General Relativity and Gravitation (2019) 51:25
https://doi.org/10.1007/s10714-019-2512-7

RESEARCH ARTICLE

Gravitational shockwaves on rotating black holes

Yoni BenTov[1,2] · Joe Swearngin[3]

Received: 26 February 2018 / Accepted: 29 January 2019 / Published online: 4 February 2019

Abstract

We present an exact solution of Einstein's equation that describes the gravitational shockwave of a massless particle on the horizon of a Kerr–Newman black hole. The backreacted metric is of the generalized Kerr–Schild form and is Type II in the Petrov classification. We show that if the background frame is aligned with shear-free null geodesics, and if the background Ricci tensor satisfies a simple condition, then all nonlinearities in the perturbation will drop out of the curvature scalars. We make heavy use of the method of spin coefficients (the Newman–Penrose formalism) in its compacted form (the Geroch–Held–Penrose formalism).

Keywords Exact solution · Rotating black hole · Differential geometry · General relativity · Kerr-Newman metric · Dray–'t Hooft · Gravitational shockwave · Newman–Penrose formalism · Geroch–Held–Penrose formalism

Contents

✉ Yoni BenTov
bentov@caltech.edu; bentov@perimeterinstitute.ca

1 Present Address: Perimeter Institute for Theoretical Physics, Waterloo, ON N2L 2Y5, Canada

2 Institute for Quantum Information and Matter, California Institute of Technology, Pasadena, CA 91125, USA

3 Department of Physics, University of California, Santa Barbara, CA 93106, USA

1 Motivation

Black holes are thermodynamic systems whose microscopic description we still do not understand. After the original work on black hole thermodynamics by Christodoulou [1], Penrose and Floyd [2], Carter [3], Bekenstein [4], and Bardeen, Carter, and Hawking [5], Hawking justified the analogy between the surface gravity[1] α and a temperature T by predicting that an isolated black hole will radiate as a black body at the expected temperature $T = \frac{\alpha}{2\pi}$ [6,7]. About 20 years later, Strominger and Vafa vindicated the analogy between the horizon area A and an entropy S by enumerating microstates in string theory to derive the expected result $S = \frac{1}{4}A$ for extremal black holes in $4+1$ dimensions [8].

We will not recount the subsequent history of microstate counting. Suffice it to say that the calculations from string theory, while eminently laudable, are restricted to black holes near extremality and may not provide enough insight into the statistical

[1] We use "α" instead of the more conventional "κ" for surface gravity because "κ" has been commandeered by Newman and Penrose (see Sect. 2.4).

mechanics behind the conventional black holes of general relativity for generic values of their parameters. It would be helpful to establish a complementary strategy for black hole statistical mechanics tailored to an expansion around the Schwarzschild solution.

One such alternative is the S-matrix approach of 't Hooft [9,10]. Motivated by this and by Shenker and Stanford's investigation of the butterfly effect [11,12], Kitaev recently proposed a quantum field theory in $0+1$ dimensions [13] whose low-energy effective action is that of dilaton gravity in $1+1$ dimensions [14,15]. Details of this model were explored further by Maldacena and Stanford [16]. Since the equations of motion derived from the effective action admit the AdS_2 black hole as a solution [17], Kitaev's calculation demonstrates that the thermodynamic limit of a quantum mechanical model[2] can produce a bona fide black hole horizon, albeit in lower-dimensional scalar-tensor gravity, not in $(3+1)$-dimensional Einstein gravity.

Foundational to all of this is an exact solution of Einstein's equation that describes the gravitational backreaction of a massless particle on the future horizon of a Schwarzschild black hole: the *Dray–'t Hooft gravitational shockwave* [19].[3]

That solution was generalized to the Reissner–Nordström (RN) black hole by Alonso and Zamorano [22] and by Sfetsos [23], who also adapted the shockwave to other static backgrounds. Kiem, Verlinde, and Verlinde [24] used a perturbative variant of the Dray–'t Hooft result to see how gravitational interactions might affect black hole evaporation. And Polchinski [25] revisited the solution to refine 't Hooft's "relation between a given black hole S-matrix element and another with an additional ingoing particle," culminating in a reformulated argument for the firewall [26,27].

In his exposition of the S-matrix framework, 't Hooft did not concern himself with more general black hole backgrounds, opining that "[c]onceptually, generalization of everything we say to these cases should be straightforward" [10]. Perhaps, but in this paper our principal ambition is to galvanize the search for a statistical mechanics underlying astrophysical black holes [28], whose equilibrium field configurations are described by the Kerr geometry. So if we intend to adapt 't Hooft's blueprint and Kitaev's recent insights to the microscopics of rotating black holes, then our very first preliminary step must be to generalize 't Hooft's formula for the transition amplitude.

That is what we do here: We generalize the Dray–'t Hooft gravitational shockwave to the Kerr–Newman background, which is the most general asymptotically flat black hole in four spacetime dimensions. Readers familiar with gravitational shockwaves and the method of spin coefficients could skip to our metric ansatz described by Eqs. (3.1) and (3.7), and then to our main results: the Ricci tensor in Eq. (7.3), the Ricci scalar Φ_{22} in Eq. (7.11), and the differential operator in Eq. (7.12). We acknowledge that this provides only the most tentative intimation toward a microscopic theory of the Kerr–Newman spacetime, but it is a new exact solution of Einstein's equation and therefore deserves to be studied in its own right.

[2] As remarked by Witten, "the average of a quantum system over quenched disorder is not really a quantum system" [18]. Strictly speaking it is only a quantum mechanical model if the average captures the physics of a single realization with fixed couplings $J_{jk\ell m}$. We thank Yonah Lemonik for a discussion about this important point.

[3] This solution can be viewed as the generalization of the Aichelburg–Sexl shockwave [20] to curved spacetime or as an application of Penrose's "scissors-and-paste" method for gluing together known solutions of Einstein's equation to form new solutions [21].

Only late in our venture did we learn that Balasin generalized the Ricci tensor for the Dray–'t Hooft solution with the express aim of including rotation in the formalism [29].[4] But he did not complete the calculation, stating only that "it would be interesting to apply it to a rotating, i.e. Kerr black hole" and that "[w]ork in this direction is currently in progress." Similar comments were made by Alonso and Zamorano [22] and by Taub [30]. We have not found later articles by any of those authors that contain our results.

In Sect. 2, we review everything required to follow the calculation—those unfamiliar with null frames will likely have to supplement this with standard references like Chandrasekhar [31] and Penrose and Rindler [32]. In Sect. 3 we recast the Dray–'t Hooft geometry as a shift of the null frame, explain how to include rotation, and compute the spin coefficients for generalized Kerr–Schild metrics. In Sects. 4 and 5 we specialize to shear-free geodesic congruences and compute the shifted curvature scalars. Section 6 is where the heavy lifting begins: We engage the rotating shockwave and compute some preliminary identities for derivatives of the shift function. This leads to Sect. 7, where we complete the calculation and announce the differential equation for the shockwave's angular profile. We offer some closing thoughts in Sect. 8, and we explain in the appendix how to change metric signature from mostly-minus to mostly-plus.

2 The Kerr–Newman black hole

To enable the reader to work through this document, we will first describe the Kerr–Newman black hole using the method of spin coefficients.

This method was invented by Newman and Penrose (NP) [33] and refined into a "compacted" version by Geroch, Held, and Penrose (GHP) [34], a refinement that has since fallen by the wayside but that we found indispensable. Beside our primary aim of generalizing the gravitational shockwave, our secondary aim is to provide a detailed example of how to use the formalism. As far as rotating black holes are concerned, the flip side of the method of spin coefficients is the madness without it.

2.1 Null frame

Our account of the spacetime will begin with a collection of *frame field 1-forms*

$$e^a \equiv e^a_\mu \, dx^\mu \equiv (-l', \, -l, \, m', \, m), \tag{2.1}$$

in terms of which the line element is

$$ds^2 = -2ll' + 2mm'. \tag{2.2}$$

[4] We found Balasin's paper after we had already computed the Ricci tensor but before we managed to express it in the relatively compact and geometrical form described by Eqs. (7.3), (7.11), and (7.12).

A tactical advantage of deploying a frame formulation is to never have to look at a line element, so we will not show ds^2 explicitly—we will always work directly with the frame. To gain our footing we will start with the "Schwarzschild-like" coordinates (t, r, θ, φ) of Boyer and Lindquist [35], which are applicable outside the black hole.

Kerr–Newman black holes have a mass M, a charge Q, and an angular momentum J. It is customary to trade J for the ratio $a \equiv J/M$ and to define the "horizon function" [31]

$$\Delta \equiv r^2 - 2Mr + a^2 + Q^2 \equiv (r - r_+)(r - r_-). \tag{2.3}$$

The inner horizon $r_- \equiv M - \sqrt{M^2 - a^2 - Q^2}$ and the outer horizon $r_+ \equiv M + \sqrt{M^2 - a^2 - Q^2}$ are defined as the solutions to $\Delta = 0$. It is useful to note that $M = \frac{1}{2}(r_+ + r_-)$ and $||(a, Q)|| \equiv (a^2 + Q^2)^{1/2} = (r_+ r_-)^{1/2}$.

We will be concerned exclusively with the region $r \geq r_+$, so when we refer to "the" horizon, we will always mean the outer one.

Since time immemorial Newman has emphasized that rotating black holes are "complex translations" of nonrotating ones [36]. Regardless of whether that means anything, it is convenient to define the complex functions

$$R \equiv r + ia\cos\theta, \quad R_0 \equiv r + ia. \tag{2.4}$$

In the above notation, the following null 1-forms describe the Kerr–Newman black hole:

$$\begin{aligned} l &= -dt + \frac{|R|^2}{\Delta} dr + a\,\sin^2\theta\, d\varphi, \quad l' = \frac{\Delta}{2|R|^2}\left(-dt - \frac{|R|^2}{\Delta} dr + a\,\sin^2\theta\, d\varphi\right), \\ m &= \frac{1}{R\sqrt{2}}\left(|R|^2\, d\theta + i|R_0|^2 \sin\theta\, d\varphi - ia\,\sin\theta\, dt\right), \quad m' = m^*. \end{aligned} \tag{2.5}$$

Given those 1-forms, we solve the matrix inversion problem

$$e_\mu^a e_a^\nu \equiv \delta_\mu^\nu, \quad e_a^\mu e_\mu^b \equiv \delta_a^b \tag{2.6}$$

for the vectors $e_a^\mu \equiv (l^\mu, l'^\mu, m^\mu, m'^\mu)$. By royal mandate we then introduce the *Newman–Penrose directional derivatives*:

$$D \equiv l^\mu \nabla_\mu, \quad D' \equiv l'^\mu \nabla_\mu, \quad \delta \equiv m^\mu \nabla_\mu, \quad \delta' \equiv m'^\mu \nabla_\mu. \tag{2.7}$$

Without loss of generality we can replace the covariant derivatives by partial derivatives and treat the operators D, D', δ, δ' as ordinary vector fields.[5] In Schwarzschild-like coordinates, we have:

[5] Once the equations of differential geometry are cast in spin coefficient form, all of the dynamical variables will be invariant under coordinate transformations on the base space, thereby becoming scalar fields.

$$D = l^\mu \partial_\mu = \frac{|R_0|^2}{\Delta}\partial_t + \partial_r + \frac{a}{\Delta}\partial_\varphi, \quad D' = l'^\mu \partial_\mu = \frac{\Delta}{2|R|^2}\left(\frac{|R_0|^2}{\Delta}\partial_t - \partial_r + \frac{a}{\Delta}\partial_\varphi\right),$$
$$\delta = m^\mu \partial_\mu = \frac{1}{R\sqrt{2}}\left(\partial_\theta + \frac{i}{\sin\theta}\,\partial_\varphi + ia\,\sin\theta\,\partial_t\right), \quad \delta' = \delta^*. \tag{2.8}$$

We will refer to the forms in Eq. (2.5) and the vectors in Eq. (2.8) as the "standard" frame. Its ubiquity derives from its utility: It is a principal basis (see Sect. 2.11) whose outgoing and ingoing null congruences are geodesic, twisting, and shear-free [see Eq. (2.33)]. Students acquainted with Reissner–Nordström but hesitant about Kerr–Newman should fiddle with the standard frame until the geometry feels less foreign.

2.2 Spin coefficients

There are two ways to express the classical field theory of gravity, distinguished by whether local invariance under $SO(3,1)$ is *imposed* or *inferred*. Drastically oversimplifying a complicated history, we will say that the former is Cartan's approach, while the latter is Einstein's.[6]

We favor the former. First introduce a frame e^a_μ and demand invariance of the action under local $SO(3,1)$ transformations:

$$e^a(x) \to O^a{}_b(x)\, e^b(x), \quad O^a{}_c(x)\, O^b{}_d(x)\, \eta_{ab} \equiv \eta_{cd}. \tag{2.9}$$

Then introduce an $SO(3,1)$ gauge field $\omega^a{}_b$, called the spin connection, to turn ordinary derivatives into covariant derivatives. As for any nonabelian gauge field, the required transformation law is

$$\omega^a{}_b(x) \to O^a{}_c(x)\left(\delta^c{}_d\, d + \omega^c{}_d\right)(O^{-1})^d{}_b(x). \tag{2.10}$$

By birthright the spin connection is antisymmetric:

$$\omega_{ab} = -\omega_{ba}. \tag{2.11}$$

The variables $e^a(x)$ and $\omega^a{}_b(x)$ are the independent classical fields in the action. Because we find it productive to work entirely within the internal, we follow Newman and Penrose and define the *spin coefficients* [33]

$$\gamma_{abc} \equiv (\omega_\mu)_{ab}\, e^\mu_c. \tag{2.12}$$

Varying the action with respect to the spin connection in a world without fermions implies the torsion-free condition

$$de_a = \gamma_{abc}\, e^b \wedge e^c. \tag{2.13}$$

[6] Penrose and Rindler [32] refer to what we call "Cartan's approach" as the "Einstein–Cartan–Sciama–Kibble theory" (see their Sect. 4.7).

Solving this gives the spin coefficients in terms of the frame:

$$\gamma_{abc} = \tfrac{1}{2}(\lambda_{abc} + \lambda_{cab} - \lambda_{bca}), \quad \lambda_{abc} \equiv -(e_a^{\mu} e_c^{\nu} - e_c^{\mu} e_a^{\nu})\partial_{\mu} e_{b\nu}. \tag{2.14}$$

While this expression is standard, the path to it depends on one's taste in formalism.

2.3 Partial gauge fixing

After Newman and Penrose invented the method of spin coefficients, Geroch, Held, and Penrose recognized that specifying a frame $e_a^{\mu} = (l^{\mu}, l'^{\mu}, m^{\mu}, m'^{\mu})$ that satisfies the normalization conditions in Eq. (2.6) only partially fixes the gauge in $SO(3,1)$.

The remaining ambiguity comprises a boost along the outgoing congruence, the corresponding inverse boost along the ingoing congruence, and a rotation of the transverse plane:

$$l^{\mu} \to r(x)\, l^{\mu}, \quad l'^{\mu} \to \frac{1}{r(x)}\, l'^{\mu}, \quad m^{\mu} \to e^{i\vartheta(x)}\, m^{\mu}, \quad m'^{\mu} \to e^{-i\vartheta(x)}\, m'^{\mu}. \tag{2.15}$$

We will say that this transformation generates the *GHP group*. It is convenient to define the complex function

$$\lambda \equiv r^{1/2} e^{i\vartheta/2} \tag{2.16}$$

and to rewrite Eq. (2.15) as

$$l^{\mu} \to \lambda\lambda^{*}\, l^{\mu}, \quad l'^{\mu} \to \lambda^{-1}\lambda^{*-1}\, l'^{\mu}, \quad m^{\mu} \to \lambda\lambda^{*-1}\, m^{\mu}, \quad m'^{\mu} \to \lambda^{-1}\lambda^{*}\, m'^{\mu}. \tag{2.17}$$

We will say that a function $f_{h,\bar{h}}$ transforms as the representation[7] $(h, \bar{h})$ of the GHP group if its transformation law under Eq. (2.17) has the form:

$$f_{h,\bar{h}} \to \lambda^{2h}\lambda^{*2\bar{h}} f_{h,\bar{h}}. \tag{2.18}$$

As shorthand for this, we will use the standard notation of representation theory:

$$f_{h,\bar{h}} \sim (h, \bar{h}). \tag{2.19}$$

The numbers $(h, \bar{h})$ are called the weights[8] of the function $f_{h,\bar{h}}$, and such a function is accordingly said to be "weighted." Borrowing group-theoretic jargon from field theory, we will say that weighted quantities *transform as matter fields*. An object that

[7] The bar is part of the name of the weight and does not denote any sort of conjugation.

