 \cos^2 \theta$.

This metric is valid up to third order in all parameters, including the interactions of all parameters with each other.

A post-linear expansion of the metric can be written as

$$V \simeq \left(1 - 2U - 2\frac{ma^2}{r^3} \cos^2 \theta \right) e^{-2\psi},$$
$$W \simeq -2\frac{J}{r} \sin^2 \theta + W_{ms}, \tag{7}$$
$$X \simeq \left(1 + 2U + 4U^2 - \frac{a^2}{r^2} \sin^2 \theta - 2\frac{ma^2}{r^3}(1 + \sin^2 \theta) \right.$$
$$\left. - 4\frac{m^2 a^2}{r^4}(2 + \sin^2 \theta) + X_{ms} \right) e^{2\chi},$$
$$Y = r^2 \left(1 + \frac{a^2}{r^2} \cos^2 \theta + \frac{Y_{ms}}{r^2} \right) e^{2\chi},$$
$$Z \simeq r^2 \sin^2 \theta \left(1 + \frac{a^2}{r^2} + 2\frac{ma^2}{r^3} \sin^2 \theta - \frac{Y_{ms}}{r^2} \right) e^{2\psi},$$

where $U = m/r$. In [10] is the complete expansion without the S_3 and M_4 terms.

3 Constructing a new Hartle–Thorne metric

Adding perturbatively some features, for example the spin octupole and the mass hexadecapole to the HT metric would be interesting, because this metric is still used as a comparison with more realistic metrics. The HT metric is an approximate solution of the EFE with three parameters, mass, angular momentum and mass quadrupole. It is given by

$$ds^2 = -F_1 dt^2 + F_2 dR^2 + R^2 F_3 \left[d\theta^2 + \sin^2\theta (d\phi - \omega dt)^2 \right] \tag{8}$$
$$= -V_{HT} dt^2 - 2W_{HT} dt d\phi + X_{HT} dR^2 + R^2 Y_{HT} \left[d\theta^2 + \sin^2\theta d\phi^2 \right],$$

where

$$F_1 = \left(1 - 2U + 2\frac{J^2}{R^4} \right) [1 + 2K_1 P_2]$$
$$\simeq \left(1 - 2U + 2\frac{J^2}{R^4} \right) e^{2\psi_1},$$
$$F_2 = [1 - 2K_2 P_2] \left(1 - 2U + 2\frac{J^2}{R^4} \right)^{-1} \tag{9}$$
$$\simeq e^{-2\psi_2} \left(1 - 2U + 2\frac{J^2}{R^4} \right)^{-1},$$
$$F_3 = 1 - 2K_3 P_2 \simeq e^{-2\psi_3},$$
$$\omega = 2\frac{J}{R^3}.$$

The functions K_1, K_2, and K_3 are given by

$$K_1 = \frac{J^2}{mR^3}(1+U) + \frac{5}{8}\left(\frac{q}{m^3} - \frac{J^2}{m^4} \right) Q_2^2 \left(\frac{R}{m} - 1 \right),$$
$$K_2 = K_1 - 6\frac{J^2}{R^4} = K_1 - 6\frac{J^2}{m^4}U^4, \tag{10}$$
$$K_3 = \left(K_1 + \frac{J^2}{R^4} \right) + \frac{5}{4}\left(\frac{q}{m^3} - \frac{J^2}{m^4} \right) \frac{U}{\sqrt{1-2U}} Q_2^1 \left(\frac{R}{m} - 1 \right),$$

where $U = M/R$. The functions Q_2^1 and Q_2^2 are Legendre functions of the second kind

$$Q_2^1(x) = \sqrt{x^2 - 1}\left(\frac{3}{2}x \ln\left(\frac{x+1}{x-1} \right) - \frac{(3x^2 - 2)}{(x^2 - 1)} \right),$$
$$Q_2^2(x) = (x^2 - 1)\left(\frac{3}{2}\ln\left(\frac{x+1}{x-1} \right) - \frac{(3x^3 - 5x)}{(x^2 - 1)^2} \right).$$

