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Weyl type N solutions with null electromagnetic fields in the Einstein–Maxwell *p*-form theory

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Abstract We consider *d*-dimensional solutions to the electrovacuum Einstein–Maxwell equations with the Weyl tensor of type N and a null Maxwell ($p + 1$)-form field. We prove that such spacetimes are necessarily aligned, i.e. the Weyl tensor of the corresponding spacetime and the electromagnetic field share the same *aligned null direction* (AND). Moreover, this AND is geodetic, shear-free, non-expanding and non-twisting and hence Einstein–Maxwell equations imply that Weyl type N spacetimes with a null Maxwell ($p + 1$)-form field belong to the Kundt class. Moreover, these Kundt spacetimes are necessarily *CSI* and the ($p + 1$)-form is *VSI*. Finally, a general coordinate form of solutions and a reduction of the field equations are discussed.

1 Introduction and summary

We study *d*-dimensional Weyl type N spacetimes with null electromagnetic fields in the context of the generalized Einstein–Maxwell *p*-form theory. The theory describes

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an interaction between a gravitational field $\boldsymbol{g}$ and an electromagnetic field $\boldsymbol{F}$ and in the electrovacuum case (no sources of the electromagnetic field are present), its action takes the form

$$S = \frac{1}{16\pi} \int \mathrm{d}^d x \sqrt{-g} \left(R - 2\Lambda - \frac{\kappa_0}{p+1} F_{a_0 \ldots a_p} F^{a_0 \ldots a_p} \right), \tag{1.1}$$

where R is the Ricci scalar, Λ is the cosmological constant, κ_0 is a coupling constant and $\boldsymbol{F}$ is a closed $(p+1)$-form, i.e.

$$\nabla_{[a} F_{b_0 \ldots b_p]} = 0. \tag{1.2}$$

Varying the action (1.1) with respect to the metric $\boldsymbol{g}$ and to the local potential p-form $\boldsymbol{A}$ of $\boldsymbol{F}$, the least action principle yields a coupled system of the generalized Einstein–Maxwell equations for the pair $(\boldsymbol{g}, \boldsymbol{F})$

$$R_{ab} - \tfrac{1}{2} R g_{ab} + \Lambda g_{ab} = 8\pi T_{ab}^{EM}, \tag{1.3}$$

$$\nabla^a F_{ab_1 \ldots b_p} = 0, \tag{1.4}$$

respectively. Here, T_{ab}^{EM} is the electromagnetic stress–energy tensor associated with the Maxwell field $\boldsymbol{F}$,

$$\frac{8\pi}{\kappa_0} T_{ab}^{EM} \equiv F_{ab_1 \ldots b_p} F_b^{\ b_1 \ldots b_p} - \frac{1}{2(p+1)} g_{ab} F_{a_0 \ldots a_p} F^{a_0 \ldots a_p}. \tag{1.5}$$

In the special case of $d = 4$ and $p = 1$, (1.1) reduces to the action of the standard electrovacuum Einstein–Maxwell theory, which has been extensively studied in the literature. We refer to the system (1.2)–(1.4) as the Einstein–Maxwell equations.

Throughout the paper, we restrict ourselves to values $0 < p < d - 2$ and $d \geq 4$. Further, we assume that the Weyl tensor is of algebraic type N in the sense of the null alignment classification (see [1,2] or [3] for a review). This condition can be reformulated in terms of the existence of a null vector $\boldsymbol{\ell}$ such that [4]

$$C_{ab[cd} \ell_{e]} = 0. \tag{1.6}$$

The vector $\boldsymbol{\ell}$ is then referred to as the *Weyl aligned null direction* (Weyl AND or WAND) of the spacetime. As in four dimensions, such a spacetime represents a transverse gravitational wave propagating along the null direction $\boldsymbol{\ell}$ (see the physical interpretation of distinct null frame components of the Weyl tensor carried out in [5]).

Regarding the electromagnetic field $\boldsymbol{F}$, we assume that $\boldsymbol{F}$ is a (non-vanishing) *null* (i.e. type N) form [1], i.e. there exists a null vector $\boldsymbol{k}$ such that [6]

$$k^a F_{ab_1 \ldots b_p} = 0, \qquad k_{[a} F_{b_0 \ldots b_p]} = 0, \tag{1.7}$$

where $\boldsymbol{k}$ is an AND of the Maxwell field $\boldsymbol{F}$. In four-dimensional spacetimes, distinguished geometrical role of null electromagnetic two-forms satisfying the source-free

Maxwell equations is well known due to the Mariot–Robinson theorem (see Theorem 7.4 in [7]). It states that *a four-dimensional spacetime admits a shear-free geodetic null congruence if and only if it admits a null Maxwell 2-form (a test field) satisfying the Maxwell equations*. In higher dimensions, there is a partial generalization of the theorem for Maxwell $(p + 1)$-form fields (see Lemmas 3 and 4 in [8]).