[8] Penrose and Rindler define $p \equiv 2h$ and $q \equiv 2\bar{h}$. Either way, the "boost weight" and the "spin weight" are defined as $\frac{1}{2}(p+q) = h + \bar{h}$ and $\frac{1}{2}(p-q) = h - \bar{h}$ respectively [34].

cannot be assigned a transformation law of the form in Eq. (2.18) for any values of $(h, \bar{h})$ will be called "nonweighted."[9] In the language of Eq. (2.19), we summarize Eq. (2.17) as

$$l^\mu \sim \left(\tfrac{1}{2}, \tfrac{1}{2}\right), \quad l'^\mu \sim \left(-\tfrac{1}{2}, -\tfrac{1}{2}\right), \quad m^\mu \sim \left(\tfrac{1}{2}, -\tfrac{1}{2}\right), \quad m'^\mu \sim \left(-\tfrac{1}{2}, \tfrac{1}{2}\right). \tag{2.20}$$

Manifest covariance under the GHP group is what defines the compacted formalism: All explicitly written quantities transform according to Eq. (2.18) for some values of h and $\bar{h}$. Only objects with the same weights can be added, and the weights of a product of objects are the sums of the weights of each object:

$$f_{h_1,\bar{h}_1} \sim (h_1, \bar{h}_1), \quad g_{h_2,\bar{h}_2} \sim (h_2, \bar{h}_2) \implies f_{h_1,\bar{h}_1} g_{h_2,\bar{h}_2} \sim (h_1 + h_2, \bar{h}_1 + \bar{h}_2). \tag{2.21}$$

From Eq. (2.18) we deduce that complex conjugation exchanges the weights:

$$f_{h,\bar{h}} \sim (h, \bar{h}) \implies (f_{h,\bar{h}})^* \sim (\bar{h}, h). \tag{2.22}$$

Beside complex conjugation, there are two discrete transformations under which the compacted formalism is covariant. The first is the *priming* transformation, which is defined to exchange primed and unprimed quantities:

$$l_\mu \leftrightarrow l'_\mu, \quad m_\mu \leftrightarrow m'_\mu. \tag{2.23}$$

In this way the notation from Eq. (2.1) becomes an operation. From Eq. (2.17) we deduce that priming flips the signs of the weights:

$$f_{h,\bar{h}} \sim (h, \bar{h}) \implies (f_{h,\bar{h}})' \sim (-h, -\bar{h}). \tag{2.24}$$

The second discrete transformation is the *Sachs operation*, which is an analog of Hodge duality:

$$(l_\mu, l'_\mu, m_\mu, m'_\mu) \to (m_\mu, -m'_\mu, -l_\mu, l'_\mu). \tag{2.25}$$

Unlike priming, the Sachs operation does not commute with complex conjugation. It is extremely convenient to streamline the spin coefficient formalism by using a notation that is manifestly covariant under priming. The Sachs operation will instead help us establish geometrical meaning.

2.4 Matter fields and gauge fields

Based on their behavior under Eq. (2.17), the 12 independent γ_{abc} fall naturally into three sets: weighted quantities associated with l_μ, weighted quantities associated with l'_μ, and nonweighted quantities that transform as gauge fields.

[9] Something invariant under Eq. (2.18) is considered to be weighted with weight zero, not nonweighted.

The weighted spin coefficients associated with l_μ, along with their weights, are

$$\kappa \equiv \gamma_{311} \sim \left(\tfrac{3}{2}, \tfrac{1}{2}\right), \quad \tau \equiv \gamma_{312} \sim \left(\tfrac{1}{2}, -\tfrac{1}{2}\right),$$
$$\sigma \equiv \gamma_{313} \sim \left(\tfrac{3}{2}, -\tfrac{1}{2}\right), \quad \rho \equiv \gamma_{314} \sim \left(\tfrac{1}{2}, \tfrac{1}{2}\right). \tag{2.26}$$

The weighted spin coefficients associated with l'_μ are defined by priming, which flips the signs of the weights[10]:

$$\kappa' \equiv \gamma_{422} \sim \left(-\tfrac{3}{2}, -\tfrac{1}{2}\right), \quad \tau' \equiv \gamma_{421} \sim \left(-\tfrac{1}{2}, \tfrac{1}{2}\right),$$
$$\sigma' \equiv \gamma_{424} \sim \left(-\tfrac{3}{2}, \tfrac{1}{2}\right), \quad \rho' \equiv \gamma_{423} \sim \left(-\tfrac{1}{2}, -\tfrac{1}{2}\right). \tag{2.27}$$

The gauge fields of the spin coefficient formalism are defined as

$$\varepsilon \equiv \tfrac{1}{2}(-\gamma_{121} + \gamma_{341}), \quad \beta \equiv \tfrac{1}{2}(-\gamma_{123} + \gamma_{343}),$$
$$\varepsilon' \equiv \tfrac{1}{2}(-\gamma_{212} + \gamma_{432}), \quad \beta' \equiv \tfrac{1}{2}(-\gamma_{214} + \gamma_{434}). \tag{2.28}$$

These are gauge fields in the sense that they combine with the NP derivatives of Eq. (2.8) to form weighted derivatives:

$$þ \equiv D + 2h\,\varepsilon + 2\bar{h}\,\varepsilon^*, \quad ð \equiv \delta + 2h\,\beta - 2\bar{h}\,\beta'^*,$$
$$þ' \equiv D' - 2h\,\varepsilon' - 2\bar{h}\,\varepsilon'^*, \quad ð' \equiv \delta' - 2h\,\beta' + 2\bar{h}\,\beta^*. \tag{2.29}$$

We will refer to the operators þ, þ′, ð, and ð′ as *GHP-covariant derivatives*. Typically the covariant derivative of a matter field transforms as the same representation as the field itself, but not so here. For a weighted function $f_{h,\bar{h}} \sim (h, \bar{h})$, we have:

$$þ f_{h,\bar{h}} \sim (h + \tfrac{1}{2}, \bar{h} + \tfrac{1}{2}), \quad þ' f_{h,\bar{h}} \sim (h - \tfrac{1}{2}, \bar{h} - \tfrac{1}{2}),$$
$$ð f_{h,\bar{h}} \sim (h + \tfrac{1}{2}, \bar{h} - \tfrac{1}{2}), \quad ð' f_{h,\bar{h}} \sim (h - \tfrac{1}{2}, \bar{h} + \tfrac{1}{2}). \tag{2.30}$$

Evidently the covariant derivatives themselves carry charge:

$$þ \sim \left(\tfrac{1}{2}, \tfrac{1}{2}\right), \quad þ' \sim \left(-\tfrac{1}{2}, -\tfrac{1}{2}\right), \quad ð \sim \left(\tfrac{1}{2}, -\tfrac{1}{2}\right), \quad ð' \sim \left(-\tfrac{1}{2}, \tfrac{1}{2}\right). \tag{2.31}$$

2.5 Null Cartan equations

Expressed in the NP hieroglyphs of Eqs. (2.26)–(2.28), the torsion-free condition of Eq. (2.13) becomes four fundamental relations:

[10] Priming acts on the $SO\,(3,1)$ indices by exchanging $1 \leftrightarrow 2$ and $3 \leftrightarrow 4$. Complex conjugation leaves 1 and 2 fixed while exchanging $3 \leftrightarrow 4$.

$$\begin{aligned} dl &= -2\,\mathrm{Re}(\varepsilon)\, l \wedge l' + 2i\,\mathrm{Im}(\rho)\, m \wedge m' \\ &\quad + \left[\left(\tau - \beta + \beta'^*\right) m' \wedge l + \kappa\, m' \wedge l' + c.c. \right], \\ dm &= (\beta + \beta'^*)\, m \wedge m' - (\tau - \tau'^*)\, l \wedge l' \\ &\quad + \left[(\rho - 2i\,\mathrm{Im}(\varepsilon))\; m \wedge l' + \sigma\, m' \wedge l' + c.c.' \right], \end{aligned} \tag{2.32}$$

and their primes. We will call these the *null Cartan equations*.

By computing the exterior derivatives of the forms in Eq. (2.5), arranging them to match the right-hand sides in Eq. (2.32), and solving the resulting equations, we can find the Kerr–Newman spin coefficients:

$$\begin{aligned} &\kappa = \kappa' = \sigma = \sigma' = 0, \quad \rho = \frac{1}{R^*}, \quad \rho' = -\frac{\Delta}{2|R|^2}\,\frac{1}{R^*}, \\ &\tau = \frac{ia\,\sin\theta}{\sqrt{2}|R|^2}, \quad \tau' = \frac{ia\,\sin\theta}{\sqrt{2}(R^*)^2}, \\ &\varepsilon = 0,\;\; \varepsilon' = \rho' + \frac{2r - r_+ - r_-}{4|R|^2}, \\ &\beta = -\frac{\cot\theta}{2\sqrt{2}R}, \quad \beta' = \tau' + \beta^*. \end{aligned} \tag{2.33}$$

Because of their noncovariance under Eq. (2.17), the above ε' and β' should be understood strictly numerically. Also note that $|\tau|^2 = |\tau'|^2$, which will be useful later.

2.6 Timelike expansion and timelike twist

Every bard recounts legends of refraction (κ), expansion (Re ρ), twist (Im ρ), and shear (σ), but nary a soul tells tales of τ.[11]

We would like to elevate the standing of τ and τ' to match the renown of their colleagues, because these neglected spin coefficients convey the relativistic effects of rotating bodies at least as directly as $\mathrm{Im}(\rho)$ and $\mathrm{Im}(\rho')$ do—a cursory assessment of Eq. (2.33), for instance, reveals the suggestive factor $a\sin\theta$. Our North Star will be the Sachs operation of Eq. (2.25).

The combinations $\tau \pm \tau'^*$, rather than τ and τ' separately, will appear front and center in the subsequent analysis, so let us consider their meaning and christen them with appropriate names. Sachs conjugation of the expansion and twist provides:

[11] Sachs, who pioneered the optical analogy for the spin coefficients, does not explain τ or τ' in his original paper [37]. Szekeres, in the paper from which we extracted the term "refraction" for κ, calls the spin coefficient τ (which he denotes Ω) the "*angular velocity* or *rotation* of the null congruence," but he does not explain why [38]. In a subsequent lecture, Sachs seems to have implicitly recognized this interpretation of τ by also choosing the symbol Ω to denote it, but he does not justify the notation [39]. An appraisal of the null Cartan equations within the formal context of lightcone kinematics as originally articulated by Dirac [40] affirms this interpretation but with τ and τ' switched.

$$\mathrm{Re}(\rho) \equiv \tfrac{1}{2}(\rho + \rho^*) \to \tfrac{1}{2}(\tau + \tau'^*) = -\frac{a^2 \sin(2\theta)}{2\sqrt{2}|R|^2 R},$$
$$\mathrm{Im}(\rho) \equiv \tfrac{1}{2i}(\rho - \rho^*) \to \tfrac{1}{2i}(\tau - \tau'^*) = \frac{ra \sin\theta}{\sqrt{2}|R|^2 R}\,. \quad (2.34)$$

Consequently, we will refer to $\tau + \tau'^*$ and $\tau - \tau'^*$ as the *timelike expansion* and *timelike twist*.

Even though we performed the Sachs operation on spin coefficients associated with l^μ, the result involved both τ and τ'. While this may be jarring at first sight, GHP covariance requires it: The spin coefficients ρ and ρ^* have the same weights and therefore can be added and subtracted at will, but τ and τ^* transform differently under Eq. (2.17). Only τ and τ'^* can be added and subtracted.

2.7 Kruskal-like coordinates

To put all this formalism to work, we will need to forge Kruskal-like coordinates. First recall the known result for the surface gravity:

$$\alpha = \frac{r_+ - r_-}{2(r_+^2 + a^2)}. \quad (2.35)$$

With that we define the null coordinates U and V outside the black hole:

$$U \equiv -e^{-\alpha u}, \quad V \equiv +e^{+\alpha v}, \quad u \equiv t - r_*, \quad v \equiv t + r_*, \quad dr_* \equiv \frac{|R_0|^2}{\Delta} dr. \quad (2.36)$$

Note that $U < 0$, which is the standard convention. We choose the integration constant in the tortoise coordinate r_* such that the product of U and V is[12]

$$UV = -\frac{\Delta}{r_+ r_-}\left(\frac{r}{r_-} - 1\right)^{-k} e^{2\alpha r}, \quad k \equiv \frac{r_-^2 + a^2}{r_+^2 + a^2} + 1. \quad (2.37)$$

Considered an implicitly defined function of U and V, the coordinate r retains its desirable property from the nonrotating case of depending only on the product UV. As written in Eq. (2.37), the ratio $\frac{\Delta}{UV}$ is manifestly finite and nonzero at $r = r_+$:

$$c \equiv -\frac{\Delta}{UV}\bigg|_{r=r_+} = r_+ r_- \left(\frac{r_+}{r_-} - 1\right)^k e^{-2\alpha r_+}. \quad (2.38)$$

[12] Since we always work with $r > r_-$, we have dropped the absolute values that emerge from integrating dr_*. Our coordinates are singular at the inner horizon, and a different set of Kruskal-like coordinates must be established to cross it.

For later convenience, we also differentiate both sides of Eq. (2.37) and rearrange to solve for the partial derivatives of $r(U, V)$:

$$U\partial_U r = V\partial_V r = \frac{\Delta}{\Delta'(r) + \left(2\alpha - \frac{k}{r-r_-}\right)\Delta}. \tag{2.39}$$

For any function $F(r)$ that depends only on the radial coordinate, we therefore have:

$$U\partial_U F(r) = V\partial_V F(r) = F'(r)\, U\partial_U r \quad \text{and} \quad U\partial_U r|_{r=r_+} = 0. \tag{2.40}$$

We will sometimes use a subscript "+" to label quantities evaluated at the horizon. For instance, $|R_+|^2 \equiv r_+^2 + a^2\cos^2\theta$ and $|R_{0+}|^2 \equiv r_+^2 + a^2$.

Finally, we define the delayed angular coordinate and the angular velocity at the horizon:

$$\chi \equiv \varphi - \Omega_H t, \quad \Omega_H = \frac{a}{r_+^2 + a^2}. \tag{2.41}$$

2.8 A smooth frame

Smooth coordinates are not enough—we also need a smooth frame. From the standard basis written in Kruskal-like coordinates, we perform the following GHP transformation:

$$l_\mu \to \hat{l}_\mu = -U l_\mu, \quad l'_\mu \to \hat{l}'_\mu = -U^{-1} l'_\mu, \quad m_\mu \to \hat{m}_\mu = m_\mu. \tag{2.42}$$

This describes the special case

$$\lambda = \lambda^* = (-U)^{1/2} \tag{2.43}$$

of the transformation in Eq. (2.17). A hatted function with weights $(h, \bar{h})$ is then related to its unhatted counterpart by

$$\hat{f}_{h,\bar{h}} = (-U)^{h+\bar{h}} f_{h,\bar{h}}. \tag{2.44}$$

The spin coefficients ρ and ρ' in the hatted basis,

$$\hat{\rho} = \frac{1}{R^*}(-U) \quad \text{and} \quad \hat{\rho}' = \frac{1}{2|R|^2}\left(\frac{\Delta}{UV}\right)\frac{1}{R^*}V, \tag{2.45}$$

go to zero at the future horizon ($U = 0$) and the past horizon ($V = 0$) respectively. These furnish local definitions for each part of the horizon.