The metric potencials are

$$V_{HT} \simeq \left(1 - 2U - \frac{2}{3}\frac{J^2}{R^4}\right) e^{2\alpha_1},$$

$$W_{HT} = -2\frac{J}{R}\sin^2\theta,$$

$$X \simeq \left(1 - 2U + 2\frac{J^2}{R^4}\right)^{-1} e^{-2\alpha_2}, \tag{11}$$

$$Y_{HT} \simeq e^{-2\alpha_3},$$

where

$$\alpha_1 = \left(K_1 + \frac{4}{3}\frac{J^2}{R^4}\right)P_2,$$

$$\alpha_2 = K_2 P_2, \tag{12}$$

$$\alpha_3 = K_3 P_2.$$

The Taylor expansion of K_1, K_2 and K_3 are

$$K_1 = \frac{q}{R^3} + 3\frac{mq}{R^4} - 2\frac{J^2}{R^4},$$

$$K_2 = \frac{q}{R^3} + 3\frac{mq}{R^4} - 8\frac{J^2}{R^4},$$

$$K_3 = \frac{q}{R^3} + \frac{5}{2}\frac{mq}{R^4} - \frac{1}{2}\frac{J^2}{R^4}.$$

The complete expansion of the HT including the second order terms in q was found in [9,10].

To guess an improvement of the HT metric, we have to find a solution of the EFE compatible with HT metric. In order to do it, we will propose an Ansatz. In [9], the second order in q for the post-linear HT was found perturbatively. A transformation that converts the post-linear Kerr-like metric (7) without S_3 and M_4 into an improved HT was obtained in [10]. The same transformation can be used to transform the post-linear Kerr-like metric (7) with S_3 and M_4 at first order into an improved HT in the post-linear form of (11) with S_3 and M_4 at first order, changing $q \to ma^2 - q$. This transformation is [10]

$$r = R\left[1 + \frac{mq}{R^4}f_1 + \frac{q^2}{R^6}f_2 + \frac{a^2}{R^2}\left(h_1 + \frac{m}{R}h_2 + \frac{m^2}{R^2}h_3\right)\right], \tag{13}$$

$$\theta = \Theta + \frac{mq}{R^4}g_1 + \frac{q^2}{R^6}g_2 + \frac{a^2}{R^2}\left(h_4 + \frac{m}{R}h_5\right),$$

where

$$f_1 = \frac{1}{9}(5P_2^2 - 4P_2 - 1),$$

$$f_2 = \frac{1}{72}(40P_2^3 - 24P_2^2 - 43),$$

$$g_1 = \frac{1}{6}(2 - 5P_2)\cos\Theta\sin\Theta,$$

$$g_2 = \frac{1}{6}P_2(2 - 5P_2)\cos\Theta\sin\Theta,$$

$$h_1 = -\frac{1}{2}\sin^2\Theta,$$

$$h_2 = -\frac{1}{2}\sin^2\Theta,$$

$$h_3 = 1 - 3\cos^2\Theta = -2P_2,$$

$$h_4 = -\frac{1}{2}\cos\Theta\sin\Theta,$$

$$h_5 = -\cos\Theta\sin\Theta,$$

$$(14)$$

with $P_2 = P_2(\cos\Theta)$

Now, considering this fact, the Jq interaction term and the terms due to the spin octupole S_3 and the mass hexadecapole M_4 at second order, the Ansatz of an improved HT metric functions is from (7)

$$V \simeq \left(1 - 2U - \frac{2}{3}\frac{J^2}{R^4}\right)e^{2\psi_1},$$

$$W \simeq -\left[2\frac{J}{R} + \left(\frac{Jq}{R^4} - \frac{7}{12}\frac{S_3}{R^3}\right)(5P_2 + 1)\right]\sin^2\Theta + \tilde{W}_{ms},$$

$$X \simeq \left(1 - 2U + 2\frac{J^2}{R^4}\right)^{-1}e^{-2\psi_2},$$

$$Y \simeq R^2 e^{-2\psi_3},$$

$$Z \simeq R^2 e^{-2\psi_3}\sin^2\Theta,$$

$$(15)$$

where

$$\psi_1 = \frac{q}{R^3}P_2 + 3\frac{mq}{R^4}P_2 - \frac{2}{3}\frac{J^2}{R^4}P_2 + \frac{M_4}{R^5}P_4 + \psi_{1ms},$$

$$\psi_2 = \frac{q}{R^3}P_2 + 3\frac{mq}{R^4}P_2 + \frac{1}{24}\frac{q^2}{R^6}[8P_2^2 - 16P_2 - 77] - 8\frac{J^2}{R^4}P_2 + \frac{M_4}{R^5}P_4 + \psi_{2ms},$$

