CrossMark

# Are there any models with homogeneous energy density?

**J. Ospino[1] · J. L. Hernández-Pastora[1] · H. Hernández[2,3] · L. A. Núñez[2,3]**

## Abstract
By applying a recent method—based on a tetrad formalism in General Relativity and the orthogonal splitting of the Riemann tensor—to the simple spherical static case, we found that the only static solution with homogeneous energy density is the Schwarzschild solution and that there are no spherically symmetric dynamic solutions consistent with the homogeneous energy density assumption. Finally, a circular equivalence is shown among the most frequent conditions considered in the spherical symmetric case: homogeneous density, isotropy in pressures, conformally flatness and shear-free conditions. We demonstrate that, due to the regularity conditions at the center of the matter distribution, the imposition of two conditions necessarily leads to the static case.

**Keywords** Relativistic fluids · Spherical and non-spherical sources · Interior solutions

## 1 Introduction

The case of a uniform density spherical matter configuration is all but the standard entry point in all textbooks of General Relativity and Relativistic Astrophysics [1–

✉ J. Ospino
j.ospino@usal.es

J. L. Hernández-Pastora
jlhp@usal.es

H. Hernández
hector@ula.ve

L. A. Núñez
lnunez@uis.edu.co

[1] Departamento de Matemática Aplicada and Instituto Universitario de Física Fundamental y Matemáticas, Universidad de Salamanca, Salamanca, Spain

[2] Escuela de Física, Universidad Industrial de Santander, Bucaramanga, Colombia

[3] Departamento de Física, Universidad de Los Andes, Mérida, Venezuela

⚭ Springer

4], presenting the most "simple" interior Schwarzschild solution. Despite its physical inconsistency—it models a fluid with an infinite sound speed—its simplicity is of a pedagogical value in illustrating the methods used in solving physical systems in different (static & dynamic) interesting scenarios [5–13].

In a recent paper [14], by using a tetrad formalism in General Relativity and the orthogonal splitting of the Riemann tensor, we proposed a full set of equations equivalent to the Einstein system which governs the evolution of self-gravitating systems. The formalism was applied to the spherical case to show, through a very simple static case, that it is possible to obtain relevant information from these self-gravitating systems.

The study of the geometric and kinematic properties of timelike congruences is fundamental in the analysis of the evolution of self-gravitating fluids and there it is common the use of a framework based on the known 1+3 formalism [15–21]. In this formalism, any tensor quantity can be split into components along a tangent vector to a timelike congruence, and in its corresponding components orthogonal to it.

The method we used consists of constructing two sets of independent equations, which contain the same information as the Einstein equations, expressed in terms of scalar functions.

As a starting point we choose an orthogonal unitary tetrad, and the first set of evolution equations is obtained from the projection of the Riemann tensor along the unit tetrad. This is equivalent to the use of Ricci identities, which will allow us to define the physical variables and the scalars of the Weyl tensor. The second set of six constrain equations is obtained directly from the Bianchi identities. For the spherical case, solving this system of first order equations will provide us with the necessary information to know the equation of state of a gravitational source with spherical symmetry.

Using this method we shall obtain three results involving isotropic and anisotropic solutions to the Einstein Equations with homogeneous energy density. First we show that the only static solution with homogeneous energy density is the Schwarzschild isotropic solution. Secondly, it is shown there are no spherically symmetric dynamic solution consistent with homogeneous energy density and, for this case the shear-free assumption is equivalent to the isotropic pressure condition. Finally, a circular equivalence is shown among the most frequent conditions considered for the spherical symmetric case: homogeneous density, isotropy in pressures, conformally flatness [22–35] and shear-free conditions [36–44]. It is shown that due to the regularity conditions at the center of the matter distribution, the imposition of any two of them necessarily leads to the static case.

The paper is organized as follows: In Sect. 2 we present the general strategy and the formalism in obtaining all relevant equations. Section 3 develops the analysis for the spherical case and also considers the particular spherical-static case. Finally, in Sect. 4 we present the final conclusions.

## 2 The strategy and general formalism

As we have mentioned above, the strategy we shall follow is to compile two independent sets of equations, expressed in terms of scalar functions, which contain the same information as the Einstein system.

