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RESEARCH ARTICLE

First law of black hole mechanics in variable background fields

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Abstract It is well known that in general theories of gravity with the diffeomorphism symmetry, the black hole entropy is a Noether charge. But what will happen if the symmetry is explicitly broken? By investigating the covariant first law of black hole mechanics with background fields, we show that the would-be Noether charge still can be identified as the black hole entropy, provided that it is a local quantity on the horizon. Moreover, motivated by the proposal that the cosmological constant behaves as a thermodynamic variable, we allow the non-dynamical background fields to be varied. To illustrate this general formalism, we study a generic static black brane in the massive gravity. Using the first law and the scaling argument, we obtain two Smarr formulas. We show that both of them can be retrieved without relying on the first law, hence providing a self-consistent check of the theory.

Keywords Black hole thermodynamics · Diffeomorphism symmetry · Smarr formula · Massive gravity

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1 Introduction

When a closed system has a differentiable dynamical symmetry, Noether's theorem indicates the existence of a corresponding conserved current. Usually, Noether's theorem is not applicable to the system coupling to the environment and the extension would be significant as shown in open quantum systems [1]. In field theories, if the environment is inert or the spontaneous breaking physics appears at a large energy scale, the system can be mimicked by coupling a background field associated with certain explicit symmetry breaking.

The gravitational models with explicit diffeomorphism (and Lorentz) breaking have a long research history. Examples include the massive gravity with a reference metric [2–6] and the Chern–Simons gravity coupling to the axions [7,8]. Of particular interest is the recent application of the massive gravity in the gauge/gravity duality [9–14], where the reference metric can imitate the mean-field disorder in realistic materials.

One of the well-known Noether charges in gravitational physics is the covariant expression of black hole entropy proposed by Wald, with respect to the diffeomorphism symmetry of the general theories of gravity [15,16]. The Wald entropy is a "local, geometrical" quantity on the Killing horizon.[1] It is identified from the covariant first law of black hole mechanics, which is a variational identity built upon the Hamiltonian that generates the evolution in the phase space of black hole solutions. Besides its elegant construction and universality, the success of the Wald entropy is that its higher curvature contribution precisely agrees with the microscopic entropy computed by state counting in string or M-theory [17].

However, as pointed out by Iyer and Wald in the appendix to Ref. [16], the diffeomorphism invariance implies the absence of "non-dynamical fields" in their Lagrangian. Nevertheless, they have applied the theories with a non-dynamical metric (such as the theories of fields in flat spacetimes) to discuss the canonical energy. But until now, little attention has been paid on the covariant first law and the Noether entropy with the background fields. In this paper, we aim to fill this gap. The essential observation is the following: the presence of background fields only appends the nonlocal volume terms to the key variational identity that leads to the first law,[2] while the black hole entropy is expected to be defined by local quantities on the horizon.[3]

[1] By "local, geometrical", the entropy is characterized by a covariant surface term made of the fields appearing in Lagrangian and their derivatives.

[2] In the path integral approach to quantum gravity, two configurations related by a diffeomorphism are physically indistinguishable and should not be double counted. If any field converts to a background, the two configurations are different. By contrast, there is no such qualitative difference which would hinder the construction of the variational identity with background fields.

[3] The seeking for local geometrical feature of black hole entropy has motivated Wald's formalism and was used to present a candidate entropy definition of dynamical black holes [15,16,18]. This feature is inherited from the Bekenstein–Hawking entropy, originally inspired by the famous teacup gedanken experiment [19,20] which suggests that any black hole horizon should be associated with the entropy to compensate the hidden information. The black hole entropy including its thermodynamic and statistical significance can be extended to more local notion of the causal horizon [21]. Moreover, the Bekenstein–Hawking entropy enlightened the holographic principle which states that the information inside a space can be encoded on its boundary. The celebrated realization of holographic principle by gauge/gravity duality reassures the local

The background fields involved in this work are specified as the fields in the Lagrangian which do not react under the diffeomorphism and whose equations of motion (EOM) are not imposed. Furthermore, we will allow them to be varied. In other words, the background fields are able to respond to the variation of the dynamical fields which are coupled to them. In this respect, our background fields are more general than the usual " prior geometry" [23] or "absolute object" [24,25], which cannot be changed by changing other fields. The motivation to study the varied background fields arises from the proposal that the cosmological constant Λ behaves as a thermodynamic variable, the pressure [26–31]. One important evidence for this proposal is that the Smarr formula integrated from the first law with variation $\delta\Lambda$ can be retrieved by the geometric method [30], hence indicating the existence of the Killing potential. Recently, in terms of the holographic duality, the origin of the Smarr formula with the pressure for AdS black holes has been understood as the fact that the free energy of a large N gauge theory only depends on the color number N via an overall factor N^2 [32]. Moreover, many interests have been attracted to study the implication of the extended phase space in black hole phase transitions from the viewpoint of chemistry [33–36]. In our formalism, one can find that the variable cosmological constant can be described by the simplest scalar background, which is constant ($\partial\Lambda = 0$) but not fixed ($\delta\Lambda \neq 0$).

As an illustration, we will study a generic static black brane in the Einstein–Maxwell–Dilaton (EMD) gravity with a reference metric and the cosmological constant. Both of them will be regarded as varied background fields. We have interests on the EMD gravity rather than the simple Einstein gravity since the former is more general: it involves three types of fields (scalar, vector and tensor) which contribute to the covariant first law with different forms. Moreover, the massive EMD gravity is very interesting in recent holographic models since it provides abundant physics in field theories. For instance, the dilaton is appealing as it features robust linear in temperature resistivity [14].

