

# Motion of charged particles in a NUTty Einstein–Maxwell spacetime and causality violation

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**Abstract** We investigate the motion of electrically charged test particles in spacetimes with closed timelike curves, a subset of the black hole or wormhole Reissner–Nordstr m–NUT spacetimes without periodic identification of time. We show that, while in the wormhole case there are closed worldlines inside a potential well, the worldlines of initially distant charged observers moving under the action of the Lorentz force can never close or self-intersect. This means that for these observers causality is preserved, which is an instance of our weak chronology protection criterion.

**Keywords** Einstein–Maxwell · NUT · Charged test particle · Causality

## 1 Introduction

A variety of solutions to the equations of general relativity describe spacetimes with closed timelike curves (CTCs). The common view regarding these is that they violate causality, “for, one could imagine that with a suitable rocketship one could travel round such a curve” [1] and eventually return to one’s original spacetime position

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after a finite proper time lapse, thus opening the possibility for time travel. In [2], Hawking examined this possibility and gave a number of arguments which strongly support the chronology protection conjecture: “The laws of physics do not allow the appearance of closed timelike curves”.

Basically, what Hawking proved is that the creation of CTCs in a finite region of an initially causal (CTC-less) spacetime is classically forbidden by the average weak energy condition. If this is waived, then quantum back-reaction effects come to the rescue to prevent the appearance of CTCs. However these arguments do not rule out the possibility of spacetimes with eternal CTCs. Such a possibility was first considered by Gödel [3]. CTCs exist in the inner region ( $r < 0$ ) of the Kerr or Kerr–Newman spacetimes. Other stationary solutions of the Einstein–Maxwell equations which present CTCs are the Taub-NUT spacetime [4,5] and its electromagnetic extensions, the Reissner–Nordström-NUT or Brill spacetimes [6]. These are characterized by a gravimagnetic or NUT charge and, similarly to the case of the magnetic monopole with its Dirac string, present one or two line metric singularities known as Misner strings. These can be removed if time is periodically identified with a period proportional to the NUT charge [7]. However there are then CTCs everywhere in the stationary sector, so that the spacetime is definitely acausal. Another drawback of this periodicity in time is that the spacetime cannot be consistently extended beyond the second (interior) horizon. The other option is to abstain from this periodic identification, and so to retain the Misner string singularities. As implied in [8], and shown explicitly in [9,10], the Brill spacetimes are then geodesically complete, i.e. the Misner string singularities do not show up in the geodesic motion. However there are still CTCs in a stationary neighborhood of the Misner strings.

The point of view advocated in [9,10] is that the presence of closed timelike curves does not in itself give rise to causality violations, which can arise only if it is possible for an observer to follow such CTCs. Freely falling observers follow geodesics, so a necessary condition for causality preservation should be that the spacetime should be free from closed timelike geodesics (CTGs). These have been shown to be absent from the Taub-NUT spacetime (without time periodicity) in [9], and from the Brill spacetimes in [10], provided the parameter fixing the strength of the Misner strings is chosen in a certain range. Another class of observers are not freely falling because they carry electric charge, and so travel in an Einstein–Maxwell spacetime as charged test particles following the Lorentz force law. Causality will be preserved if their worldlines are not closed or self-intersecting. Such a question, i.e. the possibility for a charged particle to move along a CTC in the Gödel universe under the effect of a weak sourceless magnetic field, was previously addressed in [11]. The existence of closed worldlines (CWLs) of a charged particle was investigated in [10] in the case of a very special Brill spacetime. It was found that CWLs do occur inside a potential well, but it was argued that the worldlines followed by charged observers which are initially distant (and thus outside the potential well) will necessarily be causal. Finally, more adventurous observers could try to follow CTCs by using a “suitable rocketship”, but one can argue that the (classical) back-reaction of the energy expended to follow a non-geodesic path (in the case of an uncharged spaceship) would ultimately be so large that it would deform the background spacetime geometry in such a way as to preserve chronology. If the arguments concerning these various instances of observer

wordlines prove to be correct, then we can formulate the weak chronology protection criterion (WCP):

Spacetimes with closed timelike curves do not classically violate causality, if no worldline followed by an initially distant observer can possibly be closed or self-intersecting.