Because $\tau \sim (\frac{1}{2}, -\frac{1}{2})$ and $\tau' \sim (-\frac{1}{2}, \frac{1}{2})$, those two spin coefficients are invariant under the rescaling in Eq. (2.42):

$$\hat{\tau} = \tau, \quad \hat{\tau}' = \tau'. \tag{2.46}$$

After changing coordinates from (t, r, θ, φ) to (U, V, θ, χ) and applying Eq. (2.42), we obtain the following frame field 1-forms:

$$
\begin{aligned}
\hat{l} &= \frac{-1}{2\alpha}\left(1 + \frac{|R|^2}{|R_0|^2} - \Omega_H\, a \sin^2\theta\right) dU \\
&\quad - \frac{U}{V}\left(1 - \frac{|R_{0+}|^2}{|R_0|^2}\right)\frac{a^2 \sin^2\theta}{2\alpha|R_{0+}|^2}\, dV - U a \sin^2\theta\, d\chi, \\
\hat{l}' &= \frac{\Delta}{2|R|^2}\left[\frac{1}{2\alpha}\left(1 + \frac{|R|^2}{|R_0|^2} - \Omega_H\, a \sin^2\theta\right)\frac{dV}{UV}\right. \\
&\quad \left. + \frac{1}{U^2}\left(1 - \frac{|R_{0+}|^2}{|R_0|^2}\right)\frac{a^2\sin^2\theta}{2\alpha|R_{0+}|^2}\, dU - \frac{a\sin^2\theta}{U}\, d\chi\right], \\
\hat{m} &= \frac{1}{R\sqrt{2}}\left[|R|^2\, d\theta + i|R_0|^2 \sin\theta\, d\chi\right. \\
&\quad \left. + \frac{ia\,\sin\theta}{2\alpha|R_{0+}|^2}\,\frac{r + r_+}{r - r_-}\,\frac{\Delta}{UV}\,(U\, dV - V\, dU)\right]. \qquad (2.47)
\end{aligned}
$$

The corresponding directional derivatives are

$$
\begin{aligned}
\hat{D} &= -2\alpha\,|R_0|^2 \frac{UV}{\Delta}\,\partial_V - a\,\frac{U}{\Delta}\left(1 - \frac{|R_0|^2}{a}\Omega_H\right)\partial_\chi, \\
\hat{D}' &= \frac{\Delta}{2|R|^2}\left[2\alpha\,\frac{|R_0|^2}{\Delta}\,\partial_U - \frac{a}{U\Delta}\left(1 - \frac{|R_0|^2}{a}\Omega_H\right)\partial_\chi\right], \\
\hat{\delta} &= \frac{1}{R\sqrt{2}}\left[\partial_\theta + \frac{i}{\sin\theta}\,\frac{|R_+|^2}{|R_{0+}|^2}\,\partial_\chi + i\alpha\, a\sin\theta\,(-U\,\partial_U + V\,\partial_V)\right]. \qquad (2.48)
\end{aligned}
$$

We will refer to the forms in Eq. (2.47) and the vectors in Eq. (2.48) as the "horizon" frame (or simply as the "hatted" one). Each component of the 1-forms in Eq. (2.47) and of the vectors in Eq. (2.48) is finite at $U = 0$ for fixed V, and at $V = 0$ for fixed U.

2.9 Spacelike and timelike curvatures

Commutators of covariant derivatives beget curvature. By composing GHP derivatives on a test function $\xi_h \sim (h, 0)$, we define the *spacelike and timelike curvatures* $\mathcal{K}$ and $\mathcal{K}_s$:

$$
\begin{aligned}
\mathcal{K}\,\xi_h &\equiv -\frac{1}{2h}\left([\eth, \eth'] + 2i\,\mathrm{Im}(\rho)\text{þ}' - 2i\,\mathrm{Im}(\rho')\text{þ}\right)\xi_h, \\
\mathcal{K}_s\,\xi_h &\equiv \frac{1}{2h}\left([\text{þ}, \text{þ}'] + (\tau - \tau'^*)\eth' + (\tau^* - \tau')\eth\right)\xi_h. \qquad (2.49)
\end{aligned}
$$

Twice the real part of $\mathcal{K}$ is the ordinary notion of intrinsic (or "Gaussian") curvature in Riemannian geometry. The imaginary part is an extrinsic quantity that we will call the extrinsic curvature.[13] For Kerr–Newman, the intrinsic and extrinsic curvatures are

$$\mathrm{Re}(\mathcal{K}) = \frac{1}{2|R|^6}\Big\{r^2(r^2+a^2) + \Big[(r_+^2+a^2) - 4(r_+r+a^2) - (r^2-r_+^2)\Big]\,a^2\cos^2\theta \\ +4\,\alpha\,(r_+^2+a^2)(r-r_+)\,a^2\cos^2\theta\Big\} \tag{2.50}$$

and

$$\mathrm{Im}(\mathcal{K}) = \frac{a\cos\theta}{|R|^6}\Big\{(r-r_+)\,r_+\,r + (2a^2+r^2)\,r - r_+a^2\cos^2\theta \\ +\alpha(r_+^2+a^2)\Big[(2r_+-r)\,r + a^2\cos^2\theta\Big]\Big\}. \tag{2.51}$$

At the horizon, the intrinsic curvature is [41]

$$\mathrm{Re}(\mathcal{K})|_{r=r_+} = \frac{|R_{0+}|^2}{2|R_+|^6}(r_+^2 - 3a^2\cos^2\theta). \tag{2.52}$$

Only at $r = r_+$ should the denominations "intrinsic" and "extrinsic" be taken literally, because only there do m^μ and m'^μ form a surface. In contrast, the real and imaginary parts of $\mathcal{K}_s$ can never be interpreted that way, because l^μ and l'^μ never form a surface.[14] So we leave the Kerr–Newman timelike curvature as a complex quantity:

$$\mathcal{K}_s = \frac{-2(3r_+-2r)rr_+ + 2iar_+(4r-r_+)\cos\theta + a^2[5r-2r_+-(2r_+-r)\cos(2\theta)] + 2ia^3\cos^3\theta}{4R^*|R|^4} \\ +\alpha\,(r_+^2+a^2)\,\frac{(3r_+-r)r - ia(2r-r_+)\cos\theta + a^2\cos^2\theta}{R^*|R|^4}. \tag{2.53}$$

Next we will summarize those remaining aspects of curvature that are pertinent but more or less standard.

2.10 Curvature scalars

The Riemann tensor in the NP frame is

$$R_{abcd} = \partial_c\gamma_{abd} - \partial_d\gamma_{abc} - \gamma_{ab}{}^{e}(\gamma_{ced} - \gamma_{dec}) + \gamma_{aec}\gamma^{e}{}_{bd} - \gamma_{aed}\gamma^{e}{}_{bc}. \tag{2.54}$$

[13] This is not to be conflated with what numerical relativists call the extrinsic curvature, which is part of the spin connection. See, for example, the discussion of contorted surfaces on p. 400 of *Spinors and Spacetime* [32].

[14] Take the hatted basis and consider the commutators of covariant derivatives on a test function of weight (0, 0): We have $[\hat{ð}, \hat{ð}'] = (\hat{\rho} - \hat{\rho}^*)\hat{þ}' - (\hat{\rho}' - \hat{\rho}'^*)\hat{þ}$ and $[\hat{þ}, \hat{þ}'] = (\hat{\tau} - \hat{\tau}'^*)\hat{ð}' + (\hat{\tau}^* - \hat{\tau}')\hat{ð}$. The right-hand side of the former vanishes at $U = V = 0$, while the right-hand side of the latter never vanishes except at the poles. We thank Leo Stein for emphasizing this to us.

From the corresponding Ricci tensor, $R_{ab} \equiv R^c{}_{acb}$, Newman and Penrose define a traceless matrix

$$\phi_{ab} \equiv \tfrac{1}{2}(R_{ab} - \tfrac{1}{4}\eta_{ab}\,\eta^{cd} R_{cd}). \tag{2.55}$$

For spinorial reasons of no concern to us, they then define the *Ricci scalars* as

$$\begin{aligned}
&\Phi_{00} \equiv \phi_{11} \sim (1,1), \quad \Phi_{01} \equiv \phi_{13} \sim (1,0), \quad \Phi_{02} \equiv \phi_{33} \sim (1,-1),\\
&\Phi_{22} \equiv \Phi'_{00} = \phi_{22} \sim (-1,-1), \quad \Phi_{21} \equiv \Phi'_{01} = \phi_{24} \sim (-1,0),\\
&\Phi_{20} \equiv \Phi'_{02} = \phi_{44} = \Phi^*_{02} \sim (-1,1), \quad \Phi_{10} \equiv \Phi^*_{01} = \phi_{14} \sim (0,1),\\
&\Phi_{12} \equiv \Phi^*_{21} = \phi_{23} \sim (0,-1), \quad \Phi_{11} \equiv \tfrac{1}{2}(\phi_{12} + \phi_{34}) \sim (0,0).
\end{aligned} \tag{2.56}$$

In the notation of the compacted formalism, we have[15]

$$\begin{aligned}
\Phi_{00} &= -\text{þ}\rho - \rho^2 - |\sigma|^2 + \eth'\kappa + \tau'\kappa + \tau\,\kappa^*,\\
\Phi_{02} &= -\eth\tau - \tau^2 - \kappa\kappa'^* + \text{þ}'\sigma + \rho'\sigma + \rho\,\sigma'^*,
\end{aligned} \tag{2.57}$$

$$\begin{aligned}
\Phi_{01} = \tfrac{1}{2}\big[&-\text{þ}\tau + \text{þ}'\kappa - \eth\rho + \eth'\sigma - (\tau - \tau'^*)\rho - (\tau^* - \tau')\sigma\\
&- (\rho - \rho^*)\tau + (\rho' - \rho'^*)\kappa\big].
\end{aligned} \tag{2.58}$$

The remaining Ricci scalars of nonzero weight can be defined by priming and conjugating the definitions already listed: $\Phi_{22} = \Phi'_{00}$, $\Phi_{21} = \Phi'_{01}$, $\Phi_{10} = \Phi^*_{01}$, $\Phi_{12} = \Phi^*_{21}$, and $\Phi_{20} = \Phi^*_{02}$. Meanwhile, the Ricci scalar of weight $(0,0)$ is defined in terms of the spacelike and timelike curvatures:

$$\Phi_{11} = \tfrac{1}{2}\left(\mathcal{K} - \mathcal{K}_s - \kappa\kappa' + \tau\tau' - \sigma\sigma' + \rho\rho'\right). \tag{2.59}$$

Tradition compels a fanciful notation for a factor times the trace of the Ricci tensor:

$$\Pi \equiv \tfrac{1}{12}(R_{12} - R_{34}) = -\tfrac{1}{24}\eta^{ab} R_{ab}. \tag{2.60}$$

Because of its role as the gravitational Lagrangian, we refer to this as the *Einstein–Hilbert curvature*. In GHP notation, it reads

$$\Pi = \tfrac{1}{6}\left[2\left(\rho\rho'^* - |\tau|^2 + \text{þ}'\rho - \eth'\tau\right) + \mathcal{K} + \mathcal{K}_s - \kappa\kappa' - \tau\tau' + \sigma\sigma' + \rho\rho'\right]. \tag{2.61}$$

Finally we are left with the completely traceless part of the curvature:

$$\begin{aligned}
C_{abcd} \equiv\ & R_{abcd} + \eta_{ad}\,\phi_{bc} + \eta_{bc}\,\phi_{ad} - \eta_{ac}\,\phi_{bd} - \eta_{bd}\,\phi_{ac}\\
&+2\,(\eta_{ac}\,\eta_{bd} - \eta_{ad}\,\eta_{bc})\,\Pi.
\end{aligned} \tag{2.62}$$

[15] The expression for Φ_{00} is in fact real but not manifestly so.

This is the Weyl tensor in the NP frame, and from it Newman and Penrose define the *Weyl scalars*:

$$\Psi_0 \equiv C_{1313} \sim (2,0), \quad \Psi_1 \equiv C_{1312} \sim (1,0), \quad \Psi_2 \equiv C_{1342} \sim (0,0),$$
$$\Psi_3 \equiv \Psi_1' = C_{2421} \sim (-1,0), \quad \Psi_4 \equiv \Psi_0' = C_{2424} \sim (-2,0). \tag{2.63}$$

In GHP notation, the first three of these are[16]

$$\Psi_0 = -\left[\text{þ} + (\rho + \rho^*)\right]\sigma + \left[\eth + (\tau + \tau'^*)\right]\kappa, \tag{2.65}$$

$$\Psi_1 = \tfrac{1}{2}\left\{-\text{þ}\tau + \text{þ}'\kappa + \eth\rho - \eth'\sigma - (\tau - \tau'^*)\rho - (\tau^* - \tau')\sigma \right.$$
$$\left. + (\rho - \rho^*)\tau - (\rho' - \rho'^*)\kappa\right\}, \tag{2.66}$$

$$\Psi_2 = \tfrac{1}{3}\left[\rho\rho'^* - |\tau|^2 + \text{þ}'\rho - \eth'\tau - (\mathcal{K} + \mathcal{K}_s)\right.$$
$$\left. - 2\kappa\kappa' + \tau\tau' + 2\sigma\sigma' - \rho\rho'\right]. \tag{2.67}$$

The remaining two are defined by priming.

2.11 Gravitational compass and Petrov classification

Szekeres conjured an elegant theoretical apparatus called the *gravitational compass* to interpret the Weyl scalars [42]. Following his insight, we will say that Ψ_2 describes a Coulomb field, Ψ_4 describes a transverse outgoing wave, and Ψ_3 describes a longitudinal outgoing wave. The primed quantities, $\Psi_0 \equiv \Psi_4'$ and $\Psi_1 \equiv \Psi_3'$, describe the corresponding ingoing waves.[17]

The Weyl scalars are not gauge invariant: A local $SO(3,1)$ transformation $e^a \to O^a{}_b\, e^b$ results in $\Psi_\alpha \to \sum_{\beta=0}^{4} Q_{\alpha\beta}\Psi_\beta$ for some matrix $Q_{\alpha\beta}$. We can ask how many Ψ_α can be simultaneously gauged away, and we can classify spacetimes based on the answer. This is Chandrasekhar's [31] account of the Petrov classification [43] of the Weyl tensor. A desire to elucidate the physics behind each Petrov type is what drove Szekeres to engineer the gravitational compass.

We will only study two Petrov types: Type D, in which all of the Weyl scalars beside Ψ_2 can be gauged away, and Type II, in which all of the Weyl scalars beside Ψ_2 and Ψ_4 can be gauged away.[18] Extending the standard terminology slightly beyond its

16 Having defined Ψ_1 and Φ_{01}, we can compose þ and ð on an arbitrarily-weighted test function and deduce the mixed commutator relation

$$[\text{þ}, \eth] + \rho^*\eth + \sigma\eth' - \tau'^*\text{þ} - \kappa\text{þ}' = -2h(\rho'\kappa - \tau'\sigma + \Psi_1) - 2\bar{h}(\sigma'^*\kappa^* - \rho^*\tau'^* + \Phi_{01}). \tag{2.64}$$

17 The Coulomb component is self-prime. We might also suggest an alternative notation to make Szekeres's interpretation manifest: $\Psi_\perp \equiv \Psi_4$, $\Psi_\parallel \equiv \Psi_3$, $\Psi_C \equiv \Psi_2$, $\Psi_\perp' \equiv \Psi_0$, and $\Psi_\parallel' \equiv \Psi_1$.

18 For a Type II spacetime, we can rotate the frame to trade a nonzero Ψ_4 for a nonzero Ψ_3. This resolves the superficial discrepancy between Chandrasekhar's [31] and Penrose and Rindler's descriptions [32]. Szekeres [42] and Griffiths [44] use the terminology of Penrose and Rindler.

ordinary usage, we will define a *principal frame* as any basis in which as many Weyl scalars as possible for a given geometry are gauged away.

The Kerr–Newman black hole is Type D, and its nonzero Weyl scalar is

$$\Psi_2 = -\frac{1}{(R^*)^3}\left(M - \frac{Q^2}{R}\right). \tag{2.68}$$

Because it carries charge, this black hole is not a vacuum solution—the Weyl scalars are no longer the whole story. Local sources of energy induce Ricci curvature, and in this case the electromagnetic field induces

$$\Phi_{11} = \frac{Q^2}{2|R|^4}. \tag{2.69}$$

2.12 Energy scalars

In the relativistic zeitgeist, the Ricci scalars are considered a stand-in for the energy tensor by means of Einstein's equation. But we find this confusing and will briefly suggest a refined presentation.