$$\psi_3 = \frac{q}{R^3}P_2 + \frac{5}{2}\frac{mq}{R^4}P_2 + \frac{1}{72}\frac{q^2}{R^6}[28P_2^2 - 8P_2 + 43] - \frac{1}{2}\frac{J^2}{R^4}P_2 + \frac{M_4}{R^5}P_4 + \psi_{3ms},$$

$$(16)$$

with

$$\psi_{1ms} = \frac{m M_4}{r^6} \tilde{h}_{12} + \frac{q M_4}{r^8} \tilde{h}_{13} + \frac{J S_3}{r^6} \tilde{h}_{14} + \frac{S_3^2}{r^8} \tilde{h}_{15},$$

$$\psi_{2ms} = \frac{m M_4}{r^6} \tilde{h}_{22} + \frac{q M_4}{r^8} \tilde{h}_{23} + \frac{J S_3}{r^6} \tilde{h}_{24} + \frac{M_4^2}{r^{10}} \tilde{h}_{25} + \frac{S_3^2}{r^8} \tilde{h}_{26}, \quad (17)$$

$$\psi_{3ms} = \frac{m M_4}{r^6} \tilde{h}_{32} + \frac{q M_4}{r^8} \tilde{h}_{33} + \frac{J S_3}{r^6} \tilde{h}_{34} + \frac{M_4^2}{r^{10}} \tilde{h}_{35} + \frac{S_3^2}{r^8} \tilde{h}_{36},$$

$$\tilde{W}_{ms} = \frac{m S_3}{r^4} \tilde{h}_{42} + \frac{q S_3}{r^6} \tilde{h}_{43} + \frac{M_4 S_3}{r^8} \tilde{h}_{44} + \frac{J S_3}{r^5} \tilde{h}_{45} + \frac{J M_4}{r^6} \tilde{h}_{46}.$$

The $\tilde{h}_{ij}$ are found solving the EFE perturbatively. These functions are

$$\tilde{h}_{12} = 3 P_4,$$

$$\tilde{h}_{13} = P_7,$$

$$\tilde{h}_{14} = \frac{3}{4} P_2 + P_4,$$

$$\tilde{h}_{15} = -\frac{7}{48} P_2 - \frac{63}{352} P_4 - \frac{175}{528} P_6 - \frac{7}{64},$$

$$\tilde{h}_{22} = 5 P_4 - \frac{110}{27},$$

$$\tilde{h}_{23} = -\frac{20}{21} P_2 + \frac{36}{77} P_4 + \frac{16}{33} P_6 + P_7 - 7,$$

$$\tilde{h}_{24} = -\frac{15}{4} P_2 + \frac{11}{2} P_4 - 5,$$

$$\tilde{h}_{25} = \frac{1000}{693} P_2 + \frac{360}{1001} P_4 + \frac{20}{99} P_6 + \frac{280}{1287} P_8 + \frac{5}{18},$$

$$\tilde{h}_{26} = -\frac{679}{144} P_2 - \frac{1505}{1056} P_4 - \frac{1085}{792} P_6 - \frac{637}{96},$$

$$\tilde{h}_{32} = \frac{245}{27} P_2^2 - \frac{70}{27} P_2 - 1,$$

$$\tilde{h}_{33} = \frac{5}{21} P_2 + \frac{12}{77} P_4 + \frac{47}{132} P_6 + P_7 + 1, \quad (18)$$

$$\tilde{h}_{34} = \frac{7}{3} P_4 + 1,$$

$$\tilde{h}_{35} = -\frac{50}{693} P_2 + \frac{45}{1001} P_4 + \frac{8}{99} P_6 + \frac{217}{1287} P_8,$$

$$\tilde{h}_{36} = \frac{259}{576} P_2 - \frac{35}{264} P_4 - \frac{10255}{12672} P_6 + 1,$$

$$\tilde{h}_{42} = \frac{5}{2} (P_2 - P_4),$$

$$\tilde{h}_{43} = \frac{1}{6} P_2 + \frac{27}{55} P_4 - \frac{25}{33} P_6 + \frac{1}{10},$$

$$\tilde{h}_{44} = \frac{5}{33}P_2 + \frac{25}{143}P_4 + \frac{35}{66}P_6 - \frac{245}{286}P_8,$$

$$\tilde{h}_{45} = 0,$$

$$\tilde{h}_{46} = 4P_2P_4 - \frac{8}{7}P_2 - \frac{20}{7}P_4.$$

This metric is solution of the EFE up to the third order in all parameters (m, q, M_4, J, S_3).