The purpose of the present paper is to study the motion of charged particles, with the question of CWLs in mind, in a class of Brill spacetimes more general than that considered in [10]. This motion was recently analyzed in [12] in the case of the electric Kerr–Newman–NUT spacetime, but the question of causality was not investigated there. Here, in order to have a simple tractable form for the effective potential, we restrict to the case of the massless magnetic Reissner–Nordström–NUT spacetime, considering the whole range of black hole, extreme black hole and wormhole solutions.

In the next section, we summarize the general results of [10] concerning the motion of an electrically charged test particle in a Brill spacetime. We then specialize to the case of the massless and electrically neutral spacetimes (with only magnetic and NUT charges) in Sect. 3, where we analyze the possible circular orbits. This analysis is used in Sect. 4 to prove that, for this class of Brill spacetimes, the Lorentz-force motion of an initially distant charged observer is always causal. Our conclusions are summarized in the last section.

## 2 Charged particle motion in the Brill spacetime

The Reissner–Nordström–NUT solution of the Einstein–Maxwell system of equations is given by

$$\begin{aligned} ds^2 &= -f(dt - 2n(\cos\theta + C)d\varphi)^2 + f^{-1}dr^2 + (r^2 + n^2)(d\theta^2 + \sin^2\theta d\varphi^2), \\ A &= \Phi(dt - 2n(\cos\theta + C)d\varphi), \end{aligned} \quad (2.1)$$

with

$$\begin{aligned} f &= \frac{(r - m)^2 + b^2}{r^2 + n^2}, \quad \Phi = \frac{qr + p(r^2 - n^2)/2n}{r^2 + n^2}, \\ (b^2 &= q^2 + p^2 - m^2 - n^2). \end{aligned} \quad (2.2)$$

This solution depends on four parameters associated with conserved charges, the mass  $m$ , the NUT or gravimagnetic charge  $n$ , the electric charge  $q$  and the magnetic charge  $p$ . We assume here  $n \neq 0$ . In this case, if time is not periodically identified the maximal analytic extension of the spacetime (2.1) is geodesically complete [10], and corresponds to a black hole for  $b^2 < 0$ , to an extreme black hole for  $b^2 = 0$  and to a traversable Lorentzian wormhole for  $b^2 > 0$ .

The above solution was first given (in the Taub form) for  $C = 0$  by Brill [6]. An unphysical local coordinate transformation  $t \rightarrow t - 2nC\varphi$  generates the family of solutions (2.1) depending on the additional parameter  $C$ , which we shall refer to as Brill spacetimes for simplicity. As shown in [10], for any  $|C| > 1$  there are Brill

spacetimes with closed null geodesics, so to ensure the possibility of weak chronology protection we restrict to  $|C| \leq 1$ . The parameter  $C$  governs the strength of the Misner string metric singularities at  $\theta = 0$  and/or  $\pi$ . The commonly preferred values are  $C = 1$  (singularity at  $\theta = 0$ ),  $C = -1$  (singularity at  $\theta = \pi$ ), or  $C = 0$  (two symmetrical singularities at  $\theta = 0$  and  $\theta = \pi$ ).

The equations of motion of a charged particle with mass  $m_c$  and charge  $q_c$  may be derived from the Lagrangian

$$L = \frac{1}{2} g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu + \kappa A_\mu \dot{x}^\mu, \tag{2.3}$$

where  $\dot{\phantom{x}} = d/d\tau$ , with  $\tau$  the proper time, and  $\kappa = q_c/m_c$ . The momenta  $p_\mu = g_{\mu\nu} \dot{x}^\nu + \kappa A_\mu$  conjugate to the cyclic variables  $t$  and  $\varphi$  are constants of the motion,  $p_t = -E$  and  $p_\varphi = J_z + 2nCE$ . Noting that the electromagnetic gauge chosen in (2.1) is such that  $A^\varphi = 0$ , we obtain the equations of motion for the time and azimuthal coordinates:

$$\dot{t} - 2n(\cos \theta + C) \dot{\varphi} = f^{-1} \mathcal{E} \quad [\mathcal{E}(r) = E + \kappa \Phi(r)], \tag{2.4}$$

$$\dot{\varphi} = \frac{J_z - 2nE \cos \theta}{(r^2 + n^2) \sin^2 \theta}. \tag{2.5}$$

