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RESEARCH ARTICLE

Hawking radiation via tunneling from a d-dimensional black hole in Gauss–Bonnet gravity

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Abstract We extend the Parikh–Wilczek method from Einstein gravity spacetime to Gauss–Bonnet modified gravity and study the tunneling radiation of particles across the event horizon of a d-dimensional Gauss–Bonnet Anti de-Sitter black hole. The emission rate of a particle is calculated. It is shown that the emission rate of massive particles takes the same functional form as that of massless particles although that their motion equations tunneling across the horizon are different. It is also shown that the emission spectrum deviates from the pure thermal spectrum but is consistent with an underlying unitary theory. In addition, significant but interesting phenomenon is demonstrated when Gauss–Bonnet term is present. The expression of the emission rate for a black hole in Gauss–Bonnet gravity differs from that for a black hole in Einstein gravity. After adopting the conventional tunneling rate, we obtain the expression of the entropy of the Gauss–Bonnet black hole, which is in accordance with the early results but does not obey the area law. So the research of tunneling radiation in this paper may serve as a new perspective of understanding the thermodynamics of black holes in Gauss–Bonnet gravity.

Keywords Tunneling radiation · Gauss–Bonnet gravity · d-Dimensional AdS black hole · Self-gravitation

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Since Hawking [1] proved that the black hole could radiate particles and the emission spectrum is exactly thermal, much effort has been devoted to the study of the black hole radiation [2–16]. The earlier work in common proves that Hawking radiation is precisely thermal and the energy spectrum is precisely thermal one [2–6]. Recently, a method to describe Hawking radiation as a tunneling process where a particle moves in dynamic geometry was developed by Kraus and Wilczek [7] and elaborated upon by Parikh and Wilczek [8]. In a coordinate system which behaves well at the event horizon, taking the self-interaction effect into account and according to the energy conservation, they have calculated the corrected emission spectrum of the spherically symmetric black holes, such as Schwarzschild black holes and Reissner–Norström black holes. After this, this method was used to calculate the emission rate of particles from different black holes [9–19]. We also made use of this technique to calculate the emission rate at which a particle tunnel from the black plane [20], black string [21] and black toroidal [22]. The derived results in all the above studies indicate that the factually radiant spectrum is not precisely thermal but is consistent with the underlying unitary theory, and that the tunneling rate is related to the change of Bekenstein-Hawking entropy. It is interesting to probe whether these results can be generalized to black holes in modified gravity or not. For this purpose, we extend in this paper the work of Parikh-Wilczek to a d-dimensional black hole in Gauss–Bonnet gravity and drive the corrected emission spectrum by calculating the rate of the Hawking radiation. Indeed we find some new features when the Gauss–Bonnet term is present.

Gauss–Bonnet gravity is one kind of modified gravity theories which include higher derivative curvature terms in the Lagrangian. These terms are of great interest since they naturally occur in the effective low-energy action of string theory. Furthermore, they can be viewed as the corrections of large N expansion of boundary CFTs according to AdS/CFT correspondence. Gauss–Bonnet gravity gains some fantastic features different from Einstein gravity. The resulting equations of motion have no more than second derivatives of metric and the theory is free of ghosts [23]. Moreover, the Gauss–Bonnet term appears as the leading correction to the effective low-energy action of the heterotic string theory [24,25].

Considering the d-dimensional Einstein–Maxwell theory with a Gauss–Bonnet term and a cosmological constant, the metric of a static black hole solution can be written as

$$ds^2 = -f(r)dt^2 + f(r)^{-1}dr^2 + r^2 h_{ij}dx^i dx^j, \tag{1}$$

where $h_{ij}dx^i dx^j$ represents the line element of a $(d-2)$-dimensional maximal symmetric Einstein space with constant curvature $(d-2)(d-3)k$ and volume $\sum_k$. Without loss of the generality, one may take $k = 1, 0$ and -1, corresponding to the spherical, Ricci flat and hyperbolic topology of the black hole horizon, respectively. The metric function f is given by [23,26–30]