2 Covariant first law

Iyer and Wald [15,16] have derived the covariant first law of black hole mechanics by constructing a variational identity. The black hole entropy is identified with the Noether charge with respect to the diffeomorphism symmetry. They also pointed out that a number of formulas and results continue to hold for theories with a non-dynamical metric. However, they focused on the canonical energy but did not mention what is the black hole entropy in that case. This can be partially understood since the only non-dynamical field in their work is the spacetime metric (not the reference metric) and in the typical non-dynamical spacetime, i.e. the flat spacetime, the black hole entropy loses physical meaning. In this section, we will not restrict on a single fixed non-dynamical metric but will extend the variational identity to involve more general background fields, which can be scalars, vectors or tensors and all of them are

Footnote 3 continued
feature of black hole entropy, which is dual to the dependence of thermal entropy on IR physics alone [22] (We thank Hong Liu for discussion on this point.). With these in mind, we will use the local feature to identify the black hole entropy.

allowed to be varied. At last, we will identify the black hole entropy from the extended variational identity in terms of its local nature.

For this purpose, we consider a Lagrangian L as a functional of some concrete field tensors and their derivatives. These fields are collectively denoted by ψ, including the metric $g^{\mu\nu}$ and various matter fields (scalar, vector, two-tensor) $\psi = \left(g^{\mu\nu}, \phi, a_\mu, b_{\mu\nu}\right)$. Each type of the matter fields can involve multiple fields, for instance, $\phi = (\phi_1, \phi_2, \ldots)$, and our formalism below can be generalized directly. The derivatives of these fields are defined as $\left(R_{\mu\nu\lambda\rho}, \nabla_\mu\phi, \nabla_\nu a_\mu, \nabla_\lambda b_{\mu\nu}\right)$. We assume that all the fields ψ are the backgrounds at the beginning. Thus, their EOM cannot be imposed. However, any of them can be set as the dynamical fields by imposing the corresponding EOM in the end.[4] Moreover, we emphersize that there are two ingredients composed in the "diffeomorphism symmetry": the general coordinate invariance and free of "prior geometry".[5] Any background field denotes certain " prior geometry". To see what will be different when the diffeomorphism symmetry is broken completely, we also assume at the beginning that the Lagrangian L is not a scalar and restore the coordinate invariance in the end.

We start from the variation of the Lagrangian 4-form

$$\delta\,(*L) = *E\delta\psi + \mathrm{d}\,(*\theta), \tag{2.1}$$

where $*$ refers to the Hodge dual and $\theta = \theta\,(\psi, \delta\psi)$ is an one-form. A sum over all variables in $E\delta\psi$ is understood and the quantity E denotes collectively the EOM $E = (E^{(g)}_{\mu\nu}, E^{(\phi)}, E^{(a)\mu}, E^{(b)\mu\nu})$ with respect to ψ. We note again that $E \neq 0$ for any background fields. In the "Appendix", we will list the explicit expressions of the EOM.

Let's stop and review a little work by Iyer and Wald. In Ref. [15,16], the Lagrangian is assumed to be diffeomorphism invariant, that is

$$\mathrm{L}(f^*\psi) = f^*\mathrm{L}\,(\psi) \tag{2.2}$$

where $\mathrm{L} = *L$ and f denotes any diffeomorphism map. Since the pullback f^* does not act on the background fields, there is no dependence of background fields in the Lagrangian. As stated in Ref. [15], the diffeomorphism invariance implies

$$\delta_\xi\,(*L) = \pounds_\xi\,(*L), \tag{2.3}$$

where ξ is a vector generating the infinitesimal diffeomorphism and $\pounds_\xi$ the Lie derivative. Equation (2.3) means that the variation of $*L$ induced by the field variation $\delta_\xi\psi = \pounds_\xi\psi$ is equal to the Lie derivative of $*L$ itself. Using Cartan's formula, Eq. (2.3) leads to

$$\delta_\xi\,(*L) = di_\xi\,(*L), \tag{2.4}$$

[4] Put differently, one can suppose ψ as the dynamical variables at the beginning, convert one or more variables in ψ to the background fields in the end, but not impose any EOM in the intermediate steps.

[5] In Appendix B of the textbook [37], one can find a wonderful discussion on the difference between the diffeomorphism invariance and the general coordinate invariance.

where i_ξ denotes the contraction. Furthermore, one can define a current

$$j_\xi = *\theta\left(\psi, \pounds_\xi \psi\right) - i_\xi\,(*L) \tag{2.5}$$

which satisfies

$$\mathrm{d}\, j_\xi = - * E \pounds_\xi \psi \tag{2.6}$$

in terms of Eqs. (2.1) and (2.4). This current is the key object to construct the variational identity and has been called as the Noether current associated with the diffeomorphism symmetry in the sense of [38].

Now we break the diffeomorphism symmetry. Since L is not a scalar, Eq. (2.3) does not hold. Equivalently, we have

$$\delta_\xi L - \xi^\mu \nabla_\mu L = P_{\mu\nu} \nabla^\nu \xi^\mu \neq 0, \tag{2.7}$$

where $\delta_\xi L$ denotes the variation of L induced by the field variation $\delta_\xi \psi$.[6] The two-tensor $P_{\mu\nu}$ is complicated, see Eq. (5.6). Although the general coordinate invariance is lost and the background fields are present, the current j_ξ still exists and we can prove

$$\theta^\beta\left(\psi, \pounds_\xi \psi\right) - \xi^\beta L = \nabla_\alpha Q_\xi^{\beta\alpha} + \xi^\mu\left(\tilde{E}_\mu^{\ \beta} + P_\mu^{\ \beta}\right), \tag{2.8}$$

where the called Noether potential is given by

$$Q_\xi^{\beta\alpha} = 2\left(X^{\alpha\beta\mu\nu}\nabla_\mu \xi_\nu - 2\xi_\nu \nabla_\mu X^{\alpha\beta\mu\nu} + \xi^\nu \tilde{Q}_\nu^{\ \beta\alpha}\right). \tag{2.9}$$