This last equation is the same as in the case of a neutral particle [8]. Also, the part of the Lagrangian (2.3) which depends explicitly on the coordinate  $\theta$  is

$$L_\theta = \frac{1}{2} g_{\varphi\varphi} \dot{\varphi}^2 + g_{t\varphi} \dot{\varphi} (\dot{t} + \kappa A^t), \tag{2.6}$$

with  $A^t = -\Phi/f$ . Owing to (2.4), this actually does not depend on  $\kappa$  [10], so that the equation of motion for  $\theta$ ,

$$[(r^2 + n^2)\dot{\theta}]' = (r^2 + n^2) \sin \theta \cos \theta \dot{\varphi}^2 - 2nE \sin \theta \dot{\varphi}, \tag{2.7}$$

is also the same as for a neutral particle. The angular equations of motion can be first integrated [8] to

$$\vec{L} + \vec{S} = \vec{J}, \tag{2.8}$$

meaning that the constant total vector angular momentum  $\vec{J} = (J_x, J_y, J_z)$  is the sum of the usual orbital angular momentum (rescaled by a factor  $(r^2 + n^2)/r^2$ )  $\vec{L} = (r^2 + n^2) \hat{r} \wedge \dot{\hat{r}}$ , where  $\hat{r}$  is a unit vector normal to the two-sphere, and a NUT charge contribution  $\vec{S} = 2nE \hat{r}$ . It follows from the orthogonality of  $\vec{L}$  and  $\vec{S}$  that

$$\vec{J} \cdot \hat{r} = 2nE. \tag{2.9}$$

so that [8] the spherical sections  $r = \text{constant}$  of the orbits are small circles (parallels)  $C$  with polar axis  $\vec{J}$  and colatitude  $\eta = \arccos(2nE/J)$ , where  $J^2 \equiv \vec{J}^2$ . Squaring (2.8) leads to

$$\vec{L}^2 = l^2 \equiv J^2 - 4n^2 E^2, \tag{2.10}$$

which can be rewritten as

$$(r^2 + n^2)^2[\dot{\theta}^2 + \sin^2 \theta \dot{\phi}^2] = l^2. \tag{2.11}$$

As discussed in [13], the system of Eqs. (2.5) and (2.11) can be completely integrated by introducing the Mino time  $\lambda$ , related to the proper time by

$$d\tau = (r^2 + n^2) d\lambda. \tag{2.12}$$

We only give here the solution for  $\theta(\lambda)$ :

$$\cos \theta = \cos \psi \cos \eta + \sin \psi \sin \eta \cos(J\lambda), \tag{2.13}$$

with  $\psi = \arccos(J_z/J)$ . This shows that the Mino period of the angular motion is  $2\pi/J$ .

The time evolution can be split [13] according to

$$t(\lambda) = t_r(\lambda) + t_\theta(\lambda), \tag{2.14}$$

where the radial and angular contributions to  $t(\lambda)$  solve the equations

$$\frac{dt_r}{d\lambda} = (r^2 + n^2) \frac{\mathcal{E}(r)}{f(r)}, \tag{2.15}$$

$$\frac{dt_\theta}{d\lambda} = \frac{2n(\cos \theta + C)(J_z - 2nE \cos \theta)}{\sin^2 \theta}. \tag{2.16}$$

The solution to Eq. (2.15) depends on the solution of the equation for radial motion (2.18). The solution to Eq. (2.16) is, up to an additive constant,

$$t_\theta(\lambda) = 4n^2 E \lambda + 2n(C + 1) \arctan \left[ \frac{\cos \psi - \cos \eta}{1 - \cos(\psi - \eta)} \tan \frac{J\lambda}{2} \right] + 2n(C - 1) \arctan \left[ \frac{\cos \psi + \cos \eta}{1 + \cos(\psi - \eta)} \tan \frac{J\lambda}{2} \right], \tag{2.17}$$

in the interval  $-\pi/J < \lambda < \pi/J$ .