To match Penrose's traceless Ricci tensor from Eq. (2.55), we define a traceless energy tensor

$$\mathcal{T}_{ab} \equiv \tfrac{1}{2}\left(T_{ab} - \tfrac{1}{4}\eta_{ab}\,\eta^{cd}T_{cd}\right). \tag{2.70}$$

From that, we define "energy scalars" analogously to the Ricci scalars: $t_{00} \equiv 8\pi\mathcal{T}_{11}$, and so on, such that Einstein's equation becomes

$$\Phi_{ij} = t_{ij} \quad \text{and} \quad \Pi = t_\Pi, \tag{2.71}$$

with $i, j \in \{0, 1, 2\}$. For the Kerr–Newman solution, the only nonzero entry is t_{11}, which can be expressed in terms of a complex number φ_1 called a Maxwell scalar[19]:

$$t_{11} = |\varphi_1|^2, \quad \varphi_1 = \frac{Q}{\sqrt{2}(R^*)^2}. \tag{2.72}$$

Our point is that the equation $t_{11} = |\varphi_1|^2$ is the statement $T_{\mu\nu} = F_{\mu\rho}F_\nu{}^\rho - \frac{1}{4}g_{\mu\nu}F_{\rho\sigma}F^{\rho\sigma}$ in the internal, and the equation $\Phi_{11} = t_{11}$ is Einstein's equation in the internal. The typically stated relation $\Phi_{11} = |\varphi_1|^2$ combines both.

Having traipsed through the background geometry, we are now ready to perturb it.

[19] The Maxwell scalars are defined as the components of the electromagnetic curvature contracted with the vectors of the null frame:

$$\varphi_0 \equiv F_{\mu\nu}\,l^\mu m^\nu, \quad \varphi_1 \equiv \tfrac{1}{2}F_{\mu\nu}\,(l^\mu l'^\nu + m'^\mu m^\nu), \quad \varphi_2 \equiv F_{\mu\nu}\,m'^\mu l'^\nu.$$

Note that because $F_{\mu\nu} = -F_{\nu\mu}$, we have $\varphi_2 = -\varphi_0'$ and $\varphi_1' = -\varphi_1$.

3 Shifted frame and Kerr–Schild form

Relative to the standard frame of the Kerr–Newman background, and in terms of a general function $S(t, r, \theta, \varphi)$, we define the *shifted frame*[20]

$$\tilde{l} \equiv l, \quad \tilde{l}' \equiv l' + S\,l, \quad \tilde{m} \equiv m. \tag{3.1}$$

It cannot be emphasized enough that the meaning of $\tilde{l}'$ in components is

$$\tilde{l}' = \tilde{l}'_\mu \, dx^\mu = (l'_\mu + Sl_\mu)\, dx^\mu, \tag{3.2}$$

not $\tilde{l}'_\mu \, d\tilde{x}^\mu$ for some shifted coordinate basis $d\tilde{x}^\mu$. Otherwise the shift would describe a change of coordinates, not a physical perturbation.

Recalling Eq. (2.2), we define the shifted line element as

$$d\tilde{s}^2 \equiv -2\tilde{l}\tilde{l}' + 2\tilde{m}\tilde{m}' = ds^2 - 2Sll. \tag{3.3}$$

Since we have chosen l^μ to be tangent to a shear-free geodesic congruence of the unshifted spacetime, the shifted line element is of the generalized Kerr–Schild form, as defined by Taub [30]. If we turn off the angular momentum and the charge and choose the ansatz

$$S = \frac{\Delta}{2r^2}\,\frac{U}{V}\,\delta(U)\, f(\theta, \varphi) \qquad (a = Q = 0) \tag{3.4}$$

then we will reproduce exactly the Dray–'t Hooft metric [19]. If we turn off the angular momentum but leave the charge nonzero and use the same functional form for the ansatz, we will reproduce the metric of Alonso–Zamorano [22] and Sfetsos [23].

3.1 From Reissner–Nordström to Kerr–Newman

To generalize to a rotating background, we will scrutinize the factors that appear in Eq. (3.4).

First, by revisiting our conventions for the unshifted frame and staring at the definition of the shifted one, we conclude that the factor $\frac{\Delta}{2r^2}$ compensates for the asymmetric normalization of l_μ relative to l'_μ. So the generalization of this factor to the rotating case is clear:

$$\frac{\Delta}{2r^2} \to \frac{\Delta}{2|R|^2}. \tag{3.5}$$

Second, we have defined the Kruskal-like coordinates so that they mimic the coordinates in the nonrotating case: The future horizon is still at $U = 0$, and the radial

[20] Only during revisions did we find the apropos work by Fels and Held [45]. While their shift is like ours, their analysis differs. Strikingly, they consider shifting Type D backgrounds but conclude that "as seeds they are not very fruitful." We disagree.

function r depends only on the product UV. So we might hope that the factor $\frac{U}{V}\,\delta(U)$ could remain unmodified.

Third, we recognize that the function $f(\theta,\varphi)$ is defined only at the origin of Kruskal-like coordinates ($U = V = 0$). Extrapolating to the Kerr–Newman spacetime should therefore entail the generalization

$$(\theta,\varphi) \to (\theta,\chi). \tag{3.6}$$

This cross-examination of the Dray–'t Hooft solution coupled with the clear geometrical underpinning of the Newman–Penrose formalism led us to the conviction that the perturbed Kerr–Newman geometry should be described by the shifted frame in Eq. (3.1) with the following ansatz:

$$S = \frac{\Delta}{2|R|^2}\,\frac{U}{V}\,\delta(U)\,f(\theta,\chi). \tag{3.7}$$

We will call S the *shift function*, and we will call $f(\theta,\chi)$ the *horizon field*. When we calculate the curvature scalars, we will work directly with the rescaled frame in Eq. (2.48), thereby enlisting the rescaled shift function

$$\hat{S} = (-U)^{-2}S = \frac{1}{2|R|^2}\,\frac{\Delta}{UV}\,\delta(U)f(\theta,\chi). \tag{3.8}$$

Like everything else in the hatted basis, this shift function is finite at the horizon.

By comparing the GHP representations $l'_\mu \sim (-\frac{1}{2},-\frac{1}{2})$ and $l_\mu \sim (+\frac{1}{2},+\frac{1}{2})$ in the context of Eq. (3.1), we deduce that the shift function must transform as

$$S \sim (-1,-1). \tag{3.9}$$

When interpreting the formulas Eqs. (3.7) and (3.8) in the GHP formalism, we assign the horizon field $f(\theta,\chi)$ the weights of the shift function:

$$f(\theta,\chi) \sim (-1,-1). \tag{3.10}$$

The remaining factors are to be treated as ordinary functions, not physical degrees of freedom, and are therefore assigned weights (0, 0).

By explicit calculation, we will indeed find that the ansatz in Eq. (3.7) results in a shifted Ricci tensor of the form

$$\tilde{R}_{\mu\nu} = R_{\mu\nu} + R^{\text{shift}}_{UU}\,\delta^U_\mu\,\delta^U_\nu \tag{3.11}$$

and therefore correctly generalizes the Dray–'t Hooft solution to a rotating background.

3.2 Preliminary commentary

Before focusing on R_{UU}, we wish to preview a miracle: If the unshifted frame is aligned with shear-free null geodesics ($\kappa = \sigma = \kappa' = \sigma' = 0$) and if the unshifted Φ_{00} is zero, the shifted Ricci tensor will depend only *linearly* on the shift function S.[21]

We will proceed step by step through the spin coefficient formalism to understand why this happens. A practical reason is to derive master formulas for the spin coefficients and curvature scalars of generalized Kerr–Schild spacetimes. For the spin coefficients we will maintain full generality in the background, but for the curvature scalars we will restrict to shear-free geodesic congruences.

3.3 Shifted spin coefficients

By shifting both sides of the null Cartan equations [Eq. (2.32)] and solving them, we can express the shifted spin coefficients in terms of their unshifted values.

Start with the equation for dl, and tilde every term[22]:

$$d\tilde{l} = -2\,\mathrm{Re}(\tilde{\varepsilon})\,\tilde{l} \wedge \tilde{l}' + 2i\,\mathrm{Im}(\tilde{\rho})\,\tilde{m} \wedge \tilde{m}' + \Big[\Big(\tilde{\tau} - \tilde{\beta} + \tilde{\beta}'^{*}\Big)\,\tilde{m}' \wedge \tilde{l} + \tilde{\kappa}\,\tilde{m}' \wedge \tilde{l}' + c.c.\Big]. \tag{3.12}$$

By inserting into the right-hand side the definition of the shifted frame in terms of the unshifted frame and recalling that $l \wedge l = 0$, we find

$$d\tilde{l} = -2\,\mathrm{Re}(\tilde{\varepsilon})\,l \wedge l' + 2i\,\mathrm{Im}(\tilde{\rho})\,m \wedge m' + \Big[\Big(\tilde{\tau} - \tilde{\beta} + \tilde{\beta}'^{*} + \tilde{\kappa}\,S\Big)\,m' \wedge l + \tilde{\kappa}\,m' \wedge l' + c.c.\Big]. \tag{3.13}$$

Since $\tilde{l} = l$, we have $d\tilde{l} = dl$, so the left-hand side can be replaced with the untilded version of Eq. (3.12). The four basis 2-forms $l \wedge l'$, $m \wedge m'$, $m' \wedge l$, and $m' \wedge l'$ are linearly independent, so we can match their coefficients on both sides to obtain the first set of shifted spin coefficient equations:

$$\mathrm{Re}(\tilde{\varepsilon}) = \mathrm{Re}(\varepsilon), \quad \mathrm{Im}(\tilde{\rho}) = \mathrm{Im}(\rho), \quad \tilde{\tau} - \tilde{\beta} + \tilde{\beta}'^{*} + \tilde{\kappa}\,S = \tau - \beta + \beta'^{*}, \quad \tilde{\kappa} = \kappa. \tag{3.14}$$

Next up, dl'. The right-hand side parallels that for dl, but since $l' = l' + Sl$ the left-hand side is more complicated. Not only do we require the untilded equations for both dl' and dl, we also require the exterior derivative of the shift function:

$$dS \equiv dx^{\mu}\,\partial_{\mu} S = e^{a}\,\partial_{a} S = -l\,D'S - l'DS + m\,\delta' S + m'\delta S. \tag{3.15}$$

[21] This was in fact noticed by Taub [30] and by Alonso and Zamorano [22].

[22] Note that, in keeping with our advisory remark below the definition of the shifted frame [Eq. (3.1)], we do not tilde the exterior derivative operator.

Matching the coefficients of the basis 2-forms gives the second set of shifted spin coefficient equations:

$$
\begin{aligned}
&\mathrm{Re}(\tilde{\varepsilon}') = \mathrm{Re}(\varepsilon') - \mathrm{Re}(\varepsilon)S + \tfrac{1}{2}DS, \quad \mathrm{Im}(\tilde{\rho}') = \mathrm{Im}(\rho') - \mathrm{Im}(\rho)S,\\
&\tilde{\tau}' - \tilde{\beta}' + \tilde{\beta}^* = \tau' - \beta' + \beta^* + \kappa^* S, \quad \tilde{\kappa}' + (\tilde{\tau}' - \tilde{\beta}' + \tilde{\beta}^*)S\\
&\quad = \kappa' + (\tau^* - \beta^* + \beta')S + \delta' S.
\end{aligned} \tag{3.16}
$$

Before moving on, it is helpful to take stock of where we are. We have already solved directly for $\mathrm{Re}(\tilde{\varepsilon})$, $\mathrm{Im}(\tilde{\rho})$, and $\tilde{\kappa}$, and may thereby observe that they remain unshifted. We have also solved for $\mathrm{Re}(\tilde{\varepsilon}')$ and $\mathrm{Im}(\tilde{\rho}')$. By inserting the third equation in Eq. (3.16) into the fourth one, we obtain the shifted κ':

$$
\tilde{\kappa}' = \kappa' + \left(\delta' - 2\beta^* + 2\beta'\right) S + (\tau^* - \tau')S - \kappa^* S^2. \tag{3.17}
$$

Recall that $S \sim (-1, -1)$ and that the GHP-covariant version of δ' is $\eth' = \delta' - 2h\beta' + 2h\beta^*$. As expected from GHP covariance, the NP derivatives and gauge fields appear in just the right combination to form a covariant derivative:

$$
\tilde{\kappa}' = \kappa' + \eth' S + (\tau^* - \tau')S - \kappa^* S^2. \tag{3.18}
$$

On the other hand, the terms involving D, ε, and ε^* in $\mathrm{Re}(\tilde{\varepsilon}')$ do not collect themselves into a GHP-covariant combination. But that too is expected: While $\tilde{\kappa}'$ is a weighted quantity, $\tilde{\varepsilon}'$ is not. By solving the matrix inversion problem in Eq. (2.6) for the shifted frame, we obtain the shifted NP derivatives:

$$
\tilde{D} = D, \quad \tilde{D}' = D' - S\,D, \quad \tilde{\delta} = \delta. \tag{3.19}
$$

We will see that $\tilde{\varepsilon}'$ will in fact combine with $\tilde{D}'$ to create a shifted $\tilde{\text{þ}}'$ that can be written in terms of GHP-covariant quantities. But to prove that, we will need to solve for the shifted $\mathrm{Im}(\tilde{\varepsilon}')$, and for that we will need to study dm.

Applying the above procedure to dm, we find the final set of shifted spin coefficient equations:

$$
\begin{aligned}
&\tilde{\beta} + \tilde{\beta}'^* = \beta + \beta'^*, \quad \tilde{\tau} - \tilde{\tau}'^* = \tau - \tau'^*, \quad \tilde{\rho} - 2i\,\mathrm{Im}(\tilde{\varepsilon}) = \rho - 2i\,\mathrm{Im}(\varepsilon),\\
&\tilde{\rho}' + 2i\,\mathrm{Im}(\tilde{\varepsilon}') + (\tilde{\rho} - 2i\,\mathrm{Im}(\tilde{\varepsilon}))S = \rho' + 2i\,\mathrm{Im}(\varepsilon'), \quad \tilde{\sigma}'^* + \tilde{\sigma} S = \sigma'^*.
\end{aligned} \tag{3.20}
$$

By solving Eqs. (3.14), (3.16), and (3.20), we learn that the weighted spin coefficients and gauge field associated with l_μ do not receive corrections:

$$
\tilde{\kappa} = \kappa, \quad \tilde{\tau} = \tau, \quad \tilde{\sigma} = \sigma, \quad \tilde{\rho} = \rho, \quad \tilde{\varepsilon} = \varepsilon. \tag{3.21}
$$

While it should not be surprising that κ, σ, ρ, and ε do not receive corrections, it may be unexpected that τ does not shift. It turns out that τ' also remains unshifted:

$$
\tilde{\tau}' = \tau'. \tag{3.22}
$$

So the timelike expansion $\tau + \tau'^{*}$ and the timelike twist $\tau - \tau'^{*}$ remain unshifted.

The weighted spin coefficients and gauge field associated with l'_μ do receive corrections:

$$\begin{aligned}\tilde{\kappa}' &= \kappa' + \left[\eth' + \left(\tau^* - \tau'\right)\right] S - \kappa^* S^2, \quad \tilde{\sigma}' = \sigma' - \sigma^* S, \quad \tilde{\rho}' = \rho' - \rho\, S, \\ \tilde{\varepsilon}' &= \varepsilon' - \varepsilon^* S + \tfrac{1}{2} D S - i\, \mathrm{Im}(\rho)\, S. \end{aligned} \tag{3.23}$$

In general, the transverse gauge fields also receive corrections:

$$\tilde{\beta} = \beta + \tfrac{1}{2}\kappa\, S, \quad \tilde{\beta}' = \beta' - \tfrac{1}{2}\kappa^* S. \tag{3.24}$$

From Eqs. (3.21)–(3.24) we conclude that if we align l^μ with background geodesics—namely if $\kappa = 0$—then not only do the formulas simplify considerably, but all nonlinearity in the shift function drops out of the spin coefficients.