4 Relativistic multipole moments

To determine if two metric are isometric, one has to compare its multipole structure. It is useful to find this structure for our spacetime. At first glance, our metric has 5 complex multipoles ($\mathcal{M}_0 = m$, $\mathcal{S}_1 = J = ma$, $\mathcal{M}_2 = q - ma^2$, $\mathcal{S}_3$, and $\mathcal{M}_4$). To see if it is true, one has to construct the Ernst potential for this metric. This potential is given by [5]

$$\mathcal{E} = f + i\Omega, \tag{19}$$

where $f = V = V_K e^{-2\psi}$ and Ω is the twist scalar. To get this scalar, the following equation has to be solved

$$\partial_\alpha \Omega = \varepsilon_{\alpha\beta\mu\nu} k^\beta \nabla^\mu k^\nu, \tag{20}$$

where k^β is the Killing vector, ∇^μ is the contravariant derivative and $\varepsilon_{\alpha\beta\mu\nu} = \sqrt{-g}\epsilon_{\alpha\beta\mu\nu}$ (g is determinant of the metric tensor). Let us take the Killing vector as in the Kerr metric $k^\beta = (1, 0, 0, 0)$. Then, the approximate solution of (20) is

$$\Omega = -2\frac{J}{\rho^2}\cos\theta + \mathcal{H}, \tag{21}$$

where

$$\mathcal{H} = \frac{S_3}{r^4}h_{61} + \frac{Jq}{r^5}h_{62} + \frac{mS_3}{r^5}h_{63} + \frac{qS_3}{r^7}h_{64} + \frac{JM_4}{r^7}h_{65} + \frac{M_4S_3}{r^9}h_{66}, \tag{22}$$

with

$$h_{61} = \frac{7}{12}(5P_2 - 2)\cos\theta,$$

$$h_{62} = 4P_2\cos\theta,$$

$$h_{63} = \frac{1}{420}(490(5P_2 - 2)\cos\theta - 96P_4), \tag{23}$$

$$h_{64} = \frac{1}{6}(4P_2 - 18P_4 - 7)\cos\theta,$$

$$h_{65} = 4P_4 \cos\theta,$$
$$h_{66} = \frac{1}{66}(-110P_2 + 54P_4 - 175P_6)\cos\theta.$$

Now, the Ernst function is given by

$$\xi = \frac{1 + \mathcal{E}}{1 - \mathcal{E}}. \tag{24}$$

It is easy to show that this Ernst function and its inverse are solutions of the Ernst equation [5]

$$(\xi\xi^\star - 1)\nabla^2\xi = 2\xi^\star[\nabla\xi]^2.$$

To calculate the relativistic multipole moments, it is better to employ the inverse function [5]. Moreover, it is custumary to employ the prolate spheroidal coordinates (t, x, y, ϕ). The transformation to these coordinates is achieved by means of

$$\sigma x = r - M, \tag{25}$$
$$y = \cos\theta,$$

where $\sigma^2 = M^2 - a^2$.

The method to obtain the relativistic multipole moments is the following [6]

(1) use the inverse Ernst function ξ^{-1} in prolate coordinates,
(2) set $y = \cos\theta = 1$ into ξ^{-1},
(3) change $\sigma x \to 1/z$ into ξ^{-1},
(4) expand in Taylor series of z the inverse Ernst function, and finally,
(5) employ the Fodor–Hoenselaers–Perjés (FHP) formulae [6].

A REDUCE program that calculates the multipole moment was written with this recipe. The first ten complex moments $\mathcal{P}_n = \mathcal{M}_n + i\mathcal{S}_n$ are