In four dimensions, it is known that Petrov type N Einstein–Maxwell fields are either non-aligned and non-null or aligned (null or non-null) and Kundt (see Tables 38.9 and 38.10 in [7] and Theorems 7.4, 28.4 and 30.3 therein) and any type III/N pure-radiational metric is necessarily aligned (see Theorem 3 of [9]). Thus, in four dimensions, it follows that *any Petrov type N null Einstein–Maxwell field is aligned and Kundt*. Here, we extend this result to any dimension $d \geq 4$ and any electromagnetic $(p + 1)$-form field such that $0 < p < d - 2$.

As a consequence of our previous result [10] it follows that *a spacetime corresponding to a Weyl type N solution of the Einstein–Maxwell equations with a null Maxwell $(p + 1)$-form $\boldsymbol{F}$ is necessarily aligned with $\boldsymbol{F}$*. In other words, the null vectors $\boldsymbol{k}$ and $\boldsymbol{\ell}$ defined above are parallel. Proposition 1.1 then states that *the common AND $\boldsymbol{\ell}$ of the Weyl tensor and of the Maxwell field $\boldsymbol{F}$ is tangent to a shear-free, expansion-free and non-twisting geodetic null congruence*, i.e. *the spacetime belongs to the Kundt class*.

Proposition 1.1 *All Weyl type N solutions of the Einstein–Maxwell equations with a non-vanishing null $(p + 1)$-form field are aligned and Kundt.*

Further, applying results of Appendix A, where it is shown that all Weyl type N Kundt spacetimes with type N traceless Ricci tensor are CSI and an arbitrary covariant derivative of their Riemann tensor is even VSI, it follows that:

Proposition 1.2 *Any Weyl type N solution of the Einstein–Maxwell equations with a non-vanishing null electromagnetic field consists of a CSI spacetime and a VSI $(p + 1)$-form.*[1] *For $\Lambda = 0$, the corresponding spacetime is V SI.*

As a consequence, such solutions possess a certain universality property [6,17].

The paper is organized as follows. In the next section, we review some basic notions and notation employed throughout the paper and briefly discuss the structure of the field equations following from (1.1) in the case of the null Maxwell field (Sect. 2). Section 3 contains proofs of Propositions 1.1 and 1.2, in which more general results of Appendix A are employed. In Sect. 4, a general coordinate form of both the metric and the Maxwell field in adapted coordinates is obtained. Subsequently, a reduction of the Einstein–Maxwell equations to a set of equations for quantities emerging in the local description of $\boldsymbol{g}$ and $\boldsymbol{F}$ follows. A class of solutions with a covariantly constant WAND, pp –waves, is briefly discussed. At the end of the paper, Appendix A devoted to the study of scalar curvature invariants in general Weyl type N Kundt spacetimes of traceless Ricci type N is included.

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[1] $\boldsymbol{T}$ is a *CSI* (constant scalar invariant) tensor if all scalar polynomial invariants constructed from $\boldsymbol{T}$ and its covariant derivatives of arbitrary order are constant. If, moreover, all these scalar invariants vanish, we say that $\boldsymbol{T}$ is a VSI (vanishing scalar invariant) tensor. *CSI/VSI* spacetime is then a spacetime whose Riemann tensor is *CSI/VSI*, see [11–13] and [14–16], respectively (see also [3] and reference therein).

2 Preliminaries

For purposes of this paper, let us recall the notation, algebraic classification and NP and GHP formalisms as were introduced in [1,2,8,18] and summarized in [3]. In a d-dimensional spacetime, we shall consider a local *null* frame $\{\mathbf{e}_{(a)}\} \equiv \{\boldsymbol{\ell}, \boldsymbol{n}, \boldsymbol{m}_{(i)}\}$ with two null vector fields $\boldsymbol{\ell}$, $\boldsymbol{n}$ and $d-2$ spacelike vector fields $\{\boldsymbol{m}_{(i)}\}$ such that they satisfy the following orthogonality relations

$$\ell_a n^a = 1, \quad \ell_a \ell^a = n_a n^a = \ell_a m^a_{(i)} = n_a m^a_{(i)} = 0, \quad m^{(i)}_a m^a_{(j)} = \delta^i_j. \tag{2.1}$$

Let us note that relations (2.1) are invariant under Lorentz transformations of $\{e_{(a)}\}$.

2.1 Ricci rotation coefficients and directional derivatives

Now, let us introduce a few basic objects of the higher-dimensional Newman–Penrose formalism. Given a null frame $\{\boldsymbol{\ell}, \boldsymbol{n}, \boldsymbol{m}_{(i)}\}$, consider covariant derivatives of the individual frame vectors

$$L_{ab} \equiv \nabla_b \ell_a, \quad N_{ab} \equiv \nabla_b n_a, \quad \overset{i}{M}_{ab} \equiv \nabla_b m^{(i)}_a. \tag{2.2}$$