This already implies $\tilde{R}_{abcd} = R_{abcd} + S\, R^{(1)}_{abcd} + S^2\, R^{(2)}_{abcd}$, i.e., there are no terms of $O(S^3)$ or higher. Furthermore, if the geodesics to which l^μ are aligned can also be taken shear-free—namely if $\sigma = 0$—then we get $\tilde{\sigma}' = \sigma'$ as well. Finally, if we also align l'^μ with background shear-free geodesics, then

$$\tilde{\kappa}' = \left[\eth' + \left(\tau^* - \tau'\right)\right] S, \quad \tilde{\rho}' = \rho' - \rho\, S, \quad \tilde{\varepsilon}' = \varepsilon' - \varepsilon^* S + \tfrac{1}{2} D S - i\, \mathrm{Im}(\rho)\, S. \tag{3.25}$$

In this case, the only GHP-covariant derivative that shifts is $þ'$. The shifted version acting on a function $f_{h,\bar{h}} \sim (h, \bar{h})$ is

$$\tilde{þ}' f_{h,\bar{h}} = \left[þ' - S þ - (h + \bar{h})(þ S) + 2i(h - \bar{h})\mathrm{Im}(\rho) S\right] f_{h,\bar{h}}. \tag{3.26}$$

This vindicates the discussion below Eq. (3.18) and completes our derivation of the shifted spin coefficients.

Dray and 't Hooft explained [19] that test particles crossing the shockwave get translated and refracted. (See also the work by Matzner [46].) In the spin coefficient formalism, these effects are described by the shifted versions of ρ' and κ'—to the physics we now turn.

3.4 Shifted horizon

Cartography of the horizon requires the hatted basis. As we discussed back in Sect. 2.8, the future horizon can be defined locally as the subspace of Kruskal-like coordinates on which the expansion of the outgoing congruence vanishes ($\hat{\rho} = 0$). Similarly, the past horizon is the subspace on which the expansion of the ingoing congruence vanishes ($\hat{\rho}' = 0$).

Recalling the unshifted $\hat{\rho}$ and $\hat{\rho}'$ from Eq. (2.45) and the shift described in Eq. (3.25), we find that the coordinate V receives a correction while the coordinate U does not:

$$\begin{aligned}\tilde{\hat{\rho}} &= \hat{\rho} \implies \tilde{U} = U,\\ \tilde{\hat{\rho}}' &= \hat{\rho}' - \hat{\rho}\,\hat{S} = \left(1 + \frac{U}{V}\,\delta(U)\,f(\theta,\chi)\right)\hat{\rho}' \equiv \left[\frac{1}{2|R|^2}\left(\frac{\Delta}{UV}\right)\frac{1}{R^*}\right]\tilde{V}\\ &\implies \tilde{V} - V = U\,\delta(U)\,f(\theta,\chi).\end{aligned} \tag{3.27}$$

This last expression implies that smooth functions of U will experience no coordinate shift, while functions that go as $\frac{1}{U}$ near $U = 0$ will experience a discontinuity in the coordinate. To see this, interpret Eq. (3.27) as a differential equation in U in the vicinity of $U = 0$, i.e., $\frac{d(\tilde{V}-V)}{dU} = \lim_{U\to 0}\frac{\tilde{V}-V}{U} = \delta(U)\,f(\theta,\chi)$. Integration then gives

$$\tilde{V} = V + \Theta(U)\,f(\theta,\chi). \tag{3.28}$$

This is the shift as described by Dray and 't Hooft [19] and by Sfetsos [23].

3.5 Refraction

Since every acolyte of Penrose knows that κ and κ' describe the refraction of light rays, the result that κ' becomes nonzero after the shift speaks for itself.

4 Petrov classification for the Kerr–Newman shockwave

Let the games begin. We will first shift the Weyl scalar $\Psi_4 \sim (-2, 0)$, or more conveniently its complex conjugate $\Psi_4^* \sim (0, -2)$. Since this is just our opening act, we will reserve intricate computational details for the main event, the shifted Ricci scalars.

4.1 Shifted Ψ_4 and physical interpretation

Aligning the background frame with shear-free geodesic congruences but assuming an arbitrary shift function S, we find:

$$\tilde{\Psi}_4^* = \Psi_4^* + ðð S + 2\tau\, ð S. \tag{4.1}$$

To obtain this we used the complex conjugate and the prime of Φ_{02} from Eq. (2.57) in the forms $ð'\tau^* = -\tau^{*2}$ and $ð'\tau' = -\tau'^2$, which hold when $\Phi_{02} = 0$.

To specialize to the shockwave, hat everything and insert the ansatz of Eq. (3.8) for the shift function. Since the calculation is laborious, it is advantageous to first enumerate conceivable terms.

Remember that the horizon field $f(\theta,\chi)$ has weights $(-1,-1)$. Since Ψ_4^* has weights $(0,-2)$, we will have to find operators of weights $(1,-1)$. Fortunately, the list of such operators that are nonzero at the Kerr–Newman horizon is short:

$$ðð\,;\quad \tau ð,\quad \tau'^{*}ð\,;\quad \tau^2,\quad \tau'^{*2},\quad \tau\tau'^{*}. \tag{4.2}$$

In principle we would also need $ð\tau$ and $ð\tau'^{*}$, but again when $\Phi_{02} = 0$ those can be traded for $-\tau^2$ and $-\tau'^{*2}$. So the result must have the form

$$\hat{\hat{\Psi}}{}_4^* = k_0\,\delta(U)\left[ððf + (k_1\,\tau + k_2\,\tau'^{*})ðf + (k_3\,\tau^2 + k_4\,\tau'^{*2} + k_5\,\tau\tau'^{*})f\right] \tag{4.3}$$

for some functions $k_i(\theta)$ that will depend on the parameters r_+, a, and α. Whether by hand or by machine we ultimately find:

$$\begin{aligned} &k_0 = -\tfrac{c}{2|R|^2}, \quad k_1 = \tfrac{\alpha|R|^2}{r}, \quad k_2 = -2\left(1 + \tfrac{\alpha|R|^2}{2r}\right), \quad k_3 = \left(\tfrac{\alpha|R|^2}{2r}\right)^2, \\ &k_4 = 1 + \left(1 + \tfrac{\alpha|R|^2}{2r}\right)^2, \quad k_5 = -\tfrac{\alpha|R|^2}{r}\left(1 + \tfrac{\alpha|R|^2}{2r}\right). \end{aligned} \tag{4.4}$$

On the way to this result, we encounter terms involving $\partial_U\delta(U)$ and $\partial_U^2\delta(U)$.[23] We interpret them according to the distributional edict of integrating by parts against an arbitrary smooth test function $\mathcal{F}(U)$:

$$\int dU\,\mathcal{F}(U)\,\partial_U{}^n\delta(U) = \int dU\,(-1)^n\,\partial_U{}^n\mathcal{F}(U)\,\delta(U). \tag{4.5}$$

It should also be understood, as required by the overall factor $\delta(U)$, that all instances of r in Eq. (4.4) actually denote r_+. Also note that numerically we have

$$\tau' = -\frac{R}{R^*}\,\tau^*, \tag{4.6}$$

so it is possible to shuffle terms among the coefficients k_3, k_4, and k_5. The particular form shown in Eq. (4.4) is what we exhumed upon performing the rituals to be disclosed in Sect. 6.

Invoking the gravitational compass from Sect. 2.11, we interpret Eqs. (4.3) and (4.4) as describing a transverse "outgoing" gravitational wave stuck to the horizon.[24]

4.2 Nonrotating limit

It is worth pausing to consider the nonrotating limit, $a \to 0$, in which case only the $ððf$ term in Eq. (4.3) survives.

As far as we know, the Weyl scalars for the shifted Reissner–Nordström geometry have not been calculated explicitly, so we will unpack the definitions of the GHP

[23] We also stumble upon the gargantuan notational implosion "$ð\delta(U) = \delta\delta(U)$."

[24] We cannot help calling the reader's attention to the following famous quotation: "Now, here, you see, it takes all the running you can do, to keep in the same place." This is originally from *Through the Looking-Glass* by Lewis Carroll, but we first encountered its application to the horizon of a black hole from the textbook on the Kerr geometry by O'Neill [47].

derivatives at the horizon. Remembering that $f \sim (-1, -1)$ and therefore $\eth f \sim (-\frac{1}{2}, -\frac{3}{2})$, and that in the nonrotating limit we have $\beta = \beta' = \beta^* = \beta'^*$, we find:

$$\eth\eth f|_{a=0} = \delta\delta f - 2\beta\,\delta f. \tag{4.7}$$

4.3 Shifted Ψ_3 and Petrov type

Our debt to Ψ_4 settled, we turn to Ψ_3. Shifting the frame (with $\kappa = \sigma = \kappa' = \sigma'$) seemingly produces this Weyl scalar:

$$\begin{aligned}\tilde{\Psi}_3^* = \Psi_3^* + \tfrac{1}{4}(\text{þ}\eth + \eth\text{þ})S + \tfrac{1}{4}(2\tau - \tau'^*)\text{þ}S + \tfrac{1}{4}(2\rho - 5\rho^*)\eth S \\ - \tfrac{1}{2}[(2\tau - \tau'^*)\rho^* + \eth\rho^*]S.\end{aligned} \tag{4.8}$$

But by hatting and specializing to Eq. (3.8), we find that each term in Eq. (4.8) goes to zero at $U = 0$ for fixed nonzero V:

$$\hat{\tilde{\Psi}}_3^* = 0. \tag{4.9}$$

Since the unshifted geometry already had a nonzero Ψ_2, we conclude that the shockwave is Petrov Type II:

$$\hat{\tilde{\Psi}}_0 = \hat{\tilde{\Psi}}_1 = \hat{\tilde{\Psi}}_3 = 0, \quad \hat{\tilde{\Psi}}_2 \neq 0, \quad \hat{\tilde{\Psi}}_4 \neq 0. \tag{4.10}$$

To quote Szekeres: "[I]t can be viewed as a Coulomb field with an outgoing wave component superimposed" [42].

4.4 Curvatures of submanifolds

Shifting both sides of the GHP commutator equations [see Eq. (2.49)], we find

$$\tilde{\mathcal{K}} = \mathcal{K} + \text{Im}(\rho)\left[2\,\text{Im}(\rho)\,S + i\,(\text{þ}S)\right], \quad \tilde{\mathcal{K}}_s = \mathcal{K}_s - \tfrac{1}{2}(\text{þ}^2 S) + i\,\text{þ}\,[\text{Im}(\rho)S]\,. \tag{4.11}$$

But if we hat everything and specialize to the shockwave ansatz, we will find that all of the corrections in Eq. (4.11) go to zero. Curiously enough, the shockwave does not alter the spacelike and timelike curvatures.

4.5 Shifted Ψ_2

Inserting Eq. (4.11) into Eq. (2.67) provides the shifted Weyl scalar of weight zero:

$$\tilde{\Psi}_2 = \Psi_2 + \tfrac{1}{6}\text{þ}^2 S - \tfrac{1}{3}(2\rho - \rho^*)\text{þ}S + \tfrac{1}{3}\left[(3\rho - 2\rho^*)\rho + \Phi_{00}\right]S. \tag{4.12}$$

To arrive at this expression, we used the relation[25]

$$\Phi_{00} = -(\text{þ}\rho + \rho^2) \qquad (\text{if } \ \kappa = \sigma = 0) \tag{4.13}$$

along with $\Phi_{00}^* = \Phi_{00}$. Just as we found for the shifted Ψ_3, we find upon disbursing hats and availing ourselves of Eq. (3.8) that the correction to Ψ_2 is zero.

Appealing again to the gravitational compass [42], we say that the Coulomb field remains unchanged by the presence of a massless particle on the future horizon.

5 Shifted Ricci scalars

Show time. We will first present the shifted Ricci scalars for the generalized Kerr–Schild geometry under the assumption $\kappa = \kappa' = \sigma = \sigma'$, and then we will specialize to the shockwave.

5.1 Ricci scalar of weight (− 1, − 1): absence of nonlinearity

After the shift from Eq. (3.1), three of the Ricci scalars will become nonzero. Of these, the apple of our eye will be $\Phi_{22} \sim (-1, -1)$.

This quantity is defined by priming the definition of Φ_{00} in Eq. (2.57):

$$\Phi_{22} = -\left(\text{þ}'\rho' + \rho'^2\right) + \eth\kappa' + \tau\kappa' + \tau'\kappa'^* - |\sigma'|^2. \tag{5.1}$$

Using the shifted ρ' from Eq. (3.25) and the shifted $\text{þ}'$ from Eq. (3.26), and using $h = \bar{h} = -\frac{1}{2}$ for ρ' [recall Eq. (2.27)], we find:

$$\begin{aligned} \tilde{\text{þ}}'\tilde{\rho}' &= \text{þ}'\rho' + (\rho'\text{þ} - \rho\text{þ}')S - (\text{þ}'\rho + \text{þ}\rho')S + (\text{þ}\rho)S^2, \\ \tilde{\rho}'^2 &= \rho'^2 - 2\rho\rho' S + \rho^2 S^2. \end{aligned} \tag{5.2}$$

It is worth keeping in mind the formula for Φ_{00} under the shear-free geodesic assumption [Eq. (4.13)]. Next, for $\sigma\sigma' = \kappa\kappa' = 0$, we have[26]:

$$\Psi_2 + 2\Pi = -(\eth'\tau + |\tau|^2) + \text{þ}'\rho + \rho'^*\rho \tag{5.3}$$

$$= -(\eth\tau' + |\tau'|^2) + \text{þ}\rho' + \rho^*\rho'. \tag{5.4}$$

It is a matter of some discretion which variables to keep and which to trade away. We are guided by comparison with the nonrotating limit, which suggests we should

25 This is Raychaudhuri's equation for null shear-free geodesic congruences. When $\Phi_{00} = 0$, it tells us that $\text{þ}\rho = -\rho^2$. Given the standard interpretation of $\text{Re}(\rho)$ as the expansion, we recognize this as the focusing theorem.

26 Attentive readers have every right to be confused by the second equality: Indeed it turns out that the combination of derivatives and products of spin coefficients in Eq. (5.3) equals its primed version in Eq. (5.4). This must be so, since both $\Psi_2 = C_{1342}$ and $\Pi = \frac{1}{12}(R_{12} - R_{34})$ are self-prime.

express as much as possible in terms of τ and τ' and their derivatives. So we will use Eqs. (5.3) and (5.4) to evict $þ'\rho$ and $þ\rho'$ from Eq. (5.2).

With our shifted κ' from Eq. (3.25), we find:

$$\begin{aligned} ð\tilde{\kappa}' &= ðð'S + (\tau^* - \tau')ðS + (ð\tau^* - ð\tau')S, \\ \tau\tilde{\kappa}' + \tau'\tilde{\kappa}'^* &= (\tau ð' + \tau' ð)S + (|\tau|^2 - |\tau'|^2)S. \end{aligned} \tag{5.5}$$

Equations (5.2)–(5.5) then supply the preliminary expression:

$$\begin{aligned} \tilde{\Phi}_{22} = \Phi_{22} &+ (\rho þ' - \rho' þ)S + \Phi_{00}S^2 + ðð'S + \tau ð'S + \tau^* ðS \\ &+ \Big[2\rho\rho' - (\rho\rho'^* + \rho^*\rho') + ð'\tau + ð\tau^* + 2|\tau|^2 + 2(\Psi_2 + 2\Pi)\Big] S. \end{aligned} \tag{5.6}$$

Behold: For a background in which $\Phi_{00} = 0$, *all nonlinear dependence on the perturbation drops out of the curvature scalars*. Terms of $O(S^2)$ could not possibly show up elsewhere, because the only curvature scalar with the appropriate weight to include a *product* of shifted quantities (in this case $þ'$ and ρ') is Φ_{22}.

To make sense of Eq. (5.6) we will rewrite it in a manifestly real form:

$$\begin{aligned} \tilde{\Phi}_{22} = \mathrm{Re}(\tilde{\Phi}_{22}) &= \Phi_{22} + \mathrm{Re}\left[(\rho þ' - \rho' þ)S\right] + \Phi_{00}S^2 \\ &+ \tfrac{1}{2}\left[ðð' + ð'ð + (\tau + \tau'^*)ð' + (\tau^* + \tau')ð + (\tau - \tau'^*)ð' + (\tau^* - \tau')ð\right] S \\ &+ \Big\{(\rho - \rho^*)(\rho' - \rho'^*) + (ð'\tau + c.c.) + 2|\tau|^2 + 2\left[\mathrm{Re}(\Psi_2) + 2\Pi\right]\Big\} S. \end{aligned} \tag{5.7}$$

Experts in the compacted formalism should recognize the combination $ðð' + (\tau + \tau'^*)ð' + c.c.$ as part of the generalized Laplacian (we will get to this in Sect. 6). Before elaborating on this, we will vanquish the remaining curvature scalars.