Here we have defined a four-tensor by the derivative $X^{\alpha\beta\mu\nu} = \partial L / \partial R_{\alpha\beta\mu\nu}$, a three-tensor $\tilde{Q}_\nu^{\ \beta\alpha}$ composed of the fields $\left(a_\mu, b_{\mu\nu}\right)$ and the derivatives $(\partial L/\partial \nabla_\nu a_\mu$, $\partial L/\partial \nabla_\lambda b_{\mu\nu})$, and a two-tensor made by:

$$\tilde{E}_{\mu\beta} = 2E^{(g)}_{\mu\beta} - E^{(a)}_\beta a_\mu - E^{(b)}_{\alpha\beta} b^\alpha_{\ \mu} - E^{(b)}_{\beta\alpha} b_\mu^{\ \alpha}. \tag{2.10}$$

The lengthy expression of $\tilde{Q}_\nu^{\ \beta\alpha}$ can be found in the "Appendix". The one-form $\theta\,(\psi, \delta\psi)$ can induce a general symplectic form

$$\Omega\,(\psi, \delta_1\psi, \delta_2\psi) = \delta_1\left[*\theta\,(\psi, \delta_2\psi)\right] - \delta_2\left[*\theta\,(\psi, \delta_1\psi)\right], \tag{2.11}$$

where the two variations are not specified. Consider a special symplectic form where $\delta_1\psi$ is arbitrary and $\delta_2\psi = \pounds_\xi \psi$. It can be recast as

$$\begin{aligned}\Omega\left(\psi, \delta\psi, \pounds_\xi \psi\right) = {} & \mathrm{d}\left[\delta\left(*Q_\xi\right) - i_\xi\,(*\theta)\right] \\ & + \delta\left(*\, i_\xi \tilde{E}\right) + i_\xi\left(*\, E\delta\psi\right) + \delta\left(*\, i_\xi P\right).\end{aligned} \tag{2.12}$$

[6] Schematically, we express $\delta_\xi L = L\left(\psi, \delta_\xi \psi\right) = L\left(\psi, \pounds_\xi \psi\right)$.

Define the integral

$$\delta H_{\xi} = \int_{\Sigma} \Omega\left(\psi, \delta\psi, £_{\xi}\psi\right), \tag{2.13}$$

where Σ is a Cauchy surface that connects the horizon cross section with the spatial infinity. Suppose that ξ is the Killing vector which generates a symmetry inducing $£_{\xi}\psi = 0$. Then one immediately has $\delta H_{\xi} = 0$, which yields a variational identity

$$\int_{\partial\Sigma} \delta\left(*Q_{\xi}\right) - i_{\xi}\left(*\theta\right) + \int_{\Sigma} \delta\left(*i_{\xi}\tilde{E}\right) + i_{\xi}\left(*E\delta\psi\right) + \delta\left(*i_{\xi}P\right) = 0. \tag{2.14}$$

Equation (2.14) is the essential result of this section. Some remarks are in order.

(1) One can recover the covariant first law of diffeomorphism-invariant theories based on Eq. (2.14), as it should be. In this case, we turn off all the background fields by imposing $E = 0$ and restore the general coordinate invariance by setting $P_{\mu\nu} = 0$. Then, Eq. (2.14) is reduced to

$$\int_{\partial\Sigma} \delta\left(*Q_{\xi}\right) - i_{\xi}\left(*\theta\right) = 0. \tag{2.15}$$

Consider a stationary black hole with the Killing vector $\xi = \mathrm{t} + \Omega_H\varphi$, where t denotes the time translation, Ω_H the angular velocity and φ the angular rotation. Equation (2.15) can be written as the first law [15,16]

$$T\delta S = \delta\mathcal{E} + \Omega_H\delta\mathcal{J}. \tag{2.16}$$

Here T is the Hawking temperature. The black hole entropy is nothing but the Noether charge, defined by local geometrical quantities,

$$S = 2\pi\int_{B} *Q_{\xi}\big|_{\xi\to 0,\ \nabla_{\mu}\xi_{\nu}\to n_{\mu\nu}}, \tag{2.17}$$

where B denotes the bifurcation horizon, $n_{\mu\nu}$ is its binormal, and any reference to the Killing vector (that is nonlocal) was eliminated. $\delta\mathcal{E}$ and $\delta\mathcal{J}$ denote the variations of energy and angular momentum, respectively.[7]

(2) In the appendix to Ref. [16], Iyer and Wald studied the theories in a non-dynamical spacetime. The spacetime metric $g^{\mu\nu}$ is the only background field, and it is fixed. From Eqs. (2.14) and (2.17), one can see that the variation of the would-be entropy of black holes is vanishing in that case. Thus the first law cannot be well defined. In the rest of our work, we will regard the spacetime metric as a dynamical variable. Our theory still breaks the diffeomorphism symmetry if some matter fields are the background fields.

[7] $\delta\mathcal{E}$ and $\delta\mathcal{J}$ are not integrable in general unless specific boundary conditions of field variables are imposed. Various black hole hairs are possibly involved in $\delta\mathcal{E}$.