Projecting these derivatives onto the null frame, one obtains a set of scalars $L_{(a)(b)}$, $N_{(a)(b)}$ and $\overset{i}{M}_{(a)(b)}$, the so-called *Ricci rotation coefficients*. We omit the parenthesis, whenever it is clear that the corresponding quantities are projections of a tensor onto the null frame. Due to the orthogonality relations (2.1) satisfied by the null frame vectors, the corresponding rotation coefficients fulfill the following equalities

$$N_{0a} + L_{1a} = 0, \quad \overset{i}{M}_{0a} + L_{ia} = 0, \quad \overset{i}{M}_{1a} + N_{ia} = 0, \quad \overset{i}{M}_{ja} + \overset{j}{M}_{ia} = 0, \tag{2.3}$$

$$L_{0a} = N_{1a} = \overset{i}{M}_{ia} = 0. \tag{2.4}$$

For transformation properties of the Ricci rotation coefficients under the Lorentz transformations, see [18]. Lastly, let us denote the null frame directional derivatives as

$$D \equiv \ell^a \nabla_a, \quad \Delta \equiv n^a \nabla_a, \quad \delta_i \equiv m^a_{(i)} \nabla_a. \tag{2.5}$$

Using these, the action of the covariant derivative ∇ can be decomposed in the following way

$$\nabla_a = n_a D + m^{(i)}_a \delta_i + \ell_a \Delta. \tag{2.6}$$

2.2 Optical matrix and optical scalars

When studying geometrical properties of null congruences, it is usually convenient to employ an adapted null frame with $\mathbf{e}_{(0)}$ being the null vector tangent to the congruence. Then, some information on geometry of the congruence is encoded in L_{ab}.

For example, the congruence is geodetic if and only if $\kappa_i \equiv L_{i0}$ vanishes. In such a case, the scalar L_{10} corresponds to an affine parametrization of the congruence—it is affinely parametrized if and only if $L_{10} = 0$.

For a geodetic congruence, some other geometrical properties are encoded in the optical matrix $\rho_{ij} \equiv L_{ij}$. Consider the following decomposition of ρ_{ij}

$$\rho_{ij} = \sigma_{ij} + \theta\delta_{ij} + A_{ij}, \tag{2.7}$$

where σ_{ij} denotes the shear part, $\theta\delta_{ij}$ is the expansion part and A_{ij} is the antisymmetric part of ρ_{ij}. Using these particular matrices, the following *optical scalars* can be defined: the *expansion* θ, *shear* $\sigma^2 \equiv \sigma_{ij}\sigma_{ij}$ and *twist* $\omega^2 \equiv A_{ij}A_{ij}$ of the corresponding congruence.

2.3 Weyl type N spacetimes and null electromagnetic fields

Recall the conditions (1.7) for $\boldsymbol{F}$ to be a null form. Employing an adapted frame $\{\boldsymbol{k}, \boldsymbol{n}, \boldsymbol{m}_{(i)}\}$ with $\boldsymbol{k}$ being the null vector emerging in (1.7), the only non-vanishing independent null frame components of $\boldsymbol{F}$ are the boost weight (-1) quantities

$$\varphi'_{k_1\ldots k_p} \equiv F_{ab\ldots c}n^a m^b_{(k_1)} \ldots m^c_{(k_p)}, \tag{2.8}$$

i.e. $\boldsymbol{F}$ is of type N with the aligned null direction (AND) $\boldsymbol{k}$ (see e.g. Sect. 3 of [8]).

For a Weyl type N spacetime, the only non-trivial independent frame components of the Weyl tensor $\boldsymbol{C}$ in an adapted frame with $\boldsymbol{e}_{(0)} \equiv \boldsymbol{\ell}$ being the corresponding WAND are

$$\Omega'_{ij} \equiv C_{abcd}n^a m^b_{(i)} n^c m^d_{(j)}. \tag{2.9}$$

If also the components Ω'_{ij} vanish, the Weyl tensor is said to be of algebraic type O and the spacetime is conformally flat.

2.4 Einstein equations for null electromagnetic fields

For a null form $\boldsymbol{F}$ (1.7), the scalar invariant $F_{a_0\ldots a_p}F^{a_0\ldots a_p}$ in (1.5) vanishes and T^{EM}_{ab} is of type N (with the same AND $\boldsymbol{k}$). In an adapted frame $\{\boldsymbol{k}, \boldsymbol{n}, \boldsymbol{m}_{(i)}\}$, its only non-vanishing null frame component reads

$$T^{EM}_{11} = \frac{\kappa_0}{8\pi} F_{1a_1\ldots a_p} F^{0a_1\ldots a_p}. \tag{2.10}$$

Thus, denoting $\tilde{\Lambda} \equiv 2\Lambda/(d-2)$, the field Eq. (1.3) takes the form

$$R_{ab} = \tilde{\Lambda}g_{ab} + 8\pi T^{EM}_{11}k_a k_b, \tag{2.11}$$

or, equivalently, in the GHP formalism (see Table 3 of [8])

$$\omega = 0, \quad \psi_i = 0, \quad \phi = \tilde{\Lambda}, \quad \phi_{ij} = \tilde{\Lambda}\delta_{ij}, \quad \psi'_i = 0, \quad \omega' = 8\pi T^{EM}_{11}. \tag{2.12}$$

Thus, the corresponding solution is of traceless Ricci type N and the multiple Ricci AND is precisely the multiple AND $\boldsymbol{k}$ of the Maxwell field $\boldsymbol{F}$. Moreover, due to the alignment result for spacetimes of Weyl and traceless Ricci type N (see Proposition 2.1 of [10]), we obtain that *the Maxwell field $\boldsymbol{F}$ is aligned with the Weyl tensor*, i.e. $\boldsymbol{k}$ is also a quadruple WAND in the spacetime, i.e. $\boldsymbol{k} \propto \boldsymbol{\ell}$ and without loss of generality, one can set $\boldsymbol{k} = \boldsymbol{\ell}$. Furthermore, $\boldsymbol{\ell}$ *is geodetic*—this follows independently either from the contracted Bianchi identity or from the source-free Maxwell equations for a null Maxwell (test) field (see [19] for the 2-form case and [8] for an arbitrary p).