$$f(r) = k + \frac{r^2}{2\tilde{\alpha}}\left(1 - \sqrt{1 + \frac{64\pi\tilde{\alpha}M}{(d-2)\sum_k r^{d-1}} - \frac{2\tilde{\alpha}Q^2}{(d-2)(d-3)r^{2d-4}} + \frac{8\tilde{\alpha}\Lambda}{(d-1)(d-2)}}\right), \tag{2}$$

with $\tilde{\alpha} = (d-3)(d-4)\alpha_{\rm GB}$, where $\alpha_{\rm GB}$ is the Gauss–Bonnet coefficient, M is the black hole mass, Q is related to the charge of the black hole and $\Lambda < 0$ is the cosmological constant.

Solving the equation $f(r) = 0$, we obtain the horizon radii and let $r_{\rm H}$ denote the position of the outer event horizon of the black hole. The mass M can be expressed in terms of the event horizon $r_{\rm H}$ as

$$M = \frac{(d-2)\sum_k r_{\rm H}^{d-3}}{16\pi}\left(k + \frac{k^2\tilde{\alpha}}{r_{\rm H}^2} - \frac{2\Lambda r_{\rm H}^2}{(d-1)(d-2)}\right) + \frac{\sum_k Q^2}{32\pi(d-3)r_{\rm H}^{d-3}}. \quad (3)$$

The Hawking temperature of the black hole can be easily obtained by requiring the absence of conical singularity at the horizon in the Euclidean sector of the black hole solution, which is given by

$$\begin{aligned} T_{\rm H} &= \frac{1}{4\pi} f'(r_{\rm H}) \\ &= \frac{1}{4\pi r_{\rm H}(r_{\rm H}^2 + 2k\tilde{\alpha})}\left((d-3)kr_{\rm H}^2 + (d-5)k^2\tilde{\alpha} - \frac{Q^2}{2(d-2)r_{\rm H}^{2d-8}} - \frac{2\Lambda r_{\rm H}^4}{(d-2)}\right). \end{aligned} \quad (4)$$

To describe tunneling, we make a Painlevé coordinate transformation

$$dT = dt - g(r)dr. \quad (5)$$

Substituting Eq. (5) into Eq. (1) yields

$$ds^2 = -f(r)dT^2 - 2f(r)g(r)dTdr + \left(\frac{1}{f(r)} - f(r)g^2(r)\right)dr^2 + r^2 hijdx^i dx^j. \quad (6)$$

Considering flat Euclidean space in radial, we get

$$\frac{1}{f(r)} - f(r)g^2(r) = 1. \quad (7)$$

So Eq. (6) can be re written as

$$ds^2 = -f(r)dT^2 \pm 2\sqrt{1-f(r)}dTdr + dr^2 + r^2 h_{ij}dx^i dx^j, \quad (8)$$

where $+$ sign denotes the space-time line element of outgoing particles at the event horizon, and—sign denotes the space-time line element of ingoing particles at the cosmological horizon.

Obviously, the line element (8) has many superior features. First, it does not have coordinate singularity and is well-behaved at the horizons. Second, the event horizon and the infinite red-shift surface are coincident with each other. Third, ∂_T is a Killing vector in the global space-time. Fourth, constant-time slices are just flat Euclidean space. Moreover, it is easy to show that the metric in this new coordinate system satisfies Landau's condition of coordinate clock synchronization which is given by [31]

$$\frac{\partial}{\partial x^j}\left(-\frac{g_{0i}}{g_{00}}\right)=\frac{\partial}{\partial x^i}\left(-\frac{g_{0j}}{g_{00}}\right),\quad (i,j=1,2,3). \tag{9}$$

That is, the coordinate clock synchronization in the Painlevé coordinates can be transmitted from one place to another though the line element is not diagonal. In quantum mechanics, it is an instantaneous process that particle tunnels across a barrier. All of these make it convenient to discuss the Hawking radiation as tunneling.