(3) One may notice that a great simplification from Eqs. (2.14) to (2.15) is the vanishing of the complicated tensor $P_{\mu\nu}$. In Refs. [15,16], it has been ascribed to the diffeomorphism symmetry of the Lagrangian. However, $P_{\mu\nu} = 0$ when the Lagrangian is a scalar under the general coordinate transformation. On the contrary, the diffeomorphism symmetry, which not only conveys the information of general coordinate invariance but also implies that the theory is free of " prior geometry", is sufficient but not necessary for that. We stress that this is a simple but important concept which has been rarely noted, if existed. In the rest of our work, we will consider L as a scalar. As a result, we still have $P_{\mu\nu} = 0$ regardless the presence of background fields.[8] Moreover, all the stuff below that will be derived based on $P_{\mu\nu} = 0$ can be dubbed as the one with respect to general coordinate invariance, instead of the diffeomorphism symmetry.

(4) Equation (2.14) exhibits that the background fields and their variations would add two volume integrals on the Cauchy surface Σ in the variational identity. In particular, the first volume integral may be nonvanishing even when all background fields are fixed. And the second one will be different for various tensor types. If one turns off all the background fields except a special scalar (i.e. the cosmological constant) that is allowed to be varied, Eq. (2.14) will be reduced to the variational identity constructed in [29].

Using Eq. (2.14), one can rewrite the first law (2.16) with the different $\delta\mathcal{E}$ and $\delta\mathcal{J}$:

$$\delta\mathcal{E} = \int_\infty \delta\big(*Q_{\mathrm{t}}\big) - i_{\mathrm{t}}\big(*\theta\big) + \int_\Sigma \delta\big(*i_{\mathrm{t}}\tilde{E}\big) + i_{\mathrm{t}}\big(*E\delta\psi\big),$$
$$\delta\mathcal{J} = \int_\infty \delta\big(*Q_\varphi\big) - i_\varphi\big(*\theta\big) + \int_\Sigma \delta\big(*i_\varphi\tilde{E}\big) + i_\varphi\big(*E\delta\psi\big).$$

The presence of the background fields does not change the Wald entropy simply because the black hole entropy is expected to be localized on the horizon but the corrections in Eq. (2.14) are volume terms. The volume terms are naturally attributed to $\delta\mathcal{E}$ and $\delta\mathcal{J}$, which is reminiscent of the known " physical process" version of the first law [39]. Actually, the " key ingredient" in that analysis, i.e. a general formula (eqs. (4) and (5) in [39]) for the variation of the mass and the angular momentum, can be deduced from Eq. (2.14) by some operations: (1) turn off all background fields, (2) turn on the energy-momentum source δT_μ^β and charge-current source δj^β, (3) suppose the variation δ in Eq. (2.14) as the linear perturbation caused by the sources, and (4) consider Σ as the unperturbed spacetime. Then one can obtain

$$E\delta\psi = E_{\text{unperturbed}}\left(\psi_{\text{perturbed}} - \psi_{\text{unperturbed}}\right) = 0, \tag{2.18}$$
$$\delta\tilde{E}_{\mu\beta} = \left[2E^{(g)}_{\mu\beta} - E^{(a)}_\beta a_\mu\right]_{\text{perturbed}} = \delta T_{\mu\beta} + a_\mu\delta j_\beta, \tag{2.19}$$

[8] Given a concrete Lagrangian that is a scalar and involves some background fields, one can check $P_{\mu\nu} = 0$ using the EOM. The difference between $P_{\mu\nu} = 0$ and $E^{(g)}_{\mu\nu} = 0$ might be interesting for some readers.

where we have used the EOM with the sources

$$2E^{(g)}_{\mu\beta} = \delta T_{\mu\beta},\ E^{(a)}_{\beta} = -\delta j_{\beta}.$$

Equation (2.14) with Eqs. (2.18) and (2.19) can recover eq. (33) in [39], which further leads to the mentioned general formula.

3 Massive gravity

In this section, we will illustrate the covariant first law in massive EMD gravity. There are five fields, including the spacetime metric $g^{\mu\nu}$, the reference metric $b_{\mu\nu}$, the gauge potential a_μ, the dilaton field ϕ, and the cosmological constant Λ. We will take $b_{\mu\nu}$ and Λ as two background fields in which one is a two-tensor and the other is a special scalar. We will assume that both of them can be varied.

Consider the gravity theory described by the EMD Lagrangian

$$L_0 = R - 2\Lambda - \frac{1}{2}(\partial\phi)^2 - \frac{1}{4}Z(\phi)F^2 - V(\phi) \tag{3.1}$$

plus a graviton mass term $L_1 = U(b_{\mu\nu}g^{\mu\nu})$ [13]. Here R is the Ricci scalar and F is the Maxwell field. The function form of the scalar potential V, the effective electromagnetic coupling Z, and the potential U for the reference metric will be specified latter. Assumed to be projected only on the spatial coordinates x^i, the reference metric is given by $b_{\mu\nu} = c^2\delta^i_\mu\delta^j_\nu\delta_{ij}$, where c is a parameter. Note we have set $16\pi G = 1$ for brevity. We will study a generic static black brane with the metric

$$ds^2 = -h(r)dt^2 + \frac{1}{f(r)}dr^2 + r^2\left(dx^2 + dy^2\right), \tag{3.2}$$

and the gauge potential $A = a_t(r)dt$. The independent EOM can be written as

$$\begin{aligned}
&\frac{c^2h}{r^2} - \frac{fh}{r^2} + \frac{1}{2}hV - \frac{3}{4}\frac{Q_e^2h}{r^4Z} + \frac{fh'}{r} + \frac{1}{2}f'h' - \frac{fh'^2}{2h} + \frac{1}{4}fh\phi'^2 + fh'' = 0,\\
&-c^2 + f + \frac{1}{2}r^2V + \frac{Q_e^2}{4r^2Z} + rf' + \frac{1}{4}r^2f\phi'^2 = 0,\\
&-V' + \frac{Q_e^2Z'}{2r^4Z^2} + \frac{2f\phi'^2}{r} + \frac{1}{2}f'\phi'^2 + \frac{fh'\phi'^2}{2h} + f\phi'\phi'' = 0,\\
&-c^2 + f + \frac{1}{2}r^2V + \frac{Q_e^2}{4r^2Z} + \frac{1}{2}rf' + \frac{rfh'}{2h} = 0,
\end{aligned} \tag{3.3}$$