$$\mathcal{P}_0 = \mathcal{M}_0 = m,$$
$$\mathcal{P}_1 = i\mathcal{S}_1 = iJ = ima,$$
$$\mathcal{P}_2 = \mathcal{M}_2 = q - ma^2,$$
$$\mathcal{P}_3 = i\mathcal{S}_3 = -i\left(ma^3 + \frac{7}{8}S_3\right),$$
$$\mathcal{P}_4 = \mathcal{M}_4 + i\mathcal{S}_4 = ma^4 + M_4 + i\left(-2Jq + \frac{4}{35}mS_3\right),$$
$$\mathcal{P}_5 = \mathcal{M}_5 + i\mathcal{S}_5 = \frac{7}{4}JS_3 - 4mM_4 + ima^5, \tag{26}$$
$$\mathcal{P}_6 = \mathcal{M}_6 + i\mathcal{S}_6 = -ma^6 + i\left(\frac{7}{4}qS_3 - 2JM_4\right),$$
$$\mathcal{P}_7 = \mathcal{M}_7 + i\mathcal{S}_7 = \left(\frac{1}{5}qM_4 + \frac{16}{429}S_3^2\right) - ima^7,$$

$$\mathcal{P}_8 = \mathcal{M}_8 + i\mathcal{S}_8 = ma^8 + i\frac{7}{4}M_4 S_3,$$
$$\mathcal{P}_9 = i\mathcal{S}_9 = ima^9,$$
$$\mathcal{P}_{10} = \mathcal{M}_{10} = -ma^{10}.$$

The real parts are the massive multipoles, $\mathcal{M}_i$ and the imaginary parts are the spin multipoles, $\mathcal{S}_i$. If one eliminates mixed terms and $S_3^2 \sim 0$ in (26) the multipole structure becomes simpler.

For neutron stars, the form of the first five multipole moments are [16,17,21]

$$\begin{aligned}
\mathcal{M}_0 &= m, \\
\mathcal{S}_1 &= S = J = ma, \\
\mathcal{M}_2 &= -\alpha ma^2, \\
\mathcal{S}_3 &= -\beta ma^3, \\
\mathcal{M}_4 &= \gamma ma^4,
\end{aligned} \tag{27}$$

where α, β, and γ are parameters.

It is easy to see that if one sets

$$\begin{aligned}
q &= (1-\alpha)ma^2, \\
S_3 &= -\frac{7}{8}(\beta+1)ma^3, \\
M_4 &= (\gamma-1)ma^4,
\end{aligned} \tag{28}$$

a similar multipole structure is obtained from (26).

Now, let us determine the multipole structure of the new HT spacetime. The twist scalar for this HT metric is

$$\begin{aligned}
\Omega = \Big[&-2Ju^2 + \frac{7}{12}S_3(5P_2 - 2)u^4 + \frac{1}{6}(-24JqP_2 + 7MS_3(5P_2 - 2))u^5 \quad (29) \\
&+\frac{1}{6}(-24JM_4P_4 + qS_3(-4P_2 + 18P_4 + 7))u^7 \\
&+\frac{1}{66}M_4 S_3(110P_2 - 54P_4 + 175P_6)u^9 \Big]\cos\theta.
\end{aligned}$$

After using (15) and (29) to construct the Ernst functions, we find that the relativistic multipole moments for this HT metric are

$$\begin{aligned}
\mathcal{M}_0 &= M, \\
\mathcal{S}_1 &= J, \\
\mathcal{M}_2 &= -q, \\
\mathcal{S}_3 &= -\frac{7}{8}S_3, \\
\mathcal{M}_4 &= M_4.
\end{aligned} \tag{30}$$

Obviously, from this multipole structure, it is possible to calculate the multipole moments of a neutron star, as well.

5 Comparison with the Pappas metric

Pappas found an approximate solution of the EFE by means of the Ernst method [17]. This spacetime has 5 parameters M, M_2, M_4, $S_1 = J$, S_3, which, by construction, represent the relativistic multipole moments. The metric is given in cylindrical Weyl–Papapetrou coordinates by

$$ds^2 = -f(dt - \omega d\phi)^2 + \frac{1}{f}\left[e^{2\gamma}(d\rho^2 + dz^2) + \rho^2 d\phi^2\right], \tag{31}$$

where

$$
\begin{aligned}
f &= 1 - 2\frac{M}{\eta} + 2\frac{M^2}{\eta^2} + \frac{1}{\eta^5}\left[(M_2 - M^3)\rho^2 - 2(M^3 + M_2)z^2\right] \\
&\quad + \frac{1}{\eta^6}\left[2z^2(M^4 - J^2 + 2MM_2) - 2MM_2\rho^2\right] + \frac{A}{28\eta^9} + \frac{B}{14\eta^{10}}, \\
\omega &= -2\frac{J}{\eta^3}\rho^2 - 2\frac{MJ}{\eta^4}\rho^2 + \frac{F}{\eta^7} + \frac{H}{2\eta^8} + \frac{G}{4\eta^{11}}, \\
\gamma &= \frac{1}{4\eta^8}\rho^2\left[J^2(\rho^2 - 8z^2) + M(M^3 + 3M_2)(\rho^2 - 4z^2)\right] - \frac{M^2}{2\eta^4}\rho^2,
\end{aligned}
\tag{32}
$$