2.5 Maxwell equations for null electromagnetic fields

Employing an adapted null frame $\{\boldsymbol{\ell}, \boldsymbol{n}, \boldsymbol{m}_{(i)}\}$, the source-free GHP Maxwell Eqs. (1.2), (1.4) for a null Maxwell field $\boldsymbol{F}$ with a geodetic AND $\boldsymbol{\ell}$ reduce to (see Sect. 3 of [8]):

$$\left(p\rho_{[k_1|i} + p\rho_{i[k_1|} - \rho\delta_{[k_1|i}\right)\varphi'_{i|k_2...k_p]} = 0, \tag{2.13}$$

$$\varphi'_{[k_1...k_p}\rho_{k_{p+1}k_{p+2}]} = 0, \tag{2.14}$$

$$\rho_{[ij]}\varphi'_{ijk_1...k_{p-2}} = 0, \tag{2.15}$$

$$\eth_i\varphi'_{ik_1...k_{p-1}} = \tau_i\varphi'_{ik_1...k_{p-1}}, \tag{2.16}$$

$$\eth_{[k_1}\varphi'_{k_2...k_{p+1}]} = \tau_{[k_1}\varphi'_{k_2...k_{p+1}]}, \tag{2.17}$$

$$2\flat\varphi'_{k_1...k_p} = \left(p\rho_{[k_1|i} - p\rho_{i[k_1|} - \rho\delta_{[k_1|i}\right)\varphi'_{i|k_2...k_p]}. \tag{2.18}$$

For $p = 1$, Eq. (2.15) does not appear. Also note that for $p > d - 4$, Eq. (2.14) is identically satisfied.

3 Proofs of the main results

Proof of Proposition 1.1 We have already argued that $\boldsymbol{\ell}$ is the common geodetic AND of the Weyl tensor and the corresponding Maxwell field. Now, we shall prove that its optical matrix $\boldsymbol{\rho}$ vanishes. In order to do that, we start from the canonical form of the optical matrix corresponding to the WAND in spacetimes of Weyl and traceless Ricci type N (given by Eq. (1.3) of [10]) and argue that it is compatible with restrictions following from the Maxwell equations (see Lemmas 3 and 4 in [8]) only if $\boldsymbol{\rho} = \boldsymbol{0}$. For simplicity, let us assume that $p > 1$. The $p = 1$ case can be easily proved in a similar way using the stronger result of Lemma 4 of [8].

Let $\mathcal{F}$ be a frame in which $\boldsymbol{\rho}$ takes the canonical form and let $S_{ij} \equiv \rho_{(ij)}, A_{ij} \equiv \rho_{[ij]},$ $\rho \equiv \mathrm{Tr}\,\boldsymbol{\rho}$. The canonical form of $\boldsymbol{\rho}$ in spacetimes of Weyl and traceless Ricci type N reads $\boldsymbol{\rho} = \mathrm{diag}(\mathcal{L}, \boldsymbol{0}, \ldots, \boldsymbol{0})$ with the only possibly non-vanishing 2×2 block

$$\mathcal{L} = s \begin{bmatrix} 1 & a \\ -a & b \end{bmatrix}, \tag{3.1}$$

where $b \neq 1$, otherwise the spacetime is Einstein, i.e. $\omega' = 0$.

First, we show that $\boldsymbol{\rho}$ is traceless, i.e. $\rho = 0$. According to Lemma 3 of [8], the eigenvalues $\{S_i\}$ of $\mathbf{S}$ can be rearranged such that

$$\sum_{i=2}^{p+1} S_i = \frac{\rho}{2} = \sum_{j=p+2}^{d-1} S_j. \tag{3.2}$$

Assume that ρ is non-vanishing. Then each of these two sums has to contain at least one non-vanishing element of S_i. In our case, the only non-vanishing eigenvalues of $\mathbf{S}$ are s and sb. Hence, each sum in (3.2) has to contain exactly one of these two elements. Since the rest of the eigenvalues of $\mathbf{S}$ vanishes, these sums reduce to $s = \rho/2$ and $sb = \rho/2$. In particular, one has that $b = 1$. This contradicts our assumption of the presence of a non-trivial Maxwell field $\boldsymbol{F}$ in the spacetime. Hence $\rho = 0$.