where the prime denotes the derivative with respect to r and the electric charge $Q_e = Zr^2\sqrt{f/h}a_t'$. The extended variational identity (2.14) includes the usual surface terms

$$\delta\left(*Q_\xi\right) - i_\xi\left(*\theta\right) = -2r\sqrt{\frac{h}{f}}\delta f - r^2\sqrt{hf}\phi'\delta\phi - a_t\delta Q_e \tag{3.4}$$

and the new volume terms

$$\begin{aligned}&\left[i_\xi \tilde{E}\right]_\beta = -E^{(b)}_{\alpha\beta} b^\alpha_t - E^{(b)}_{\beta\alpha} b_t{}^\alpha = 0,\\&\left[i_\xi\,(*E\delta\psi)\,\right]_{\nu\lambda\rho} = \varepsilon_{t\nu\lambda\rho}\Big[2U' r^{-2}\delta c^2 - \Big(2+\frac{\partial V}{\partial\Lambda}\Big)\delta\Lambda\Big].\end{aligned} \tag{3.5}$$

As a result, we have a conserved quantity independent with r:

$$\begin{aligned}\delta H \; &= -2r\sqrt{\frac{h}{f}}\delta f - r^2\sqrt{hf}\phi'\delta\phi - a_t\delta Q_e\\&\quad + \int\left[2U'\delta c^2 - r^2\left(2+\frac{\partial V}{\partial\Lambda}\right)\delta\Lambda\right]\sqrt{\frac{h}{f}}dr.\end{aligned} \tag{3.6}$$

Although not necessarily, the potential of the reference metric in holographic models is usually assumed to be $L_1 = (\mathrm{Tr}K)^2 - \mathrm{Tr}K^2$ [9], where the matrix K is defined by a matrix square root $K^\mu_\nu = \sqrt{g^{\mu\lambda}b_{\lambda\nu}}$, following the same form as in the standard dRGT massive gravity [5,6]. In the following, we will use this special potential, which is equivalent to set $U' = 1$ in the current situation. The more general potential will not change our results qualitatively. To go ahead, we need to specify the behavior of the dynamical fields near horizon and boundary. This is enough to derive the first law by applying the variational identity, even though the explicit solutions are not known. Similar process can be found, for instance, in [40]. The solutions near the horizon can be given by

$$\begin{aligned}&f(r) = f_1(r-r_0)+\cdots,\ \ h(r) = h_1(r-r_0)+\cdots,\\&\phi(r) = \phi_0+\cdots,\end{aligned} \tag{3.7}$$

where r_0 denotes the horizon location. From the EOM, the solutions at the boundary can be solved with the form[9]:

$$\begin{aligned}f(r) &= \frac{r^2}{l^2} + c^2 + \frac{f_1}{r^{1-\sigma}} - \frac{f_2}{r} + \frac{f_3}{r^{1+\sigma}} + \frac{Q_e^2}{4r^2} + \cdots,\\h(r) &= \frac{r^2}{l^2} + c^2 - \frac{\mu}{r} + \frac{Q_e^2}{4r^2} + \cdots,\\\phi(r) &= \frac{\phi_1}{r^{(3-\sigma)/2}} + \frac{\phi_2}{r^{(3+\sigma)/2}} + \cdots,\end{aligned} \tag{3.8}$$

where μ and $\phi_{1,2}$ are some free parameters and

$$\sigma = \sqrt{4m^2l^2+9}, \qquad f_1 = (3-\sigma)\,\phi_1^2/(8l^2),$$

[9] Here we focus on the " standard" case with $0<\sigma<1$, and assume $Z(\phi) = 1 + Z_1\phi^2 + \cdots$, $V(\phi) = \frac{1}{2}m^2\phi^2 + \gamma_4\phi^4 + \cdots$ for simplicity. Note that the negative m^2 is allowed provided that it does not violate the Breitenlohner–Freedman bound [41].

$$f_2 = \mu - \frac{\left(9-\sigma^2\right)\phi_1\phi_2}{12l^2}, \;\; f_3 = (3+\sigma)\,\phi_1^2/(8l^2). \tag{3.9}$$

Close to the horizon and the boundary, the variational identity can be expanded. At leading order, they are

$$\begin{aligned}\delta H|_{\text{horizon}} &= T\delta S + \delta\Lambda\Phi_\Lambda,\\ \delta H|_{\text{boundary}} &= \delta M - \Phi_e\delta Q_e + \Phi_\phi\delta\Lambda - \frac{\sigma}{96\pi l^2}\big[(3-\sigma)\phi_1\delta\phi_2 - (3+\sigma)\phi_2\delta\phi_1\big],\end{aligned} \tag{3.10}$$

where we have defined the gravitational mass (density), entropy, temperature

$$M = 2\mu, \;\; S = 4\pi r_0^2, \;\; T = \frac{1}{4\pi}\sqrt{f'(r_0)h'(r_0)}, \tag{3.11}$$

and two local potentials as well as two nonlocal potentials

$$\begin{aligned}&\Phi_e = a_t(\infty), \qquad \Phi_\phi = \frac{9-\sigma^2}{576\pi}\phi_1\phi_2,\\ &\Phi_c = \int_{r_0} 2\sqrt{\frac{h}{f}}dr, \;\; \Phi_\Lambda = \int_{r_0}\left(2+\frac{\partial V}{\partial\Lambda}\right)\sqrt{\frac{h}{f}}r^2 dr.\end{aligned} \tag{3.12}$$