with

$$
\begin{aligned}
\eta &= \sqrt{\rho^2 + z^2}, \\
A &= 8\rho^2 z^2(24MJ^2 + 17M^2 M_2 + 21M_4) + \rho^4(7M^5 - 10MJ^2 + 32M^2 M_2 - 21M_4) \\
&\quad + 8z^4(-7M^5 + 20MJ^2 - 22M_2 M^2 - 7M_4), \\
B &= \rho^4(10M^2 J^2 + 10M^3 M_2 + 21MM_4 + 7M_2^2) \\
&\quad + 4z^4(7M^6 - 40M^2 J^2 - 14JS_3 + 30M^3 M_2 + 14MM_4 + 7M_2^2) \\
&\quad - 4\rho^2 z^2(7M^6 + 27J^2 M^2 - 21JS_3 + 48M^3 M_2 + 42MM_4 + 7M_2^2), \\
F &= \rho^4(S_3 - M^2 J) - 4\rho^2 z^2(M^2 J + S_3), \\
G &= \rho^2(-J^3(\rho^4 + 8z^4 - 12\rho^2 z^2) + MJ((M^3 + 2M_2)\rho^4 \\
&\quad - 8(3M^3 + 2M_2)z^4 + 4(M^3 + 10M_2)\rho^2 z^2) + M^2 S_3(3\rho^4 - 40z^4 + 12\rho^2 z^2)), \\
H &= 4\rho^2 z^2(J(M_2 - 2M^3) - 3MS_3) + \rho^4(JM_2 + 3MS_3).
\end{aligned}
\tag{33}
$$

To see which form has this metric in spherical-like coordinates, we use the Kerr mapping (2) with $a = 0$. Then, the function η^2 is

$$\eta^2 = \Delta + (M^2 - a^2)\cos^2\theta = r(r - 2M) + M^2 \cos^2\theta. \tag{34}$$

Substituting (34) in the metric functions (32) and expanding in Taylor series up to $\mathcal{O}(r^{-6})$, the metric potentials take the form

$$
\begin{aligned}
V &= f \\
&= 1 - 2Mu - 2M_2 P_2 u^3 - \frac{2}{3}\left[J^2(2P_2 + 1) + 3MM_2 P_2\right]u^4 \\
&\quad + \frac{1}{63}\left[-3MJ^2(14P_2^2 + 20P_2 + 14) - MM_2(70P_2^2 + 88P_2 - 14)\right. \\
&\quad \left. + 7M_4(-35P_2^2 + 10P_2 + 7)\right]u^5,
\end{aligned}
$$

$$
\begin{aligned}
W &= f\omega \\
&= \left[-2Ju - \frac{2}{3}S_3 u^3(5P_2 + 1) + \frac{1}{3}[3JM_2 - 5MS_3]u^4(5P_2 + 1)\right. \\
&\quad + \frac{1}{3}\left[J^3(-7P_2^2 + 8P_2 + 5) + 6MJM_2(-3P_2^2 + 7P_2 + 2)\right. \\
&\quad \left.\left. + M^2 S_3(7P_2^2 - 50P_2 - 11)\right]u^5\right]\sin^2\theta,
\end{aligned}
$$

$$
\begin{aligned}
X &= \frac{1}{f\Delta}[(r - M)^2\sin^2\theta + \Delta\cos^2\theta]\,e^{2\gamma} \\
&= 1 + 2Mu + 4M^2 u^2 + 2(4M^3 + M_2 P_2)u^3 \\
&\quad + \frac{2}{3}\left[24M^4 + 3J^2 P_2^2 + MM_2(5P_2^2 + 11P_2 - 1)\right]u^4 \\
&\quad + \frac{1}{63}\left[2016M^5 + 3MJ^2(266P_2^2 - 36P_2 - 14)\right. \\
&\quad \left. + M^2 M_2(1330P_2^2 + 1096P_2 - 266) + 7M_4(35P_2^2 - 10P_2 - 7)\right]u^5,
\end{aligned}
\tag{35}
$$