Let us choose an orthogonal unitary tetrad:

$$e_\alpha^{(0)} = V_\alpha, \ e_\alpha^{(1)} = K_\alpha, \ e_\alpha^{(2)} = L_\alpha \ \text{and} \ e_\alpha^{(3)} = S_\alpha. \tag{1}$$

As usual, $\eta_{(a)(b)} = g_{\alpha\beta} e_{(a)}^\alpha e_{(b)}^\beta$, with $a = 0, \ 1, \ 2, \ 3$, i.e. latin indices label different vectors of the tetrad. Thus, the tetrad satisfies the standard relations:

$$V_\alpha V^\alpha = -K_\alpha K^\alpha = -L_\alpha L^\alpha = -S_\alpha S^\alpha = -1 \,,$$
$$V_\alpha K^\alpha = V_\alpha L^\alpha = V_\alpha S^\alpha = K_\alpha L^\alpha = K_\alpha S^\alpha = S_\alpha L^\alpha = 0 \,.$$

With the above tetrad (1) we shall also define the corresponding directional derivatives operators

$$f^\bullet = V^\alpha \partial_\alpha f; \quad f^\dagger = K^\alpha \partial_\alpha f \quad \text{and} \quad f^* = L^\alpha \partial_\alpha f. \tag{2}$$

The first set can be considered purely geometrical and emerges from the projection of the Riemann tensor along the tetrad [45], i.e.

$$2V_{\alpha\,;[\beta;\gamma]} = R_{\delta\alpha\beta\gamma} V^\delta \,, \quad 2K_{\alpha\,;[\beta;\gamma]} = R_{\delta\alpha\beta\gamma} K^\delta \,,$$
$$2L_{\alpha\,;[\beta;\gamma]} = R_{\delta\alpha\beta\gamma} L^\delta \quad \text{and} \quad 2S_{\alpha\,;[\beta;\gamma]} = R_{\delta\alpha\beta\gamma} S^\delta \,; \tag{3}$$

where $e_{\alpha\,;\beta\gamma}^{(a)}$ are the second covariant derivatives of each tetrad (6) vector indicated with $a = 0, 1, 2, 3$.

The second set emerges from the Bianchi identities:

$$R_{\alpha\beta[\gamma\delta\,;\mu]} = R_{\alpha\beta\gamma\delta\,;\mu} + R_{\alpha\beta\mu\gamma\,;\delta} + R_{\alpha\beta\delta\mu\,;\gamma} = 0 \tag{4}$$

## 3 Spherical case

In this section we shall present the relevant equations, for the spherically symmetric, locally anisotropic, dissipative, collapsing matter configuration, written in terms of the kinematical variables: the four acceleration $a_\alpha$, the expansion scalar $\Theta$, the shear tensor $\sigma$ and some scalars functions (the structure scalars related to the splitting of the Riemann Tensor).

### 3.1 The tetrad, the source and kinematical variables

To proceed with the above objective we shall restrict to a spherically symmetric line element given by

$$ds^2 = -A^2 dt^2 + B^2 dr^2 + R^2 (d\theta^2 + \sin^2(\theta) d\phi^2) \,, \tag{5}$$

where the coordinates are: $x^0 = t$, $x^1 = r$, $x^2 = \theta$, and $x^3 = \phi$; with $A(t, r)$, $B(t, r)$ and $R(t, r)$ functions of their arguments.

For this case the tetrad can written as:

$$V_\alpha = (-A, 0, 0, 0), \ K_\alpha = (0, B, 0, 0), \ L_\alpha = (0, 0, R, 0),$$
$$\text{and} \quad S_\alpha = (0, 0, 0, R\sin(\theta)). \tag{6}$$

The covariant derivatives of the orthonormal tetrad are:

$$V_{\alpha;\beta} = -a_1 K_\alpha V_\beta + \sigma_1 K_\alpha K_\beta + \sigma_2 (L_\alpha L_\beta + S_\alpha S_\beta),$$
$$K_{\alpha;\beta} = -a_1 V_\alpha V_\beta + \sigma_1 V_\alpha K_\beta + J_1 (L_\alpha L_\beta + S_\alpha S_\beta),$$
$$L_{\alpha;\beta} = \sigma_2 V_\alpha L_\beta - J_1 K_\alpha L_\beta + J_2 S_\alpha S_\beta \quad \text{and}$$
$$S_{\alpha;\beta} = \sigma_2 V_\alpha S_\beta - J_1 K_\alpha S_\beta - J_2 L_\alpha S_\beta. \tag{7}$$

where $J_1$, $J_2$, $\sigma_1$, $\sigma_2$ and $a_1$ are expressed in terms of the metric functions and their derivatives as:

$$J_1 = \frac{1}{B}\frac{R'}{R}, \quad J_2 = \frac{1}{R}\cot(\theta), \quad \sigma_1 = \frac{1}{A}\frac{\dot{B}}{B}, \quad \sigma_2 = \frac{1}{A}\frac{\dot{R}}{R}$$
$$\text{and} \quad a_1 = \frac{1}{B}\frac{A'}{A}, \tag{8}$$

with primes and dots representing radial and time derivatives, respectively.