5.2 Other Ricci scalars

The Ricci scalar of weight $(-1, 0)$ is corrected by the general shift[27]:

$$\begin{aligned} \tilde{\Phi}_{21} = \Phi_{21} &+ \tfrac{1}{4}(þð' + ð'þ)S + \tfrac{1}{4}(2\tau^* - \tau')þS + \tfrac{1}{4}(3\rho - 2\rho^*)ð'S \\ &+ \tfrac{1}{2}(\tau'\rho - 2\tau^*\rho^* + ð'\rho)S. \end{aligned} \tag{5.8}$$

For the Kerr–Newman background, we have $\Phi_{21} = 0$. After hatting and specializing to Eq. (3.8), we find that each would-be contribution from S to Eq. (5.8) is zero.

Next we have the Ricci scalar of weight $(0, 0)$:

$$\tilde{\Phi}_{11} = \Phi_{11} + \tfrac{1}{4}þ^2 S - \tfrac{1}{2}\Big[|\rho|^2 + (\rho - \rho^*)^2\Big] S. \tag{5.9}$$

[27] The steps leading to this expression parallel closely those that led to $\tilde{\Psi}_3^*$.

Here too we find no correction to the unshifted value after hatting both sides of the equation and specializing to the shockwave: $\hat{\tilde{\Phi}}_{11} = \hat{\Phi}_{11} = \Phi_{11}$.

The Einstein–Hilbert curvature also superficially becomes nonzero as a result of the shift:

$$\tilde{\Pi} = \Pi - \tfrac{1}{6}\left\{\tfrac{1}{2}\text{þ}^2 S + (\rho + \rho^*)\text{þ}S + (|\rho|^2 - 2\Phi_{00})\, S\right\}. \tag{5.10}$$

But we know that Π is proportional to the Lagrangian of general relativity, so its first order variation must comport with the standard formula

$$S_{\text{GR}}[g+h] - S_{\text{GR}}[g] = \tfrac{1}{2}\int d^4x\, |\det(g)|^{1/2}\, T^{\mu\nu} h_{\mu\nu} + O(h^2). \tag{5.11}$$

The shift from Eq. (3.1) effects the metric variation

$$h_{\mu\nu} = -2S\, l_\mu l_\nu. \tag{5.12}$$

So varying the action with respect to S will result in something proportional to $T^{\mu\nu} l_\mu l_\nu = \mathcal{T}^{\mu\nu} l_\mu l_\nu = (8\pi)^{-1}\, t_{00}$ (recall Sect. 2.12). Because the only nonzero energy scalar for the background spacetime is $t_{11} \propto (l_\mu l'_\nu + l'_\mu l_\nu + m_\mu m'_\nu + m'_\mu m_\nu)\mathcal{T}^{\mu\nu}$, we know that $t_{00} = 0$ and thereby expect the $O(S)$ term in Eq. (5.11) to equal zero.[28]

The nonzero $O(S)$ term in Eq. (5.10) might invite consternation, but we have been cavalierly ignoring possible boundary terms in the action. So all we require is that the $O(h)$ term in Eq. (5.11) should be zero, not necessarily that the shift in Π itself should be zero.

For Kerr–Newman, we have[29] $|\det(g)|^{1/2} = i\, \varepsilon^{\mu\nu\rho\sigma} l_\mu l'_\nu m_\rho m'_\sigma = |R|^2 \sin\theta$. After integrating by parts, dropping total derivatives, and using D and ρ from Eqs. (2.8) and (2.33), we indeed obtain

$$\int d^4x\, |\det(g)|^{1/2}\, \tilde{\Pi} = 0. \tag{5.13}$$

This completes our account of the shifted curvature scalars for the generalized Kerr–Schild geometry. (The Ricci scalars not explicitly enumerated in this section do not shift.) Now we will specialize the shifted Φ_{22} to the shockwave.

6 Derivatives of the shift

The spacetime Laplacian $\nabla^2 = \nabla_\mu \nabla^\mu$ finds refuge in the compacted spin coefficient formalism within a more general operator

$$\Box = -\Box_\parallel + \Box_\perp, \tag{6.1}$$

[28] We thank Alexei Kitaev for suggesting this check on our work.

[29] This expression for $|\det(g)|^{1/2}$ makes clear that it does not receive a correction from Eq. (3.1).

where

$$\Box_{\parallel} \equiv [\text{þ} + 2\,\mathrm{Re}(\rho)]\text{þ}' \;+\; ', \quad \Box_{\perp} \equiv [\eth + (\tau + \tau'^{*})]\eth' \;+\; '. \tag{6.2}$$

The operator $\Box_{\perp}$ will be called the "transverse box." Evaluating its action on the shift function is the most technically cumbersome aspect of computing $\tilde{\Phi}_{22}$.

We will do our best to show how the sausage is made without belaboring mindless algebra.

6.1 Key facts

To set up the calculation we will first collect some useful formulas.

From what may seem like a lifetime ago, we recall that $U\partial_U r = V\partial_V r$ (which can be traced back to the relation $-U\partial_U + V\partial_V = \frac{1}{\alpha}\partial_t$). Therefore, acting on a weight-(0, 0) function $F(r)$, we have:

$$\eth F(r) = 0. \tag{6.3}$$

This is our first key fact.

Next we recall the explicit formulas for the timelike expansion and the timelike twist [Eq. (2.34)]. They will compose our basic mnemonic for making sense of complicated algebraic expressions: The trigonometric functions $\sin(2\theta)$ and $\sin\theta$ should evoke $\tau + \tau'^{*}$ and $\tau - \tau'^{*}$ respectively.

We will use this to establish additional useful formulas. Treating $\delta(U)$ as having weight (0, 0) and summoning the NP derivatives in Kruskal-like coordinates [Eq. (2.48)], we find:

$$\eth\delta(U) = -\alpha\,\frac{ia\sin\theta}{R\sqrt{2}}\,U\partial_U\delta(U) = -\,\frac{\alpha|R|^2}{2r}\,(\tau - \tau'^{*})\,U\partial_U\delta(U). \tag{6.4}$$

This is our second key fact.

Finally, we must bear in mind that although functions of r can be treated as constants, the generalized radial function $R = r + ia\cos\theta$ is also a function of θ. Treating this too as a function of weight (0, 0), we compute the following:

$$\eth\left(\frac{1}{|R|^2}\right) = -\,\frac{1}{(|R|^2)^2}\,\eth\Big(|R|^2\Big) = +\,\frac{a^2\sin(2\theta)}{\sqrt{2}R|R|^4} = -\,\frac{1}{|R|^2}(\tau + \tau'^{*}). \tag{6.5}$$

This is our third key fact.

6.2 Integration by parts

We described back in Eq. (4.5) the standard integration-by-parts procedure that defines the delta function. Here it will be useful to study two special cases of that formula.

First consider a distribution $\mathcal{O}(U)\, U\partial_U \delta(U)$ (where the conditions on $\mathcal{O}(U)$ will be specified shortly), and integrate it against a test function $\mathcal{F}(U)$ that falls off quickly enough to merit dropping the boundary term:

$$\int dU\, \mathcal{O}(U)\, U\partial_U \delta(U)\, \mathcal{F}(U) = -\int dU\ [\mathcal{O}(U)\mathcal{F}(U) \\ + U\partial_U(\mathcal{O}(U)\mathcal{F}(U))]\, \delta(U). \tag{6.6}$$

If $\partial_U(\mathcal{O}(U)\mathcal{F}(U)) \sim U^n$ with $n \geq 0$ near $U = 0$, then the second term evaluates to zero. We then obtain the following distributional equality:

$$\mathcal{O}(U)\, U\partial_U \delta(U) = -\mathcal{O}(U)\, \delta(U). \tag{6.7}$$

Along similar lines, we will obtain a second distributional equality:

$$\mathcal{O}(U)\, U\partial_U(U\partial_U \delta(U)) = +\mathcal{O}(U)\, \delta(U). \tag{6.8}$$

Equipped with the key facts in Eqs. (6.3)–(6.5) and the above distributional equalities, we are ready to face the transverse box.

6.3 First-derivative terms

We warm up with a first-derivative term. Specializing to the shockwave ansatz in Eq. (3.8) and applying our key facts, we obtain the preliminary expression

$$\begin{aligned} ð\hat{S} &= \frac{1}{2}\frac{\Delta}{UV}\left[ð(\tfrac{1}{|R|^2})\delta(U)f(\theta,\chi) + \tfrac{1}{|R|^2}ð\delta(U)f(\theta,\chi) + \tfrac{1}{|R|^2}\delta(U)ðf(\theta,\chi)\right] \\ &= \frac{1}{2|R|^2}\frac{\Delta}{UV}\left\{\delta(U)\left[ð - (\tau + \tau'^*)\right] f(\theta,\chi) - \tfrac{\alpha|R|^2}{2r}(\tau - \tau'^*)\, U\partial_U\delta(U)\, f(\theta,\chi)\right\}. \end{aligned} \tag{6.9}$$

Before integrating by parts against a test function, we need to multiply by $\tau^* + \tau'$ to obtain the term $(\tau^* + \tau')ð\hat{S}$ that appears in the transverse box.[30]

Note that since $|\tau|^2 = |\tau'|^2$ for the Kerr–Newman spacetime, we have

$$(\tau^* + \tau')(\tau - \tau'^*) = 2i\, \mathrm{Im}(\tau\tau'). \tag{6.10}$$

Using this and the distributional equality in Eq. (6.7), we obtain [also recall $c \equiv -\frac{\Delta}{UV}\big|_{r=r_+}$ from Eq. (2.38)]

$$(\tau^* + \tau')\, ð\hat{S} = -\frac{c}{2|R|^2}\,\delta(U)\left\{(\tau^* + \tau')\, ð - |\tau + \tau'^*|^2 + i\,\tfrac{\alpha|R|^2}{r}\,\mathrm{Im}(\tau\tau')\right\} f(\theta,\chi). \tag{6.11}$$

[30] Since $\tau^* + \tau'$ depends on U and V only through $r = r(UV)$, and since we have already said such functions can be treated as constants with respect to $U\partial_U$ for our calculation, it does not matter in this particular instance whether we integrate by parts before or after multiplying by $\tau^* + \tau'$.

6.4 Second-derivative terms

Returning to Eq. (6.9), we act with $\eth'$ (and skip a few steps now that the method is presumably clear) to obtain

$$\eth'\eth\hat{S} = \tfrac{1}{2|R|^2}\tfrac{\Delta}{UV}\Big\{\delta(U)\,\eth'\eth f - \Big[(\tau+\tau'^*)\,\delta(U) + \tfrac{\alpha|R|^2}{2r}(\tau-\tau'^*)\,U\partial_U\delta(U)\Big]\eth' f - \Big[(\tau^*+\tau')\,\delta(U) + \tfrac{\alpha|R|^2}{2r}(\tau^*-\tau')\,U\partial_U\delta(U)\Big]\eth f + \mathscr{C} f\Big\}, \tag{6.12}$$

where

$$\mathscr{C} = |\tau+\tau'^*|^2\,\delta(U) - i\tfrac{\alpha|R|^2}{r}\mathrm{Im}(\tau\tau')\,U\partial_U\delta(U) - \Big(\delta(U) + \tfrac{\alpha|R|^2}{2r}\,U\partial_U\delta(U)\Big)\eth'\tau - \Big(\delta(U) - \tfrac{\alpha|R|^2}{2r}\,U\partial_U\delta(U)\Big)(\eth\tau')^* + \Big(\tfrac{\alpha|R|^2}{2r}\Big)^2|\tau-\tau'^*|^2\,U\partial_U\,[U\partial_U\delta(U)]\,. \tag{6.13}$$

Note that in Eq. (6.12) the coefficient of $\eth' f$ is the complex conjugate of the coefficient of $\eth f$. This did not have to be so, because we are computing $\eth'\eth\hat{S}$ right now, not $\eth'\eth\hat{S} + c.c.$, and in general $\eth'\eth \neq \eth\eth'$.

This quantity $\eth'\eth\hat{S}$ will be integrated directly against a test function (because it appears directly in the transverse box, which in turn appears directly in $\hat{\hat{\Phi}}_{22}$), so we can use the distributional equalities in Eqs. (6.7) and (6.8), loosely expressed as $U\partial_U\delta(U) \to -\delta(U)$ and $U\partial_U[U\partial_U\delta(U)] \to +\delta(U)$. Applying these to Eq. (6.12), we obtain:

$$\eth'\eth\hat{S} = -\frac{c}{2|R|^2}\,\delta(U)\Big\{\eth'\eth - \Big[(\tau+\tau'^*) - \tfrac{\alpha|R|^2}{2r}(\tau-\tau'^*)\Big]\eth' - \Big[(\tau^*+\tau') - \tfrac{\alpha|R|^2}{2r}(\tau^*-\tau')\Big]\eth + |\tau+\tau'^*|^2 + i\tfrac{\alpha|R|^2}{r}\mathrm{Im}(\tau\tau') - \Big(1-\tfrac{\alpha|R|^2}{2r}\Big)\eth'\tau - \Big(1+\tfrac{\alpha|R|^2}{2r}\Big)(\eth\tau')^* + \Big(\tfrac{\alpha|R|^2}{2r}\Big)^2|\tau-\tau'^*|^2\Big\} f(\theta,\chi). \tag{6.14}$$

6.5 Transverse box

Now we can finish the job. Returning to the first-derivative term in Eq. (6.9) and adding its complex conjugate, we obtain:

$$(\tau^*+\tau')\eth\hat{S} + c.c. = -\frac{c}{2|R|^2}\,\delta(U) \times\Big\{(\tau+\tau'^*)\eth' + (\tau^*+\tau')\eth - 2|\tau+\tau'^*|^2\Big\} f(\theta,\chi). \tag{6.15}$$

Next we obtain the anticommutator of GHP derivatives by taking Eq. (6.12) plus its complex conjugate:

$$\begin{aligned}\eth'\eth\hat{S} + c.c. = &-\frac{c}{2|R|^2}\,\delta(U)\Big\{(\eth'\eth + \eth\eth')\\ &+\Big[-\Big(2(\tau+\tau'^*) - \tfrac{\alpha|R|^2}{r}(\tau-\tau'^*)\Big)\,\eth' + c.c.\Big]\\ &+2|\tau+\tau'^*|^2 - \Big(1-\tfrac{\alpha|R|^2}{2r}\Big)(\eth'\tau + c.c.) - \Big(1+\tfrac{\alpha|R|^2}{2r}\Big)(\eth\tau' + c.c.)\\ &+2\Big(\tfrac{\alpha|R|^2}{2r}\Big)^2|\tau-\tau'^*|^2\Big\}\,f(\theta,\chi). \end{aligned} \tag{6.16}$$

We then add Eqs. (6.15) and (6.16) to obtain the transverse box. For reasons morally unbeknownst to us, the $|\tau+\tau'^*|^2$ term will cancel out. Also, for Kerr–Newman, we have

$$\eth\tau' = \eth'\tau. \tag{6.17}$$

There is probably a good reason for this, but it escapes us. At any rate, it implies that the α-dependent parts of the coefficients of $\eth'\tau + c.c.$ and $\eth\tau' + c.c.$ drop out.

Therefore, the transverse box acting on the shift function, expressed in terms of GHP derivatives at the horizon, simplifies to:

$$\begin{aligned}\Box_\perp\hat{S} = -\frac{c}{2|R|^2}\,\delta(U)\Big\{&(\eth'\eth+\eth\eth') + \Big[-\Big((\tau+\tau'^*) - \tfrac{\alpha|R|^2}{r}(\tau-\tau'^*)\Big)\,\eth' + c.c.\Big]\\ &-2(\eth'\tau + c.c.) + 2\Big(\tfrac{\alpha|R|^2}{2r}\Big)^2|\tau-\tau'^*|^2\Big\}\,f(\theta,\chi). \end{aligned} \tag{6.18}$$

This completes the most arduous part of the calculation. It bears repeating that all quantities in Eq. (6.18) are understood to be evaluated at $r = r_+$, as mandated by the overall delta function.

6.6 Laplacian on the squashed sphere

We could leave the result for $\Box_\perp\hat{S}$ in the form of Eq. (6.18), but those familiar with the Dray–'t Hooft solution expect 2d Laplacians.