Now, we prove that this already implies $\boldsymbol{\rho} = \boldsymbol{0}$. From $\rho = 0$, we have that $s = 0$ or $b = -1$. Of course, if $s = 0$, we are done. Hence, let us suppose that $s \neq 0$ and $b = -1$ instead. Note that, at the moment, $\boldsymbol{\rho}$ can be non-trivial only if both the shear and the twist of the corresponding geodetic null congruences are non-vanishing, otherwise the Sachs equation implies $\boldsymbol{\rho} = \boldsymbol{0}$ (see (15b) in [18]). Hence both s and a in (3.1) are non-vanishing. Let us define auxiliary GHP scalars $\boldsymbol{T}$, $\boldsymbol{U}$ and $\boldsymbol{V}$ as

$$T_{k_1\ldots k_p} \equiv S_{[k_1|i}\varphi'_{i|k_2\ldots k_p]} = S_{[k_1|2}\varphi'_{2|k_2\ldots k_p]} + S_{[k_1|3}\varphi'_{3|k_2\ldots k_p]}, \tag{3.3}$$

$$U_{k_1\ldots k_{p+2}} \equiv \varphi'_{[k_1\ldots k_p}A_{k_{p+1}k_{p+2}]}, \tag{3.4}$$

$$V_{k_1\ldots k_{p-2}} \equiv A_{ij}\varphi'_{ijk_1\ldots k_{p-2}} = 2sa\varphi'_{23k_1\ldots k_{p-2}} \tag{3.5}$$

($\boldsymbol{V}$ is defined only for $p > 1$). The Maxwell Eqs. (2.13), (2.14) and (2.15) then read

$$\boldsymbol{T} = \boldsymbol{0}, \quad \boldsymbol{U} = \boldsymbol{0}, \quad \boldsymbol{V} = \boldsymbol{0}. \tag{3.6}$$

In view of (3.5), the last equation of (3.6) immediately implies $\varphi'_{23k_1\ldots k_{p-2}} = 0$ for every $k_1, \ldots, k_{p-2} > 3$. Let us further examine $\boldsymbol{T}$ and $\boldsymbol{U}$ for a particular choice of indices. For every $k_1, \ldots, k_p > 3$, we have the following

$$s\varphi'_{2k_2\ldots k_p} \propto T_{2k_2\ldots k_p}, \tag{3.7}$$

$$s\varphi'_{3k_2\ldots k_p} \propto T_{3k_2\ldots k_p}, \tag{3.8}$$

$$sa\varphi'_{k_1\ldots k_p} \propto U_{23k_1\ldots k_p}. \tag{3.9}$$

Thus, according to the first and the second equation of (3.6), also $\varphi'_{2k_2\ldots k_p}$, $\varphi'_{3k_2\ldots k_p}$ and $\varphi'_{k_1\ldots k_p}$ vanish. Hence $\boldsymbol{\varphi}'$ vanishes completely. Since the Maxwell form $\boldsymbol{F}$ is non-vanishing, we arrive at a contradiction. Thus, $\boldsymbol{\rho}$ has to be zero and the spacetime is Kundt. □

Moreover, since the corresponding spacetime is of the Weyl type N with the Kundt AND $\boldsymbol{\ell}$ and its Ricci tensor takes the form (2.11), we conclude that it is a degenerate Kundt spacetime (see Sect. 7.1.2 in [3] and references therein).

Proof of of Proposition 1.2 Suppose that $(\boldsymbol{g}, \boldsymbol{F})$ is a solution of the Einstein–Maxwell equations satisfying our assumptions. The *CSI* property of $\boldsymbol{g}$ follows immediately from Theorem 1.1 and Corollary A.5. In the case of $\Lambda = 0$, the spacetime is of Weyl and Ricci type N. Hence its Riemann tensor is also of type N and hence the spacetime is *VSI* (see Theorem 1 of [15]).

To prove that $\boldsymbol{F}$ is *VSI*, recall that the spacetime is a degenerate Kundt spacetime and $\boldsymbol{F}$ is a closed form of type N. Then, employing the characterization of VSI electromagnetic $(p + 1)$-forms (Theorem 1.5 of [6]), we arrive at the desired result. □

Theorems 1.1 and 1.2 suggest that such solutions may be of interest also in various generalized theories. In particular, $\boldsymbol{F}$ corresponding to any solution $(\boldsymbol{g}, \boldsymbol{F})$ under consideration is a *universal* [17] test solution to generalized electrodynamics on the fixed background metric $\boldsymbol{g}$, i.e. it is a solution to any electrodynamics with the field equations of the form

$$\mathrm{d}\boldsymbol{F} = \boldsymbol{0}, \quad *\mathrm{d} * \tilde{\boldsymbol{F}} = \boldsymbol{0}, \tag{3.10}$$

where $*$ denotes the Hodge dual in $\boldsymbol{g}$ and $\tilde{\boldsymbol{F}}$ is any $(p + 1)$-form constructed as a polynomial of $\boldsymbol{F}$ and its covariant derivatives of an arbitrary order (see Sect. 2.4 of [6]).[2] Thus, $(\boldsymbol{g}, \boldsymbol{F})$ is also a solution to the Einstein Eq. (1.3) coupled with the electrodynamics (3.10).