$$
\begin{aligned}
Y &= \frac{1}{f}[(r - M)^2\sin^2\theta + \Delta\cos^2\theta]\,e^{2\gamma} \\
&= r^2\left[1 + 2M_2 u^3 P_2 + \frac{2}{3}\left[3J^2 P_2^2 + MM_2(5P_2^2 + 5P_2 - 1)\right]u^4\right. \\
&\quad + \frac{1}{63}\left[3MJ^2(182P_2^2 - 36P_2 - 14) + M^2 M_2(910P_2^2 + 172P_2 - 182)\right. \\
&\quad \left.\left. + 7M_4(35P_2^2 - 10P_2 - 7)\right]u^5\right],
\end{aligned}
$$

$$
\begin{aligned}
Z &= \frac{\rho^2}{f} - f\omega^2 \\
&= r^2\sin^2\theta\left[1 + 2M_2 P_2 u^3 + 2(J^2(2P_2 - 1) + 3MM_2 P_2)u^4\right. \\
&\quad + \frac{1}{63}\left[3MJ^2(14P_2^2 + 188P_2 - 70) + M^2 M_2(70P_2^2 + 844P_2 - 14)\right. \\
&\quad \left.\left. - 7M_4(35P_2^2 - 10P_2 - 7)\right]u^5\right],
\end{aligned}
$$

where $u = 1/r$. By means of a REDUCE program, we checked that the metric potentials fulfill the EFE. According to Pappas the multipole structure of his spacetime is

$\mathcal{M}_{2n} = M$, M_2, M_4 for $n = 0$, 1, 2 and $\mathcal{S}_{2n+1} = S_1$, S_3 for $n = 0$, 1. The twist potential for the expanded Pappas spacetime is

$$\Omega = -\frac{1}{3}\left[6Ju^2 + S_3u^4(10P_2 - 4) + u^5\left(12JM_2P_2 + MS_3(20P_2 - 8)\right)\right]\cos\theta. \tag{36}$$

Using (35) and (36) in our program, the multipole structure is as expected

$$\begin{aligned}
\mathcal{M}_0 &= M, \\
\mathcal{S}_1 &= J, \\
\mathcal{M}_2 &= M_2, \\
\mathcal{S}_3 &= S_3, \\
\mathcal{M}_4 &= M_4.
\end{aligned} \tag{37}$$

Then, our metric (3) have not the same multipole structure, therefore they are not isometric. The multipole structures of the new HT and the Pappas metrics are similar. Setting $q = -M_2$ and rescaling S_3 in (30) both metrics become isometric. From (35), the Pappas metric contains the post-linear versions of the Schwarzschild, Erez–Rosen (up to MM_2) and the Lense–Thirring metrics. Our metric contains the Kerr metric and the post-linear version of the Erez–Rosen metric (up to second order in q).

6 Conclusions

We have found an approximate solution of the EFE by means of perturbing the Kerr metric. This approximate solution has five parameters, mass, angular momentum, mass quadrupole, spin octupole and mass hexadecapole. The mass quadrupole, spin octupole and mass hexadecapole were included perturbatively. It is valid up to the third order in these parameters.

By finding the twist scalar, we found the multipole structure employing the FHP formalism. It is possible to choose the multipole parameters, so that the first five multipole moments of a neutron star are similar. The simple form of our spacetime does it easy to implement computationaly. Including more relativistic multipole moments to our metric is easy using the procedure described here. Our metric presents an advantage over the Pappas metric, because it does not contain the Kerr spacetime as a limiting case.

Through a transformation we guessed an improved HT metric at first order in the spin octupole and mass hexadecapole. Solving the EFE, a new version of the HT, including mixed and cuadratic terms was also found. This is important, because the original HT can be matched with interior solutions and is used to validate spacetimes. The twist scalars and the multipole structures were also found for this improved HT and the Pappas metric. A comparison reveals that this HT metric is isometric with the Pappas metric after doing a transformation.

Our spacetime has potentialy many applications. It could be used to infer the properties of the structure of a neutron star from astrophysical observations. Another task is to find the ISCO as a function of the mass, mass quadrupole, mass hexadecapole, spin and spin octupole, as an extention of [4]. An interesting future work is to find an interior solution for our spacetime.

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