According to Eq. (8), the radial outgoing null geodesic at the event horizon can be represented as

$$\dot{r}=\frac{dr}{dT}=1-\sqrt{1-f(r)}. \tag{10}$$

Equation (10) is the motion equation of a massless particle when it tunnels across the horizon. The world-line of a massive quanta is timelike, so it does not follow radial-lightlike geodesic (10). Similar to Ref. [10,11], we treat the outgoing massive particle as a de Broglie wave and we can easily obtain its motion equation

$$\dot{r}=-\frac{g_{00}}{g_{01}}=\frac{f(r)}{2\sqrt{1-f(r)}}. \tag{11}$$

If the particle self-gravitation, energy conservation and angular momentum conservation are taken into account, when a particle of energy ω is emitted, the black hole's energy will become $M-\omega$, all of the equations which are mentioned above and related with $r_{\mathrm{H}}(M)$ should be used with $M\to M-\omega$. Since the metric is of spherical symmetry, so regarding the outgoing particle as an s-wave, i.e. a shell of energy is reasonable. Assuming that the outgoing wave is traced back toward the horizon, its wave-length, as measured by local fiducial observers, will be blue-shifted. Near the horizon, the radial wave number approaches infinity, so that the Wentzel–Kramers–Brillouin (WKB) approximation is appropriate [8].

The action of the outgoing particle which crosses the horizon outwards from r_{i} to r_{f} could be expressed as.

$$Z=\int_{r_{\mathrm{i}}}^{r_{\mathrm{f}}}P_r dr=\int_{r_{\mathrm{i}}}^{r_{\mathrm{f}}}\int_0^{P_r}dP_r dr, \tag{12}$$

where P_r is canonical momentum conjugate to r, r_{i} and r_{f} represent the locations of the event horizon before and after the particle emission. Taking the Hamilton equation into account, we have

$$\dot{r}=\left.\frac{dH}{dP_r}\right|_r=\frac{dM}{dP_r}. \tag{13}$$

Substituting Eqs. (10),(11) and (13) into Eq. (12), Changing the variable from the momentum to the mass and switching the order of integration, we obtain

$$Z=\int_{M_i}^{M_f}\int_{r_i}^{r_f}\frac{dr}{\dot{r}}dM=\begin{cases}\int_{M_i}^{M_f}\int_{r_i}^{r_f}\frac{1+\sqrt{1-f(r)}}{f(r)}drdM & \text{(massless particle)},\\ \int_{M_i}^{M_f}\int_{r_i}^{r_f}\frac{2\sqrt{1-f(r)}}{f(r)}drdM & \text{(massive particle)},\end{cases} \tag{14}$$

where $M_i=M$; $M_f=M-\omega$.

It is easy to find that the integrand is singular at the point $r=r_H$. The integral can be evaluated by deforming the contour around the pole, so as to ensure that positive energy solution decay in time. Note that all real parts, divergent or not, can be discarded since they only contribute a phase. Doing the r integral, we find

$$Z=-i2\pi\int_{M_i}^{M_f}\frac{1}{f'(r_H)}dM. \tag{15}$$

From Eq. (3), we have

$$dM=\frac{(d-2)\sum_k r_H^{d-6}}{16\pi}\left((d-3)kr_H^2+(d-5)k^2\tilde{\alpha}-\frac{2\Lambda r_H^4}{(d-2)}-\frac{Q^2}{2(d-2)r_H^{2d-8}}\right)dr_H. \tag{16}$$

Substituting Eqs. (4) and (16) into Eq. (15) yields

$$Z=-i\frac{(d-2)\sum_k}{8}\int_{r_i}^{r_f}(r_H^{d-3}+2k\tilde{\alpha}r_H^{d-5})dr_H=-i\frac{\sum_k r_H^{d-2}}{8}\left(1+\frac{2(d-2)k\tilde{\alpha}}{(d-4)r_H^2}\right)\Bigg|_{r_i}^{r_f} \tag{17}$$