4 General form of the solution

In this section, we discuss the general form of a Weyl type N metric $\boldsymbol{g}$ and a null Maxwell $(p + 1)$-form field $\boldsymbol{F}$ satisfying the Einstein–Maxwell Eqs. (1.2)–(1.4).

Since $(\boldsymbol{g}, \boldsymbol{F})$ is a degenerate Kundt spacetime with a VSI form, the discussion on the local form of a solution in Sects. 2 and 3 of the paper [6] applies. In particular, $\boldsymbol{g}$ and $\boldsymbol{F}$ take the local coordinate form (9) and (13) of [6], respectively. These are then subject to the field Eqs. (15), (21), (22), (24) and (25) of the corresponding paper. In particular, negative boost weight field equations $\omega = 0$, $\psi_i = 0$ are automatically satisfied.

However, such a spacetime is, in general, of the Weyl type II. Thus, in order to obtain a solution of the Weyl type N, conditions on an algebraic type of the Weyl tensor need to be imposed. This can be achieved by employing conditions II(a)–III(b) of [20], where an explicit algebraic classification of the general Kundt line element was carried out. Doing so, the local form of both $\boldsymbol{g}$ and the field equations further reduces.

It is convenient to discuss the coordinate form of the metric first. Then, we briefly discuss the general coordinate form of the electromagnetic field.

As has already been said, the local form of the metric in adapted Kundt coordinates $\{r, u, x^\alpha\}$ with r being an affine parameter of the multiple WAND $\boldsymbol{\ell} \equiv \partial_r$ is given by (9) of [6]. Employing Weyl type N conditions II(a)–III(b) of [20], the local form of $\boldsymbol{g}$ further reduces to

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[2] For $\tilde{\boldsymbol{F}} = \boldsymbol{F}$, equations (3.10) reduce to the Maxwell equations.

$$ds^2 = 2H(r, u, x)du^2 + 2dudr + 2W_\alpha(r, u, x)dx^\alpha du + g_{\alpha\beta}(u, x)dx^\alpha dx^\beta, \quad (4.1)$$

where $g^\perp \equiv g_{\alpha\beta}dx^\alpha dx^\beta$ is a Riemannian metric on a $(d-2)$-dimensional transverse space spanned by $\{x^\alpha\}$, which is of constant sectional curvature K depending on the cosmological constant and the dimension of the spacetime as

$$K = \frac{\tilde{\Lambda}}{d-1} = \frac{2\Lambda}{(d-1)(d-2)} \quad (4.2)$$

and the metric functions H and W_α take the form

$$W_\alpha(r, u, x) = r W_\alpha^{(1)}(u, x) + W_\alpha^{(0)}(u, x), \quad (4.3)$$

$$H(r, u, x) = r^2 H^{(2)}(u, x) + r H^{(1)}(u, x) + H^{(0)}(u, x). \quad (4.4)$$

As a consequence of the fact that $g^\perp$ is a constant curvature metric with K given by (4.2), the field Eqs. (22), (21) of [6] corresponding to the boost weight zero equations $\phi = \tilde{\Lambda}, \phi_{ij} = \tilde{\Lambda}\delta_{ij}$ reduce to

$$2H^{(2)} = \frac{1}{4} W_\alpha^{(1)} W^{(1)\alpha} + K, \quad (4.5)$$

$$W_{(\alpha||\beta)}^{(1)} - \frac{1}{2} W_\alpha^{(1)} W_\beta^{(1)} = 2K g_{\alpha\beta}, \quad (4.6)$$

respectively. Here, || denotes the covariant derivative in the transverse space with the metric $g^\perp$. The contracted Bianchi identity (see (23) of [6]) for the transverse metric $g^\perp$ is automatically satisfied provided Weyl type condition (IId) of [20] holds

$$W_{[\alpha||\beta]}^{(1)} = 0. \quad (4.7)$$

Further, employing condition III(a) of [20], Eq. (24) of [6] corresponding to the boost weight (-1) field equation $\psi_i' = 0$ reads

$$2H_{,\alpha}^{(1)} = W_{\alpha,u}^{(1)} - W_{[\alpha||\beta]}^{(0)} W^{(1)\beta} + \frac{1}{2} g_{\alpha\beta,u} W^{(1)\beta} + \frac{1}{2} W_\beta^{(1)} W^{(0)\beta} W_\alpha^{(1)} + 2K W_\alpha^{(0)}. \quad (4.8)$$

Conditions III(a), III(b) of [20] reduce to

$$H_{,\alpha}^{(2)} - H^{(2)} W_\alpha^{(1)} = 0, \quad (4.9)$$

$$W_{[\alpha||\beta]||\gamma}^{(0)} = \frac{1}{2} \left(W_{\gamma\beta}^{(0)} W_\alpha^{(1)} - W_{\gamma\alpha}^{(0)} W_\beta^{(1)} \right) - K \left(g_{\gamma\beta} W_\alpha^{(0)} - g_{\gamma\alpha} W_\beta^{(0)} \right) - g_{\gamma[\beta,u||\alpha]}, \quad (4.10)$$

respectively, where the auxiliary geometrical quantity $W_{\alpha\beta}^{(0)}$ is defined as

$$W_{\alpha\beta}^{(0)} \equiv W_{(\alpha||\beta)}^{(0)} - \frac{1}{2} g_{\alpha\beta,u}. \quad (4.11)$$