As we mentioned before we shall assume our source as a bounded, spherically symmetric, locally anisotropic, dissipative, collapsing matter configuration, described by a general energy momentum tensor, written in the "canonical" form, as:

$$T_{\alpha\beta} = (\rho + P)V_\alpha V_\beta + P g_{\alpha\beta} + \Pi_{\alpha\beta} + \mathcal{F}_\alpha V_\beta + \mathcal{F}_\beta V_\alpha. \tag{9}$$

It is immediately seen that the physical variables can be defined—in the Eckart frame where fluid elements are at rest—as:

$$\rho = T_{\alpha\beta}V^\alpha V^\beta, \quad \mathcal{F}_\alpha = -\rho V_\alpha - T_{\alpha\beta}V^\beta, \quad P = \frac{1}{3}h^{\alpha\beta}T_{\alpha\beta}$$
$$\text{and} \quad \Pi_{\alpha\beta} = h^\mu_\alpha h^\nu_\beta \left(T_{\mu\nu} - P h_{\mu\nu}\right), \tag{10}$$

with $h_{\mu\nu} = g_{\mu\nu} + V_\nu V_\mu$.

As can be seen from the condition $\mathcal{F}^\mu V_\mu = 0$, and the symmetry of the problem, Einstein Equations imply $T_{03} = 0$, thus:

$$\mathcal{F}_\mu = \mathcal{F}K_\mu \quad \Leftrightarrow \quad \mathcal{F}_\mu = \left(0, \frac{\mathcal{F}}{B}, 0, 0\right). \tag{11}$$

Clearly $\rho$ is the energy density (the eigenvalue of $T_{\alpha\beta}$ for eigenvector $V^\alpha$), $\mathcal{F}_\alpha$ represents the energy flux four vector; $P$ corresponds to the isotropic pressure, and $\Pi_{\alpha\beta}$ is the anisotropic tensor, which can be expressed as

$$\Pi_{\alpha\beta} = \Pi_1 \left( K_\alpha K_\beta - \frac{h_{\alpha\beta}}{3} \right), \tag{12}$$

with

$$\Pi_1 = \left( 2K^\alpha K^\beta + L^\alpha L^\beta \right) T_{\alpha\beta}. \tag{13}$$

Finally, we shall express the kinematical variables (the four acceleration, the expansion scalar and the shear tensor) for a self-gravitating fluid as:

$$a_\alpha = V^\beta V_{\alpha;\beta} = aK_\alpha = \left( 0, \frac{A'}{A}, 0, 0 \right), \tag{14}$$

$$\Theta = V^\alpha_{;\alpha} = \frac{1}{A} \left( \frac{\dot{B}}{B} + \frac{2\dot{R}}{R} \right), \tag{15}$$

$$\sigma = \frac{1}{A} \left( \frac{\dot{B}}{B} - \frac{\dot{R}}{R} \right). \tag{16}$$

### 3.2 The orthogonal splitting of the Riemann tensor and structure scalars

In this section we shall introduce a set of scalar functions—the structure scalars—obtained from the orthogonal splitting of the Riemann tensor (see [46–48]) which has proven to be very useful in expressing the Einstein Equations.

Following [46], we can express the splitting of the Riemann tensor as:

$$
\begin{aligned}
R_{\alpha\beta\mu\nu} &= 2V_\mu V_{[\alpha} Y_{\beta]\nu} + 2h_{\alpha[\nu} X_{\mu]\beta} + 2V_\nu V_{[\beta} Y_{\alpha]\mu} \\
&\quad + h_{\beta\nu}(X_0\, h_{\alpha\mu} - X_{\alpha\mu}) + h_{\beta\mu}(X_{\alpha\nu} - X_0\, h_{\alpha\nu}) \\
&\quad + 2V_{[\nu} Z^\gamma_{\mu]} \varepsilon_{\alpha\beta\gamma} + 2V_{[\beta} Z^\gamma_{\alpha]} \varepsilon_{\mu\nu\gamma},
\end{aligned}
\tag{17}
$$

with $\varepsilon_{\mu\nu\gamma} = \eta_{\phi\mu\nu\gamma} V^\phi$, and $\eta_{\phi\mu\nu\gamma}$ the Levi-Civita 4-tensor. The corresponding Ricci contraction for the above Riemann tensor can also be written as:

$$
\begin{aligned}
R_{\alpha\mu} &= Y_0\, V_\alpha V_\mu - X_{\alpha\mu} - Y_{\alpha\mu} + X_0\, h_{\alpha\mu} + Z^{\nu\beta} \varepsilon_{\mu\nu\beta} V_\alpha \\
&\quad + V_\mu Z^{\nu\beta} \varepsilon_{\alpha\nu\beta};
\end{aligned}
\tag{18}
$$

where the quantities: $Y_{\alpha\beta}$, $X_{\alpha\beta}$ and $Z_{\alpha\beta}$ can be expressed as

$$Y_{\alpha\beta} = \frac{1}{3} Y_0\, h_{\alpha\beta} + Y_1 \left[ K_\alpha K_\beta - \frac{1}{3} h_{\alpha\beta} \right], \tag{19}$$

$$X_{\alpha\beta} = \frac{1}{3} X_0\, h_{\alpha\beta} + X_1 \left[ K_\alpha K_\beta - \frac{1}{3} h_{\alpha\beta} \right] \tag{20}$$