Here "$\int_{r_0}$" means to drop any terms relevant to the integral upper limit ∞. These terms are absent in the first law because they are divergent and exactly cancel other divergent terms from Eq. (3.4). Note $\partial V/\partial\Lambda = V/\Lambda \neq 0$ since we have fixed the expansion coefficients in $V(\phi)$ multiplying l^2 (like m^2l^2) as dimensionless constants. Then the first law can be obtained by matching the variational identity near the horizon and at boundary

$$\begin{aligned}T\delta S = \delta M - \Phi_e\delta Q_e + \left(\Phi_\phi - \Phi_\Lambda\right)\delta\Lambda + \Phi_c\delta c^2\\ - \frac{\sigma}{96\pi l^2}\big[(3-\sigma)\phi_1\delta\phi_2 - (3+\sigma)\phi_2\delta\phi_1\big].\end{aligned} \tag{3.13}$$

The first law (3.13) implies some interesting relations among the variables. In terms of the dimensional analysis, it is easy to see that the black brane solution is scale invariant, i.e. the field configurations are homogeneous functions (with order zero), if the radial coordinate and the parameters transform as[10]

$$\begin{aligned}&r \to \lambda r, \;\; \mu \to \lambda\mu, \;\; Q_e \to \lambda Q_e,\\ &l \to \lambda l, \;\; \phi_1 \to \lambda^{\frac{3-\sigma}{2}}\phi_1, \;\; \phi_2 \to \lambda^{\frac{3+\sigma}{2}}\phi_2.\end{aligned} \tag{3.14}$$

[10] Note that we are studying the static and isotropic solution so the scaling of coordinates t, x, and y are not relevant.

In terms of the scale invariance and Eq. (3.13), one can take the gravitational mass as a homogeneous function

$$M\left(\lambda^2 S, \lambda Q_e, \lambda^{-2}\Lambda, \lambda^0 c^2, \lambda^{\frac{3-\sigma}{2}}\phi_1, \lambda^{\frac{3+\sigma}{2}}\phi_2\right)$$
$$= \lambda M\left(S, Q_e, \Lambda, c^2, \phi_1, \phi_2\right). \tag{3.15}$$

Acting the derivative ∂_λ and setting $\lambda = 1$ at last gives

$$2S\frac{\partial M}{\partial S} + Q_e\frac{\partial M}{\partial Q_e} - 2\Lambda\frac{\partial M}{\partial \Lambda} + \frac{3-\sigma}{2}\phi_1\frac{\partial M}{\partial \phi_1} + \frac{3+\sigma}{2}\phi_2\frac{\partial M}{\partial \phi_2} = M. \tag{3.16}$$

Using Eq. (3.13) again, we obtain a Smarr formula

$$M = 2TS + Q_e\Phi_e + 2\Lambda\left(\Phi_\phi - \Phi_\Lambda\right). \tag{3.17}$$

Note that since c^2 does not transform under the rescaling, Eq. (3.17) remains the same if one sets $\delta c^2 = 0$ at the beginning. Interestingly, there is another scale invariance for which Λ is not rescaled:

$$r \to \lambda r,\ \mu \to \lambda^3\mu,\ Q_e \to \lambda^2 Q_e,$$
$$c \to \lambda c,\ \phi_1 \to \lambda^{\frac{3-\sigma}{2}}\phi_1,\ \phi_2 \to \lambda^{\frac{3+\sigma}{2}}\phi_2. \tag{3.18}$$

Under this scaling transformation, the field configurations are homogeneous functions and the EOM are not changed. Compared with Eq. (3.15), now the gravitational mass behaves as a different homogeneous function

$$M\left(\lambda^2 S, \lambda^2 Q_e, \lambda^0\Lambda, \lambda^2 c^2, \lambda^{\frac{3-\sigma}{2}}\phi_1, \lambda^{\frac{3+\sigma}{2}}\phi_2\right)$$
$$= \lambda^3 M\left(S, Q_e, \Lambda, c^2, \phi_1, \phi_2\right), \tag{3.19}$$

which yields a different Smarr formula

$$3M = 2\left(TS + Q_e\Phi_e - c^2\Phi_c\right). \tag{3.20}$$

Similarly, one can set $\delta\Lambda = 0$ at the beginning which will not change Eq. (3.20).

In the previous derivation of Eqs. (3.17) and (3.20), either $\delta\Lambda$ or δc^2 has to be nonvanishing in the first law. As a self-consistent check, we will prove both of the Smarr formulas without using the first law. Before doing this, we note that one can use the explicit solution found in a special case [42] to check Eqs. (3.17) and (3.20).

Now consider the Einstein equation and the Killing equation which can lead to

$$\nabla_\mu\left[\left(T^{\mu\nu} - \frac{1}{2}g^{\mu\nu}T\right)\xi_\nu\right] = \nabla_\mu\left(2R^{\mu\nu}\xi_\nu\right) = 0. \tag{3.21}$$

Following the geometric method to derive the Smarr formula [30], where the key ingredient is the construction of the Killing potential, one can build up a new conserved tensor, at least locally:

$$Q_1^{\mu\nu} = \nabla^\mu \xi^\nu - \Phi_1^{\mu\nu}. \tag{3.22}$$

Here we have defined the generalized Killing potential which is determined by

$$2\nabla_\nu \Phi_1^{\mu\nu} = \left(T^{\mu\nu} - \frac{1}{2} g^{\mu\nu} T\right) \xi_\nu. \tag{3.23}$$