Finally, after eliminating in the Hamiltonian  $H$  associated with (2.3) the angular and temporal velocities in terms of the constants of the motion  $l$  and  $E$ , and conventionally normalizing proper time by setting  $H = -1/2$ , we obtain the effective radial equation [10]

$$\dot{r}^2 + W(r) = 0, \quad \left[ W(r) \equiv f(r) \left( 1 + \frac{l^2}{r^2 + n^2} \right) - \mathcal{E}^2(r) \right], \tag{2.18}$$

where  $\mathcal{E}(r)$  has been defined in (2.4). In the stationary sector ( $f(r) > 0$ ),  $\mathcal{E}^2(r)$  must remain positive, so that the sign of  $\mathcal{E}(r)$  is a constant of the motion. From (2.15), a necessary condition for coordinate time and proper time to have the same orientation is  $\mathcal{E}(r) > 0$ . This condition, which ensures that the radial contribution  $t_r(\lambda)$  is an

increasing function, is not sufficient, because the angular contribution  $t_\theta(\lambda)$  is not generically increasing. Indeed, as shown in [10], for parallels  $\mathcal{C}$  which do not circle a Misner string,  $t_\theta(\lambda)$  is in the large a decreasing function of Mino time for all values of the parameter  $C$ . So it could be possible for the time coordinate  $t = t_r + t_\theta$  to take again the same value after a finite lapse of Mino or proper time. If after this lapse the radial and angular coordinates also took again the same values, then the worldline would be closed (or self-intersecting). In order to investigate this possibility, it is first necessary to discuss the possible radial motions determined by (2.18).

### 3 Circular orbits: case of the massless magnetic Brill spacetime

The discussion of the radial motion depends on the position of the stationary points of the effective potential  $W(r)$ ,

$$W(r) = W'(r) = 0. \quad (3.1)$$

which correspond to circular orbits. The equation  $W(r) = 0$  is generically of fourth order [12]. However it can be reduced to a second-order equation if  $W(r)$  is even in  $r$ , i.e. in the special case  $m = 0$ ,  $q = 0$  of a massless magnetic Brill spacetime, to which we now restrict. The effective energy can then be written

$$\mathcal{E}(r) = E_0 + \frac{\beta}{2}y \quad \left( 0 < y = \frac{n^2}{r^2 + n^2} \leq 1 \right), \quad (3.2)$$

with  $\beta = -2\kappa p/n$ , and

$$E_0 = \mathcal{E}(\infty) = E - \beta/4. \quad (3.3)$$

Note that the choice  $\mathcal{E}(r) > 0$  implies  $E_0 > 0$  in the case of orbits extending to infinity, i.e. if  $E_0^2 > 1$ . On account of (3.2) the effective potential (2.18) reduces to

$$W(y) = Ay^2 + By + D \quad (3.4)$$

with

$$A = \alpha \bar{l}^2 - \beta^2/4, \quad B = \alpha + \bar{l}^2 - \beta E_0, \quad D = 1 - E_0^2, \quad (3.5)$$

where we have put  $\bar{l} = l/n$ ,  $\alpha = \bar{b}^2 - 1$  with  $\bar{b}^2 = b^2/n^2 \geq -1$  (the very special case  $p^2 = n^2$  treated in [10] corresponds to  $\alpha = 0$ ).

#### 3.1 Circular orbits with $r = 0$

The equation  $W'(r) = 0$  has two solutions. The first is  $r = 0$ , with effective energy  $\mathcal{E}(0) = E_0 + \beta/2$  given by the positive root of

$$W(y = 1) = A + B + D = (\alpha + 1)(\bar{l}^2 + 1) - \mathcal{E}(0)^2 = 0. \quad (3.6)$$

The corresponding orbit is stable or unstable, depending on whether the second derivative

$$W''_r(0) = -\frac{2}{n^2} W'_y(1) = -\frac{2}{n^2} (2A + B) = -\frac{2}{n^2} [\alpha + \bar{l}^2 + 2\alpha\bar{l}^2 - \beta\mathcal{E}(0)] \tag{3.7}$$

is positive or negative. In the black-hole or extreme-black-hole case,  $-2 \leq \alpha \leq -1$ ,  $W(0) < 0$  for all  $\mathcal{E}(0) > 0$ , so that  $r = 0$  cannot be a stationary point. In the wormhole case  $\alpha > -1$ , the effective energy is

$$\mathcal{E}(0) = \bar{b}\gamma \quad \left( \bar{b} \equiv \sqrt{\alpha + 1}, \quad \gamma \equiv \sqrt{\bar{l}^2 + 1} \right). \tag{3.8}$$

If

$$\beta > \frac{\alpha + \bar{l}^2 + 2\alpha\bar{l}^2}{\bar{b}\gamma} = \alpha\frac{\gamma}{\bar{b}} + \bar{l}^2\frac{\bar{b}}{\gamma} = 2\bar{b}\gamma - \frac{\gamma}{\bar{b}} - \frac{\bar{b}}{\gamma}, \tag{3.9}$$

$r = 0$  is a minimum of  $W$  (stable orbit). On the contrary, if the inequality (3.9) is reversed,  $r = 0$  is a maximum and this orbit is unstable.