and

$$Z_{\alpha\beta} = Z \left( L_\alpha S_\beta - L_\beta S_\alpha \right), \tag{21}$$

with

$$
\begin{aligned}
Y_0 &= 4\pi(\rho + 3P), \quad Y_1 = \mathcal{E}_1 - 4\pi\Pi_1, \quad X_0 = 8\pi\rho, \\
X_1 &= -(\mathcal{E}_1 + 4\pi\Pi_1) \quad \text{and} \quad Z = 4\pi\mathcal{F},
\end{aligned}
\tag{22}
$$

and the electric part of the Weyl tensor is written as

$$
E_{\alpha\beta} = C_{\alpha\nu\beta\delta}V^\nu V^\delta = \mathcal{E}_1\left[K_\alpha K_\beta - \frac{1}{3}h_{\alpha\beta}\right].
\tag{23}
$$

### 3.3 Projections of Riemann tensor

From the above system (3) (by using the covariant derivative of equations (7) and the projections of the orthogonal splitting of the Riemann tensor) we can obtain the first set of independent equations, for the spherical case, in terms of $J_1$, $J_2$, $\sigma_1$, $\sigma_2$, and $a_1$, (defined in (8)) and their directional derivatives, i.e.

$$
\sigma_1^\bullet - a_1^\dagger - a_1^2 + \sigma_1^2 = -\frac{1}{3}(Y_0 + 2Y_1),
\tag{24}
$$

$$
\sigma_2^\bullet + \sigma_2^2 - a_1 J_1 = \frac{1}{3}(Y_1 - Y_0),
\tag{25}
$$

$$
\sigma_2^\dagger + J_1(\sigma_2 - \sigma_1) = -Z,
\tag{26}
$$

$$
J_1^\bullet + J_1\sigma_2 - a_1\sigma_2 = -Z,
\tag{27}
$$

$$
J_1^\dagger + J_1^2 - \sigma_1\sigma_2 = \frac{1}{3}(X_1 - X_0),
\tag{28}
$$

$$
J_2^\bullet + J_2\sigma_2 = 0,
\tag{29}
$$

$$
J_2^\dagger + J_1 J_2 = 0 \quad \text{and}
\tag{30}
$$

$$
J_1^2 - \frac{1}{R^2} - \sigma_2^2 = -\frac{1}{3}(X_0 + 2X_1).
\tag{31}
$$

### 3.4 Equations from Bianchi identities

The second set of equations for the spherical case, emerge from the independent Bianchi identities (4), and can be written as:

$$
\begin{aligned}
&a_1[-X_0 + X_1 - Y_0 + Y_1] + 3J_1 Y_1 + 3Z^\bullet \\
&\quad + 6Z\sigma_1 + 3Z\sigma_2 - Y_0^\dagger + Y_1^\dagger = 0,
\end{aligned}
\tag{32}
$$

$$
\begin{aligned}
&X_0^\bullet - X_1^\bullet - 6a_1 Z - 3J_1 Z + [Y_0 - Y_1 - X_1]\sigma_1 \\
&\quad + [Y_0 + 2Y_1 - X_1]\sigma_2 + X_0[\sigma_1 + \sigma_2] - 3Z^\dagger = 0,
\end{aligned}
\tag{33}
$$

$$
\begin{aligned}
&X_0^\bullet + 2X_1^\bullet + 2X_0\sigma_2 - 6J_1 Z \\
&\quad + [4X_1 + 2Y_0 - 2Y_1]\sigma_2 = 0,
\end{aligned}
\tag{34}
$$

$$
X_0^\dagger + 2X_1^\dagger + 6J_1 X_1 + 6Z\sigma_2 = 0.
\tag{35}
$$

### 3.5 The static case

In the line element (5) we can assume, without any loss of generality, $R = r$ and integrate (25) to obtain:

$$A = C_1 e^{\int B^2 r(Y_0 - Y_1) dr} .$$ (36)

Next, from (27) it follows at once that:

$$B^2 = \frac{1}{1 - \frac{r^2}{3}(X_0 + 2X_1)} ,$$ (37)

where $C_1$ is a constant of integration. These metric elements (36) and (37) expressed in terms of the structure scalars $X_1$ and $Y_0 - Y_1$, describe any static anisotropic sphere (see reference [49]).

### 3.6 Models with homogeneous energy density

It is easy to check that the Schwarzschid interior solution corresponds to the case $X_1 = Y_1 = 0$, and it follows clearly from (22) that $\mathcal{E}_1 = \Pi_1 = 0$.