For the massive gravity, Q_1 has the nonvanishing components

$$Q_1^{tr} = -Q_1^{rt} = -\frac{fh'}{2h} + \frac{1}{4r^2}\sqrt{\frac{f}{h}}\left(Q_e a_t - 2\Lambda\tilde{\Phi}_\Lambda\right), \tag{3.24}$$

where $\tilde{\Phi}_\Lambda = \int (2 + V/\Lambda)\sqrt{h/f} r^2 dr$. Since $\partial_r\left(\sqrt{-g}Q_1^{tr}\right) = 0$, one can match $\sqrt{-g}Q_1^{tr}$ at horizon and boundary. By calculating

$$\begin{aligned}
\sqrt{-g}Q_1^{tr}|_{\text{horizon}} &= -\frac{1}{2}ST + \frac{1}{2}\Lambda\Phi_\Lambda, \\
\sqrt{-g}Q_1^{tr}|_{\text{boundary}} &= -\frac{1}{4}M + \frac{1}{4}Q_e\Phi_e + \frac{1}{2}\Lambda\Phi_\phi,
\end{aligned} \tag{3.25}$$

we retrieve Eq. (3.17).

On the other hand, there is a global scaling symmetry for static gravity theories [43,44] which can be uncovered easily with a different metric ansatz

$$ds^2 = -u(\rho)\, dt^2 + d\rho^2 + v(\rho)\left(dx^2 + dy^2\right). \tag{3.26}$$

One can show that $L = \sqrt{-g}(L_0 + L_1)$ with this metric is invariant up to a total derivative Q_0', under the global rescaling

$$u \to \lambda^{-2}u,\ v \to \lambda v,\ a_t \to \lambda^{-1}a_t. \tag{3.27}$$

Hence there is a Noether charge satisfied with $Q_2' = 0$:

$$\begin{aligned}
Q_2 = Q_0 &- \left(-2u\partial_{u'}L + v\partial_{v'}L - a_t\partial_{a_t'}L\right) \\
&- 2\left(-2u'\partial_{u''}L + v'\partial_{v''}L - a_t'\partial_{a_t''}L\right) \\
&+ \left(-2u\partial_{u''}L + v\partial_{v''}L - a_t'\partial_{a_t''}L\right)'.
\end{aligned} \tag{3.28}$$

Note that the later two brackets are necessary since the Lagrangian depends on higher derivative u'' and v''. Substituting the concrete L of massive gravity, we have

$$Q_2 = 2\left[-r^2\sqrt{fh}\left(\frac{2}{r} - \frac{h'}{h}\right) - Q_e a_t + c^2\tilde{\Phi}_c\right], \tag{3.29}$$

where $\tilde{\Phi}_c = \int 2\sqrt{h/f}\,dr$. Equating Q_2 at horizon and boundary, which are

$$\begin{aligned} Q_2|_{\text{horizon}} &= 2TS - 2c^2\Phi_c, \\ Q_2|_{\text{boundary}} &= 3M - 2Q_e\Phi_e, \end{aligned} \tag{3.30}$$

leads to Eq. (3.20) again.

4 Conclusion and discussion

We derived the covariant form of the first law of black hole mechanics in the presence of variable background fields. Due to its local nature, the general expression of black hole entropy previously identified with a Noether charge is still applicable, although the relevant diffeomorphism symmetry is broken. The current situation is distinct from two important works concerning the symmetry for the Noether entropy [45,46]. The former focused on the Chern–Simons term, where the bare affine connection breaks not only the diffeomorphism symmetry but also the general coordinate invariance, up to a total derivative. The latter pointed out that in the frame formalism, both diffeomorphism and Lorentz symmetries should be invoked. Combining these results, one might argue that the diffeomorphism symmetry is not necessary nor sufficient to identify what is the black hole entropy.[11]

We illustrated the general formalism by a static black brane in the massive EMD gravity. We derived the first law using the variational identity with two variable background fields—the cosmological constant and the reference metric. One can find that the conjugate variable of the cosmological constant, the called black hole volume [30,31,33–36], involves the local and nonlocal potentials. Such a separation suggests that these potentials may have different physical origins by the gauge/gravity duality. We identified two kinds of the scale invariance of the black brane solutions. Using these together with the first law, we obtained two Smarr formulas. We also proved both of them without invoking the first law.

Two implications from the present work deserve to be stressed. One is that we provided a rare case for which the general coordinate invariance, needless of the complete diffeomorphism symmetry, would induce nontrivial physics: the general coordinate invariance has been used to greatly simplify the covariant first law with background fields. We argue this is rare, since the diffeomorphism symmetry that fathered the general relativity is always taken seriously [23], but not the coordinate

[11] What is the essential feature of black hole entropy is a long-standing mystery. Various studies are in progress. For instance, in a recent work [47], it was argued that the black hole entropy is a Hamiltonian generator but not necessarily a property associated with the horizon.

invariance alone which any semi-respectable theory of physics can be required to respect [37]. In this regard, one may notice that the nonrelativistic version of the general coordinate invariance developed in [48,49] has been applied to construct the low-energy effective field theories of condensed matter, such as the unitary Fermi gas and fractional quantum Hall systems. Another example that is more alike to ours is the existence of the possible conflicts between the dynamical and geometrical constraints for a theory with explicit diffeomorphism breaking, which stems exactly from the fact that the general coordinate invariance still holds [50,51]. Our work also suggests to regard the reference metric and the cosmological constant in massive gravity as thermodynamic variables, since the variation of each one corresponds to a Smarr formula that has an explicit physical interpretation: one indicates the existence of the generalized Killing potential and the other comes from a scaling symmetry of the reduced action. We expect that the extension of the phase space in massive gravity and the associated Smarr formulas would imply interesting results in the black hole chemistry and holographic duals.

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5 Appendix: Some tensors

Here we list the explicit expressions of some tensors. They are obtained following the derivation given in the appendix to Ref. [52].