Adopting the WKB approximation, the tunneling probability of the particle is related to the imaginary part of the action via $\Gamma\sim\exp(-2ImZ)$. So

$$\Gamma\sim\exp(-2ImZ)=\exp\left\{\frac{\sum_k r_H^{d-2}}{4}\left(1+\frac{2(d-2)k\tilde{\alpha}}{(d-4)r_H^2}\right)\Bigg|_{r_i}^{r_f}\right\}=\exp(\Delta F), \tag{18}$$

where

$$\Delta F=\frac{\sum_k r_f^{d-2}}{4}\left(1+\frac{2(d-2)k\tilde{\alpha}}{(d-4)r_f^2}\right)-\frac{\sum_k r_i^{d-2}}{4}\left(1+\frac{2(d-2)k\tilde{\alpha}}{(d-4)r_i^2}\right). \tag{19}$$

The emission spectrum (18) obviously deviates from the pure thermal spectrum but is consistent with an underlying unitary theory. It should be noted that the emission spectrums of massless particles and massive ones have the same functional forms.

Compared with the conventional tunneling rate $\Gamma\sim e^{\Delta S_{BH}}$ which has been shown in all of the early references about tunneling radiation, where S_{BH} is the black hole Bekenstein–Hawking (BH) entropy and ΔS_{BH} the difference of the BH entropy before and after the particle emission, the entropy at the event horizon of a d-dimensional Gauss–Bonnet Anti-de Sitter black hole may be expressed as

$$S_{\text{BH}}^{\text{GB}} = F(M) = \frac{\sum_k r_{\text{H}}^{d-2}}{4}\left(1 + \frac{2(d-2)k\tilde{\alpha}}{(d-4)r_{\text{H}}^2}\right). \tag{20}$$

The result is in accord with that given by Ref. [28], where the entropy was obtained by integrating the first law and the physical assumption was imposed that the entropy vanishes when the horizon of black holes shrinks to zero. When $k = 1$, the entropy (20) is in complete agreement with the one in [32], where the entropy of the Gauss–Bonnet black holes without the cosmological constant is obtained by calculating the Euclidean action of black holes. But, it is obvious that the entropy $S_{\text{BH}}^{\text{GB}}$ is not proportional to the horizon area $A = \sum_k r_{\text{H}}^{d-2}$ and does not obey the area law $S_{\text{BH}}^{\text{GB}} = A/4$ unless $k = 0$. We think that the peculiarity can be regarded as the effects on black hole thermodynamic quantities due to Gauss–Bonnet modified gravity. In fact, it was pointed by Kanti and Tamvakis [33] that the Gauss–Bonnet term has an effect both on the temperature and the entropy of the Gauss–Bonnet black hole.

We expand the $F(M - \omega)$ in terms of the energy of the emitted particle ω, i.e.

$$F(M-\omega) = F(M) + \sum_{n=1}^{\infty} a_n \omega^n \tag{21}$$

where

$$a_n = \frac{1}{n!}\frac{d^n F(M-\omega)}{d\omega^n}\bigg|_{\omega=0} \qquad n = 1, 2, 3, \ldots, \infty. \tag{22}$$

Then

$$\Delta S_{\text{BH}}^{\text{GB}} = F(M-\omega) - F(M) = -\beta\omega + \sum_{n=2}^{\infty} a_n \omega^n \tag{23}$$

and

$$\Gamma \sim \exp(\Delta S_{\text{BH}}^{\text{GB}}) = \exp\left(-\beta\omega + \sum_{n=2}^{\infty} a_n \omega^n\right), \tag{24}$$

where $\beta = 1/T_{\text{H}}$ is the inverse of the Hawking temperature.

In Eq. (24), the first term gives the familiar thermal Boltzmann factor $\exp(-\beta\omega)$ for the emanating radiation, the others are the corrections resulting from the response of the background geometry to the emission of a quantum, which can easily be calculated to any desired order in $\omega \to 0$ and are indicative of a "greybody" factor in the emission spectrum. The existence of these correction terms means that it is probable that we can obtain other information from the spectrum in addition to the temperature, that is, the corrected spectrum is not purely thermal [34].

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