Our shift function S and our horizon field f have GHP weight $(-1,-1)$. In general, a weighted function $f_{h,h} \sim (h,h)$ has *spin-weight* $s \equiv h - \bar{h} = h - h = 0$. The shockwave has $h = -1$, but without much fuss we can understand the situation for $s = 0$ but arbitrary h.[31]

By explicit computation on a function $f_{h,h}(\theta,\chi)$ of the Kruskal-like angular coordinates only, we find that the following combination of NP derivatives and GHP gauge

[31] Since complex conjugation exchanges h and $\bar{h}$, only functions with $s = 0$ can be taken real. We therefore assume $f^*_{h,h} = f_{h,h}$ for simplicity.

fields reproduces the Laplacian on the *squashed sphere*[32]:

$$\delta'\delta + \delta\delta' - (\beta + \beta'^*)\delta' - (\beta' + \beta^*)\delta = \nabla^2_{2\text{d}}. \tag{6.19}$$

So unpacking the GHP derivatives according to their original definitions back in Eq. (2.29) provides the desired expression:

$$(\eth'\eth + \eth\eth') f_{h,h}(\theta,\chi) = \Big\{\nabla^2_{2\text{d}} + \big[4h(\beta - \beta'^*)\delta' + c.c.\big] \\ +2h\Big[(\delta'\beta - \delta\beta' + c.c.) + 2(|\beta'|^2 - |\beta|^2) + 4h|\beta - \beta'^*|^2\Big]\Big\} f_{h,h}(\theta,\chi). \tag{6.20}$$

That is how our coveted 2d spatial Laplacian manifests in our story. Its tragedy is that while we may find temporary solace in a familiar face, this yearning for camaraderie cost us the guidance of GHP covariance, without which we are hopelessly lost.

7 Ricci tensor

The trace-reversed Ricci tensor, being necessary to the gravitational field of a localized Source, the propensity of a massless particle to generate Curvature, shall now be realized.

7.1 Relation to curvature scalars

We emerge from the chrysalis of the internal by translating the usual prescription $R_{\mu\nu} = e_\mu^{\ a} e_\nu^{\ b} R_{ab}$ into NP notation:

$$\begin{aligned}\tfrac{1}{2}R_{\mu\nu} = {} & l'_\mu l'_\nu\,\Phi_{00} + l_\mu l_\nu\,\Phi_{22} + \big[m'_\mu m'_\nu\,\Phi_{02} - (l'_\mu m'_\nu + m'_\mu l'_\nu)\,\Phi_{01} \\ & - (l_\mu m_\nu + m_\mu l_\nu)\,\Phi_{21} + c.c.\big] \\ & + (l_\mu l'_\nu + l'_\mu l_\nu + m_\mu m'_\nu + m'_\mu m_\nu)\,\Phi_{11} \\ & + (l_\mu l'_\nu + l'_\mu l_\nu - m_\mu m'_\nu - m'_\mu m_\nu)\,3\Pi.\end{aligned} \tag{7.1}$$

To evaluate the right-hand side, we first need to tilde everything (to calculate shifted quantities), and then we need to hat everything (to work in the horizon basis).

We will specialize directly to the shockwave, so the only Ricci scalar that will shift is Φ_{22}. Meanwhile, the unshifted geometry has only a nonzero Φ_{11}. Therefore, we have for the full (i.e., including the unshifted part) Ricci tensor:

[32] A squashed sphere of radius r has line element $ds^2 = |R|^2 d\theta^2 + \frac{|R_0|^4}{|R|^2}\sin^2\theta\, d\chi^2 \equiv h_{ij}\,dx^i\,dx^j$, and the Laplacian derived from that is $\nabla^2_{2\text{d}} = \frac{1}{|R|^2}\left[\partial_\theta^2 + \left(\frac{|R_0|^2 + a^2\sin^2\theta}{|R|^2}\right)\cot\theta\,\partial_\theta + \frac{|R|^4}{|R_0|^4}\frac{1}{\sin^2\theta}\partial_\chi^2\right]$. If $R^{2\text{d}}_{ij}$ is the Ricci tensor derived from h_{ij}, then $\frac{1}{4}h^{ij}R^{2\text{d}}_{ij} = \text{Re}(\mathcal{K})|_{r=r_+}$, the intrinsic curvature from Eq. (2.52).

$$\tfrac{1}{2}\tilde{R}_{\mu\nu} = \hat{\tilde{l}}_\mu \hat{\tilde{l}}_\nu \, \hat{\tilde{\Phi}}_{22} + (\hat{\tilde{l}}_\mu \hat{\tilde{l}}'_\nu + \hat{\tilde{l}}'_\mu \hat{\tilde{l}}_\nu + \hat{\tilde{m}}_\mu \hat{\tilde{m}}'_\nu + \hat{\tilde{m}}'_\mu \hat{\tilde{m}}_\nu)\, \hat{\tilde{\Phi}}_{11}$$
$$= \hat{l}_\mu \hat{l}_\nu \, \hat{\tilde{\Phi}}_{22} + (\hat{l}_\mu \hat{\tilde{l}}'_\nu + \hat{\tilde{l}}'_\mu \hat{l}_\nu + m_\mu m'_\nu + m'_\mu m_\nu)\, \Phi_{11}. \tag{7.2}$$

In the second line we have removed the tildes for quantities that equal their unshifted counterparts, and we have removed the hats from quantities that do not get rescaled by factors of U when passing from the standard frame to the horizon one.[33]

Recalling from Eq. (3.1) the premise that launched this travail in the first place, we isolate the part of the Ricci tensor that results from the shift:

$$R^{\text{shift}}_{\mu\nu} = 2\,\hat{l}_\mu \hat{l}_\nu \,(\hat{\tilde{\Phi}}_{22} + 2\hat{S}\,\Phi_{11}). \tag{7.3}$$

Returning to our explicit expressions for the 1-forms in Eq. (2.47), we find:

$$\hat{l}\Big|_{U=0} = \frac{|R_+|^2}{\alpha |R_{0+}|^2}\, dU. \tag{7.4}$$

So we learn first of all that $R^{\text{shift}}_{\mu\nu} = R^{\text{shift}}_{UU}\,\delta_\mu^{\ U}\delta_\nu^{\ U}$, as promised.

7.2 Relation to energy scalars

Meanwhile, the energy tensor also admits an expansion analogous to Eq. (7.1):

$$4\pi T_{\mu\nu} = l'_\mu l'_\nu\, t_{00} + l_\mu l_\nu\, t_{22} + \big[m'_\mu m'_\nu\, t_{02} - (l'_\mu m'_\nu + m'_\mu l'_\nu)\, t_{01}$$
$$- (l_\mu m_\nu + m_\mu l_\nu)\, t_{21} + c.c.\big]$$
$$+ (l_\mu l'_\nu + l'_\mu l_\nu + m_\mu m'_\nu + m'_\mu m_\nu)\, t_{11}$$
$$+ (l_\mu l'_\nu + l'_\mu l_\nu - m_\mu m'_\nu - m'_\mu m_\nu)\, 3t_\Pi. \tag{7.5}$$

Anticipating the required energy tensor term by term, we conclude:

$$8\pi T^{\text{shift}}_{\mu\nu} = 2\,\hat{l}_\mu \hat{l}_\nu\,(\hat{\tilde{t}}_{22} + 2\hat{S}\, t_{11}). \tag{7.6}$$

Given that the background Einstein equation is, by construction, $\Phi_{11} = t_{11}$ [recall Eq. (2.71)], all we need is a t_{22} such that

$$\hat{\tilde{\Phi}}_{22} = \hat{\tilde{t}}_{22}. \tag{7.7}$$

The whole point of this tale is that the correction to the left-hand side can be interpreted as the backreaction from a *massless particle on the future horizon*, so that is what will populate the right-hand side. In this paper we focus on the geometry instead of the field theory, so let us leave that aside and press on.

[33] There is no need to place a hat on the Ricci *tensor*, because by construction it is invariant under GHP transformations of the frame.

7.3 Final result for Φ_{22}

Returning to our earlier calculation of $\eth\hat{S}$ [Eq. (6.9)], multiplying by $\tau^*-\tau'$, integrating by parts, and adding the complex conjugate, we obtain the remaining first-derivative terms:

$$(\tau-\tau'^*)\eth'\hat{S}+c.c. = -\frac{c}{2|R|^2}\,\delta(U)\left\{(\tau-\tau'^*)\eth'+(\tau^*-\tau')\eth \right.$$
$$\left. +\tfrac{\alpha|R|^2}{r}|\tau-\tau'^*|^2\right\} f(\theta,\chi). \tag{7.8}$$

Next take the general shifted Φ_{22} from Eq. (5.7), hat it, and recognize that $\hat{\rho}'\,\hat{þ}\hat{S}$ and $(\hat{\rho}-\hat{\rho}^*)(\hat{\rho}'-\hat{\rho}'^*)$ go to zero at $U=0$.

But $\hat{\rho}\,\hat{þ}'\hat{S}$ is more subtle, since within $\hat{D}'$ lurks ∂_U. Applying Eq. (6.7), we obtain

$$\mathrm{Re}(\hat{\rho}\hat{D}'\hat{S}) = \alpha r_+\frac{|R_{0+}|^2}{|R_+|^4}\hat{S} = -\tfrac{1}{2}\left[\eth\tau+\eth'\tau^*+2|\tau|^2+2\,\mathrm{Re}(\Psi_2)\right]\Big|_{r=r_+}\hat{S}. \tag{7.9}$$

Because $\hat{\rho}\big|_{U=0}=0$, the terms involving $\hat{\varepsilon}'$ and $\hat{\varepsilon}'^*$ drop out, leaving us with $\mathrm{Re}(\hat{\rho}\hat{þ}'\hat{S})=\mathrm{Re}(\hat{\rho}\hat{D}'\hat{S})=-\mathrm{Re}(\hat{þ}'\hat{\rho})\hat{S}$.

Putting all this together (and using $\Phi_{00}=\Pi=0$), we reduce our shifted Φ_{22} to the relatively compact form:

$$\hat{\hat{\Phi}}_{22} = \tfrac{1}{2}\left\{\Box_\perp+(\tau-\tau'^*)\eth'+(\tau^*-\tau')\eth+(\eth'\tau+c.c.)+2|\tau|^2+2\,\mathrm{Re}(\Psi_2)\right\}\hat{S}. \tag{7.10}$$

Enlisting our result for $\Box_\perp\hat{S}$ in Eq. (6.18) and the relation $4|\tau|^2=|\tau+\tau'^*|^2+|\tau-\tau'^*|^2$, we finally obtain the beautiful, exquisite, magical expression

$$\hat{\hat{\Phi}}_{22} = -\frac{c}{4|R|^2}\,\delta(U)\,\mathscr{D}f(\theta,\chi), \tag{7.11}$$

where the differential operator $\mathscr{D}$ is

$$\mathscr{D} = \eth'\eth+\eth\eth'+\left[-(\tau+\tau'^*)+\left(1+\tfrac{\alpha|R|^2}{r}\right)(\tau-\tau'^*)\right]\eth'$$
$$+\left[-(\tau^*+\tau')+\left(1+\tfrac{\alpha|R|^2}{r}\right)(\tau^*-\tau')\right]\eth$$
$$+2\,\mathrm{Re}(\Psi_2)-(\eth'\tau+\eth\tau^*)+\tfrac{1}{2}|\tau+\tau'^*|^2+\tfrac{1}{2}\left(1+\tfrac{\alpha|R|^2}{r}\right)^2|\tau-\tau'^*|^2. \tag{7.12}$$

This is our final result.

It is expressed in terms of quantities that have innate geometrical significance, in that each operator has a definite GHP weight. When $a = 0$, we obtain[34]

$$\mathscr{D}|_{a=0} = \eth'\eth + \eth\eth' + 2\,\mathrm{Re}(\Psi_2). \tag{7.13}$$

As could be anticipated from the Type D character of the background, we see that it is part of the Weyl tensor, $\mathrm{Re}(\Psi_2)$, not the intrinsic curvature, $\mathrm{Re}(\mathcal{K})$, that appears most naturally in the GHP-covariant form of the shifted Φ_{22} for generic values of the angular momentum.

On the other hand, the intrinsic curvature presents itself when we trade the GHP-covariant derivatives for the 2d Laplacian plus its associated ejecta. We first expand $\eth f = [\delta + 2(-1)\beta - 2(-1)\beta'^{*}]f$ and specialize Eq. (6.20) to $h = -1$. Then we shuffle the terms around using numerical relations like[35]

$$\beta' - \beta^{*} = \tau' \qquad \text{(Kerr–Newman)} \tag{7.14}$$

and

$$|\beta'|^2 - |\beta|^2 = \frac{a^2}{2|R|^4} \qquad \text{(Kerr–Newman).} \tag{7.15}$$

In this way we obtain the following alternative form for Eq. (7.12):

$$\begin{aligned}\mathscr{D} = \nabla^2_{2\mathrm{d}} &+ \tfrac{1-\alpha R}{r}(\tau - \tau'^{*})R^{*}\,\delta' + \tfrac{1-\alpha R^{*}}{r}(\tau^{*} - \tau')R\,\delta \\ &+ 2r\alpha\left\{-2\left(\frac{|R|^2}{|R_0|^2}\mathrm{Re}(\mathcal{K}) + \frac{2a^2}{|R|^4}\right) + |\tau + \tau'^{*}|^2\right. \\ &\left. + \left[1 + r\alpha\left(\frac{|R|^2}{2r^2}\right)^2\right]|\tau - \tau'^{*}|^2\right\}.\end{aligned} \tag{7.16}$$

We will refer to the coefficient of $f(\theta, \chi)$ in $\hat{\hat{\Phi}}_{22}$, encapsulated by the term in Eq. (7.16) without any derivatives, as the "mass term." It is organized in terms of the intrinsic curvature at the horizon [recall Eq. (2.52)] and quantities proportional to some power of the angular momentum. Expressed in this way, the mass term reeks of Kaluza–Klein, but we will leave that for another day. Regardless, this form shows clearly which terms go to zero as we turn off the rotation.

When $a = 0$ (but $Q \neq 0$), we recover the known spherically symmetric answer[36]:

$$\hat{\hat{\Phi}}_{22}\Big|_{a=0} = -\frac{c}{4r_+^2}\,\delta(U)\left(\nabla^2_{2\mathrm{d}} - \frac{2\alpha}{r_+}\right)f(\theta, \varphi). \tag{7.17}$$

[34] At $r = r_+$, we have $\mathrm{Re}(\Psi_2)|_{a=0} = -\frac{\alpha}{r_+}$.

[35] It is possible that these relations embody some hidden meaning. But the two sides of Eq. (7.14) do not transform in the same way under Eq. (2.17), so we hesitate to dig deeper.

[36] At $r = r_+$, we have $\mathrm{Re}(\mathcal{K})|_{a=0} = \frac{1}{2r_+^2}$. Also, when $a = 0$ the delayed angle χ becomes the ordinary azimuthal angle φ.

While the geometrical significance of the mass term in Eq. (7.16) eludes us, the physical significance of the overall factor of α in shockwave geometries has been emphasized by others.[37] In the extremal limit, which in this case is $a^2 + Q^2 = M^2$ and hence $r_- = r_+$, the surface gravity α goes to zero (as usual), and the entire mass term vanishes.

As far as we know, the first to point this out in the spherically symmetric situation was Sfetsos, who interpreted it as a breakdown of the solution [23]. The effect was recently revisited by Leichenauer in the context of entanglement between the conformal field theories dual to the asymptotically-AdS generalization of the Reissner–Nordström black hole [48]. And in the context of scattering, the vanishing of the mass term in the operator $\mathscr{D}$ is what Maldacena and Stanford call the "βJ enhancement" of the amplitude [16].

But let us not get ahead of ourselves. In this paper we are concerned exclusively with the single-shockwave geometry and its interpretation within general relativity. The sun will rise tomorrow, and we will have another opportunity to traverse that wormhole.

8 Discussion

Inspired by 't Hooft's S-matrix approach to quantum gravity and Kitaev's recent revival thereof, we have generalized the Dray–'t Hooft gravitational shockwave to the Kerr–Newman black hole using the method of spin coefficients.