Now, let us show that if we have the homogeneous energy density premise, the only possible outcome is the isotropic Schwarzschild solution. Thus, let us consider models with homogeneous energy density

$$X_0 = 8\pi\rho = cte$$ (38)

and study the consequences derived from this assumption, under certain physically reasonable circumstances. First, taking into account (38) and integrating Eq. (35), we obtain

$$X_1 = \frac{C}{r^3}.$$ (39)

Next consider the regularity condition at $r = 0$ via (39), which implies $C = 0$, then:

$$\mathcal{E}_1 + 4\pi\Pi_1 = 0.$$ (40)

Also from Eq. (32), taking into account (25) and (37) we get

$$P_r' = -4\pi r \frac{\left(P_r^2 + \frac{4}{3}\rho P_r + \frac{1}{3}\rho^2\right)}{1 - \frac{8\pi}{3}\rho r^2} - \frac{2}{r}\Pi_1 .$$ (41)

Clearly, if the anisotropic term ($\Pi_1$) is zero at a point other than the origin, it will be zero at all points [50]. If $\Pi_1(r)$ does not vanish it must be positive or negative and, from Eq. (40), the same thing will be true for Weyl's scalar $\mathcal{E}_1$. But, given that $\mathcal{E}_1(0) = 0$, by the conditions of regularity at the origin, and $\mathcal{E}_1(R_\Sigma) = P_t > 0$, by the boundary conditions, we find that $\mathcal{E}_1(r) > 0$. On the other hand, if $\Pi_1 = -\Delta < 0$, from (41) we find that there is a $r_c < r_\Sigma$ given by

$$r_c^2 = \frac{3\Delta}{2\pi(3P_r^2 + 4\rho P_r + \rho^2 + 4\rho\Delta)}, \tag{42}$$

for which $P_r' = 0$, showing that the minimum radial pressure is reached in a $r_c$ smaller than $r_\Sigma$. Thus, we conclude that $\Pi_1(r)$ and $\mathcal{E}_1(r)$ must vanish, i.e.

$$\Pi_1(r) > 0 \text{ and } \mathcal{E}_1(r) > 0 \implies \Pi_1(r) = \mathcal{E}_1(r) = 0. \tag{43}$$

Therefore, we can see that the only static solution with homogeneous energy density, under the above considerations, is necessarily the Schwarzschild solution. More over, if we require that the circumference $2\pi R$ of an infinitesimal sphere about the origin be just $2\pi$ times its proper radius $Bdr$, that is [51]

$$B(t, r) = R'(t, r) \text{ when } r \to 0 \tag{44}$$

Now, replacing (44) into (31), in the static case, we get

$$X_0(0) = 0!!! \tag{45}$$

From (45) we conclude that the models with homogeneous energy density do not satisfy the Euclidean condition (44).

### 3.7 The non-static case

#### 3.7.1 Regularity on the origin

To guarantee a good asymptotic behavior of the metric (5), in the vicinity of the origin, we must demand that the functions $A(r, t)$, $B(r, t)$ and $R(r, t)$, have the following analytical form, from the standard Taylor expansion:

$$\begin{aligned} A(r, t) &= \alpha_0 + \alpha_1(t)r + \alpha_2(t)r^2 + \cdots \\ B(r, t) &= 1 + \beta_1(t)r + \beta_2(t)r^2 + \cdots \\ R(r, t) &= r + \gamma_1(t)r^2 + \gamma_2(t)r^3 + \cdots \end{aligned} \tag{46}$$

#### 3.7.2 The case $X_0 = X_0(t)$ and $Z = 0$

In this case, we obtain from Eq. (35) that $X_1 = 0$ and Eqs. (32)–(34) become:

$$a_1[Y_1 - X_0 - Y_0] + 3J_1Y_1 = [Y_0 - Y_1]^\dagger, \tag{47}$$

$$X_0^\bullet + [X_0 + Y_0 - Y_1]\sigma_1 + [X_0 + Y_0 + 2Y_1]\sigma_2 = 0, \tag{48}$$

$$X_0^\bullet + 2[X_0 + Y_0 - Y_1]\sigma_2 = 0. \tag{49}$$

Notice that if $X_0 =$ cte.,then from the Eq. (49), it follows that $\sigma_2 = 0$, and we get the static case (iii) analyzed in [14].

Next, combining Eqs. (48) and (49) we find that

$$[Y_0 + X_0 - Y_1][\sigma_1 - \sigma_2] + 3Y_1\sigma_2 = 0, \tag{50}$$

or

$$[\rho + P_r][\sigma_1 - \sigma_2] = 2\Pi_1\sigma_2, \tag{51}$$

where from equation (22) and $X_1 = 0$. From the Eq. (51), it follows that

$$\sigma_1 - \sigma_2 = 0 \iff \Pi_1 = 0. \tag{52}$$

In other words, the shear-free and isotropic pressure conditions are equivalent, for non-dissipative fluids with homogeneous energy density.

Evaluating Eq. (49) at $r = 0$ and taking into account (46) we obtain

$$\sigma_1 = \sigma_2 = 0 \implies X_0 = cte., \tag{53}$$

therefore it follows that there are no spherically symmetric dynamic solutions with homogeneous energy density.