Finally, Eq. (25) of [6] corresponding to the boost weight (-2) field equation $\omega' = \kappa_0 f_{\alpha...\beta} f^{\alpha...\beta}$ simplifies to a slightly simpler form

$$
\triangle_{LB} H^{(0)} + W^{(1)\alpha} H^{(0)}_{,\alpha} + \left(4H^{(2)} + 2(d-3)K\right) H^{(0)} = 2W^{(0)}_\alpha W^{(0)\alpha} H^{(2)}
$$
$$
+ g^{\alpha\beta} W^{(0)}_{\alpha\beta} H^{(1)} + W^0_{[\alpha||\beta]} W^{(0)[\alpha||\beta]} + W^{(0)}_{\alpha,u}{}^{||\alpha} - W^{(0)}_{[\alpha||\beta]} W^{(1)\alpha} W^{(0)\beta} \qquad (4.12)
$$
$$
+ \frac{1}{2} g_{\alpha\beta,u} \left(W^{(1)\alpha} W^{(0)\beta} - \frac{1}{2} g^{\alpha\beta}{}_{,u} \right) - \frac{1}{2} g^{\alpha\beta} g_{\alpha\beta,uu} - \kappa_0 f_{\alpha...\beta} f^{\alpha...\beta},
$$

where $\triangle_{LB}$ denotes the Laplace–Beltrami operator in the transverse space, i.e. in coordinates, it acts on a scalar as $\triangle_{LB} f \equiv g^{\alpha\beta} f_{,\alpha||\beta}$.

The coordinate form (4.1) of the metric is preserved under the coordinate transformations (81) of [21]. In particular, since the spacetime is a *CSI* spacetime, without loss of generality one can assume that components $g_{\alpha\beta}$ are independent of u (see Theorem 4.1 of [11]). Hence, in the case of a *VSI* spacetime ($\Lambda = 0$), the coordinates can be chosen such that $g_{\alpha\beta} = \delta_{\alpha\beta}$ and (4.1) reduces to the standard form of the VSI line element (8) of [16] for which most of the above equations simplify.

Since the metric g is a degenerate Kundt metric and the Maxwell field F is a closed form of type N, F takes the coordinate form (see (13) of [6])

$$
F = \frac{1}{p!} f_{\alpha_1...\alpha_p}(u, x^\alpha) \mathrm{d}u \wedge \mathrm{d}x^{\alpha_1} \wedge \cdots \wedge \mathrm{d}x^{\alpha_p} \qquad (4.13)
$$

in coordinates (r, u, x^α) adopted in the previous section. Here, $f_{\alpha_1...\alpha_p} \equiv F_{u\alpha_1...\alpha_p}$. For F in the form (4.13), the Maxwell Eqs. (1.2) and (1.4) reduce to the effective Maxwell equations

$$
\left(\sqrt{g^\perp} f^{\beta\alpha_2...\alpha_p} \right)_{,\beta} = 0, \quad f_{[\alpha_1...\alpha_p,\beta]} = 0, \qquad (4.14)
$$

for a p-form f in the $(d-2)$-dimensional transverse Riemannian space (see Sect. 2.2 of [6]).

In [6], it was pointed out that, under suitable conditions, solutions (g, F) of the Einstein–Maxwell equations with a *VSI* metric g and a VSI Maxwell field F are *universal* (in the sense of Sect. 3.2 in [6]) and thus also simultaneously solve various Einstein-generalized Maxwell theories. Weyl type N universal solutions considered in [6] are necessarily *VSI pp*-waves, i.e. spacetimes admitting a covariantly constant null vector field (see [3,6] and references therein), for which the field equations simplify considerably.

Let us conclude with a brief summary of Sect. 4. Any Weyl type N spacetime with a metric g corresponding to a solution of the electrovacuum Einstein–Maxwell equations with a null Maxwell field F can be expressed in the coordinate form (4.1) while the null Maxwell form F takes the coordinate form (4.13). The transverse metric $g^\perp$ in the line element (4.1) is a metric on a Riemannian space of constant curvature K given by (4.2) and both g and F have to satisfy (4.5)–(4.10), (4.12), and (4.14).

Appendix A: Curvature invariants in Kundt spacetimes of Weyl and traceless Ricci type N

Let us study scalar curvature invariants in the Kundt subclass of Weyl type N spacetimes with the Ricci tensor of the form

$$R_{ab} = \tilde{\Lambda} g_{ab} + \eta k_a k_b, \tag{A.1}$$

where $\tilde{\Lambda}$ is a constant, η is a scalar and $\boldsymbol{k}$ is a null vector. In general, a Kundt vector in a Ricci type II spacetime is necessarily a multiple WAND (see e.g. Proposition 2 of [18]). Thus, in our case, it must coincide with the common AND $\boldsymbol{\ell}$ of the Weyl and the Ricci tensor. Hence, the spacetime is of Riemann type II (i.e. the Riemann tensor is of type II) with the Kundt AND $\boldsymbol{\ell}$. In addition to this, all boost weight zero null frame components of the Riemann tensor are constant (depending on $\tilde{\Lambda}$).