**3.2 Circular orbits with  $r \neq 0$**

The second solution is  $dW/dy = 0$ , i.e.  $y = y_0$ , with

$$y_0 = -\frac{B}{2A}. \tag{3.10}$$

Then  $W = 0$  if  $B^2 = 4AD$ , i.e.

$$4\alpha\bar{l}^2 E_0^2 - 2\beta(\alpha + \bar{l}^2)E_0 + \beta^2 + (\alpha - \bar{l}^2)^2 = 0, \tag{3.11}$$

with discriminant

$$\Delta = (\beta^2 - 4\alpha\bar{l}^2)(\alpha - \bar{l}^2)^2. \tag{3.12}$$

This is clearly non-negative if  $\alpha \leq 0$ . If  $\alpha > 0$ , the discriminant is non-negative provided

$$\beta^2 \geq 4\alpha\bar{l}^2 \tag{3.13}$$

( $A \leq 0$ ). Then (3.11) is solved by

$$E_0 = \frac{\beta(\alpha + \bar{l}^2) - \delta(\alpha - \bar{l}^2)}{4\alpha\bar{l}^2}, \tag{3.14}$$

where we have put

$$\delta \equiv \text{sign}(\alpha - \bar{l}^2)\sqrt{\beta^2 - 4\alpha\bar{l}^2}. \tag{3.15}$$

Further defining

$$z \equiv \frac{\beta - \delta}{2\bar{l}^2}, \tag{3.16}$$

we can rewrite (3.14) as

$$E_0 = \frac{1 + z^2}{2z}. \tag{3.17}$$

It follows from (3.17) that  $E_0^2 > 1$  (the limiting value  $E_0^2 = 1$  for  $z^2 = 1$  generically corresponding from (A.2), see the ‘‘Appendix’’, to an orbit at infinity), so that  $W(r_0) = 0 > W(\infty) = 1 - E_0^2$ , i.e. the extremum at  $r = r_0$ , if it exists, is a local maximum (another way to see this is to compute  $W_r''(\pm r_0) = (4r_0^2 y_0^4/n^4)W_y''(y_0) = (8r_0^2 y_0^4/n^4)A \leq 0$ ).

From its definition,  $y_0 = n^2/(r_0^2 + n^2)$  given by (3.10) must be positive and smaller than 1. As shown in the ‘‘Appendix’’, this means that circular orbits with  $r_0 \neq 0$  exist only if the test particle charge-to-mass ratio  $\kappa$  and scaled orbital angular momentum  $\bar{l}$  satisfy the relations

$$\frac{\alpha + \bar{l}^2 + 2\alpha\bar{l}^2}{\bar{b}\sqrt{\bar{l}^2 + 1}} < -\frac{2\kappa p}{n} < \alpha + \bar{l}^2 \tag{3.18}$$

(the lower bound being irrelevant in the black-hole case  $\bar{b}^2 < 0$ ).

In the very special limiting case  $\bar{l}^2 = \alpha = \beta/2$ , (3.15) and (3.16) lead to  $z = 1$ , so that also  $E_0 = 1$ , and  $A = B = D = 0$ , leading to  $W(r) \equiv 0$ . This means that a charged test particle with parameters fine tuned to those of the Brill spacetime so that  $\kappa \equiv q_e/m_e = (2n^2 - p^2)/np$  can follow a (metastable) circular orbit with any given radius.

### 4 Causality

Knowing the stationary points of the effective potential, one can classify the possible worldlines of a charged test particle. We shall not go here into the details of this analysis, which leads to results qualitatively similar to those discussed in [10] for  $\alpha = 0$ . The possible orbits are contained in the range of values of  $r$  such that  $W(r) \leq 0$ . This range can be finite (in the wormhole case), corresponding to bound orbits in the potential well around the minimum  $r = 0$  of  $W(r)$  if the inequality (3.9) is satisfied. Or it can be infinite, with either scattering orbits reflected on the potential barrier (if the maximum of the effective potential  $W(0)$  or  $W(\pm r_0)$  is positive), or traversing orbits going from  $r = +\infty$  to  $r = -\infty$  (if the maximum is negative).