### 3.8 Circular conditions

In this section we shall prove the equivalence of the following circular conditions taken two by two:

- Homogeneous energy density, $X_0 = X_0(t)$ ①
- Isotropy in the pressures, $\Pi_1 = 0$ ②
- Conformally flat, $\mathcal{E} = 0$ ③
- Shear-free condition, $\sigma_1 = \sigma_2$ ④

① **and** ② ↔ ③ **and** ④. . If we assume ① and ②, then from (35) we find that $X_1 = 0 \implies Y_1 = 0 \implies$ ③, and by using Eq. (50) we get $\sigma_1 = \sigma_2, \implies$ ④

On the other hand, if we assume ③ and ④, with the result obtained in [14],

$$\sigma_1 = \sigma_2 \implies a_1 = 0, \tag{54}$$

and from the subtraction of (24) from (25), we find $Y_1 = X_1 = 0 \implies$ ②, and from (50) we obtain $X_0 = X_0(t) \implies$ ①.

① **and** ③ ↔ ② **and** ④. Now if we assume ① and ③, we find from (35) that $X_1 = 0 \implies Y_1 = 0 \implies$ ②. Again, by using Eq. (50) we get $\sigma_1 = \sigma_2, \implies$ ④. On the other hand, if we assume ② and ④, using (54) and again, substracting (24) from (25), we find $Y_1 = X_1 = 0 \implies$ ③, now with (50), we obtain $X_0 = X_0(t) \implies$ ①.

① **and** ④ ↔ ② **and** ③. If we assume① and ④, then from (35) and (50) we obtain $X_1 = Y_1 = 0 \implies$ ② and ③. On the other hand, if we assume ② and ③,then

$$X_1 = Y_1 = 0, \tag{55}$$

by substituting (55) into (35) we obtain

$$X_0 = X_0(t). \Rightarrow ①$$ (56)

Next, from (56) and by replacing (55) in (50) we find $\sigma_1 = \sigma_2 \Rightarrow$ ④. Notice that if we also take into account (46), in all cases previously considered, we only get the static case.

## 4 Final remarks

We have found that, despite its simplicity and pedagogical interest, the uniform density spherical matter configuration is a very restricted and unphysical solution to the Einstein Equations.

As we have stated above in this short paper we have presented several results concerning the homogeneous energy density assumption for isotropic and anisotropic solutions to the Einstein Equations. First, we have shown that if the regularity condition at the center of the distribution and some other physical reasonable boundary condition at the surface of the distribution are to be satisfied, then the only static solution for a spherically symmetric matter distribution with homogeneous energy density is the Schwarzschild isotropic solution. This rules out any anisotropic generalization for $\rho = const$ found in the literature [8,11] and complements the proof for the classic problem that a static perfect fluid star should be spherically symmetric for physically reasonable isotropic equation of state [52–55]. More over, we have shown that even for the static homogeneous Schwarschild solution the center of the matter distribution has to be excluded because it does not satisfy the Euclidean condition. Clearly, is possible to obtain viable solutions if this condition is relaxed assuming a core-envelope model (see [56] and references therein).

Secondly, it is shown there are no spherically symmetric dynamic solution consistent with homogeneous energy density and, for this case the shear-free assumption is equivalent to the isotropic pressure condition.

Finally, we have considered the most frequent conditions assumed in a spherical symmetric case: homogeneous density, isotropy in pressures, conformally flatness and shear-free conditions (see [5–13,49,57–60] and reference therein). It is found that the two of these assumptions are necessarily equivalent to the other remainder two. Additionally, it is demonstrated that, due to the regularity conditions at the center of the matter distribution, the imposition of two of them necessarily leads to the static case.

Again, we have shown that the most simple and "pedagogic" spherical matter solution—$\rho = const$—is very restricted and unphysical, but there has been much recent work with variable energy densities, satisfying all physical criteria, that seems to correspond to more realistic matter configurations [61,62].