We have not solved the resulting Green's function equation, $\mathscr{D} f \propto \delta^2(\mathbf{x}_\perp)$. Since $\mathscr{D}$ is analytic near $a = 0$, we could perturb around the Dray–'t Hooft integral formula [19]. Or maybe we should expand in spheroidal harmonics, but we would probably have to resort to numerics for anything beyond a rudimentary understanding.[38] On a different tack, we could perturb other backgrounds by shifting the frame: Shockwaves on Kerr–AdS might eventually lead to precise statements about chaos in a putative dual field theory.[39]

We will conclude with a pedantic remark about the effective action for the horizon field. Given a classical equation of motion, we should ask what variational principle could lead to it. Since the Ricci tensor is linear in $f(\theta, \chi)$, our equation of motion is linear in the field, so we might expect a quadratic action.

But the Lagrangian is proportional to the Einstein–Hilbert curvature Π, which we have already seen is *linear* in f. What to make of this? Recall that if the "equation of motion" is actually a constraint—which in this case it is—then it should be implemented in the calculus of variations by introducing a Lagrange multiplier.

37 We thank Douglas Stanford for explaining this to us.

38 Dray and 't Hooft themselves "have not attempted to perform the integration explicitly" for their result [19]. Sfetsos, for his part, did elaborate somewhat on his solutions in Appendix D of his paper [23].

39 We thank Nick Hunter-Jones for encouragement in this direction.

Consider a path integral over all classical fields $f(\theta, \chi)$ that satisfy $\mathscr{D} f = 0$[40]:

$$\mathcal{Z} \equiv \int \mathcal{D}f\, \delta(\mathscr{D} f) = \int \mathcal{D}f\, \mathcal{D}f'\, e^{i \int d^2x\, f' \mathscr{D} f}. \tag{8.1}$$

We have used the Fourier representation of the delta function and thereby concocted a classical field f', which serves as a Lagrange multiplier for the equation $\mathscr{D} f = 0$.

The argument of the exponential in Eq. (8.1) is 't Hooft's effective action [9]. This straightforward interpretation of the constraint for the horizon field provides a path-integral sense in which the two shockwaves are canonically conjugate variables.

Acknowledgements We thank Jan Willem Dalhuisen, Aaron Zimmerman, David Nichols, Christopher White, Dave Aasen, Nick Hunter-Jones, Alex Rasmussen, Yonah Lemonik, Douglas Stanford, and Saul Teukolsky for insightful discussions at various points in this endeavor. Y. B. especially thanks Justin Wilson, Leo Stein, and Alexei Kitaev. J. S. especially thanks Dirk Bouwmeester. Y. B. is funded by the Institute for Quantum Information and Matter (NSF Grant PHY-1125565) with support from the Simons Foundation (Award No. 376205). J. S. is funded by a UCSB Central Fellowship.

A Signature change

In this appendix we sail from West to East, scrupulously marking all signs in our wake. Relics will be tagged by overbars.

A.1 Basic assumptions

We begin by flipping the signs of both the base space and internal metrics:

$$g_{\mu\nu} \equiv \zeta\, \bar{g}_{\mu\nu}, \quad \eta_{ab} \equiv \zeta\, \bar{\eta}_{ab}, \quad \zeta \equiv -1. \tag{A.1}$$

In terms of the corresponding frames, we have $g_{\mu\nu} = \eta_{ab}\, e^a_\mu e^b_\nu$, $\bar{g}_{\mu\nu} = \bar{\eta}_{ab}\, \bar{e}^a_\mu \bar{e}^b_\nu$, $\eta_{ab} = g_{\mu\nu} e^\mu_a e^\nu_b$, and $\bar{\eta}_{ab} = \bar{g}_{\mu\nu} \bar{e}^\mu_a \bar{e}^\nu_b$. Defining $e_{a\mu} \equiv \eta_{ab}\, e^b_\mu$ and $\bar{e}_{a\mu} \equiv \bar{\eta}_{ab}\, \bar{e}^b_\mu$, we obtain from Eq. (A.1):

$$e^a_\mu e_{a\nu} = \zeta\, \bar{e}^a_\mu \bar{e}_{a\nu}, \quad e^\mu_a e_{b\mu} = \zeta\, \bar{e}^\mu_a \bar{e}_{b\mu}. \tag{A.2}$$

Relative to Chandrasekhar [31], our null vectors $(l^\mu, l'^\mu, m^\mu, m'^\mu)$ will not flip sign, in which case our null forms $(l_\mu, l'_\mu, m_\mu, m'_\mu) \equiv (g_{\mu\nu}\, l^\nu, g_{\mu\nu}\, l'^\nu, g_{\mu\nu}\, m^\nu, g_{\mu\nu}\, m'^\nu)$ will. This is a choice.

From this—with attention to the fact that the basis is *null*—we infer:

$$e^\mu_a \equiv \bar{e}^\mu_a, \quad e^a_\mu \equiv \bar{e}^a_\mu. \tag{A.3}$$

40 For the sake of brevity we are only considering the gravitational part of the action. More generally there should be an f-independent function on the right-hand side of the constraint.

Neither e_a^μ nor e_μ^a flips sign. What does flip sign is the quantity with both indices lowered:

$$e_{a\mu} = \zeta\, \bar{e}_{a\mu}. \tag{A.4}$$

A.2 Spin coefficients flip sign

If we insert the above definitions into $de^a + \omega^a{}_b \wedge e^b = 0$, we will find that the spin connection *with one index up and one index down* does not flip sign:

$$\omega^a{}_b = \bar{\omega}^a{}_b. \tag{A.5}$$

So ω_{ab} *does* flip sign. Unpacking the 1-form index and recognizing that $dx^\mu = d\bar{x}^\mu$, we find $(\omega_\mu)_{ab} = \zeta\, (\bar{\omega}_\mu)_{ab}$. Recalling Eq. (2.12), we conclude that *the spin coefficients flip sign*:

$$\gamma_{abc} = \zeta\, \bar{\gamma}_{abc}. \tag{A.6}$$

Meanwhile, because of Eq. (A.4), *the null Cartan equations are the same in either signature*. So Eq. (2.32) looks exactly the same as Eq. (4.13.44) in *Spinors and Spacetime* [32].

A.3 Curvature scalars do not flip sign

Next up, curvature. Since $\omega^a{}_b$ does not flip sign, neither does $\Omega^a{}_b \equiv d\omega^a{}_b + \omega^a{}_c \wedge \omega^c{}_b$:

$$\Omega^a{}_b = \bar{\Omega}^a{}_b. \tag{A.7}$$

So Ω_{ab} *does* flip sign. As an unavoidable consequence, the Riemann tensor in the internal with all indices down, $R_{abcd} \equiv (\Omega_{\mu\nu})_{ab}\, e_c^\mu e_d^\nu$, flips sign:

$$R_{abcd} = \zeta\, \bar{R}_{abcd}. \tag{A.8}$$

It is misleading to simply assert that the Newman–Penrose equations remain fixed upon changing the metric signature, as if it were to follow as night the day.

Crucially, the Weyl scalars are defined from C_{abcd}, which in turn is defined from R_{abcd} [recall Eq. (2.62)]—*this quantity flips sign under a change of signature*:

$$C_{abcd} = \zeta\, \bar{C}_{abcd}. \tag{A.9}$$

Should we fashion an extra sign in the definition of the Weyl scalars to obviate this? No. Beside the sign from Eq. (A.8), there is also an overall sign choice in the *definition* of the curvature scalars—by sheer happenstance, our conventions in Eq. (2.63) automatically cancel this additional sign compared to the GHP equations as traditionally written [34].

Meanwhile, since $R^a{}_{bcd} = \eta^{ae} R_{ebcd}$ and $R_{abcd} = \zeta\, \bar{R}_{abcd}$, the Ricci tensor in the internal does *not* flip sign:

$$R_{ab} \equiv R^c{}_{acb} = \bar{R}^c{}_{acb} \equiv \bar{R}_{ab}. \tag{A.10}$$

For the Ricci scalars in Eq. (2.56) we do commission a sign relative to the standard references.

The Einstein–Hilbert curvature sprouts yet another sign:

$$\eta^{ab} R_{ab} = \zeta\, \bar{\eta}^{ab} \bar{R}_{ab}. \tag{A.11}$$

To maintain the sanctity of the GHP equations, we must begrudgingly define

$$\Pi \equiv -\frac{1}{24}\eta^{ab} R_{ab} = -\zeta \frac{1}{24}\bar{\eta}^{ab}\bar{R}_{ab} = +\bar{\Pi}. \tag{A.12}$$

A.4 Extra sign in GHP derivatives

Before docking we must ensure that the Icelandic runes make sense. Consider the GHP derivatives as defined by Penrose and Rindler [32]:

$$\begin{aligned} \bar{\text{þ}} &\equiv \bar{D} - 2h\,\bar{\varepsilon} - 2\bar{h}\,\bar{\varepsilon}^*, \quad \bar{\eth} \equiv \bar{\delta} - 2h\,\bar{\beta} + 2\bar{h}\,\bar{\beta}'^*, \\ \bar{\text{þ}}' &\equiv \bar{D}' + 2h\,\bar{\varepsilon}' + 2\bar{h}\,\bar{\varepsilon}'^*, \quad \bar{\eth}' \equiv \bar{\delta}' + 2h\,\bar{\beta}' - 2\bar{h}\,\bar{\beta}^*. \end{aligned} \tag{A.13}$$

Explicitly verifying their GHP covariance on a weighted test function, we see that a certain crucial sign emerges as a result of whether $l^\mu l'_\mu = -m^\mu m'_\mu$ is $+1$ or -1. It is this sign that determines the extra signs in Eq. (A.13) relative to those in Eq. (2.29).

References

1. Christodoulou, D.: Reversible and irreversible transformations in black-hole physics. Phys. Rev. Lett. **25**(22), 1596–1597 (1970)
2. Penrose, R., Floyd, R.M.: Extraction of rotational energy from a black hole. Nat. Phys. Sci. **229**, 177–179 (1971)
3. Carter, B.: Rigidity of a black hole. Nat. Phys. Sci. **238**, 71–72 (1972)
4. Bekenstein, J.D.: Black holes and entropy. Phys. Rev. D **7**(8), 2333–2346 (1973)
5. Bardeen, J.M., Carter, B., Hawking, S.W.: The four laws of black hole mechanics. Commun. Math. Phys. **31**, 161–170 (1973)
6. Hawking, S.: Black hole explosions? Nature **248**, 30–31 (1974)
7. Hawking, S.: Particle creation by black holes. Commun. Math. Phys. **43**, 199–220 (1975)
8. Strominger, A., Vafa, C.: Microscopic origin of the Bekenstein–Hawking entropy. Phys. Lett. B **379**, 99–104 (1996)
9. ’t Hooft, G.: The black hole interpretation of string theory. Nucl. Phys. B **1**, 138–154 (1990)
10. ’t Hooft, G.: The scattering matrix approach for the quantum black hole, an overview. Int. J. Mod. Phys. A **11**, 4623–4688 (1996)
11. Shenker, S.H., Stanford, D.: Black holes and the butterfly effect. JHEP **03**, 67 (2014)
12. Shenker, S.H., Stanford, D.: Stringy effects in scrambling. JHEP **05**, 132 (2015)

13. Kitaev, A.: A simple model of quantum holography. Talk given at the Kavli Institute for Theoretical Physics at the University of California, Santa Barbara (Feb. 12, 2015; Apr. 7, 2015; May 27, 2015)
14. Maldacena, J., Stanford, D., Yang, Z.: Conformal symmetry and its breaking in two-dimensional nearly anti-de Sitter space. Prog. Theor. Exp. Phys. **2016**(12), 12C104 (2016)
15. Kitaev, A., Suh, S.J.: The soft mode in the Sachdev–Ye–Kitaev model and its gravity dual. JHEP **05**, 183 (2018)
16. Maldacena, J., Stanford, D.: Remarks on the Sachdev–Ye–Kitaev model. Phys. Rev. D **94**, 106002 (2016)
17. Strominger, A.: AdS_2 quantum gravity and string theory. JHEP **01**, 8 (1999)
18. Witten, E.: An SYK-like model without disorder. arXiv:1610.09758 [hep-th]
19. Dray, T., 't Hooft, G.: The gravitational shock wave of a massless particle. Nucl. Phys. B **253**, 173–188 (1985)
20. Aichelburg, P.C., Sexl, R.U.: On the gravitational field of a massless particle. Gen. Relativ. Gravit. **2**(4), 303–312 (1971)
21. Penrose, R.: In: O'Raifeartaigh, L. (ed.) General Relativity: Papers in honour of J. L. Synge. Clarendon, Oxford (1972)
22. Alonso, R., Zamorano, N.: Generalized Kerr–Schild metric for a massless particle on the Reissner–Nordström horizon. Phys. Rev. D **35**(6), 1798–1801 (1987)
23. Sfetsos, K.: On gravitational shock waves in curved spacetimes. Nucl. Phys. B **436**, 721–745 (1995)
24. Kiem, Y., Verlinde, H., Verlinde, E.: Black hole horizons and complementarity. Phys. Rev. D **52**(12), 7053–7065 (1995)
25. Polchinski, J.: Chaos in the black hole S-matrix. arXiv:1505.08108 [hep-th]
26. Almheiri, A., Marolf, D., Polchinski, J., Sully, J.: Black holes: complementarity or firewalls? JHEP **2**, 062 (2013)
27. Marolf, D., Polchinski, J.: Gauge-gravity duality and the black hole interior. Phys. Rev. Lett. **111**, 171301 (2013)
28. Abbott, B.P., et al.: Observation of gravitational waves from a binary black hole merger. Phys. Rev. Lett. **116**, 061102 (2016)
29. Balasin, H.: Generalized Kerr–Schild metrics and the gravitational field of a massless particle on the horizon. Class Quantum Grav. **17**, 1913–1920 (2000)
30. Taub, A.H.: Generalized Kerr–Schild space–times. Ann. Phys. **134**, 326–372 (1981)
31. Chandrasekhar, S.: The Mathematical Theory of Black Holes. Oxford University Press, Oxford (1983)
32. Penrose, R., Rindler, W.: Spinors and Space–Time. Cambridge University Press, Cambridge (1987)
33. Newman, E., Penrose, R.: An approach to gravitational radiation by a method of spin coefficients. J. Math. Phys. **3**(566), 566–578 (1962)
34. Geroch, R., Held, A., Penrose, R.: A space–time calculus based on pairs of null directions. J. Math. Phys. **14**(874), 874–881 (1973)
35. Boyer, R.H., Lindquist, R.W.: Maximal analytic extension of the Kerr metric. J. Math. Phys. **8**, 265–281 (1967)
36. Newman, E.T., Janis, A.I.: Note on the Kerr spinning-particle metric. J. Math. Phys. **6**, 915–917 (1965)
37. Sachs, R.K.: Gravitational waves in general relativity VI The outgoing radiation condition. Proc. R. Soc. **264**(1318), 309–338 (1961)
38. Szekeres, P.: On the propagation of gravitational fields in matter. J. Math. Phys. **7**, 751–761 (1966)
39. Sachs, R.K.: Gravitational radiation. In: Relativity Groups and Topology. Lectures Delivered at Les Houches During the 1963 Session of the Summer School of Theoretical Physics, pp. 521–562. DeWitt and DeWitt, published by Gordon & Breach, Science Publishers, Inc. (1964)
40. Dirac, P.A.M.: Forms of relativistic dynamics. Rev. Mod. Phys. **21**(3), 392–399 (1949)
41. Smarr, L.: Surface geometry of charged rotating black holes. Phys. Rev. D **7**(2), 289–295 (1973)
42. Szekeres, P.: The gravitational compass. J. Math. Phys. **6**(9), 1387–1391 (1965)
43. Petrov, A.Z.: The classification of spaces defining gravitational fields. Gen. Rel. Gravit. **32**(8), 1665–1685 (2000) [This is an updated version of: A. Z. Petrov. On spaces defining gravitational fields. *Dokl. Akad. Nauk* SSSR, XXXI (1951) 149–152]
44. Griffiths, J.B.: Colliding Plane Waves in General Relativity. Clarendon Press, Oxford (1991). (Dover reprint 2016)
45. Fels, M., Held, A.: Kerr–Schild rides again. Gen. Rel. Gravit. **21**, 61–68 (1989)
46. Matzner, R.A.: Behavior of ray optics in the Dray–'t Hooft geometry. Nucl. Phys. B **266**, 661–668 (1986)

47. O'Neill, B.: The Geometry of Kerr Black Holes. A K Peters Ltd, Wellesley (1995)
48. Leichenauer, S.: Disrupting entanglement of black holes. Phys. Rev. D **90**, 046009 (2014)