The only possibly causality violating wordlines, in the sense of our WCP criterion, correspond to scattering orbits followed by a charged observer. These wordlines can self-intersect at an event  $M_1$  if:

1. the Mino time delay

$$\Delta\lambda = 2 \int_{r_{turn}}^{r_1} [-W(r)]^{-1/2} \frac{dr}{r^2 + n^2} \tag{4.1}$$



(where  $r_{turn}$  is the turning point,  $W(r_{turn}) = 0$ ) is an integer multiple of the period  $2\pi/J$ , so that the angular coordinates take again the same values, and

2. the coordinate time delay

$$\Delta t = 2 \int_{r_{turn}}^{r_1} \frac{dt}{d\lambda} [-W(r)]^{-1/2} \frac{dr}{r^2 + n^2} \tag{4.2}$$

vanishes. As discussed in [10], for an initially distant observer this will generically not occur, because at distances (to the black hole horizon or to the wormhole neck, according to the sign of  $b^2$ ) large before the characteristic length  $n$ , the potentially negative angular contribution  $dt_\theta/d\lambda$  (which does not depend on the distance  $r$ ) will easily be balanced by the positive definite radial contribution  $dt_r/d\lambda \simeq E_0 r^2$  [see (2.15) and (2.16)].

But this argument breaks down if the turning point is close to the maximum, i.e. if  $W'(r_{turn})$  is small. Then the observer can spend a long proper time near the turning point, making a large number of turns  $N$  before returning to infinity, so that the integral in (4.2) will be of the order of

$$\Delta t \simeq N \Delta_{1t}(r_{max}), \tag{4.3}$$

where  $\Delta_{1t}(r_{max}) = \Delta_{1t_r}(r_{max}) + \Delta_{1t_\theta}$ , the time lapse during a Mino period  $2\pi/J$  for a particle on an unstable circular orbit, is not obviously positive definite. From (2.15) and (2.17), the radial and angular components of  $\Delta_{1t}(r)$  are given by

$$\Delta_{1t}(r) = \frac{2\pi}{J} (r^2 + n^2) \frac{\mathcal{E}(r)}{f(r)}, \tag{4.4}$$

$$\Delta_{1t_\theta} = 2\pi n \left[ \frac{4nE}{J} + (C + 1)\text{sgn}(J_z - 2nE) + (C - 1)\text{sgn}(J_z + 2nE) \right], \tag{4.5}$$

For  $|C| \leq 1$ , a lower bound for  $\Delta_{1t_\theta}$  is

$$\Delta_{1t_\theta} \geq 2\pi n \left[ \frac{4nE}{J} - 2 \right], \tag{4.6}$$

the lower bound being attained for all orbits if  $C = \pm 1$ , and for orbits with  $-2nE < J_z < 2nE$  for other values of  $C$ . Thus the lower bound for the net time lapse during one period is

$$\Delta_{1t}(r) \geq \frac{2\pi n^2}{J} \left( \Psi(r) + 4E - 2\frac{J}{n} \right), \tag{4.7}$$

where

$$\Psi(r) = \frac{\mathcal{E}(r)}{yf(r)} > 0, \tag{4.8}$$

and  $E$  is related to  $E_0$  by (3.3). The lower bound (4.7) is not positive definite because  $J \geq 2nE$ . It is positive provided<sup>1</sup>

$$\Delta(r) = (\Psi(r) + 4E)^2 - \left(\frac{2J}{n}\right)^2 = \Psi(r)^2 + 8E\Psi(r) - 4\bar{l}^2 > 0 \quad (4.9)$$

[where we have used (2.10)]. We now investigate whether  $\Delta(r)$  can vanish in the two cases of circular orbits ( $r = 0$  and  $r = r_0 \neq 0$ ) discussed in Sect. 3.