## References

1. Weinberg, S.: Gravitation and Cosmology: Principles and Applications of the General Theory of Relativity. Wiley, New York (1972)
2. Shapiro, S.L., Teukolsky, S.A.: The Physics of Compact Objects. Wiley, New York (1983)
3. Schutz, B.F.: A First Course in General Relativity. Cambridge University Press, Cambridge (2009)
4. Misner, C.W., Thorne, K.S., Wheeler, J.A.: Gravitation. Princeton University Press, Princeton (2017)
5. Wyman, M.: Radially symmetric distributions of matter. Phys. Rev. **75**(12), 1930–1936 (1949)
6. Bonnor, W.B., Faulkes, M.C.: Exact solutions for oscillating spheres in general relativity. Mon. Not. R. Astron. Soc. **137**, 239 (1967)
7. Misra, R.M., Srivastava, D.C.: Dynamics of fluid spheres of uniform density. Phys. Rev. D **8**(6), 1653 (1973)
8. Bowers, R.L., Liang, E.P.T.: Anisotropic spheres in general relativity. Astrophys. J. **188**, 657 (1974)
9. Ponce de Leon, J.: Fluid spheres of uniform density in general relativity. J. Math. Phys. **27**(1), 271–276 (1986)
10. Herrera, L., Núñez, L.A.: Propagation of a shock wave in a radiating spherically symmetric distribution of matter. Astrophys. J. **319**, 868–884 (1987)
11. Maharaj, S.D., Maartens, R.: Anisotropic spheres with uniform energy density in general relativity. Gen. Relativ. Gravit. **21**(9), 899–905 (1989)
12. Herrera, L., Núñez, L.A.: Modeling "hydrodynamic phase transitions" in a radiating spherically symmetric distribution of matter. Astrophys. J. **339**, 339–353 (1989)
13. Rueda, J.A., Núñez, L.A.: General relativistic radiant shock waves in the post-quasistatic approximation. In Apostolopoulos, P., Bona, C., Carot, J., Mas, Ll., Sintes, A.M., Stela, J. (eds.) Einstein's Legacy: From the Theoretical Paradise to Astrophysical Observations, volume 66 of Journal of Physics: Conference Series, p. 012042, London UK, 2007. XXIXth Spanish Relativity Meeting (ERE 2006), Institute of Physics Publishing (2007)
14. Ospino, J., Hernández-Pastora, J. L., Núñez, L. A.: An equivalent system of Einstein equations. In Journal of Physics Conference Series, volume 831 of Journal of Physics, Conference Series, p. 012011, March (2017)
15. Ellis, G.F.R.: Relativistic cosmology. In: Sachs, R.K. (ed.) General Relativity and Cosmology, pp. 104–182. Academic Press, New York (1971)
16. Ellis, G.F.R., Bruni, M.: Covariant and gauge-invariant approach to cosmological density fluctuations. Phys. Rev. D **40**, 1804–1818 (1989)
17. Ehlers, J.: Contributions to the relativistic mechanics of continuous media. Gen. Relativ. Gravit. **25**, 1225–1266 (1993)
18. Wainwright, J., Ellis, G.F.R.: Dynamical Systems in Cosmology. Cambridge University Press, Cambridge (2005)
19. Ellis, G.F.R., van Elst, H.: Cosmological models (Cargèse lectures 1998). In: Lachièze-Rey, M. (ed.) NATO Advanced Science Institutes (ASI) Series C, volume 541 of *Series C*, pp. 1–116 (1999)
20. Tsagas, C.G., Challinor, A., Maartens, R.: Relativistic cosmology and large-scale structure. Phys. Rep. **465**, 61–147 (2008)
21. Ellis, G.F.R., Maartens, R., MacCallum, M.A.H.: Relativistic Cosmology. Cambridge University Press, Cambridge (2012)
22. Stewart, B.W.: Flat anisotropic spheres in general relativity. J. Phys. A Math. Gen. **15**(8), 2419 (1982)
23. Wainwright, J.: Power law singularities in orthogonal spatially homogeneous cosmologies. Gen. Relativ. Gravit. **16**(7), 657 (1984)
24. Goode, S.W., Wainwright, J.: Isotropic singularities in cosmological models. Class. Quantum Gravity **2**(1), 99 (1985)
25. Bonnor, W.B.: The gravitational arrow of time. Phys. Lett. A **112**, 26 (1985)
26. Bonnor, W.B.: Arrow of time for a collapsing radiating sphere. Phys. Lett. A **122**, 305 (1987)
27. Ponce de Leon, J.: Gravitational repulsion in sources of the Reissner–Nordstrom field. J. Math. Phys. **29**(1), 197 (1988)