The equations of motion are

$$
\begin{aligned}
E^{(g)}_{\mu\nu} &= \frac{\partial L}{\partial g^{\mu\nu}} - \frac{1}{2} g_{\mu\nu} L - X^{\alpha\beta\rho}{}_{(\mu} R_{\nu)\rho\beta\alpha} \\
&\quad - 2\nabla^{\rho}\nabla^{\lambda} X_{\lambda(\mu\nu)\rho} + \nabla^{\beta} A_{(\mu\nu)\beta} + \nabla^{\beta} B_{(\mu\nu)\beta}, \\
E^{(\phi)} &= \frac{\partial L}{\partial \phi} - \nabla_{\mu} Y^{\mu}, \\
E^{(a)\mu} &= \frac{\partial L}{\partial a_{\mu}} - \nabla_{\nu} Z^{\mu\nu}, \\
E^{(b)\mu\nu} &= \frac{\partial L}{\partial b_{\mu\nu}} - \nabla_{\lambda} W^{\mu\nu\lambda},
\end{aligned} \tag{5.1}
$$

where the derivative of L is involved, including

$$
\begin{aligned}
W^{\mu\nu\lambda} &= \frac{\partial L}{\partial \nabla_{\lambda} b_{\mu\nu}}, \quad X^{\mu\nu\lambda\rho} = \frac{\partial L}{\partial R_{\mu\nu\lambda\rho}}, \\
Y^{\mu} &= \frac{\partial L}{\partial \nabla_{\mu}\phi}, \quad Z^{\mu\nu} = \frac{\partial L}{\partial \nabla_{\nu} a_{\mu}}.
\end{aligned} \tag{5.2}
$$

Two three-tensors are given by

$$A_{\mu\nu\beta} = \frac{1}{2}\big(a_\beta Z_{\mu\nu} - a_\mu Z_{\beta\nu} - a_\nu Z_{\mu\beta}\big), \tag{5.3}$$

$$\begin{aligned} B_{\mu\nu\beta} = \frac{1}{2}\big(&W_{\alpha\mu\nu} b^\alpha_\beta - W_{\alpha\beta\nu} b^\alpha_\mu - W_{\alpha\mu\beta} b^\alpha_\nu \\ &+ W_{\mu\alpha\nu} b^\alpha_\beta - W_{\beta\alpha\nu} b^\alpha_\mu - W_{\mu\alpha\beta} b^\alpha_\nu\big). \end{aligned} \tag{5.4}$$

Note that we have arranged Eq. (5.4) so that the first line and second line in the parentheses are equal when $b_{\mu\nu}$ and $W^{\mu\nu\lambda}$ are symmetric or antisymmetric on the $\mu\nu$ index.

The one-form $\theta = \theta\,(\psi, \delta\psi)$ is given by

$$\begin{aligned} \theta^\beta = {}& 2X_{(\mu}{}^{\alpha\beta}{}_{\nu)} \nabla_\alpha \delta g^{\mu\nu} - 2\nabla_\alpha X_{(\mu}{}^{\alpha\beta}{}_{\nu)} \delta g^{\mu\nu} \\ &+ Y^\beta \delta\phi + Z^{\mu\beta} \delta a_\mu + W^{\mu\nu\beta} \delta b_{\mu\nu} \\ &- A_{\mu\nu}{}^\beta \delta g^{\mu\nu} - B_{\mu\nu}{}^\beta \delta g^{\mu\nu}. \end{aligned} \tag{5.5}$$

The two-tensor $P_{\mu\nu}$ resulted from the general coordinate invariance is

$$\begin{aligned} P_{\mu\nu} = {}& -2\frac{\partial L}{\partial g^{\mu\nu}} + 4R_{\alpha\beta\rho\mu} X^{\alpha\beta\rho}{}_\nu + Y_\nu \nabla_\mu \phi \\ &+ \left(Z^\lambda{}_\nu \nabla_\mu a_\lambda + Z_\nu{}^\lambda \nabla_\lambda a_\mu + \frac{\partial L}{\partial a_\lambda} a_\mu g_{\lambda\nu}\right) \\ &+ \big(W^{\alpha\beta}{}_\nu \nabla_\mu b_{\alpha\beta} + W_\nu{}^{\alpha\beta} \nabla_\beta b_{\mu\alpha} + W^\alpha{}_\nu{}^\beta \nabla_\beta b_{\alpha\mu}\big) \\ &+ \frac{\partial L}{\partial b_{\lambda\rho}} \big(b_{\mu\rho} g_{\nu\lambda} + b_{\lambda\mu} g_{\nu\rho}\big). \end{aligned} \tag{5.6}$$

The three-tensor $\tilde{Q}_\nu{}^{\beta\alpha}$ in the Noether potential $Q_\xi^{\beta\alpha}$ is

$$\begin{aligned} \tilde{Q}_\nu{}^{\beta\alpha} = \frac{1}{2}\Big(& a_\nu Z^{[\alpha\beta]} + a^{[\beta} Z^{\alpha]}{}_\nu + a^{[\beta} Z_\nu{}^{\alpha]} \\ &+ W_{\nu\mu}{}^{[\alpha} b^{\beta]\mu} + W^{[\alpha}{}_\mu{}^{\beta]} b_\nu{}^\mu + W^{[\alpha}{}_{\mu\nu} b^{\beta]\mu} \\ &+ W^\mu{}_\nu{}^{[\alpha} b_\mu{}^{\beta]} + W^{\mu[\alpha\beta]} b_{\mu\nu} + W^{\mu[\alpha}{}_\nu b_\mu{}^{\beta]}\Big), \end{aligned} \tag{5.7}$$

where the last two lines are arranged similar to Eq. (5.4).

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