### 4.1 $r = 0$

In this case,  $\Psi(0) = \mathcal{E}(0)/\bar{b}^2 = \gamma/\bar{b}$ ,  $E = \bar{b}\gamma - \beta/4$ , leading to

$$\Delta(0) = \bar{b}^{-2} \left[ (1 + 4\bar{b}^2)\gamma^2 - 2\beta\bar{b}\gamma + 4\bar{b}^2 \right]. \quad (4.10)$$

$\Delta(0)$  is positive definite if  $\beta < 0$ . If  $\beta > 0$ ,  $\Delta(0)$  vanishes for

$$\gamma = \gamma_{\pm} \equiv \frac{\bar{b}}{4\bar{b}^2 + 1} \left[ -\beta \pm \sqrt{\beta^2 - 4(4\bar{b}^2 + 1)} \right]. \quad (4.11)$$

From (3.8),  $\gamma = \gamma_+$  is possible only if  $\gamma_+ \geq 1$ , which is ensured if

$$\beta \geq 4\bar{b} + \frac{1}{2\bar{b}} \geq 2\sqrt{2} \quad (4.12)$$

(with equality for  $\bar{b}^2 = 1/8$ ). For  $\beta \geq 4\bar{b} + 1/\bar{b}$ ,  $\Delta(0)$  can also vanish for  $\gamma = \gamma_-$ .

Thus, the worldlines of charged particles circling the wormhole neck  $r = 0$  can be closed if their orbital angular momentum matches the value(s) (4.11). However these circular orbits are stable, because  $\Delta(0) = 0$  can be written

$$\beta = 2\bar{b}\gamma + \frac{\gamma}{2\bar{b}} + \frac{2\bar{b}}{\gamma} \geq 2(\bar{b}\gamma + 1), \quad (4.13)$$

which is stronger than the stability condition (3.9).<sup>2</sup>

<sup>1</sup> The positivity condition (4.9), derived under the assumption  $\Psi(r) + 4E > 0$ , is sufficient because  $\Psi(r) + 4E \leq 0$  is possible only if  $E < 0$ , so that  $4(\bar{l}^2 - E\Psi(r)) > 0 \geq \Psi(r)(\Psi(r) + 4E)$ .

<sup>2</sup> Note that (4.13) means  $\mathcal{E}(\infty) = \bar{b}\gamma - \beta/2 \leq -1$ , leading to  $W(\infty) \leq 0$ , so that the CWLs at  $r = 0$  could be accessed by quantum tunnelling from infinity, but only if negative effective energies were allowed, which we have excluded.

## 4.2 $r = r_0$

Using (A.4), we find

$$\Psi(r_0) = \frac{\mathcal{E}(r_0)}{y_0(1 + \alpha y_0)} = \frac{1}{y_0 z}. \quad (4.14)$$

Using this together with (3.17), (3.3) and (A.1), we obtain<sup>3</sup>

$$\Delta(r_0) = \frac{1}{y_0^2 z^2} + \frac{4(1 + z^2)}{y_0 z^2} + \frac{2(\alpha + \bar{l}^2 z^2)}{y_0 z^2} - 4\bar{l}^2 \quad (4.15)$$

$$= \frac{1}{y_0^2 z^2} + \frac{2(\alpha + \bar{l}^2 z^2)}{y_0 z^2} + \frac{8}{y_0} - \frac{4\alpha}{z^2} \quad (4.16)$$

$$= \frac{(1 + \alpha y_0)^2}{y_0^2 z^2} - \frac{\alpha(\alpha + 4)}{z^2} + \frac{2(4 + \bar{l}^2)}{y_0}, \quad (4.17)$$

which is positive definite for  $-2 \leq \alpha \leq 0$ .

To cover the complementary range  $\alpha \geq 0$ , we use another expression of  $\Delta(r_0)$ , obtained by inserting (A.3) into (4.15) or (4.17),

$$(1 + \alpha y_0) y_0^2 \Delta(r_0) = 4\alpha \bar{l}^2 y_0^2 (1 - y_0) + 4\alpha y_0^2 + (2\alpha + 3\bar{l}^2 + 8)y_0 + 1, \quad (4.18)$$

which is positive definite for  $y_0 \in [0, 1]$  if  $\alpha \geq 0$ .

So the only possible CWLs with circular orbits have  $r = 0$ , but these have  $\mathcal{E}(\infty) < 0$  and so cannot attract charged test particles coming from infinity.

## 5 Conclusion

We have investigated the motion of electrically charged test particles in a spacetime with closed timelike curves, the Brill or Reissner–Nordström–NUT spacetime with only magnetic and gravimagnetic (NUT) charges, and without periodic identification of time. We have argued that causality violations can be observed by an initially distant charged observer only if he can be attracted by an unstable closed worldline with circular orbit. It turns out that the only circular orbits around which the net coordinate time lapse vanishes, corresponding to a closed worldline, are stable. It follows that no worldline followed by an initially distant charged observer moving under the action of the Lorentz force can possibly self-intersect, meaning that, in this specific dynamical framework, causality is preserved in the sense of our weak chronology protection criterion.