28. Goode, S.W., Coley, A.A., Wainwright, J.: The isotropic singularity in cosmology. Class. Quantum Gravity **9**(2), 445 (1992)
29. Herreran, L., Di Prisco, A., Ospino, J., Fuenmayor, E.: Conformally flat anisotropic spheres in general relativity. J. Math. Phys. **42**, 2129 (2001)
30. Herrera, L.: The Weyl tensor and equilibrium configurations of selfgravitating fluids. Gen. Relativ. Gravit. **35**, 437 (2003)
31. Herrera, L., Di Prisco, A., Ospino, J., Fuenmayor, E.: Spherically symmetric dissipative anisotropic fluids: a general study. Phys. Rev. D **69**, 084026 (2004)
32. Grøn, Ø., Johannesen, S.: Conformally flat spherically symmetric spacetimes. Eur. Phys. J. Plus **128**, 92 (2013)
33. Manjonjo, A.M., Maharaj, S.D., Moopanar, S.: Conformally vectors and stellar models. Eur. Phys. J. Plus **132**, 62 (2017)
34. Manjonjo, A.M., Maharaj, S.D., Moopanar, S.: Static models with conformal symmetry. Class. Quantum Gravity **35**(4), 045015 (2018)
35. Ivanov, B.V.: Conformally flat realistic anisotropic model for a comppact star. Eur. Phys. J. C **78**, 332 (2018)
36. Collins, C.B., Wainwright, J.: Role of the shear in general-relativistic cosmological and stellar models. Phys. Rev. D **27**, 1209 (1983)
37. Glass, E.N.: Shearfree gravitational collapse. J. Math. Phys. **20**, 1508 (1979)
38. Stephani, H., Wolf, T.: Spherically symmetric perfect fluids in shear-free motion—the symmetry approach. Class. Quantum Gravity **13**, 1261 (1996)
39. Govinder, K.S., Govender, M., Maartens, R.: On radiating stellar collapse with shear. Mon. Not. R. Astron. Soc. **299**(3), 809 (1998)
40. Joshi, P.S., Dadhich, N., Maartens, R.: Why do naked singularities form in gravitational collapse ? Phys. Rev. D **65**, 101501 (2002)
41. Joshi, P.S., Goswami, R., Dadhich, N.: The critical role of shear in gravitational collapse. gr-qc/0308012
42. Herrera, L., Le Denmat, G., Santos, N.O., Wang, A.: Shear-free radiating collapse and conformal flatness. Int. J. Modern Phys. D **13**(04), 583–592 (2004)
43. Herrera, L., Santos, N.O., Wang, A.: Shearing expansion-free spherical anisotropic fluid evolution. Phys. Rev. D **78**, 084026 (2008)
44. Herrera, L., Di Prisco, A., Ospino, J.: Shear-free axially symmetric dissipative fluids. Phys. Rev. D **89**(12), 127502 (2014)
45. Wald, R.M.: General Relativity. University of Chicago Press, Chicago (2010)
46. Gómez-Lobo, A.García-Parrado: Dynamical laws of superenergy in general relativity. Class. Quantum Gravity **25**(1), 015006 (2007)
47. Herrera, L., Ospino, J., Di Prisco, A., Fuenmayor, E., Troconis, O.: Structure and evolution of self-gravitating objects and the orthogonal splitting of the Riemann tensor. Phys. Rev. D **79**(6), 064025 (2009)
48. Herrera, L., Di Prisco, A., Ibáñez, J., Ospino, J.: Dissipative collapse of axially symmetric, general relativistic sources: a general framework and some applications. Phys. Rev. D **89**(8), 084034 (2014)
49. Herrera, L., Ospino, J., Di Prisco, A.: All static spherically symmetric anisotropic solutions of Einstein's equations. Phys. Rev. D **D77**, 027502 (2008)
50. Herrera, L., Santos, N.O.: Local anisotropy in self-gravitating systems. Phys. Rep. **286**(2), 53–130 (1997)
51. Misner, C.W., Sharp, D.H.: Relativistic equations for adiabatic, spherically symmetric gravitational collapse. Phys. Rev. **136**, 571–576 (1964)
52. Lindblom, L.: Static uniform-density stars must be spherical in general relativity. J. Math. Phys. **29**(2), 436–439 (1988)
53. Lindblom, L., Masood-ul Alam, A.K.M.: On the spherical symmetry of static stellar models. Commun. Math. Phys. **162**(1), 123–145 (1994)
54. Masood-ul Alam, A.K.M.: Proof that static stellar models are spherical. Gen. Relativ. Gravit. **39**(1), 55–85 (2007)
55. Pfister, H.: A new and quite general existence proof for static and spherically symmetric perfect fluid stars in general relativity. Class. Quantum Gravity **28**(7), 075006 (2011)
56. Takisa, P.M., Maharaj, S.D.: Anisotropic charged core envelope star. Astrophys. Space Sci. **361**(8), 262 (2016)

57. Lake, K.: All static spherically symmetric perfect-fluid solutions of Einstein's equations. Phys. Rev. D **67**(10), 104015 (2003)
58. Rahman, S., Visser, M.: Spacetime geometry of static fluid spheres. Class. Quantum Gravity **19**(5), 935 (2002)
59. Herrera, L., Di Prisco, A., Ospino, J.: On the stability of the shear-free condition. Gen. Relativ. Gravit. **42**(7), 1585–1599 (2010)
60. Herrera, L., Di Prisco, A., Ospino, J., Fuenmayor, E.: Conformally flat anisotropic spheres in general relativity. J. Math. Phys. **42**, 2129–2143 (2001)
61. Kileba Matondo, D., Maharaj, S.D., Ray, S.: Relativistic stars with conformal symmetry. Eur. Phys. J. C **78**(6), 437 (2018)
62. Maurya, S.K., Maharaj, S.D.: New anisotropic fluid spheres from embedding. Eur. Phys. J. A **54**, 1–11 (2018)