If $\tilde{\Lambda} = 0$, the Riemann tensor is of type N and it is aligned with the Kundt vector $\boldsymbol{\ell}$. Thus, according to Theorem 1 of [15] on characterization of VSI spacetimes, the corresponding spacetime is VSI.

If $\tilde{\Lambda}$ is non-vanishing, then the spacetime is at least CSI_0, i.e. all curvature invariants constructed solely from the Riemann tensor (without incorporating its covariant derivatives) are constant. Indeed, any full contraction of a tensor $\boldsymbol{T}$ is completely determined by its boost weight zero part $\boldsymbol{T}_{(0)}$. Since in our case, the boost weight zero part $\boldsymbol{R}_{(0)}$ of the Riemann tensor $\boldsymbol{R}$ possesses only constant null frame components, also any full contraction of a tensor given by a series of tensor products of the Riemann tensor with itself or with the metric (which consists only of its boost weight zero part) is necessarily constant. See also Sect. 2.3 of [11].

In order to study higher-order invariants constructed using also covariant derivatives of $\boldsymbol{R}$, we make use of the balanced scalar approach introduced in [14], see also [22]. First, let us define k-balancedness of a tensor.

Definition A.1 We say that a tensor $\boldsymbol{T}$ is k-balanced if there exists a null vector $\boldsymbol{\ell}$ such that $\mathrm{bo}_{\boldsymbol{\ell}}(\boldsymbol{T}) < -k$ and for any of its null frame components η with boost weight $b < -k$, the derivative $D^{-b-k}\eta$ is zero.

Now, let us prove the following result on k-balancedness of covariant derivatives $\nabla^{(n)}\boldsymbol{R}$ of the Riemann tensor.

Proposition A.2 *In a Weyl type N Kundt spacetime with the Ricci tensor of the form (A.1), an arbitrary covariant derivative of the Riemann tensor is 1-balanced and thus it is VSI.*

Proof Let us stress out again that such a spacetime is necessarily aligned [10], and hence without loss of generality one can assume that $\boldsymbol{k} = \boldsymbol{\ell}$. Also, using the notation of the GHP formalism [8], the boost weight (-2) component of the Ricci tensor reads $\omega' = \eta$.

Consider the Ricci decomposition of the Riemann tensor [7],

$$\boldsymbol{R} = \boldsymbol{G} + \boldsymbol{E} + \boldsymbol{C}, \tag{A.2}$$

where $\boldsymbol{G}$ is the so-called scalar part ($G_{abcd} \propto R g_{a[c} g_{d]b}$), $\boldsymbol{E}$ is the semi-traceless part ($E_{abcd} \propto g_{a[c} S_{d]b} - g_{b[c} S_{d]a}$, S_{ab} being the traceless part of the Ricci tensor) and the Weyl tensor $\boldsymbol{C}$ is the fully traceless part of the Riemann tensor.

Since the Ricci scalar R is constant ($R \propto \tilde{\Lambda}$), we have $\nabla \boldsymbol{G} = \boldsymbol{0}$ and hence

$$\nabla \boldsymbol{R} = \nabla \boldsymbol{C} + \nabla \boldsymbol{E}. \tag{A.3}$$

For Weyl type N Kundt spacetimes with the Ricci tensor of the form (A.1), Eqs. (28)–(30) in [23] are still valid together with $D\omega' = 0$ and thus the following generalizations of Lemmas 4.2 and 4.3 in [23] are also valid.[3]

Lemma A.3 *In a Weyl type N Kundt spacetime with the Ricci tensor of the form (A.1), for a 1-balanced scalar η, scalars $L_{11}\eta$, $\tau_i\eta$, $L_{1i}\eta$, $\kappa'_i\eta$, $\rho'_{ij}\eta$, $\overset{i}{M}_{j1}\eta$, $\overset{i}{M}_{kl}\eta$ and $D\eta$, $\delta_i\eta$, $\triangle\eta$ are also 1-balanced scalars.*

Lemma A.4 *In a Weyl type N Kundt spacetime with the Ricci tensor of the form (A.1), a covariant derivative of a 1-balanced tensor is again a 1-balanced tensor.*

Since the Weyl tensor and the traceless part of the Ricci tensor, $\eta \ell_a \ell_b$, as well as $\boldsymbol{E}$ are 1-balanced, their arbitrary derivative is also 1-balanced and so is any covariant derivative of the Riemann tensor, i.e. $\nabla^{(k)} \boldsymbol{R}$ is 1-balanced for any $k \in \mathbb{N}$. $\square$

Thus, although the Riemann tensor is not VSI, its first covariant derivative is. In particular, this means that the only non-trivial scalar curvature invariants are precisely those constructed from the Riemann tensor itself. However, we already know that all these invariants are necessarily constant. Therefore, we immediately obtain the following result.

Corollary A.5 *A Weyl type N Kundt spacetime with the Ricci tensor of the form (A.1) is CSI.*

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[3] As in [23], we consider behaviour of scalars with respect to constant boosts.

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