This work should be extended in several directions. First, the same problem should be investigated in the general case of the Brill spacetime with all four charges non-vanishing. Our guess is that weak causality should also hold in that case, although a complete analytical proof seems very difficult, unless methods more powerful than

<sup>3</sup> To transform (4.15) into (4.16), we have combined the second and fourth term of (4.15) and used (A.2).

those of the present paper are used. Also, the possible existence of closed or self-intersecting worldlines should be similarly investigated in the case of other spacetimes with CTCs, in order to see whether they satisfy weak causality. The simplest cases are presumably those of three-dimensional spacetimes, such as BTZ [14, 15], and warped AdS black hole spacetimes [16–18], which both admit CTCs. The latter are self-consistent solutions of the three-dimensional Einstein–Maxwell theory with gravitational and electromagnetic Chern–Simons terms [19], so that Lorentz-force motion of charged observers could be investigated for causality in a fashion similar to that of the present paper.

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### Appendix: Circular orbits with $r \neq 0$

From (3.15) and (3.16),  $\beta$  and  $\delta$  can be expressed in terms of  $z$  as

$$\beta = \frac{\alpha + \bar{l}^2 z^2}{z}, \quad \delta = \frac{\alpha - \bar{l}^2 z^2}{z}. \quad (\text{A.1})$$

Using this, we obtain from (3.10)

$$y_0 = \frac{-\delta(\alpha + \bar{l}^2) + \beta(\alpha - \bar{l}^2)}{2\alpha\bar{l}^2\delta} = \frac{1 - z^2}{\bar{l}^2 z^2 - \alpha}, \quad (\text{A.2})$$

which can be inverted to

$$z^2 = \frac{1 + \alpha y_0}{1 + \bar{l}^2 y_0}. \quad (\text{A.3})$$

We also obtain from (A.1) and (A.2) the value of the effective energy  $\mathcal{E}(r_0) = E_0 + (\beta/2)y_0$ :

$$\mathcal{E}(r_0) = \frac{(\bar{l}^2 - \alpha)z}{\bar{l}^2 z^2 - \alpha} = (1 + \bar{l}^2 y_0)z, \quad (\text{A.4})$$

so that the effective energy is positive provided

$$z > 0. \quad (\text{A.5})$$

In the black-hole case or extreme-black-hole case,  $-2 \leq \alpha \leq -1$ , (A.3) is positive definite provided

$$0 < y_0 < y_h = -\frac{1}{\alpha}, \quad (\text{A.6})$$

so that the circular orbits must be outside the horizon ( $r_0 > r_h$ ). The allowed range of  $z$  is then from (A.2)

$$0 < z < 1, \quad (\text{A.7})$$

leading from (A.1) to the condition for the existence of these circular orbits:

$$\bar{l}^2 > \beta - \alpha. \quad (\text{A.8})$$

In the wormhole case,  $\alpha > -1$ ,  $y_0$  can vary in the full range  $0 < y_0 < 1$ , leading to the allowed range of  $z$

$$\begin{aligned} \frac{\bar{b}}{\gamma} < z < 1 & \quad \text{if } \bar{l}^2 > \alpha, \\ 1 < z < \frac{\bar{b}}{\gamma} & \quad \text{if } \bar{l}^2 < \alpha \end{aligned} \quad (\text{A.9})$$

[where  $\bar{b}$  and  $\gamma$  are related to  $\alpha$  and  $\bar{l}^2$  by (3.8)]. Both cases lead to the same bounds for the existence of an unstable circular orbit of radius  $r = \pm r_0$ ,

$$\frac{\alpha + \bar{l}^2 + 2\alpha\bar{l}^2}{\bar{b}\gamma} < \beta < \alpha + \bar{l}^2. \quad (\text{A.10})$$

For  $\alpha > 0$ , the lower bound ensures that the first existence condition (3.13) is satisfied, due to the identity

$$(\alpha + \bar{l}^2 + 2\alpha\bar{l}^2)^2 = 4\alpha\bar{l}^2\bar{b}^2\gamma^2 + (\alpha - \bar{l}^2)^2. \quad (\text{A.11})$$

Note that in the parameter range (A.10) there is also from (3.9) a stable circular orbit at  $r = 0$ .

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